

## CLASSIFICATION OF ALL PARABOLIC SUBGROUP-SCHEMES OF A REDUCTIVE LINEAR ALGEBRAIC GROUP OVER AN ALGEBRAICALLY CLOSED FIELD

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**ABSTRACT.** Let  $G$  be a reductive linear algebraic group over an algebraically closed field  $K$ . The classification of all parabolic subgroups of  $G$  has been known for many years. In that context subgroups of  $G$  have been understood as varieties, i.e. as reduced schemes. Also several nontrivial nonreduced subgroup schemes of  $G$  are known, but until now nobody knew how many there are and what their structure is. Here I give a classification of all parabolic subgroup schemes of  $G$  in  $\text{char}(K) > 3$ .

### INTRODUCTION

In the special case  $G = \text{Sl}_2$ , the  $2 \times 2$  matrices  $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$  and determinant 1, it can easily be verified that

$$P_n = \text{Spec} \frac{K[x, y, z, w]}{(z^{p^n}, xw - yz - 1)}$$

is a parabolic subgroup scheme of  $\text{Sl}_2$  for each  $n \in \mathbb{N}$ , if  $\text{char}(K) = p > 0$ . Furthermore  $P_n$  is not reduced whenever  $n \neq 0$ .

In the general case of an arbitrary  $G$  the question for all parabolic subgroup schemes of  $G$ , and their structure, has been asked, but until now nobody has given an answer to this question. Virtually nothing was known so far.

In  $\text{char}(K) = 0$  all parabolic subgroup schemes are known to be reduced, so there is nothing new. In  $\text{char}(K) = 2, 3$  the problem is more complicated due to the vanishing of certain coefficients. Henceforth  $K$  will denote a fixed algebraically closed field of characteristic  $p > 0$ , and  $G$  will denote a linear connected, reductive linear algebraic group over  $K$ ,  $T$  a maximal torus of  $G$ , and  $B$  a Borel subgroup of  $G$  containing  $T$ . Now let  $\phi$  denote the corresponding set of roots, and  $\Delta$  the set of simple roots. For any  $K$ -algebra  $S$ , and for any subgroup scheme  $H$  of  $G$ ,  $H(S)$  will always mean the  $S$ -points of  $H$ , and  $H_{\text{red}}$  will denote the reduced part of  $H$ , i.e.  $K[H_{\text{red}}] = K[H]/\text{nilradical}$ .  $H$  is said to be reduced, if  $H = H_{\text{red}}$ .

**1. Definition.** Let  $P$  be a subgroup scheme of  $G$ .  $P$  is said to be a parabolic subgroup scheme of  $G$ , if it contains a Borel subgroup.

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All Borel subgroups are known to be conjugate, so it suffices to classify all subgroup schemes containing  $B$ . I will analyze the structure of a supposedly given parabolic subgroup scheme  $P$  containing  $B$ .

Let  $G_a$  denote the 1-dimensional additive linear algebraic group  $\text{Spec}(K[T])$ . For each  $n \in \mathbb{N}_0$ , let  $\alpha_{p^n}$  be the subscheme of  $G_a$  defined by  $T^{p^n}$ ; they are known to be the only closed connected subgroup schemes of  $G_a$  different from  $G_a$ . I set  $\alpha_{p^\infty} = G_a$ . By abuse of notation, we sometimes write  $\alpha_n$  for the local group scheme  $\alpha_{p^n}$ . Let  $U$  denote the unipotent part of  $B$ , and let  $\{\beta_1, \beta_2, \dots, \beta_m\} = \phi^+$  be the set of positive roots. Then it is known that there exist morphisms of algebraic groups  $x_{\beta_i}: G_a \rightarrow U$ ,  $i \in \{1, m\}$ , such that

$$\begin{aligned} G_a^m &\rightarrow U \\ (\xi_1, \dots, \xi_m) &\rightarrow \prod x_{\beta_i}(\xi_i) \end{aligned}$$

is an isomorphism of varieties.

Let  $w_0$  denote the element of maximal length in the Weyl group  $W$ . There is an equivalent statement for  $U^- = w_0 U w_0^{-1}$ , where we use  $x_{-\beta_i}$ 's instead. It is usual to write  $U_\beta$  for  $x_\beta(G_a)$ ,  $\beta \in \phi$ .

