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CLASSIFICATION OF ALL PARABOLIC SUBGROUP-SCHEMES OF A REDUCTIVE LINEAR ALGEBRAIC GROUP OVER AN ALGEBRAICALLY CLOSED FIELD

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ABSTRACT. Let G be a reductive linear algebraic group over an algebraically closed field K. The classification of all parabolic subgroups of G has been known for many years. In that context subgroups of G have been understood as varieties, i.e. as reduced schemes. Also several nontrivial nonreduced subgroup schemes of G are known, but until now nobody knew how many there are and what there structure is. Here I give a classification of all parabolic subgroup schemes of G in $\mathrm{char}(K) > 3$.

INTRODUCTION

In the special case $G = \mathrm{Sl}_2$, the 2×2 matrices $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ and determinant 1, it can easily be verified that

$$P_n = \mathrm{Spec}\, \frac{K[x, y, z, w]}{(z^{p^n}, xw - yz - 1)}$$

is a parabolic subgroup scheme of Sl_2 for each $n \in \mathbb{N}$, if $\mathrm{char}(K) = p > 0$. Furthermore P_n is not reduced whenever $n \neq 0$.

In the general case of an arbitrary G the question for all parabolic subgroup schemes of G, and their structure, has been asked, but until now nobody has given an answer to this question. Virtually nothing was known so far.

In $\mathrm{char}(K) = 0$ all parabolic subgroup schemes are known to be reduced, so there is nothing new. In $\mathrm{char}(K) = 2, 3$ the problem is more complicated due to the vanishing of certain coefficients. Henceforth K will denote a fixed algebraically closed field of characteristic $p > 0$, and G will denote a linear connected, reductive linear algebraic group over K, T a maximal torus of G, and B a Borel subgroup of G containing T. Now let ϕ denote the corresponding set of roots, and Δ the set of simple roots. For any K-algebra S, and for any subgroup scheme H of G, $H(S)$ will always mean the S-points of H, and H_{red} will denote the reduced part of H, i.e. $K[H_{\mathrm{red}}] = K[H]/\,\text{nilradical}$. H is said to be reduced, if $H = H_{\mathrm{red}}$.

1. **Definition.** Let P be a subgroup scheme of G. P is said to be a parabolic subgroup scheme of G, if it contains a Borel subgroup.

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All Borel subgroups are known to be conjugate, so it suffices to classify all subgroup schemes containing B. I will analyze the structure of a supposedly given parabolic subgroup scheme P containing B.

Let G_a denote the 1-dimensional additive linear algebraic group $\mathrm{Spec}(K[T])$. For each $n \in \mathbb{N}_0$, let α_{p^n} be the subscheme of G_a defined by T^{p^n}; they are known to be the only closed connected subgroup schemes of G_a different from G_a. I set $\alpha_{p^\infty} = G_a$. By abuse of notation, we sometimes write α_n for the local group scheme α_{p^n}. Let U denote the unipotent part of B, and let $\{\beta_1, \beta_2, \dots, \beta_m\} = \phi^+$ be the set of positive roots. Then it is known that there exist morphisms of algebraic groups $x_{\beta_i}\colon G_a \to U$, $i \in \langle 1, m\rangle$, such that

$$\begin{gathered} G_a^m \to U \\ (\xi_1, \dots, \xi_m) \to \prod x_{\beta_i}(\xi_i) \end{gathered}$$

is an isomorphism of varieties.

Let w_0 denote the element of maximal length in the Weyl group W. There is an equivalent statement for $U^- = w_0 U w_0^{-1}$, where we use $x_{-\beta_i}$'s instead. It is usual to write U_β for $x_\beta(G_a)$, $\beta \in \phi$.

I make the following notation for a parabolic subgroup scheme P of G: Let $R_u^-(P_{\mathrm{red}})$ denote the opposite of $R_u(P_{\mathrm{red}})$ (replacing U_β by $U_{-\beta}$), and $U_P^- = P \cap R_u^-(P_{\mathrm{red}})$.

