DIVISORS ON SYMMETRIC PRODUCTS OF CURVES

ALEXIS KOUVIDAKIS

ABSTRACT. For a curve with general moduli, the Neron-Severi group of its symmetric products is generated by the classes of two divisors $x$ and $\theta$. In this paper we give bounds for the cones of effective and ample divisors in the $x\theta$-plane.

1. Introduction

Let $C$ be a smooth irreducible curve of genus $g$, $J(C)$ its Jacobian variety and $C_d$ its $d$-fold symmetric product. Fixing a point $P_0$ on $C$, we define the maps

$$u_d : C_d \to J(C) \quad \text{by} \quad u_d(D) = \Theta(D - dP_0),$$
$$i_{d-1} : C_{d-1} \to C_d \quad \text{by} \quad i_{d-1}(D) = D + P_0.$$

On $J(C)$ we denote by $\Theta$, the theta divisor and by $\theta$, its class in the Neron-Severi group. On $C_d$ we denote by $\theta_d$, or simply (again) $\theta$, the class of $u_d^*(\Theta)$ and by $x_d$, or simply $x$, the class of $i_{d-1}^*(C_{d-1})$ in $C_d$. For curves $C$ with general moduli, it is known that the Neron-Severi group of the symmetric product $C_d$ is generated by $\theta$ and $x$, see [A-C-G-H]. In this paper we give estimates for the cones of the effective and ample divisors on $C_d$, in the $\theta, x$-plane.

By standard theory, we know the following things about the map $u_d$:

1. Abel's Theorem. The fiber of the map $u_d$ containing the divisor $D$, is exactly the set of divisors belonging to the complete series of $D$.
2. Jacobi's Inversion Theorem. The map $u_g : C_g \to J(C)$ is onto.
3. Poincaré's Formula. We denote by $W_d$ the image of $C_d$ in the Jacobian by the map $u_d$ and by $w_d$ its class. For $0 \leq d \leq g$ we have

$$w_d = \frac{\theta^{g-d}}{(g - d)!}.$$

In particular, for $d = g - 1$ we have $w_{g-1} = \theta$. 

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4. If \( d \leq g \), the map \( u_d \) is birational to its image, reflecting the fact that \( h^0(C, D) = 1 \), for \( D \) general point in \( C_d \). If \( g + 1 \leq d \leq 2g - 2 \), the generic fiber of \( u_d \) has dimension \( d - g \).

If \( 2g - 1 \leq d \), the map is a \( \mathbb{P}^{d-g} \)-fibration with tautological class \( x \).

**Lemma 1** (Intersections on \( C_d \)). For \( 0 \leq r \leq d \leq g \) we have on \( C_d \)

\[
\theta^r x^{d-r} = \frac{g!}{(g-r)!},
\]

**Proof.** Indeed, \( x = \text{class}(i_{d-1}(C_{d-1})) = \text{class}(D + P_1, D \in C_{d-1}, P_1 \) a fixed point in \( C) \) and so,

\[
x^{d-r} = \text{class}(D + P_1 + \cdots + P_{d-r}, D \in C_r, P_1, \ldots, P_{d-r} \text{ fixed in } C).
\]

Therefore \( u_d \ast x^{d-r} = w_r \). Since the map \( u_d \) is generically 1-1, projection formula implies that \( \theta^r x^{d-r} \) on \( C_d \), is equal to \( \theta^r w_r \) on \( J(C) \). Poincaré's Formula and the fact that \( \theta^g = g! \) complete the proof. Q.E.D.

We denote by \( H^{2n}_{an}(C_d, \mathbb{Q}) \) the algebraic part of \( H^{2n}(C_d, \mathbb{Q}) \). Given an algebraic cycle \( Z \) in \( C_d \), we define the maps

\[A_k : H^{2m}_{an}(C_d, \mathbb{Q}) \to H^{2m}_{an}(C_{d+k}, \mathbb{Q})\]

by

\[A_k(Z) = \{E \in C_{d+k}, E - D \geq 0 \text{ for some } D \in Z\}\]

and

\[B_k : H^{2m}_{an}(C_d, \mathbb{Q}) \to H^{2m-2k}_{an}(C_{d-k}, \mathbb{Q})\]

by

\[B_k(Z) = \{E \in C_{d-k}, D - E \geq 0 \text{ for some } D \in Z\}.
\]

We have the standard formulas, see [A-C-G-H, p. 367].