I make the following notation for a parabolic subgroup scheme  $P$  of  $G$ : Let  $R_u^-(P_{\text{red}})$  denote the opposite of  $R_u(P_{\text{red}})$  (replacing  $U_\beta$  by  $U_{-\beta}$ ), and  $U_P^- = P \cap R_u^-(P_{\text{red}})$ .

**2. Lemma.** *Let  $P$  be a (not necessarily reduced) parabolic subgroup scheme of  $G$ . Then  $U^- \cdot P_{\text{red}} = R_u^-(P_{\text{red}}) \cdot P_{\text{red}} \cong R_u^-(P_{\text{red}}) \times P_{\text{red}}$  as varieties, and  $R_u^-(P_{\text{red}}) \cap P_{\text{red}} = \{e\}$ . This follows from [Sp, 10.3.1 and 10.3.2].*

**3. Lemma.** *Let  $P$  be a (not necessarily reduced) parabolic subgroup scheme of  $G$ . Then  $P$  is a closed subscheme of  $U^- \cdot P_{\text{red}}$ .*

*Proof.* We have  $U^- \cdot P_{\text{red}} \supset U^- \cdot B = w_0 U w_0^{-1} \cdot B$ , the big cell in  $G$ , which is open and dense in  $G$ . Hence  $U^- \cdot P_{\text{red}}$  is also dense in  $G$ . Furthermore  $U^- \cdot P_{\text{red}} = \bigcup_{g \in P_{\text{red}}} U^- B g$ , hence  $U^- \cdot P_{\text{red}}$  is also open in  $G$ . Moreover  $U^- \cdot P_{\text{red}} = R_u^-(P_{\text{red}}) \cdot P_{\text{red}}$ , and  $R_u^-(P_{\text{red}}) \cap P_{\text{red}} = \{e\}$  by Lemma 2. Hence  $U^- \cdot P_{\text{red}} \cong R_u^-(P_{\text{red}}) \times P_{\text{red}}$  as varieties, and thus  $U^- \cdot P_{\text{red}}$  is an affine, irreducible variety.

Let  $A = K[G]$ , the coordinate-ring of  $G$ . The complement of  $U^- \cdot P_{\text{red}}$  in  $G$  is a finite union of divisors, which are principal in the simply-connected cover of  $G$ ; hence there is some  $f \in A$  so that  $K[U^- \cdot P_{\text{red}}] = A_f$ , see [P, Introduction]. We have the following commutative diagrams:

$$\begin{array}{ccc} U^- \cdot P_{\text{red}} & \xrightarrow{\text{open}} & G & & A_f & \longleftarrow & A \\ \text{closed} \uparrow & & \uparrow \text{closed} & & \downarrow & & \downarrow \\ P_{\text{red}} & \xrightarrow{\text{closed}} & P & & K[P_{\text{red}}] & \longleftarrow & K[P] \end{array}$$

It follows that the class of  $f$  is a unit in  $K[P_{\text{red}}]$ ; i.e. there is a  $g \in A$  so that the class of  $f \cdot g$  in  $K[P_{\text{red}}]$  is 1. Now  $K[P_{\text{red}}] = K[P]/\text{nilradical}$ , hence  $f \cdot g = 1 + u$  in  $K[P]$ , where  $u \in \text{nilrad}(K[P])$ . But  $1 + u$  is a unit in  $K[P]$ ,

and so the class of  $f$  is a unit in  $K[P]$ . Hence there is a commutative diagram:

$$\begin{array}{ccc}
 A_f & \longleftarrow & A \\
 \downarrow & \searrow & \downarrow \\
 K[P_{\text{red}}] & \longleftarrow & K[P]
 \end{array}$$

This means that  $P$  is a closed subscheme of the variety  $U^- \cdot P$ .  $\square$

**4. Proposition.** *Let  $P$  be a (not necessarily reduced) parabolic subgroup scheme of  $G$ . Then  $P = U_P^- \cdot P_{\text{red}}$  and  $U_P^- \cap P_{\text{red}} = \{e\}$ , as scheme-theoretic intersection.*