2. **Lemma.** *Let P be a (not necessarily reduced) parabolic subgroup scheme of G. Then $U^- \cdot P_{\mathrm{red}} = R_u^-(P_{\mathrm{red}}) \cdot P_{\mathrm{red}} \cong R_u^-(P_{\mathrm{red}}) \times P_{\mathrm{red}}$ as varieties, and $R_u^-(P_{\mathrm{red}}) \cap P_{\mathrm{red}} = \{e\}$. This follows from* [Sp, 10.3.1 *and* 10.3.2].

3. **Lemma.** *Let P be a (not necessarily reduced) parabolic subgroup scheme of G. Then P is a closed subscheme of $U^- \cdot P_{\mathrm{red}}$.*

Proof. We have $U^- \cdot P_{\mathrm{red}} \supset U^- \cdot B = w_0 U w_0^{-1} \cdot B$, the big cell in G, which is open and dense in G. Hence $U^- \cdot P_{\mathrm{red}}$ is also dense in G. Furthermore $U^- \cdot P_{\mathrm{red}} = \bigcup_{g \in P_{\mathrm{red}}} U^- B g$, hence $U^- \cdot P_{\mathrm{red}}$ is also open in G. Moreover $U^- \cdot P_{\mathrm{red}} = R_u^-(P_{\mathrm{red}}) \cdot P_{\mathrm{red}}$, and $R_u^-(P_{\mathrm{red}}) \cap P_{\mathrm{red}} = \{e\}$ by Lemma 2. Hence $U^- \cdot P_{\mathrm{red}} \cong R_u^-(P_{\mathrm{red}}) \times P_{\mathrm{red}}$ as varieties, and thus $U^- \cdot P_{\mathrm{red}}$ is an affine, irreducible variety.

Let $A = K[G]$, the coordinate-ring of G. The complement of $U^- \cdot P_{\mathrm{red}}$ in G is a finite union of divisors, which are principal in the simply-connected cover of G; hence there is some $f \in A$ so that $K[U^- \cdot P_{\mathrm{red}}] = A_f$, see [P, Introduction]. We have the following commutative diagrams:

$$\begin{array}{ccc} U^- \cdot P_{\mathrm{red}} & \xrightarrow{\text{open}} & G \\ {\scriptstyle\text{closed}}\uparrow & & \uparrow{\scriptstyle\text{closed}} \\ P_{\mathrm{red}} & \xrightarrow{\text{closed}} & P \end{array} \qquad \begin{array}{ccc} A_f & \longleftarrow & A \\ \downarrow & & \downarrow \\ K[P_{\mathrm{red}}] & \longleftarrow & K[P] \end{array}$$

It follows that the class of f is a unit in $K[P_{\mathrm{red}}]$; i.e. there is a $g \in A$ so that the class of $f \cdot g$ in $K[P_{\mathrm{red}}]$ is 1. Now $K[P_{\mathrm{red}}] = K[P]/\,\mathrm{nilradical}$, hence $f \cdot g = 1 + u$ in $K[P]$, where $u \in \mathrm{nilrad}(K[P])$. But $1 + u$ is a unit in $K[P]$, and so the class of f is a unit in $K[P]$. Hence there is a commutative diagram:

$$\begin{tikzcd} A_f \arrow[d] \arrow[dr] & A \arrow[l] \arrow[d] \\ K[P_{\mathrm{red}}] & K[P] \arrow[l] \end{tikzcd}$$

This means that P is a closed subscheme of the variety $U^- \cdot P$. $\square$

4. **Proposition.** *Let P be a (not necessarily reduced) parabolic subgroup scheme of G. Then $P = U_P^- \cdot P_{\mathrm{red}}$ and $U_P^- \cap P_{\mathrm{red}} = \{e\}$, as scheme-theoretic intersection.*