**Lemma 2** (Push-pull formulas for symmetric products).

\[
A_k(x^\alpha \theta^\beta) = \sum_{i=0}^{k} \binom{\beta}{i} \left(\begin{array}{c}g - \beta + i \\ i \end{array}\right) \left(\begin{array}{c}d + k - \alpha - 2\beta \\ k - i \end{array}\right) i! x^{\alpha+i} \theta^{\beta-i}
\]

and

\[
B_k(x^\alpha \theta^\beta) = \sum_{j=0}^{k} \binom{\alpha}{k-j} \left(\begin{array}{c}\beta \\ j \end{array}\right) \left(\begin{array}{c}g - \beta + j \\ j \end{array}\right) j! x^{\alpha-k+j} \theta^{\beta-j}.
\]

### 2. Ample and Effective Divisors

We will use often the following criterion for ampleness, see [Ha]:

**Lemma 3** (Nakai-Moishezon). Let \( D \) be a Cartier divisor on a variety \( X \). Then \( D \) is ample on \( X \), if and only if, for every subvariety \( Y \) in \( X \) of dimension \( r \), we have that \( D \cdot Y > 0 \).

In particular, when \( X \) is a smooth surface this says that the cone of effective and the cone of ample divisors are dual under the intersection pairing.

If \( X \) is a smooth surface, we have the following numerical criterion, for checking effectivity of a divisor \( D \) on \( X \), see [Ha]:
Lemma 4. Let $X$ be a smooth surface, $H$ an ample divisor on $X$ and $D$ a divisor with $D^2 > 0$ and $DH > 0$.

Then, for $n$ sufficiently large, $nD$ is an effective divisor.

Proof. We prove first that for $n$ large enough we have $h^2(X, nD) = 0$; Indeed, $h^2(X, nD) = h^0(X, K_X - nD)$, and if we assume that $h^2(X, nD) > 0$ for all $n$, then $K_X - nD$ is effective and so, $(K_X - nD)H > 0$ (Lemma 3) i.e. $K_XH > nDH$. Since $DH > 0$, taking $n$ big enough—namely $n > K_XH$—we get $K_XH > n$. A contradiction. Therefore there exists an $n$ with $h^2(X, nD) = 0$ and the Riemann-Roch theorem completes the proof. Q.E.D.

In the higher dimensional case i.e. if $X$ is a smooth variety of dimension $d$, the Riemann-Roch theorem gives that

$$h^0(X, D) - \chi(X, D) + h^2(X, D) + \cdots + (-1)^d h^d(X, D) = \chi(-D, D) + \text{(terms containing strictly lower powers of } c_1).$$

If $H$ is again an ample divisor on $X$ and $D$ a divisor satisfying $D^rH^{d-r} > 0$ for all $0 \leq r \leq d$, then it is not known if

$$h^0(X, nD) > 0 \quad \text{for } n \text{ big enough}$$

In order to have such a conclusion, we have to impose extra condition, for example, the restriction of $\mathcal{O}(D)$ on $H$ to be an ample divisor. We have

Theorem 1. Let $X$ be a d-dimensional variety, $H$ an ample effective divisor on $X$ and $D$ a divisor with $D^d > 0$ and $\mathcal{O}(D)|_H$ is an ample line bundle on $H$.

Then,

$$h^0(nD) > 0 \quad \text{for } n \text{ big enough}.$$

Proof. We use the following lemmas:

Lemma 5 (Serre). Let $X$ be a proper scheme over a Noetherian ring. If $N$ is an invertible sheaf on $X$, then the following conditions are equivalent:

(i) $N$ is ample.

(ii) For each coherent sheaf $E$ on $X$, there exists an integer $n_0$ depending on $E$ s.t. for each $i \geq 1$ and each $n \geq n_0$

$$H^i(X, E \otimes N^n) = 0.$$

Lemma 6. Let $S$ be a proper scheme over a Noetherian ring, $\mathcal{L}$ an ample line bundle on $S$ and $\mathcal{F}$ a line bundle generated by global sections. Then there exists an $n_0$ s.t.

$$\forall n \geq n_0 \quad H^i(S, \mathcal{L}^n \otimes \mathcal{F}^k) = 0 \quad \forall k \geq 0, \forall i \geq 1.$$

Acknowledgment. I would like to thank J.-F. Burnol, for showing me Lemma 6 above and the following proof.

Proof. We have an exact sequence $\bigoplus_{i=1}^m \mathcal{O}_S \to \mathcal{F} \to 0$. Tensoring by $\mathcal{F}$ we get $\bigoplus_{i=1}^m \mathcal{F} \to \mathcal{O}_S \to 0$. Hence (Koszul):

$$0 \to (\mathcal{F})^r \to \bigoplus_i (\mathcal{F})^{r-1} \to \bigoplus_i (\mathcal{F})^{r-2} \to \cdots \to \bigoplus_i \mathcal{F} \to \mathcal{O}_S \to 0.$$
And so, tensoring by \( F^r \) we get

\[
0 \to \mathcal{O}_S \to \bigoplus_r \mathcal{F} \to \bigoplus_r (\mathcal{F})^2 \to \cdots \to \bigoplus_r (\mathcal{F})^{r-1} \to (\mathcal{F})^r \to 0.
\]

By Lemma 5 we can choose \( n_0 \) so that \( H^i(S, \mathcal{L}^n \otimes \mathcal{F}^k) = 0 \) for all \( i \geq 1, n \geq n_0, k = 0, 1, \ldots, r \). Then this \( n_0 \) works: suppose the claim is true for \( k \leq l \). Tensoring the above exact sequence by \( \mathcal{L}^n \otimes \mathcal{F}^{l+1-r} \), all sheaves have zero \( H^i \)'s, \( i \geq 1, n \geq n_0 \), except maybe the last one which is \( \mathcal{L}^n \otimes \mathcal{F}^{l+1} \). Therefore the last one has also zero \( H^i \)'s, \( i \geq 1, n \geq n_0 \). Q.E.D.