*Proof.* By Lemmas 2 and 3 we have  $P \subset U^- \cdot P_{\text{red}} = R_u^-(P_{\text{red}}) \cdot P_{\text{red}}$ . Let  $S$  be any  $K$ -algebra. Let  $g \in P(S)$ . Then there is an element  $u$  in  $(R_u^-(P_{\text{red}}))(S)$ , and an element  $h$  in  $P_{\text{red}}(S)$  such that  $g = u \cdot h$ . Then  $u = g \cdot h^{-1} \in P(S) \cdot P_{\text{red}}(S) = P(S)$ . So  $u \in (R_u^-(P_{\text{red}})(S)) \cap P(S) = (P \cap R_u^-(P_{\text{red}}))(S) = U_P^-(S)$ , and we have  $P(S) \subset U_P^-(S) \cdot P_{\text{red}}(S)$ . By definition,  $U_P^-(S)$  and  $P_{\text{red}}(S)$  are both contained in  $P(S)$ , and so we also have the other inclusion  $P(S) \supset U_P^-(S) \cdot P_{\text{red}}(S)$ , and thus the equality  $P(S) = U_P^-(S) \cdot P_{\text{red}}(S) = (U_P^- \times P_{\text{red}})(S) = (U_P^- \cdot P_{\text{red}})(S)$  for any  $K$ -algebra  $S$ . Hence  $P = U^- \cdot P_{\text{red}}$ .  $U_P^- \cap P_{\text{red}} = \{e\}$  follows from the last equality in Lemma 2:  $R_u^-(P_{\text{red}}) \cap P_{\text{red}} = \{e\}$ , and from the definition of  $U_P^-$  as  $U_P^- = P \cap R_u^-(P_{\text{red}})$ .  $\square$

Thus  $P$  is the product of two closed subgroup schemes, with trivial intersection. Notice that  $\dim(P_{\text{red}}) = \dim(P)$ , hence  $\dim(U_P^-) = 0$ . Furthermore  $U_P^-$  is connected, since  $P$  is connected. Thus  $U_P^-$  is a local unipotent closed subgroup scheme of  $G$ .

**5. Lemma.** *Let  $\alpha$  and  $\beta$  be two linearly independent roots in  $\phi^+$ . Then there is some  $t \in T$  with  $\alpha(t) = -1$  and  $\beta(t) \neq -1$ .*

*Proof.* Because  $W\Delta = \phi$  we may assume that  $\alpha$  is simple. Write  $\beta = \sum_{\gamma \in \Delta} n_\gamma \gamma$ . There is at least one  $\delta \in \Delta \setminus \{\alpha\}$  with  $n_\delta \neq 0$ .

The simple roots are linearly independent, thus we can choose  $t \in T$  with  $\alpha(t) = -1$ ,  $\delta(t)^{n_\delta} \neq \pm 1$ ,  $\gamma(t) = 1$  if  $\gamma \neq \alpha, \delta$ . Then  $t$  is as required.  $\square$

**6. Remark.** Lemma 5 is also true if we take two distinct roots in  $\phi^-$  instead of  $\phi^+$ .

We may choose the  $\beta_1, \dots, \beta_m \in \phi^+$  such that  $\{\beta_1, \dots, \beta_m\} = \Delta$ , the set of simple roots, and such that  $\text{ht}(\beta_1) \leq \text{ht}(\beta_2) \leq \dots \leq \text{ht}(\beta_m)$ , where  $\text{ht}(\beta)$  is the height of  $\beta \in \phi^+$ :  $\text{ht}(\beta) = \sum_{i=1}^m c_i$ , where  $\beta = \sum_{i=1}^m c_i \cdot \beta_i$ ,  $c_i \geq 0$ . We write  $x_1(a_1) \cdots x_m(a_m)$  for an element in  $U^-(A)$ ,  $A$  any  $K$ -algebra,  $a_i \in A$ ,  $x_i: G_a \rightarrow G$  morphisms of algebraic groups, and  $x_i = x_{-\alpha_i}$ .

For further reference I give the following formula for any two roots  $\alpha, \beta$ , with  $\alpha + \beta \neq 0$ , and for any  $a, b \in A$  (for a proof see [SL], or [Sp, 10.1.4]:

$$(*) \quad (x_\alpha(a), x_\beta(b)) = \prod_{\substack{i, j > 1 \\ i\alpha + j\beta \in \phi}} x_{i\alpha + j\beta}(c_{ij} \cdot a^i \cdot b^j).$$

**7. Proposition.** *If  $x = x_i(a_i) \cdot x_{i+1}(a_{i+1}) \cdots x_m(a_m) \in U_P^-(A)$ ,  $i \in \langle 1, m \rangle$ , then  $x_i(a_i) \in U_P^-(A)$ .*

*Proof.* It suffices to prove the following: if  $a_1, \dots, a_m \in A$  are such that  $x = x_i(a_i) \cdot x_j(a_j) \cdot x_{j+1}(a_{j+1}) \cdots x_m(a_m) \in U_P^-(A)$ , where  $1 \leq i < j \leq m$ , then there exist  $a'_{j+1}, \dots, a'_m \in A$  such that  $x' = x_i(a_i) \cdot x_{j+1}(a'_{j+1}) \cdots x_m(a'_m) \in U_P^-(A)$ . In fact repeated application for  $j = i + 1, \dots, m$  will then prove the lemma.

