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NONWANDERING STRUCTURES AT THE PERIOD-DOUBLING LIMIT IN DIMENSIONS 2 AND 3

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ABSTRACT. A Cantor set supporting an adding machine is the simplest nonwandering structure that can occur at the conclusion of a sequence of period-doubling bifurcations of plane homeomorphisms. In some families this structure is persistent. In this manuscript it is shown that no plane homeomorphism has nonwandering Knaster continua on which the homeomorphism is semiconjugate to the adding machine. Using a theorem of M. Brown, a three-space homeomorphism is constructed which has an invariant set, Λ, the product of a Knaster continuum and a Cantor set. Λ is chainable, supports positive entropy but contains only power-of-two periodic orbits. And the homeomorphism restricted to Λ is semiconjugate to the adding machine. Lastly, a zero topological entropy C^∞ disk diffeomorphism is constructed which has large nonwandering structures over a generalized adding machine on a Cantor set.

In 1976 R. Bowen and J. Franks (partially) answered a question of S. Smale by constructing a C^1 diffeomorphism of the two-sphere that is Kupka-Smale and has no periodic sources or sinks [B-F, S1]. J. Franks and L-S. Young subsequently constructed a C^2 model [F-Y], and more recently J. Gambaudo, S. van Strien, and C. Tresser have applied renormalization theory to demonstrate the existence of a C^∞ example [G-vS-T].

Each of these examples has a periodic hyperbolic saddle of period 2^n for each $n \geq 0$, no other periodic orbits, and an invariant Cantor set contained in the closure of the periodic set. On this Cantor set the diffeomorphism acts as a minimal "adding machine." This is the simplest nonwandering structure on a two-sphere or two-disk that can occur at the conclusion of a sequence of period-doubling bifurcations. And at least in some families, this structure is persistent [G-vS-T].

In the next several paragraphs we discuss our motivations for studying the following question: Can large compact connected sets (continua) be "shuttled" about over an adding machine by a two- or three-space homeomorphism or diffeomorphism? In §1, we construct a positive entropy homeomorphism, F, of three-space having an invariant set, Λ_0, which is an imbedded product of the Cantor set and the Knaster "bucket-handle" continuum. (In the usual Smale horseshoe [S2], the closure of any of the unstable manifolds of the periodic saddles is such a continuum.) Furthermore, $F|\Lambda_0$ is semiconjugate to the adding machine on two symbols, and the set of periodic orbits, $\operatorname{per} F|\Lambda_0$, contains

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only power-of-2 periodics. In §2 we show that two-space cannot support such dynamics; in fact, no plane homeomorphism shuttles Knaster continua over an adding machine (Theorem 2.2). However, large continua can be shuttled about over other minimal actions on the Cantor set by a (zero entropy) C^∞ plane diffeomorphism (§3).

In 1980 A. Katok proved a more general version of the following theorem [Ka]:

Theorem. *If f is a $C^{1+\alpha}$ surface diffeomorphism with positive topological entropy, then there exists an invariant Cantor set on which some power of f is conjugate to a full two-shift.*

So in particular, positive entropy smooth enough plane diffeomorphisms have periodic orbits of period other than a power of 2. Whether or not C^1 is sufficient in the above theorem remains open. M. Rees has constructed a positive entropy homeomorphism of the two-torus that has no periodic orbits [R]. But an interesting question remains: what can be said about the periodic structure (or nonwandering structure) of positive entropy *homeomorphisms* of the two-disk or two-sphere? For example, is it possible to construct a positive entropy homeomorphism of the two-disk that has the periodic structure of the Bowen-Franks model by blowing up the minimal set? After blowing up, the new model will have the same periodic behavior as the Bowen-Franks model and thus contain no shift maps. A partial answer is provided by Theorem 2.2; the new invariant set cannot be the imbedded product of a Cantor set with a Knaster continuum (or other continua such as the interval which have a similar property). But it is not known, even in the C^0 case, whether points of the Bowen-Franks minimal set can be blown up into Cantor set fibers having diameters bounded above zero.

A smooth version of our three-space homeomorphism would shed light on the nature of a three-dimensional analogue to Katok's theorem.

"Generalized adding machines" are minimal actions at the limits of sequences of cyclic permutations of successively smaller subblocks of the Cantor set. In §3, we construct a C^∞ plane diffeomorphism which shuttles continua with diameters bounded above zero over a generalized adding machine. The invariant set of this zero entropy diffeomorphism contains arbitrarily close fibers which have different topological type. (We do not know whether positive entropy C^0 analogues exist.)

Conjecture. No plane homeomorphism can have an invariant set that is the product of a Cantor set with a nondegenerate continuum such that, restricted to this invariant set, the homeomorphism is semiconjugate to a generalized adding machine.

Our three-space construction settles a question interesting from the point of view of dynamics on continua. A small perturbation of the Bowen-Franks disk diffeomorphism, $f: D \to D$, has an attracting set, S, with particularly nice properties. S is the closure of the unstable manifold of a fixed Mobius saddle, and consists of the union of the unstable manifolds of all periodic points together with a minimal Cantor set (see Figure 1).