Going back to the proof of Theorem 1, we use first the following fact: “A line bundle \( L \) is ample on \( H \) iff \( L_{\text{red}} \) is ample on \( H_{\text{red}} \).” By this fact, we can replace \( H \) by \( mH \) without changing hypothesis on \( D \), and so, we can assume that \( H \) is in fact very ample on \( X \). We denote by \( L \) the line bundle \( \mathcal{O}(D) \).

We have that \( L_1 = L \otimes \mathcal{O}_H \) is ample on \( H \). Let \( H_1 \) be the restriction of \( \mathcal{O}(H) \) to \( H \); then, \( H_1 \) is generated by global sections. Also \( L_1 \) is ample on \( H \) and so, by the above Lemma 6, there exists an integer \( n_0 \) s.t.

\[
(4) \quad h^i(H, L^n \otimes H^k) = 0 \quad \text{for all } i \geq 1, n \geq n_0, k \geq 0.
\]

On the other hand \( H \) is ample on \( X \) and so, given the coherent sheaf \( L^n \) there exists by Lemma 5 an integer \( m_n \) s.t. \( H^i(X, L^n \otimes H^{m_n}) = 0 \). Consider now the exact sequence:

\[
0 \to \mathcal{O}_X(L^n \otimes H^{(l-1)}) \to \mathcal{O}_X(L^n \otimes H^l) \to \mathcal{O}_H(L^n \otimes H^l_1) \to 0.
\]

The corresponding long exact sequence gives

\[
0 \to H^0(X, L^n \otimes H^{(l-1)}) \to H^0(X, L^n \otimes H^l) \to H^0(H, L^n \otimes H^l_1)
\]

\[
\to H^1(X, L^n \otimes H^{(l-1)}) \to H^1(X, L^n \otimes H^l) \to H^1(H, L^n \otimes H^l_1)
\]

\[
\to H^2(X, L^n \otimes H^{(l-1)}) \to H^2(X, L^n \otimes H^l) \to H^2(H, L^n \otimes H^l_1)
\]

\[
\vdots
\]

\[
\to H^k(X, L^n \otimes H^{(l-1)}) \to H^k(X, L^n \otimes H^l) \to H^k(H, L^n \otimes H^l_1)
\]

\[
\vdots
\]

and so, we get for each \( n \geq n_0 \) that

\[
h^i(X, L^n \otimes H^{(l-1)}) = h^i(X, L^n \otimes H^l) \quad \text{for all } i \geq 2, l \geq 1.
\]

Therefore, for each \( n \geq n_0 \) we have that

\[
h^i(X, L^n) = h^i(X, L^n \otimes H) = \cdots = h^i(X, L^n \otimes H^{m_n}) = 0 \quad \text{for all } i \geq 2.
\]

For each \( n \geq n_0 \) the Riemann-Roch theorem gives

\[
h^0(X, L^n) = h^1(X, L^n) + \frac{c_1(L^n)}{d!} + \text{(terms containing strictly lower powers of } c_1)\]

and so, since \( c_1(L) > 0 \), we get that there exists an \( n \) big enough s.t. \( h^0(X, L^n) > 0 \). Q.E.D.
3. The class of the diagonal and of $\Gamma_n(g^r_d)$’s

We recall some theory from [A-C-G-H]. Consider the diagonal map

$$\phi_d = \phi: C_{d-2} \times C \to C_d$$

defined by

$$\phi(D, p) = D + 2p.$$  

The image of this map is the diagonal $\Delta_d$ in $C_d$. A special case of Proposition 5.1 on p. 358 in [A-C-G-H] gives that

**Lemma 7 (MacDonald).** The class $\delta_d$ of the diagonal $\Delta_d$ in $C_d$ is given by

$$\delta_d = 2((d + g - 1)x - \theta).$$

We denote now by $g^r_d$ a base point free linear system of degree $d$ and dimension $r$ on $C$. Given such a $g^r_d$, then for each $n$ with $r < n \leq d$, we can construct in $C_n$ the following cycle

$$\Gamma_n(g^r_d) = \{D \in C_n \text{ s.t. } D < E \text{ for some } E \in g^r_d\}.$$  

The standard way to calculate the class $\gamma_n(g^r_d)$ of the above cycle is given by the following lemma, see [A-C-G-H, Lemma 3.2, p. 342].