Let  $x$  be as above and choose  $t \in T$  such that  $\beta_i(t) \neq -1$ , and  $\beta_j(t) = -1$ , and put  $x'' = tx t^{-1}x$ . Since  $T(A)$  acts on  $U_P^-(A)$  by conjugation, we have  $x'' \in U_P^-(A)$ . Recalling that  $T(A)$  acts on  $U_\beta(A)$  by  $t \cdot x_\beta(a) \cdot t^{-1} = x_\beta(\beta(t) \cdot a)$  we deduce from (\*) that there exist  $a''_{j+1}, \dots, a''_m \in A$  with  $x'' = ((1 + \beta_i(t)) \cdot a_i) \cdot x_{j+1}(a''_{j+1}) \cdots x_m(a''_m) \in U_P^-(A)$ . Since  $1 + \beta_i(t) \neq 0$  we can choose  $t' \in T$  with  $\beta_i(t') = 1 + \beta_i(t)$ . Then  $x' = (t')^{-1} \cdot x'' \cdot (t')$  is as required.  $\square$

**8. Proposition.** *If  $x = x_1(a_1) \cdot x_2(a_2) \cdots x_m(a_m) \in U_P^-(A)$ , then  $x_i(a_i) \in U_P^-(A)$  for all  $i \in \langle 1, m \rangle$ .*

*Proof.* By Proposition 7, we have  $x_1(a_1) \in U_P^-(A)$ , and hence  $x_1(-a_1) \in U_P^-(A)$ , and  $x_2(a_2) \cdots x_m(a_m) = x_1(-a_1) \cdot x_1(a_1) \cdot x_2(a_2) \cdots x_m(a_m) \in U_P^-(A)$ . Repeating this argument successively for  $i = 2, 3, \dots, m$ , we obtain  $x_i(a_i) \in U_P^-(A)$  for all  $i \in \langle 1, m \rangle$ .  $\square$

**9. Notation.** Let  $\tilde{\Delta}$  be the set of maps from  $\Delta$  to  $\mathbb{N}_0 \cup \{\infty\}$ , and let  $\tilde{\phi}^+$  be the set of maps from  $\phi^+$  to  $\mathbb{N}_0 \cup \{\infty\}$ .

Let  $\phi^+ = \{\beta_1, \dots, \beta_m\}$  be the set of positive roots, and  $\Delta = \{\beta_1, \dots, \beta_l\}$  the set of simple roots. I make the following definition for  $i \in \langle 1, m \rangle$ :

$$E(\beta_i) = \left\{ \beta_j \in \Delta \mid c_j \neq 0 \text{ in the expression } \beta_i = \sum_{s=1}^l c_s \cdot \beta_s \text{ with } c_s \in \mathbb{N}_0 \right\},$$

i.e.  $E(\beta_i)$  is the set of simple roots occurring with nonnegative coefficients in the expression of  $\beta_i$  in terms of simple roots. We also define  $E(-\beta_i) = E(\beta_i)$ .

Recall also that we write  $x_1(a_1) \cdots x_m(a_m)$  for an element in  $U^-(A)$ ,  $A$  any  $K$ -algebra,  $a_i \in A$ ,  $x_i: G_a \rightarrow G$  morphisms of algebraic groups, and  $x_i = x_{-\beta_i}$ . Now given a parabolic subgroup scheme  $P$  of  $G$  containing  $B$ , we define  $\varphi \in \tilde{\phi}^+$  by  $U_{-\beta} \cap P = x_{-\beta}(\alpha_{\varphi(\beta)})$  ( $\alpha_n$  being the local group scheme  $\alpha_{p^n}$  as defined above).

**10. Theorem.** *Let  $P$  and  $\varphi$  be as above. Then*

(i)  $U_P^- \cong \prod x_i(\alpha_{\varphi(\beta_i)})$ , where the product is taken over all  $\beta_i \in \phi^+$  with  $\varphi(\beta_i) \neq \infty$  (the isomorphism being an isomorphism of schemes);

(ii) If  $\beta \in \phi^+$ , then  $\varphi(\beta) = \infty$  if and only if  $U_{-\beta} \subseteq P_{\text{red}}$ ;

(iii) If  $\beta \in \phi^+$ , then  $\varphi(\beta) = \min\{\varphi(\gamma) \mid \gamma \in E(\beta)\}$ , provided that  $p = \text{char } K > 3$ , or that  $G$  is simply laced.