Proof. By Lemmas 2 and 3 we have $P \subset U^- \cdot P_{\mathrm{red}} = R_u^-(P_{\mathrm{red}}) \cdot P_{\mathrm{red}}$. Let S be any K-algebra. Let $g \in P(S)$. Then there is an element u in $(R_u^-(P_{\mathrm{red}}))(S)$, and an element h in $P_{\mathrm{red}}(S)$ such that $g = u \cdot h$. Then $u = g \cdot h^{-1} \in P(S) \cdot P_{\mathrm{red}}(S) = P(S)$. So $u \in (R_u^-(P_{\mathrm{red}})(S)) \cap P(S) = (P \cap R_u^-(P_{\mathrm{red}}))(S) = U_P^-(S)$, and we have $P(S) \subset U_P^-(S) \cdot P_{\mathrm{red}}(S)$. By definition, $U_P^-(S)$ and $P_{\mathrm{red}}(S)$ are both contained in $P(S)$, and so we also have the other inclusion $P(S) \supset U_P^-(S) \cdot P_{\mathrm{red}}(S)$, and thus the equality $P(S) = U_P^-(S) \cdot P_{\mathrm{red}}(S) = (U_P^- \times P_{\mathrm{red}})(S) = (U_P^- \cdot P_{\mathrm{red}})(S)$ for any K-algebra S. Hence $P = U^- \cdot P_{\mathrm{red}}$. $U_P^- \cap P_{\mathrm{red}} = \{e\}$ follows from the last equality in Lemma 2: $R_u^-(P_{\mathrm{red}}) \cap P_{\mathrm{red}} = \{e\}$, and from the definition of U_P^- as $U_P^- = P \cap R_u^-(P_{\mathrm{red}})$. $\square$

Thus P is the product of two closed subgroup schemes, with trivial intersection. Notice that $\dim(P_{\mathrm{red}}) = \dim(P)$, hence $\dim(U_P^-) = 0$. Furthermore U_P^- is connected, since P is connected. Thus U_P^- is a local unipotent closed subgroup scheme of G.

5. **Lemma.** *Let α and β be two linearly independent roots in ϕ^+. Then there is some $t \in T$ with $\alpha(t) = -1$ and $\beta(t) \neq -1$.*

Proof. Because $W\Delta = \phi$ we may assume that α is simple. Write $\beta = \sum_{\gamma \in \Delta} n_\gamma \gamma$. There is at least one $\delta \in \Delta \backslash \{\alpha\}$ with $n_\delta \neq 0$.

The simple roots are linearly independent, thus we can choose $t \in T$ with $\alpha(t) = -1$, $\delta(t)^{n_s} \neq \pm 1$, $\gamma(t) = 1$ if $\gamma \neq \alpha, \delta$. Then t is as required. $\square$

6. *Remark.* Lemma 5 is also true if we take two distinct roots in ϕ^- instead of ϕ^+.

We may choose the $\beta_1, \dots, \beta_m \in \phi^+$ such that $\{\beta_1, \dots, \beta_l\} = \Delta$, the set of simple roots, and such that $\mathrm{ht}(\beta_1) \leq \mathrm{ht}(\beta_2) \leq \cdots \leq \mathrm{ht}(\beta_m)$, where $\mathrm{ht}(\beta)$ is the height of $\beta \in \phi^+$: $\mathrm{ht}(\beta) = \sum_{i=1}^{1} c_i$, where $\beta = \sum_{i=1}^{1} c_i \cdot \beta_i$, $c_i \geq 0$. We write $x_1(a_1) \cdot \dots \cdot x_m(a_m)$ for an element in $U^-(A)$, A any K-algebra, $a_i \in A$, $x_i \colon G_a \to G$ morphisms of algebraic groups, and $x_i = x_{-\alpha_i}$.

For further reference I give the following formula for any two roots α, β, with $\alpha + \beta \neq 0$, and for any $a, b \in A$ (for a proof see [SL], or [Sp, 10.1.4]:

$$(*) \qquad (x_\alpha(a), x_\beta(b)) = \prod_{\substack{i,j>1 \\ i\alpha + j\beta \in \phi}} x_{i\alpha + j\beta}(c_{ij} \cdot a^i \cdot b^j).$$

7. **Proposition.** *If $x = x_i(a_i) \cdot x_{i+1}(a_{i+1}) \cdot \dots \cdot x_m(a_m) \in U_P^-(A)$, $i \in \langle 1, m \rangle$, then $x_i(a_i) \in U_P^-(A)$.*

Proof. It suffices to prove the following: if $a_1, \dots, a_m \in A$ are such that $x = x_i(a_i)\cdot x_j(a_j)\cdot x_{j+1}(a_{j+1})\cdot\dots\cdot x_m(a_m) \in U_P^-(A)$, where $1 \le i < j \le m$, then there exist $a'_{j+1}, \dots, a'_m \in A$ such that $x' = x_i(a_i)\cdot x_{j+1}(x'_{j+1})\cdot\dots\cdot x_m(a'_m) \in U_P^-(A)$. In fact repeated application for $j = i+1, \dots, m$ will then prove the lemma.