**Lemma 8.** For integers $d \geq n > r$ the class $\gamma_n(g^r_d)$ in $C_n$ is given by

$$\gamma_n(g^r_d) = \sum_{k=0}^{n-r} \binom{d - g - r}{k} x^k \frac{\theta^{n-r-k}}{(n-r-k)!}.$$  

In the particular case where $n = r + 1$ and so, $\Gamma_{r+1}(g^r_d)$ is a divisor in $C_{r+1}$, we can find the class as following:

We denote by $C^{\times(r+1)}$ the $(r+1)$th Cartesian product of $C$, by $f_1, \ldots, f_{r+1}$ the class of the coordinate planes and by $\delta_C$ the class of the sum of the diagonals in the product. Also we define $\gamma_C = \pi^*(\gamma_{r+1}(g^r_d))$, where $\pi: C^{\times(r+1)} \to C_{r+1}$ the canonical map. We have the following relations:

$$\pi^* x = f_1 + \cdots + f_{r+1}, \quad \pi^* \delta_{r+1} = 2\delta_C, \quad \delta_{r+1} = 2((g + r)x - \theta).$$

Given a $g^r_d$ on $C$ we have a canonical map $\phi: C \to \mathbb{P}^r = \mathbb{P}$ and an induced (product) map $\Phi: C^{\times(r+1)} \to \mathbb{P}^{\times(r+1)}$. We denote by $\delta_{\mathbb{P}}$ the class of the sum of the diagonals in $\mathbb{P}^{\times(r+1)}$. Observe that

$$\Phi^*(\delta_{\mathbb{P}}) = \delta_C + \gamma_C.$$  

We have $\delta_{\mathbb{P}} = f_1^\mathbb{P} + \cdots + f_{r+1}^\mathbb{P}$, where $f_i^\mathbb{P}$’s are the classes of the coordinate planes in $\mathbb{P}^{\times(r+1)}$. Therefore, $\Phi^*(\delta_{\mathbb{P}}) = \Phi^*(f_1^\mathbb{P} + \cdots + f_{r+1}^\mathbb{P}) = d(f_1 + \cdots + f_{r+1}) = df$ and so, $\gamma_C = df - \delta_C$.

Now, $\pi^*(\gamma_{r+1}(g^r_d)) = \gamma_C = df - \delta_C = d\pi^* x - \frac{1}{2} \pi^*(\delta_{r+1})$ and so, $\gamma_{r+1}(g^r_d) = dx - \delta_{r+1}/2$. Using relation (5) we conclude that

$$\gamma_{r+1}(g^r_d) = \theta - (g - d + r)x.$$  

4. First bounds for the cones

We examine the case $d \leq g$. If $D$ is an effective divisor on $C_d$, then $u_d^\theta d^{d-1} \cdot D = \theta^{d-1} \cdot u_d \cdot D$, where $u_d$ the Abel-Jacobi map. Since $\theta$ is an ample
divisor on $J(C)$, we get by Lemma 3, that $\theta^{d-1} \cdot u_d, D \geq 0$, where equality holds iff $u_d, D = 0$. This gives the first naive bound for the effective cone in $C_d$:

Suppose that $D$ is divisor with class $a \theta - bx$, $a, b > 0$ i.e. it “lies” in the fourth quarter of the $\theta, x$-plane. We define slope $m$ of $D$ to be $m = \frac{b}{a}$.

If $D$ is effective, then by the above discussion we have $(\theta - mx)^d > 0$ which implies

$$\frac{g!}{(g-d)!} - m\frac{g!}{(g-d+1)!} \geq 0 \quad \text{i.e.} \quad m \leq g - d + 1.$$  

If $D$ is ample, Lemma 3 implies that $(\theta - mx)^d > 0$. Equivalently

$$\sum_{k=0}^{d} \binom{d}{k} m^{d-k} x^{d-k} > 0,$$

i.e.

$$(7) \quad \sum_{k=0}^{d} \binom{d}{k} m^{d-k} \frac{g!}{(g-k)!} > 0.$$  

Since for $m = 0$ this is positive, we must have $m < (\text{min. posit. root of } (7)).$

For a divisor $D$ with class $ax - b\theta$, $a, b > 0$, i.e. it “lies” in the second quarter of the $\theta, x$-plane we define the slope $\overline{m}$ of $D$ to be $\overline{m} = \frac{g}{a}$. Similar argument gives that if $D$ is effective then $\overline{m} \geq g - d + 1$, and if $D$ is ample then $\overline{m}$ satisfies a similar relation as in (7). For example if $D$ is ample then for $d = 2$ we have $m < g - \sqrt{g}$ and $\overline{m} > g + \sqrt{g}$.