*Proof.* From Proposition 8 we get that

$$U_P^-(A) \cong \prod_{i=1}^m x_i(A) \cap U_P^-(A)$$

for any  $K$ -algebra  $A$ , hence

$$U_P^- \cong \prod_{i=1}^m x_i(G_a) \cap U_P^- \cong \prod U_{-\beta_i} \cap P \cong \prod x_i(\alpha_{\varphi(\beta_i)})$$

where the last two products are taken over all  $\beta_i \in \phi^+$  with  $\varphi(\beta) \neq \infty$ . This proves (i). And (ii) follows from our definition of  $\varphi$ . Now we prove (iii). If  $\beta \in \Delta$ , then  $E(\beta) = \{\beta\}$  and (iii) is trivially true. Now suppose  $\beta \in \phi^+ \setminus \Delta$ . If  $\varphi(\beta) = \infty$ , then  $U_{-\beta} \subseteq P_{\text{red}}$  and so are all  $U_{-\gamma}$  with  $\gamma \in E(\beta)$ . Hence  $\varphi(\gamma) = \infty$  for all  $\gamma \in E(\beta)$  and (iii) follows. Now suppose  $\varphi(\beta) < \infty$ . There is  $\gamma_0 \in \Delta$  such that  $\delta = \beta - \gamma_0 \in \phi^+$ . Assume  $x_{-\beta}(a) \in U_P^-(A)$ . We have  $(x_\delta(1), x_{-\beta}(a)) = \prod x_{i\delta-j\beta}(c_{ij}a^j) \in U_P^-(A)$ . By Proposition 8 one concludes that  $x_{\delta-\beta}(c_{11}a) \in U_P^-(A)$ . Recall that in general for any  $\alpha, \beta \in \phi^+$ ,  $\exists r, s \in \mathbb{N}_0$  such that  $\beta - r\alpha, \dots, \beta, \dots, \beta + q\alpha$  is the  $\alpha$ -string through  $\beta$ . We define  $N_{\alpha, \beta} = r + 1$ . It is known that  $0 \leq r \leq 3$ , hence  $1 \leq N_{\alpha, \beta} \leq 4$ . It is also known that  $c_{ij} = N_{\alpha, \beta}$  (see [SL, p. 22]). In our case  $c_{11} = N_{\delta, -\beta}$ . If  $p > 3$  or  $G$  is simply laced, then  $c_{11}$  is a nonzero integer. Thus  $x_{-\gamma_0}(a) = x_{\delta-\beta}(a) \in U_P^-(A)$ . So  $x_{-\gamma_0}(a) \in U_P^-(A)$  whenever  $x_{-\beta}(a) \in U_P^-(A)$ , i.e.  $a^{p^{\varphi(\gamma_0)}} = 0$  whenever  $a^{p^{\varphi(\beta)}} = 0$  for any  $K$ -algebra  $A$ . This implies  $\varphi(\beta) \leq \varphi(\gamma_0)$ . Similarly  $(x_{\gamma_0}(1), x_{-\beta}(a)) \in P(A)$ , whence  $x_{-\delta}(a) \in U_P^-(A)$ , and  $\varphi(\beta) \leq \varphi(\delta)$ . By induction on the height we may assume that  $\varphi(\delta)$  is given by (iii), and one concludes that  $\varphi(\beta) \leq \min\{\varphi(\gamma) \mid \gamma \in E(\beta)\}$ .

It remains to prove the reverse inequality. This can be done by induction on the height of  $\beta$ . The statement is trivially true for  $\text{ht}(\beta) = 1$ . Assume  $\text{ht}(\beta) > 1$ . There is  $\gamma_0 \in \Delta$  such that  $\beta - \gamma_0 \in \phi^+$ . Let  $\delta = \beta - \gamma_0$ . Then  $\text{ht}(\delta) < \text{ht}(\beta)$ . Let  $x_{-\delta}(a)$  and  $x_{-\gamma_0}(b) \in U_P^-(A)$ . Then

$$(x_{-\delta}(a), x_{-\gamma_0}(b)) = \prod x_{-i\delta-j\gamma_0}(c_{ij}a^i b^j) \in U_P^-(A).$$