Let x be as above and choose $t \in T$ such that $\beta_i(t) \neq -1$, and $\beta_j(t) = -1$, and put $x'' = txt^{-1}x$. Since $T(A)$ acts on $U_P^-(A)$ by conjugation, we have $x'' \in U_P^-(A)$. Recalling that $T(A)$ acts on $U_\beta(A)$ by $t\cdot x_\beta(a)\cdot t^{-1} = x_\beta(\beta(t)\cdot a)$ we deduce from $(*)$ that there exist $a''_{j+1}, \dots, a''_m \in A$ with $x'' = ((1+\beta_i(t))\cdot a_i)\cdot x_{j+1}(a''_{j+1})\cdot\dots\cdot x_m(a''_m) \in U_P^-(A)$. Since $1+\beta_i(t) \neq 0$ we can choose $t' \in T$ with $\beta_i(t') = 1 + \beta_i(t)$. Then $x' = (t')^{-1}\cdot x''\cdot(t')$ is as required. □

8. **Proposition.** *If* $x = x_1(a_1)\cdot x_2(a_2)\cdot\dots\cdot x_m(a_m) \in U_P^-(A)$, *then* $x_i(a_i) \in U_P^-(A)$ *for all* $i \in \langle 1, m\rangle$.

Proof. By Proposition 7, we have $x_1(a_1) \in U_P^-(A)$, and hence $x_1(-a_1) \in U_P^-(A)$, and $x_2(a_2)\cdot\dots\cdot x_m(a_m) = x_1(-a_1)\cdot x_1(a_1)\cdot x_2(a_2)\cdot\dots\cdot x_m(a_m) \in U_P^-(A)$. Repeating this argument successively for $i = 2, 3, \dots, m$, we obtain $x_i(a_i) \in U_P^-(A)$ for all $i \in \langle 1, m\rangle$. □

9. **Notation.** Let $\widetilde{\Delta}$ be the set of maps from Δ to $\mathbb{N}_0 \cup \{\infty\}$, and let $\tilde{\phi}^+$ be the set of maps from ϕ^+ to $\mathbb{N}_0 \cup \{\infty\}$.

Let $\phi^+ = \{\beta_1, \dots, \beta_m\}$ be the set of positive roots, and $\Delta = \{\beta_1, \dots, \beta_l\}$ the set of simple roots. I make the following definition for $i \in \langle 1, m\rangle$:

$$E(\beta_i) = \left\{\beta_j \in \Delta \mid c_j \neq 0 \text{ in the expression } \beta_i = \sum_{s=1}^{l} c_s\cdot\beta_s \text{ with } c_s \in \mathbb{N}_0\right\},$$

i.e. $E(\beta_i)$ is the set of simple roots occurring with nonnegative coefficients in the expression of β_i in terms of simple roots. We also define $E(-\beta_i) = E(\beta_i)$.

Recall also that we write $x_1(a_1)\cdot\dots\cdot x_m(a_m)$ for an element in $U^-(A)$, A any K-algebra, $a_i \in A$, $x_i\colon G_a \to G$ morphisms of algebraic groups, and $x_i = x_{-\beta_i}$. Now given a parabolic subgroup scheme P of G containing B, we define $\varphi \in \tilde{\phi}^+$ by $U_{-\beta} \cap P = x_{-\beta}(\alpha_{\varphi(\beta)})$ (α_n being the local group scheme α_{p^n} as defined above).