5. Effective and ample cones for $C_2$

By the previous discussion we have the first bounds for the effective and ample cones for $C_2$. On the other hand using the Lemma 4 we know that every

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Bounds for the cones in $C_2$}
\end{figure}
class between the thick lines in Figure 1 is effective. Since the ample and effective cones are dual, it is enough to describe the effective cone. We make the observation:

**Observation.** If $D$ is an irreducible effective divisor with slope $m$ (resp. $\bar{m}$) between

$$g - \sqrt{g} < m \leq g - 1 \quad \text{(resp. } g + \sqrt{g} > \bar{m} \geq g - 1),$$

then it is unique with this property.

Indeed, if $D'$ is another irreducible effective divisor with slope $m'$ (resp. $\bar{m}'$) in the above range then we get $DD' < 0$, a contradiction.

If we are able to find such a divisor, then this describes the cone. In the second quarter such a divisor exists, namely the diagonal: Recall that $\delta_2 = 2((g + 1)x - \theta)$ and so $\bar{m} = g + 1$ i.e. $g + \sqrt{g} > \bar{m} > g - 1$. Therefore the slope of the effective cone in the second quarter is given by

$$\bar{m}_{ef} = g + 1$$

and (by duality) of the ample cone by

$$m_a = 2g.$$

(Note that a divisor with slope $m_a$, has positive intersection with all the effective divisors in the first or the fourth quarter.)

Also, if $C$ is of genus 2, then it is hyperelliptic and the corresponding $g_2^1$ gives an effective class in $C_2$ belonging in the fourth quarter with slope $m = 1$, see formula (6). By the above observation we get that $m_{ef} = 1$ and by duality $m_a = 0$. For general genus, we have the following:

**Theorem 2.** Let $C$ be a curve of genus $g \geq 3$ with general moduli. For the slopes of the cones in $C_2$, in the fourth quarter of the $\theta$, $x$-plane, we have

1. If $g$ is a square, then $m_{ef} = m_a = g - \sqrt{g}$.
2. If $g$ is not a square and $g > 3$, then $g - \sqrt{g} + 1 \geq m_{ef} \geq g - \sqrt{g}$ and so

$$g - \sqrt{g} \geq m_a \geq g - \frac{g}{\sqrt{g} - 1}.$$

3. If $g = 3$, then $m_{ef} = \frac{4}{3}$ and $m_a = \frac{6}{5}$.

**Proof.** We use degenerations to special curves: From the formula (6) we have that the class of $\gamma_2(g_1^d)$ in $C_2$ is given by

$$\gamma_2(g_1^d) = \theta - (g - d + 1)x.$$

Therefore if $d < \sqrt{g} + 1$, the slope of the divisor is

$$m^0 = g - d + 1 \geq g - \sqrt{g}.$$

Take a smooth curve $C^0$ with a $g_1^d$, $d < \sqrt{g} + 1$. If in addition, we choose the curve to be "general" having such a $g_1^d$, then the corresponding divisor $\Gamma_2(g_1^d)$ is irreducible in $C_2$. Since $C^0$ is special, the $H^*_a(C^0, \mathbb{Q})$ may not be generated by $x$, $\theta$. Consider in $H^*_a(C^0, \mathbb{Q})$ the plane $\Pi$ spanned by $x$, $\theta$. By the previous analysis, since $m^0 > g - \sqrt{g}$ the intersection of the effective cone with the plane $\Pi$ is given by the slopes $\bar{m}_{ef}^0 = g + 1$ and $m_{ef}^0 = g - d + 1$.

Since $\mathcal{M}_g$ is connected, we can find a flat family of smooth curves $\mathbb{C} \to \Delta$, $\Delta$ a disk, with central fiber $C^0$ and the other fiber curves with general moduli.
Figure 2. The genus 3 case

Let $\Phi: \mathbb{C}^2 \to \Delta$ be the flat family with fibers the 2-symmetric products of the fibers of $\mathbb{C} \to \Delta$.

Suppose that the general curve has an irreducible divisor with slope $m$, $g - 1 \geq m > g - \sqrt{g}$. By the above observation, this is unique and so, it gives rise to an effective divisor in $\mathbb{C}^2 \backslash \Phi^{-1}(0)$. Degenerating to the central fiber $C_0^0$, we get an effective divisor $D^0$. Since the degeneration "preserves" the algebraic equivalence we have that $D^0$ belongs to $\Pi$ and so, since it is effective, the slope $m$ satisfies $g - d + 1 \geq m \geq g - \sqrt{g}$.

Therefore if $g$ is a square i.e. $g = k^2$, then choosing $d = k - 1 = \sqrt{g} - 1$ we get that $m$ has to satisfy $m < g - \sqrt{g}$.

Since for slopes smaller than $g - \sqrt{g}$ we are in the effective cone by Lemma 4, we conclude that

$$m_{ef} = m_a = g - \sqrt{g}.$$ 

If $g$ is not a square, choosing $d = \lfloor \sqrt{g} \rfloor$ we have 

$$g - \sqrt{g} \leq m_{ef} \leq g - \lfloor \sqrt{g} \rfloor + 1.$$ 

The estimation of $m_a$ comes from the duality of the ample and effective cone. Of course as $g \to \infty$ we have $m_{ef} \sim m_a \sim g - \sqrt{g}$. Q.E.D.