By Proposition 8 we have  $x_{-\beta}(c_{11}ab) = x_{-\delta-\gamma_0}(c_{11}ab) \in U_P^-(A)$ . Hence  $x_{-\beta}(ab) \in U_P^-(A)$ , i.e.  $(ab)^{p^{\varphi(\beta)}} = 0$  for all  $a, b \in A$  with  $a^{p^{\varphi(\delta)}} = b^{p^{\varphi(\gamma_0)}} = 0$ , and for any  $K$ -algebra  $A$ . So  $\varphi(\beta) \geq \min\{\varphi(\delta), \varphi(\gamma_0)\} = \min\{\varphi(\gamma) \mid \gamma \in E(\beta)\}$ , and the theorem is proven.  $\square$

**11. Corollary.** *Let  $\pi: G \rightarrow G'$  be a surjective morphism of connected reductive  $K$ -groups with central kernel. Then  $\pi$  induces a bijection of the set of parabolic subgroup schemes of  $G$  onto the same set for  $G'$ .*

*Proof.* This follows from the fact that  $\pi$  is an isomorphism on  $U^-$ , the decomposition  $P = U_P^- \cdot P_{\text{red}}$ , and that the statement holds for reduced parabolic subgroup schemes.  $\square$

**12. Corollary.** *If  $P_\varphi$  and  $P_\psi$  exist, then so does  $P_{\text{inf}(\varphi, \psi)}$ .*

*Proof.* The intersection of  $P_\varphi$  and  $P_\psi$  is a parabolic subgroup scheme of  $G$  containing  $B$ , and  $P_\varphi \cap P_\psi = P_{\text{inf}(\varphi, \psi)}$ .  $\square$

Let  $F$  be the Frobenius morphism on  $G$ , and denote the local subgroup scheme  $(F^n)^{-1}(e)$  of  $G$  by  $G_n$  for each  $n \in \mathbb{N}_0$ . Let  $\beta \in \Delta$  and denote by  $P_\beta$  the maximal reduced parabolic subgroup scheme of  $G$  containing  $B$  and not containing  $U_{-\beta}$ . Then  $P_{n, \beta} = G_n \cdot P_\beta$  is a parabolic subgroup scheme of  $G$  containing  $B$  and equals  $P_\varphi$ , where  $\varphi(\beta) = n$  and  $\varphi(\gamma) = \infty$  for  $\gamma \in \Delta \setminus \{\beta\}$ . Thus we obtain

**13. Theorem.** *For each  $\varphi \in \tilde{\Delta}$ , there exists the parabolic subgroup scheme  $P_\varphi$ .*

*Proof.* The intersection of all  $P_{\varphi(\beta), \beta}$ ,  $\beta \in \Delta$ , is a parabolic subgroup scheme and by Corollary 12 it equals  $P_\varphi$ .  $\square$

Now I can state the main theorem, giving the desired classification:

**14. Theorem.** *Let  $K$  be an algebraically closed field of characteristic  $p > 0$ . Let  $G$  be a reductive linear algebraic group defined over  $K$ . There is an injective map from  $\tilde{\Delta}$  to  $\mathfrak{P}$ , the set of all parabolic subgroup schemes containing  $B$ , given by*

$$\begin{aligned} \tilde{\Delta} &\rightarrow \mathfrak{P} \\ \varphi &\rightarrow P_\varphi, \end{aligned}$$

where  $P_\varphi = U_\varphi \cdot P_{I(\varphi)}$ ,  $I(\varphi) = \{\alpha \in \Delta \mid \varphi(\alpha) = \infty\}$ ,  $U_\varphi = \prod_{\beta \in \phi^+ - \phi_I} x_{-\beta}(\alpha_{\varphi(\beta)})$ ,  $\varphi$  being extended to all of  $\phi^+$  by  $\varphi(\beta) = \min\{\varphi(\gamma) \mid \gamma \in E(\beta)\}$ ,  $E(\beta) = \{\beta_i \in \Delta \mid \beta = \sum c_j \cdot \beta_j, \text{ with all } c_j \geq 0 \text{ and } c_i \neq 0\}$ ,  $\phi_I$  the roots generated by  $I = I(\varphi)$ .