10. **Theorem.** *Let* P *and* φ *be as above. Then*

(i) $U_P^- \cong \prod x_i(\alpha_{\varphi(\beta_i)})$, *where the product is taken over all* $\beta_i \in \phi^+$ *with* $\varphi(\beta_i) \neq \infty$ (*the isomorphism being an isomorphism of schemes*);

(ii) *If* $\beta \in \phi^+$, *then* $\varphi(\beta) = \infty$ *if and only if* $U_{-\beta} \subseteq P_{\text{red}}$;

(iii) *If* $\beta \in \phi^+$, *then* $\varphi(\beta) = \min\{\varphi(\gamma) \mid \gamma \in E(\beta)\}$, *provided that* $p = \operatorname{char} K > 3$, *or that* G *is simply laced.*

Proof. From Proposition 8 we get that

$$U_P^-(A) \cong \prod_{i=1}^{m} x_i(A) \cap U_P^-(A)$$

for any K-algebra A, hence

$$U_P^- \cong \prod_{i=1}^{m} x_i(G_a) \cap U_P^- \cong \prod U_{-\beta_i} \cap P \cong \prod x_i(\alpha_{\varphi(\beta_i)})$$

where the last two products are taken over all $\beta_i \in \phi^+$ with $\varphi(\beta) \neq \infty$. This proves (i). And (ii) follows from our definition of φ. Now we prove (iii). If $\beta \in \Delta$, then $E(\beta) = \{\beta\}$ and (iii) is trivially true. Now suppose $\beta \in \phi^+\backslash\Delta$. If $\varphi(\beta) = \infty$, then $U_{-\beta} \subseteq P_{\text{red}}$ and so are all $U_{-\gamma}$ with $\gamma \in E(\beta)$. Hence $\varphi(\gamma) = \infty$ for all $\gamma \in E(\beta)$ and (iii) follows. Now suppose $\varphi(\beta) < \infty$. There is $\gamma_0 \in \Delta$ such that $\delta = \beta - \gamma_0 \in \phi^+$. Assume $x_{-\beta}(a) \in U_P^-(A)$. We have $(x_\delta(1), x_{-\beta}(a)) = \prod x_{i\delta - j\beta}(c_{ij}a^j) \in U_P^-(A)$. By Proposition 8 one concludes that $x_{\delta-\beta}(c_{11}a) \in U_P^-(A)$. Recall that in general for any $\alpha, \beta \in \phi^+$, $\exists r, s \in \mathbb{N}_0$ such that $\beta - r\alpha, \ldots, \beta, \ldots, \beta + q\alpha$ is the α-string through β. We define $N_{\alpha,\beta} = r+1$. It is known that $0 \leq r \leq 3$, hence $1 \leq N_{\alpha,\beta} \leq 4$. It is also known that $c_{ij} = N_{\alpha,\beta}$ (see [SL, p. 22]). In our case $c_{11} = N_{\delta,-\beta}$. If $p > 3$ or G is simply laced, then c_{11} is a nonzero integer. Thus $x_{-\gamma_0}(a) = x_{\delta-\beta}(a) \in U_P^-(A)$. So $x_{-\gamma_0}(a) \in U_P^-(A)$ whenever $x_{-\beta}(a) \in U_P^-(A)$, i.e. $a^{p^{\varphi(\gamma_0)}} = 0$ whenever $a^{p^{\varphi(\beta)}} = 0$ for any K-algebra A. This implies $\varphi(\beta) \leq \varphi(\gamma_0)$. Similarly $(x_{\gamma_0}(1), x_{-\beta}(a)) \in P(A)$, whence $x_{-\delta}(a) \in U_P^-(A)$, and $\varphi(\beta) \leq \varphi(\delta)$. By induction on the height we may assume that $\varphi(\delta)$ is given by (iii), and one concludes that $\varphi(\beta) \leq \min\{\varphi(\gamma) \mid \gamma \in E(\beta)\}$.

It remains to prove the reverse inequality. This can be done by induction on the height of β. The statement is trivially true for $\operatorname{ht}(\beta) = 1$. Assume $\operatorname{ht}(\beta) > 1$. There is $\gamma_0 \in \Delta$ such that $\beta - \gamma_0 \in \phi^+$. Let $\delta = \beta - \gamma_0$. Then $\operatorname{ht}(\delta) < \operatorname{ht}(\beta)$. Let $x_{-\delta}(a)$ and $x_{-\gamma_0}(b) \in U_P^-(A)$. Then

$$(x_{-\delta}(a), x_{-\gamma_0}(b)) = \prod x_{-i\delta - j\gamma_0}(c_{ij}a^i b^j) \in U_P^-(A).$$