The case $g = 3$. In this case we can obtain for the general curve an irreducible divisor with slope $\frac{4}{3}$. (See Figure 2.) It is known that a general smooth curve $C$ of genus 3, can be represented as a smooth plane quartic. We construct the following divisor in $C^2$: For each point $P$ in $C$ consider the tangent at that point. This intersects the curve at two additional points $Q, R$. Take now in $C^2$ the divisor $D$ consisting of all the sums $Q + R$ (with $P$ moving on $C$). In order to calculate the class of $D$, we find its intersections with $x$ and $\delta_2$.

Intersection with $x$. The degree of the dual curve is 12. Fixing a point $Q$ on $C$, we ask for the number of tangents to the curve passing through $Q$ (excluding the tangent to the curve at $Q$). These are 10. So $Dx = 10$.

Intersection with $\delta_2$. This is twice the number of bitangents to $C$, so $D\delta_2 = 28 \times 2 = 56$.

Therefore if $D \sim a\theta - bx$ then, $Dx = 10$ i.e. $3a - b = 10$ and $D\theta = 56$ i.e. $2(a\theta - bx)(4x - \theta) = 56$ so $6a - b = 28$. This gives $a = 6$, $b = 8$, i.e. $m_{ef} = \frac{4}{3}$, $m_a = \frac{6}{3}$.

6. One side slope of effective cone in $C^d$

Theorem 3. The boundary of the effective cone in $C^d$, in the second quarter of the $\theta, x$-plane, is given by the class $\delta_d$ of the diagonal $\Delta_d$

$$\delta_d = 2((d + g - 1)x - \theta) \quad \text{i.e.} \quad \overline{m}_{ef} = d + g - 1.$$
Proof (Induction on $d$). For $d = 2$ it has been proved. Assuming it is true for $d$, we prove it for $d + 1$, i.e. we prove that in $C_{d+1}$, $m_{ef} = d + g$.

Suppose that there exists an irreducible divisor $D$ on $C_{d+1}$ with class $mx_{d+1} - \theta_{d+1}$ and $m < d + g$ (we add subindices for avoiding confusion). Fixing a point in $C$, we define the canonical embedding $i_k: C_k \rightarrow C_{d+1}$. Note that $i_k^*(x_{d+1}) = x_k$ and $i_k^*(\theta_{d+1}) = \theta_k$. The image of $C_d$ is an ample divisor on $C_{d+1}$, see [A-C-G-H, p. 310] and so, $i_d^*(D)$ is effective nonzero on $C_d$. Therefore the slope must satisfy $m \geq d + g - 1$.

Since $\Delta_{d+1}$, $D$ are irreducible the intersection $\Delta_{d+1}D$ is nonempty effective. Indeed, $D$, $\Delta$ are not disjoint. Otherwise,
\[m(d + g)x^2_{d+1} - (m + d + g)x_{d+1}x_{d+1} + 8d+1 = 0.\]
Applying $i_2^*$ we get $m(d + g)x^2 - (m + d + g)x_{d+1}x_{d+1} + 8d+1 = 0$. Then formula (1) implies $md = (d + 1)g$. Since $i_2^*(\Delta_{d+1})$ contains $\Delta_2$, $\Delta_2$ must be disjoint from $i_2^*(D) = D_2$. Therefore, $\Delta_2D_2 = 0$ i.e. $(mx_2 - \theta_2)d_2 = 0$ i.e. $m = 2g$, and the above relation becomes $2gd = (d + 1)g$ i.e. $d = 1$ a contradiction.

Therefore $B_1(D\Delta_{d+1})$, see §1 for definition of $B_1$, lies in the effective cone of $C_d$, i.e.
\[m(d + g)x^2_{d+1} - (m + d + g)x_{d+1}x_{d+1} + \theta^2_{d+1} = 0.\]
Applying $i_2^*$ we get $m(d + g)x^2 - (m + d + g)x_{d+1}x_{d+1} + \theta^2 = 0$. Then formula (1) implies $md = (d + 1)g$. Since $i_2^*(\Delta_{d+1})$ contains $\Delta_2$, $\Delta_2$ must be disjoint from $i_2^*(D) = D_2$. Therefore, $\Delta_2D_2 = 0$ i.e. $(mx_2 - \theta_2)d_2 = 0$ i.e. $m = 2g$, and the above relation becomes $2gd = (d + 1)g$ i.e. $d = 1$ a contradiction.