If  $\text{char } K > 3$ , or if  $G$  is simply laced, then this map is also surjective.  $\square$

**15. Remark.** It is known to the author that the map in Theorem 14 is not surjective in  $\text{char}(K) = 2, 3$  for certain  $G$ ; for example for  $G = SO_5$  in  $\text{char } K = 2$ , and for  $G$  with root system of type  $G_2$  in  $\text{char } K = 3$ .

Now we can also derive a theorem about the algebra of distributions  $\text{Dist}(G)$  on  $G$ . For detailed information see [H].  $\text{Dist}(G) = \bigoplus K \cdot X_{c^-} \cdot \binom{H}{h} \cdot X_c$  as a  $K$ -vector space, where

$$\begin{aligned} X_{c^-} &= X_{-m}^{[c_{-m}]} \cdots X_1^{[c_{-1}]}, & c_{-i} \in \mathbb{N}_0 \text{ for all } i \in \langle 1, m \rangle, & X_i^{[c_i]} = (X_i^{c_i})/(i!), \\ X_c &= X_1^{[c_1]} \cdots X_m^{[c_m]}, & c_i \in \mathbb{N}_0 \text{ for all } i \in \langle 1, m \rangle, \\ \binom{H}{h} &= \binom{H_1}{h_1} \cdots \binom{H_l}{h_l}, & H_i = H_{\beta_i}, h_i \in \mathbb{N}_0 \text{ for all } i \in \langle 1, l \rangle, \end{aligned}$$

and where the sum is taken over all possible  $c^-, c, h$ .

Let  $A = K[G]$ . Suppose  $D$  is a subalgebra and subcoalgebra of  $\text{Dist}(G)$  of the following type:

$$D = \sum K \cdot X_{-m}^{[c_{-m}]} \cdots X_{-1}^{[c_{-1}]} \cdot \binom{H_1}{h_1} \cdots \binom{H_l}{h_l} \cdot X_1^{[c_1]} \cdots X_m^{[c_m]},$$

where the sum is taken over all terms with  $c_j < c_{j0}$ ,  $h_i < h_{i0}$ , for some fixed  $c_{j0}, h_{i0}$ ,  $j \in \langle -m, m \rangle$ ,  $i \in \langle 1, l \rangle$ .

Then  $D \subset \text{Dist}(G)$ , and  $D \cap \text{Dist}_n(G) \subset \text{Dist}_n(G)$ . So we obtain natural surjections for the linear dual:

$$\text{Dist}_n(G)^* \rightarrow (D \cap \text{Dist}_n(G))^*, \quad \varinjlim \text{Dist}_n(G)^* \rightarrow \varinjlim (D \cap \text{Dist}_n(G))^*.$$

We have

$$K[U^- \cdot B] = A_f = K[x_{-m}, \dots, x_{-1}, h_1, h_1^{-1}, \dots, h_l, h_l^{-1}, y_1, \dots, y_m],$$

for some  $f \in A$ , and

$$\varinjlim \text{Dist}_n(G)^* = \widehat{A} = K[[x_{-m}, \dots, x_{-1}, z_1, \dots, z_l, x_1, \dots, x_m]]$$

(see [H, 1.2]), where  $z_i = h_i - 1$  for all  $i \in \langle 1, l \rangle$ . Furthermore  $\widehat{A} = (\widehat{A}_f)$ . Let  $C = \varinjlim (D \cap \text{Dist}_n(G))^*$ . The surjection  $\widehat{A} \rightarrow C$  is a morphism of  $K$ -algebras

and coalgebras. Let  $\tilde{I}$  be its kernel. From our description of  $D$  and from [H2, 1.2], it follows that  $\tilde{I}$  is generated over  $\hat{A}$  by the  $x_j^{c_j}, z^{i_0}$ ,  $j \in \langle -m, m \rangle$ ,  $i \in \langle 1, l \rangle$ .

Define  $I' = A_f \cap \tilde{I}$ . Then  $I'$  is an ideal of  $A_f$ , and it is generated over  $A_f$  by the  $x_j^{c_j}, z^{i_0}$ ,  $j \in \langle -m, m \rangle$ ,  $i \in \langle 1, l \rangle$ . It is obvious that  $\tilde{I} = I' \cdot \hat{A}$ .