By Proposition 8 we have $x_{-\beta}(c_{11}ab) = x_{-\delta-\gamma_0}(c_{11}ab) \in U_P^-(A)$. Hence $x_{-\beta}(ab) \in U_P^-(A)$, i.e. $(ab)^{p^{\varphi(\beta)}} = 0$ for all $a, b \in A$ with $a^{p^{\varphi(\delta)}} = b^{p^{\varphi(\gamma_0)}} = 0$, and for any K-algebra A. So $\varphi(\beta) \geq \min\{\varphi(\delta), \varphi(\gamma_0)\} = \min\{\varphi(\gamma) \mid \gamma \in E(\beta)\}$, and the theorem is proven. □

11. **Corollary.** *Let $\pi\colon G \to G'$ be a surjective morphism of connected reductive K-groups with central kernel. Then π induces a bijection of the set of parabolic subgroup schemes of G onto the same set for G'.*

Proof. This follows from the fact that π is an isomorphism on U^-, the decomposition $P = U_P^- \cdot P_{\text{red}}$, and that the statement holds for reduced parabolic subgroup schemes. □

12. **Corollary.** *If P_φ and P_ψ exist, then so does $P_{\inf(\varphi,\psi)}$.*

Proof. The intersection of P_φ and P_ψ is a parabolic subgroup scheme of G containing B, and $P_\varphi \cap P_\psi = P_{\inf(\varphi,\psi)}$. □

Let F be the Frobenius morphism on G, and denote the local subgroup scheme $(F^n)^{-1}(e)$ of G by G_n for each $n \in \mathbb{N}_0$. Let $\beta \in \Delta$ and denote by P_β the maximal reduced parabolic subgroup scheme of G containing B and not containing $U_{-\beta}$. Then $P_{n,\beta} = G_n \cdot P_\beta$ is a parabolic subgroup scheme of G containing B and equals P_φ, where $\varphi(\beta) = n$ and $\varphi(\gamma) = \infty$ for $\gamma \in \Delta\backslash\{\beta\}$. Thus we obtain

13. **Theorem.** *For each $\varphi \in \widetilde{\Delta}$, there exists the parabolic subgroup scheme P_φ.*

Proof. The intersection of all $P_{\varphi(\beta),\beta}$, $\beta \in \Delta$, is a parabolic subgroup scheme and by Corollary 12 it equals P_φ. $\square$

Now I can state the main theorem, giving the desired classification:

14. **Theorem.** *Let K be an algebraically closed field of characteristic $p > 0$. Let G be a reductive linear algebraic group defined over K. There is an injective map from $\widetilde{\Delta}$ to $\mathfrak{P}$, the set of all parabolic subgroup schemes containing B, given by*

$$\widetilde{\Delta} \to \mathfrak{P}$$
$$\varphi \to P_\varphi,$$

where $P_\varphi = U_\varphi \cdot P_{I(\varphi)}$, $I(\varphi) = \{\alpha \in \Delta \mid \varphi(\alpha) = \infty\}$, $U_\varphi = \prod_{\beta \in \phi^+ - \phi_I} x_{-\beta}(\alpha_{\varphi(\beta)})$, φ being extended to all of ϕ^+ by $\varphi(\beta) = \min\{\varphi(\gamma) \mid \gamma \in E(\beta)\}$, $E(\beta) = \{\beta_i \in \Delta \mid \beta = \sum c_j \cdot \beta_j$, with all $c_j \geq 0$ and $c_i \neq 0\}$, ϕ_I the roots generated by $I = I(\varphi)$.

If $\operatorname{char} K > 3$, *or if G is simply laced, then this map is also surjective.* $\square$

15. *Remark.* It is known to the author that the map in Theorem 14 is not surjective in $\operatorname{char}(K) = 2, 3$ for certain G; for example for $G = SO_5$ in $\operatorname{char} K = 2$, and for G with root system of type G_2 in $\operatorname{char} K = 3$.