Therefore $B_1(D\Delta_{d+1})$, see §1 for definition of $B_1$, lies in the effective cone of $C_d$, i.e.
\[\text{slope of } B_1(D\Delta_{d+1}) \geq d + g - 1.\]
Now,
\[\text{class}(D\Delta_{d+1}) = m(d + g)x^2_{d+1} - (m + d + g)x_{d+1}\theta_{d+1} + \theta^2_{d+1}.\]
By Lemma 2 we have
\[B_1(x^2_{d+1}) = 2x_d, \quad B_1(x_{d+1}\theta_{d+1}) = \theta_d + g\theta_d, \quad B_1(\theta^2_{d+1}) = 2(g - 1)\theta_d.\]
Therefore,
\[B_1(D\Delta_{d+1}) = (2md + mg - dg - g^2)x_d - (m + d - g + 2)d.\]
Since $m \geq d + g - 1$, both coefficients are positive and so,
\[\text{slope of } B_1(D\delta_{d+1}) = \frac{2md + mg - dg - g^2}{m + d - g + 2}.\]
Relation (11) implies that
\[\frac{2md + mg - dg - g^2}{m + d - g + 2} \geq d + g - 1\]
i.e. $m(d + 1) \geq dg + d^2 + d + 3g - 2$. Since $m < d + g$ we get $(d + g)(d + 1) > dg + d^2 + d + 3g - 2$ or $1 > g$, a contradiction. Q.E.D.

7. Bounds for the effective cone in $C_r, r \geq 3$

We start with $C_3$. Let $D$ be a divisor in $C_3$ with class $\theta - mx$, $m \geq 0$, i.e. it “lies” in the fourth quarter. Since $x$ is the class of $i_2(C_2)$ in $C_3$ and
\[\frac{d}{dm}(\theta - mx)^3 = -3(\theta - mx)^2x,\]
we conclude that
\[\frac{d}{dm}(\theta - mx)^3 = 0 \text{ in } C_3 \iff (\theta - mx)^2 = 0 \text{ in } C_2,\]
i.e. when $m = g + \sqrt{g}$ or $m = g - \sqrt{g}$.
The graph of the $(\theta - mx)^3$ considered as function of $m$ is given by Figure 3.

**Theorem 4.** We denote by $r_1$ the root of $(\theta - mx)^3 = 0$ closest to 0. Then for $m < r_1$, the class $\theta - mx$ is effective.

**Proof.** Note that this equation has three positive roots, see Figure 3 above. Approximately as $g \to \infty$, $r_1$ goes to $g - \sqrt{3g}$. Say $D$ a divisor with class $\theta - mx$ where $m < r_1$. By Theorem 2 the restriction of $D$ to $C_2$ is ample. Since $D^3 > 0$, Theorem 1, applied to $H = C_2$, gives the result. Q.E.D.

It is difficult to continue the above method for higher $r$'s, since we do not have a good estimate for the ample cone in $C_3$. For these cases we have

**Theorem 5.** Let $C$ be a smooth curve of genus $g \geq 1$ with general moduli. For $r \geq 3$ we have the following estimates for the effective and ample cone of $C_r$ in the fourth quarter of the $\theta, x$-plane

1. If $r \geq g + 1$ then the boundary of the effective and ample cone is given by the $\theta$ line.
2. If $3 \leq r \leq g$ then we have the following bounds for the effective cone.
   
   **Bound from inside:** for each rational $m$ with
   
   \[ 0 \leq m \leq \max(\lceil \frac{g}{r} \rceil, g - (r - 1)\sqrt{g}) \]

   there is an effective divisor with slope $m$.

   **Bound from outside:** for each $m$ with $m > g - \lceil \sqrt{r - 1} \sqrt{g} \rceil$ there is no effective divisor with slope $m$.

**Remarks.**

1. For $r = g$ the slope of the boundary of the effective cone is equal to 1. Indeed in this case $\lceil \frac{g}{r} \rceil = 1 = g - \sqrt{r - 1} \sqrt{g}$.

2. For $r \leq \sqrt{g}$ we have that

   \[ \max(\lfloor \frac{g}{r} \rfloor, g - (r - 1)\sqrt{g}) = g - (r - 1)\sqrt{g}. \]

For $r \geq \sqrt{g}$ we have that

\[ \max(\lceil \frac{g}{r} \rceil, g - (r - 1)\sqrt{g}) = \lceil \frac{g}{r} \rceil. \]

**Proof.** The proof of the first part of the theorem is easy: For $r \geq g + 1$ we have that $\theta' = u_\theta^*(-\theta)' = 0$ and so, the class $\theta$ is not ample. On the other hand, since $\theta$ is ample on the Jacobian and $x$ is ample on $C_r$, see [A-C-G-H, p. 310], projection formula and Lemma 3, imply that the class $\theta + \varepsilon x$ is ample, for each $\varepsilon > 0$. Therefore the bound for the ample cone is given by the line $\theta$. Also any divisor with class $\theta - \sigma x$ cannot be effective since there is an $\varepsilon > 0$ small enough with $(\theta + \varepsilon x)^{r-1}(\theta - \sigma x) < 0$; a contradiction by Lemma 3. Therefore the bound of the effective cone is given by the $\theta$ line too.