Define  $I = A \cap I'$ . Then the elements  $x_j^{c_j}, z^{i_0}$ ,  $j \in \langle -m, m \rangle$ ,  $i \in \langle 1, l \rangle$ , multiplied by a sufficient power of  $f$  are contained in  $A$ . Thus  $I' = I \cdot A_f$ . Hence

$$I \cdot \hat{A} = I \cdot A_f \cdot \hat{A} = I' \cdot \hat{A} = \tilde{I}.$$

**16. Proposition.** *Let  $D \subseteq \text{Dist}(G)$  and  $I \subseteq A$  be as above, then  $I$  defines a closed subgroup scheme of  $G$  whose algebra of distributions is  $D$ .*

*Proof.* Let  $\mu: A \rightarrow A \otimes A$  be the comultiplication on the coordinatizing  $A$  of  $G$ , let  $\sigma: A \rightarrow A$  be the coinverse, and let  $\varepsilon: A \rightarrow K$  be the coidentity. Let  $\hat{\mu}, \hat{\sigma}, \hat{\varepsilon}$  be the extensions of  $\mu, \sigma, \varepsilon$  respectively on the formal group scheme  $\hat{A}$ . Then we have

$$\begin{array}{ccc} \tilde{I} & \longrightarrow & \tilde{I} \hat{\otimes} \hat{A} + \hat{A} \hat{\otimes} \tilde{I} \\ \downarrow & & \downarrow \\ \hat{A} & \xrightarrow{\hat{\mu}} & \hat{A} \hat{\otimes} \hat{A} \\ \downarrow & & \downarrow \\ C & \longrightarrow & C \hat{\otimes} C \end{array}$$

So

(a)  $\mu(I) \subset \hat{\mu}(\tilde{I}) \cap A \otimes A \subset (\tilde{I} \hat{\otimes} \hat{A} + \hat{A} \hat{\otimes} \tilde{I}) \cap A \otimes A = I \otimes A + A \otimes I.$

(b)  $\hat{\sigma}(\tilde{I}) = \tilde{I}$  and so we get  
 $\sigma(I) = \sigma(\tilde{I} \cap A) \subset \hat{\sigma}(\tilde{I}) \cap \sigma(A) = \tilde{I} \cap A = I.$

(c)  $\hat{\varepsilon}(\tilde{I}) = 0$  and so we get  
 $\varepsilon(I) = \varepsilon(\tilde{I} \cap A) \subset \hat{\varepsilon}(\tilde{I}) \cap \varepsilon(A) = 0.$

Now (a)–(c) show exactly that  $\mu, \sigma, \varepsilon$  as defined on  $A$  induce the corresponding structure on  $A/I$ , i.e.  $\text{Spec}(A/I)$  is a subgroup scheme of  $G$ . Now  $\text{Dist}(\text{Spec}(A/I)) = D$  is obvious.  $\square$

Let  $\varphi \in \tilde{\Delta}$ . Then  $\varphi$  can be extended to  $\phi^+$  by defining

$$\varphi(\beta) = \min\{\varphi(\alpha) \mid \alpha \in E(\beta)\}.$$

Now we can introduce the notation  $c^- < p^\varphi$  to stand for  $c_{-1} < p^{\varphi(\alpha_1)}, \dots, c_{-m} < p^{\varphi(\alpha_m)}$ .

**17. Theorem.** *Let  $G$  be as above. For each  $\varphi \in \tilde{\Delta}$ , let*

$$D_\varphi = \bigoplus_{b^- < p^\varphi} K \cdot X_{b^-} \cdot \begin{pmatrix} H \\ h \end{pmatrix} \cdot X_b.$$

Then  $D_\varphi$  is a subalgebra and a subcoalgebra of  $\text{Dist}(G)$ . Furthermore, if  $\text{char } K > 3$  or if  $G$  is simply laced, then these are all subalgebras and subcoalgebras of  $\text{Dist}(G)$  containing  $\text{Dist}(B)$ .

*Proof.* We have

$$\text{Dist}(P_\varphi) = \text{Dist}(U_{P_\varphi}^- \cdot P_{\varphi \text{ red}}) = \text{Dist}(U_{P_\varphi}^-) \otimes \text{Dist}(P_{\varphi \text{ red}}) = D_\varphi,$$

which proves the first part. For the second part we apply the proposition above.  $\square$

18. *Remark.* Theorem 13 establishes the existence of the  $P_\varphi$  using the Frobenius morphism and the observation of Corollary 12. From these  $P_\varphi$  one obtains the algebra of distributions  $D_\varphi$ . This can also be done the other way around: One can prove directly that the  $D_\varphi$  are indeed subalgebras and subcoalgebras of  $\text{Dist}(G)$  (but the proof is long, complicated and involves several induction arguments, so I have not included it here), and then easily derive the  $P_\varphi$  by Proposition 16.

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