Now we can also derive a theorem about the algebra of distributions $\operatorname{Dist}(G)$ on G. For detailed information see [H]. $\operatorname{Dist}(G) = \bigoplus K \cdot X_{c^-} \cdot \binom{H}{h} \cdot X_c$ as a K-vector space, where

$$X_{c^-} = X_{-m}^{[c_{-m}]} \cdot \dots \cdot X_1^{[c_{-1}]}, \qquad c_{-i} \in \mathbb{N}_0 \text{ for all } i \in \langle 1, m \rangle, \ X_i^{[c_i]} = (X_i^{c_i})/(i!),$$
$$X_c = X_1^{[c_1]} \cdot \dots \cdot X_m^{[c_m]}, \qquad c_i \in \mathbb{N}_0 \text{ for all } i \in \langle 1, m \rangle,$$
$$\binom{H}{h} = \binom{H_1}{h_1} \cdot \dots \cdot \binom{H_1}{h_1}, \qquad H_i = H_{\beta_i}, \ h_i \in \mathbb{N}_0 \text{ for all } i \in \langle 1, l \rangle,$$

and where the sum is taken over all possible c^-, c, h.

Let $A = K[G]$. Suppose D is a subalgebra and subcoalgebra of $\operatorname{Dist}(G)$ of the following type:

$$D = \sum K \cdot X_{-m}^{[c_{-m}]} \cdot \dots \cdot X_{-1}^{[c_{-1}]} \cdot \binom{H_1}{h_1} \cdot \dots \cdot \binom{H_1}{h_1} \cdot X_1^{[c_1]} \cdot \dots \cdot X_m^{[c_m]},$$

where the sum is taken over all terms with $c_j < c_{j0}$, $h_i < h_{i0}$, for some fixed c_{j0}, h_{i0}, $j \in \langle -m, m \rangle$, $i \in \langle 1, l \rangle$.

Then $D \subset \operatorname{Dist}(G)$, and $D \cap \operatorname{Dist}_n(G) \subset \operatorname{Dist}_n(G)$. So we obtain natural surjections for the linear dual:

$$\operatorname{Dist}_n(G)^* \to (D \cap \operatorname{Dist}_n(G))^*, \quad \varprojlim \operatorname{Dist}_n(G)^* \to \varprojlim (D \cap \operatorname{Dist}_n(G))^*.$$

We have

$$K[U_{,}^- \cdot B] = A_f = K[x_{-m}, \dots, x_{-1}, h_1, h_1^{-1}, \dots, h_l, h_l^{-1}, y_1, \dots, y_m],$$

for some $f \in A$, and

$$\varprojlim \operatorname{Dist}_n(G)^* = \widehat{A} = K[[x_{-m}, \dots, x_{-1}, z_1, \dots, z_l, x_1, \dots, x_m]]$$

(see [H, 1.2]), where $z_i = h_i - 1$ for all $i \in \langle 1, l \rangle$. Furthermore $\widehat{A} = (\widehat{A_f})$. Let $C = \varprojlim (D \cap \operatorname{Dist}_n(G))^*$. The surjection $\widehat{A} \to C$ is a morphism of K-algebras and coalgebras. Let $\widetilde{I}$ be its kernel. From our description of D and from [H2, 1.2], it follows that $\widetilde{I}$ is generated over $\widehat{A}$ by the $x_j^{c_{j0}}$, z^{i_0}, $j \in \langle -m, m\rangle$, $i \in \langle 1, l\rangle$.

Define $I' = A_f \cap \widetilde{I}$. Then I' is an ideal of A_f, and it is generated over A_f by the $x_j^{c_{j0}}$, z^{i_0}, $j \in \langle -m, m\rangle$, $i \in \langle 1, l\rangle$. It is obvious that $\widetilde{I} = I' \cdot \widehat{A}$.