To prove the second part of the theorem we use again degenerations to special curves. We start with a lemma:
Lemma 9. If a curve $C$ has a “general” $g_d^{-1}$, $r \geq 3$ (i.e. without base points and not composed with an involution), then for any $d \geq r$ the divisor $\Gamma_r(g_d^{-1})$ in $C_r$ is irreducible.

Proof. This is an application of the fact that the monodromy acts as the full symmetric group on the generic divisor of $g_d^r$. Q.E.D.

Let us now do some calculations. Recall from relation (6), that the class of $\gamma_r(g_d^{-1})$ is given by $\theta - (g - d + r - 1)x$. Formula (1) gives that

$$x^k \theta^{r-k} = \frac{g!}{(g-r+k)!} \quad \text{and} \quad x^{k+1} \theta^{r-k-1} = \frac{g!}{(g-r+k+1)!}.$$

Using Lemma 8 we have for $d + 1 \geq 2r$ that

$$\gamma_r(g_d^{r+2}) = \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \frac{x^k \theta^{r-k-1}}{(r-k-1)!}.$$

Therefore intersection number $I = \gamma_r(g_d^{r-1}) \cdot \gamma_r(g_d^{r+2})$ is

$$I = \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \frac{g!}{(g-r+k)!(r-k-1)!}$$

$$- (g-d+r-1) \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \frac{g!}{(g-r+k+1)!(r-k-1)!}$$

$$= g \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \binom{g-1}{r-k-1}$$

$$- (g-d+r-1) \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \binom{g}{r-k-1}$$

$$= g \binom{d-r}{r-1} - (g-d+r-1) \binom{d-r+1}{r-1}$$

$$= A(d^2 - 2d(r-1) - (r-1)(g-r-1)) \quad (A \text{ a positive constant}).$$

This implies that

$$I \leq 0 \Leftrightarrow r-1 - \sqrt{r-1} \sqrt{g} \leq d \leq r-1 + \sqrt{r-1} \sqrt{g}.$$

Take now a smooth curve $C^0$ having a “general” $g_{d_0}^{r-1}$ with $d_0 = r + 1 + \lceil \sqrt{r-1} \sqrt{g} \rceil$ (note that $d_0 \geq r$). We claim that the irreducible divisor $D = \Gamma_r(g_{d_0}^{r-1})$ with class $\gamma_r(g_{d_0}^{r-1}) = \theta - (g - \lceil \sqrt{r-1} \sqrt{g} \rceil)x$ gives the bound for the effective cone in the $\theta$, $x$-plane for $(C^0)_r$. Indeed, note first that the divisor $D$ is covered by a family of curves $\mathcal{E}_{d_0-r+2}$ with class $\gamma_r(g_{d_0-r+2})$. These curves correspond to the various $g_{d_0-r+2}$'s obtained by the one-parameter family of hyperplane sections of the image of the curve in $P^{r-1}$, through collections of $r-2$ fixed points on this curve. Suppose that there exists another irreducible effective divisor $D'$ with slope $m'$ strictly greater than the slope of $D$. Then relation (13) implies that $D' \cdot \gamma_r(g_{d_0-r+2}) < 0$ and so, since $D'$ is irreducible we get that all the members of $\mathcal{E}_{d_0-r+2}$ are contained in $D'$. But since the divisor
D' is "covered" by the family \( \mathcal{C}_{d_0-r+2} \) this implies that \( D \) is contained in \( D' \). A contradiction. The rest of the proof for the bound from outside goes, using degenerations after a possible base change, as the proof of Theorem 2.

To prove the case for the bound from inside, we use the maps \( A_k \) defined in the introduction of this paper. From formula (2) we have

\[
(14) \quad A_{r-2}(x_r) = (r-1)x_2 \quad \text{and} \quad A_{r-2}(\theta_r) = \theta_2 + g(r-2)x_2.
\]

Since in \( C_2 \) the slope \( m \) with \( m < g - \sqrt{g} \) is "effective", pulling back by the \( A_{r-2} \) map we get in \( C_r \) that

\[
(15) \quad A_{r-2}(\theta_r - (g - \sqrt{g})x_r) = \theta_2 - (g - (r-1)\sqrt{g})x_2.
\]

On the other hand we have that for \( d = g + r - 1 - \lfloor \frac{g}{r} \rfloor \), a general curve has a \( g_d^{-1} \) (this is the minimum \( d \) for which the Brill-Noether number \( \rho \) is nonnegative). Using formula (6) we obtain the following class of an effective divisor in \( C_r \):

\[
(16) \quad \gamma_r(g_d^{-1}) = \theta - \lfloor \frac{g}{r} \rfloor x
\]

and this concludes the proof. Q.E.D.

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References


Department of Mathematics, University of Pennsylvania, David Rittenhouse Laboratory, Philadelphia, Pennsylvania 19104-6395

E-mail address: alexk@pennsas.upenn.edu