Define $I = A \cap I'$. Then the elements $x_j^{c_{j0}}$, z^{i_0}, $j \in \langle -m, m\rangle$, $i \in \langle 1, l\rangle$, multiplied by a sufficient power of f are contained in A. Thus $I' = I \cdot A_f$. Hence

$$I \cdot \widehat{A} = I \cdot A_f \cdot \widehat{A} = I' \cdot \widehat{A} = \widetilde{I}.$$

16. **Proposition.** *Let $D \subseteq \mathrm{Dist}(G)$ and $I \subseteq A$ be as above, then I defines a closed subgroup scheme of G whose algebra of distributions is D.*

Proof. Let $\mu\colon A \to A \otimes A$ be the comultiplication on the coordinating A of G, let $\sigma\colon A \to A$ be the coinverse, and let $\varepsilon\colon A \to K$ be the coidentity. Let $\hat{\mu}$, $\hat{\sigma}$, $\hat{\varepsilon}$ be the extensions of μ, σ, ε respectively on the formal group scheme $\widehat{A}$. Then we have

$$\begin{array}{ccc}
\widetilde{I} & \longrightarrow & \widetilde{I}\mathbin{\widehat{\otimes}}\widehat{A}+\widehat{A}\mathbin{\widehat{\otimes}}\widetilde{I} \\
\downarrow & & \downarrow \\
\widehat{A} & \xrightarrow{\hat{\mu}} & \widehat{A}\mathbin{\widehat{\otimes}}\widehat{A} \\
\downarrow & & \downarrow \\
C & \longrightarrow & C\mathbin{\widehat{\otimes}}C
\end{array}$$

So

(a) $$\mu(I) \subset \hat{\mu}(\widetilde{I}) \cap A \otimes A \subset (\widetilde{I}\widehat{\otimes}\widehat{A} + \widehat{A}\widehat{\otimes}\widetilde{I}) \cap A \otimes A = I \otimes A + A \otimes I.$$

(b) $$\hat{\sigma}(\widetilde{I}) = \widetilde{I} \text{ and so we get}$$
$$\sigma(I) = \sigma(\widetilde{I} \cap A) \subset \hat{\sigma}(\widetilde{I}) \cap \sigma(A) = \widetilde{I} \cap A = I.$$

(c) $$\hat{\varepsilon}(\widetilde{I}) = 0 \text{ and so we get}$$
$$\varepsilon(I) = \varepsilon(\widetilde{I} \cap A) \subset \hat{\varepsilon}(\widetilde{I}) \cap \varepsilon(A) = 0.$$

Now (a)–(c) show exactly that μ, σ, ε as defined on A induce the corresponding structure on A/I, i.e. $\mathrm{Spec}(A/I)$ is a subgroup scheme of G. Now $\mathrm{Dist}(\mathrm{Spec}(A/I)) = D$ is obvious. □

Let $\varphi \in \widetilde{\Delta}$. Then φ can be extended to ϕ^+ by defining

$$\varphi(\beta) = \min\{\varphi(\alpha) \mid \alpha \in E(\beta)\}.$$

Now we can introduce the notation $c^- < p^\varphi$ to stand for $c_{-1} < p^{\varphi(\alpha_1)}, \dots, c_{-m} < p^{\varphi(\alpha_m)}$.

17. **Theorem.** *Let G be as above. For each $\varphi \in \widetilde{\Delta}$, let*

$$D_\varphi = \bigoplus_{b^- < p^\varphi} K \cdot X_{b^-} \cdot \binom{H}{h} \cdot X_b.$$

Then D_φ *is a subalgebra and a subcoalgebra of* $\mathrm{Dist}(G)$. *Furthermore, if* $\mathrm{char}\, K > 3$ *or if* G *is simply laced, then these are all subalgebras and subcoalgebras of* $\mathrm{Dist}(G)$ *containing* $\mathrm{Dist}(B)$.

Proof. We have

$$\mathrm{Dist}(P_\varphi) = \mathrm{Dist}(U^-_{P_\varphi} \cdot P_{\varphi\,\mathrm{red}}) = \mathrm{Dist}(U^-_{P_\varphi}) \otimes \mathrm{Dist}(P_{\varphi\,\mathrm{red}}) = D_\varphi\,,$$

which proves the first part. For the second part we apply the proposition above. □

18. *Remark*. Theorem 13 establishes the existence of the P_φ using the Frobenius morphism and the observation of Corollary 12. From these P_φ one obtains the algebra of distributions D_φ. This can also be done the other way around: One can prove directly that the D_φ are indeed subalgebras and subcoalgebras of $\mathrm{Dist}(G)$ (but the proof is long, complicated and involves several induction arguments, so I have not included it here), and then easily derive the P_φ by Proposition 16.

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