One can show that S is "chainable;" that is, S can be mapped onto an arc such that the diameters of point pre-images are as small as desired. $f|S$ has zero entropy. It is known that shift homeomorphisms with positive entropy of

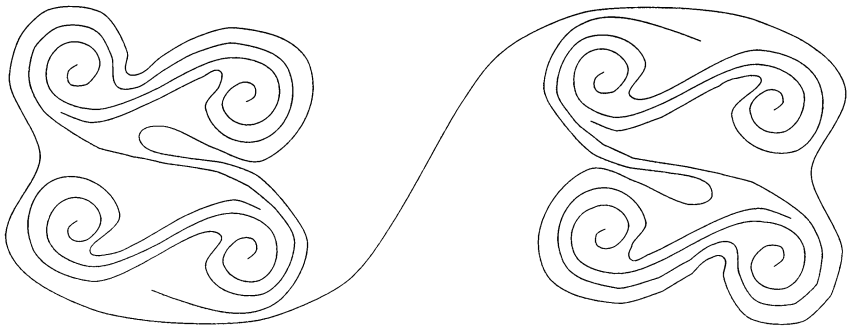

FIGURE 1. The attracting set in the Bowen-Franks model.

inverse limit spaces of maps of the interval (these are chainable continua) must contain periodic points of period other than power of 2. M. Barge conjectured at the 1987 Spring Topology Conference in Birmingham, Alabama, that such was also the case for arbitrary homeomorphisms of chainable continua.

Our three-space homeomorphism, F, has an attracting continuum, $\Lambda \subset \mathbb{R}^3$, that resembles S except that the minimal Cantor set is replaced by Λ_0, the invariant product of a Knaster continuum with a Cantor set. F is an extension of the shift map on an inverse limit space with base space a Cantor comb. Λ is chainable, contains periodic points only of period a power of 2, and $F|\Lambda$ has positive topological entropy. Being chainable, Λ can be imbedded in the plane. But by Theorem 2.2, $F|\Lambda_0$ cannot be extended onto the plane as a homeomorphism thus neither may $F|\Lambda$.

1. CONSTRUCTION OF THE EXAMPLE IN $\mathbb{R}^3$

Notation and definitions. Let Σ be the Cantor set

$$\{(a_1, a_2, \dots): a_n \in \{0, 1\}, n \in \mathbf{N}\}$$

equipped with the product topology. Denote by

$$C = \left\{x \in [0, 1]: x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, b_n \in \{0, 2\}\right\}$$

the standard middle-thirds Cantor set. The map $\phi: \Sigma \to C$ given by

$$\phi(a_1, a_2, \dots) = \sum_{n=1}^{\infty} \frac{b_n}{3^n},$$

where $b_n = 0$ if $a_n = 1$ and $b_n = 2$ if $a_n = 0$ is a homeomorphism. Given $(a_1, a_2, \dots, a_n) \in \{0, 1\}^n$, let $[a_1, a_2, \dots, a_n] \subset \Sigma$ be the cylinder set or subblock,

$$[a_1, a_2, \dots, a_n] = \{(b_1, b_2, \dots) \in \Sigma: b_i = a_i \text{ for } i = 1, 2, \dots, n\},$$

and let $C[a_1, a_2, \dots, a_n] = \phi([a_1, \dots, a_n])$. The convex hull of

$$C[a_1, a_2, \dots, a_n] \subset [0, 1]$$

will be denoted by $C\langle a_1, a_2, \dots, a_n\rangle$. Note that $C\langle a_1, a_2, \dots, a_n\rangle$ is the closed interval

$$[\phi(a_1, \dots, a_n, 1, 1, \dots), \phi(a_1, \dots, a_n, 0, 0, \dots)]$$

of length $\frac{1}{3^n}$.

We will use the following notation for the "gaps" of the Cantor set C:

$$G\langle 2\rangle = [\tfrac{1}{3}, \tfrac{2}{3}],$$

$$G\langle a_1, a_2, \dots, a_n, 2\rangle = [\phi(a_1, \dots, a_n, 1, 0, 0, \dots), \phi(a_1, \dots, a_n, 0, 1, 1, \dots)], \quad \text{for } n \geq i \text{ and } (a_1, \dots, a_n) \in \{0, 1\}^n.$$

Note that

$$C\langle a_1, \dots, a_n\rangle = C\langle a_1, \dots, a_n, 1\rangle \cup G\langle a_1, \dots, a_n, 2\rangle \cup C\langle a_1, \dots, a_n, 0\rangle.$$

The adding machine and "lightning bolt". For each $n \geq 1$ let $A_n\colon \{0, 1\}^n \to \{0, 1\}^n$ be the cyclic permutation given by $A_n(a_1, \dots, a_n) = (b_1, \dots, b_n)$ where $b_1 = a_1 + 1 \pmod 2$, and

$$b_i = \begin{cases} a_i + 1 \pmod 2; & \text{if } a_{i-1} = 1 \text{ and } b_{i-1} = 0, \\ a_i; & \text{otherwise}, \end{cases}$$

for $2 \leq i \leq n$. Let $A\colon \Sigma \to \Sigma$ be given by $A(a_1, a_2, \dots) = (b_1, b_2, \dots)$ where $(b_1, b_2, \dots, b_n) = A_n(a_1, a_2, \dots, a_n)$ for all $n \geq 1$; A is a minimal homeomorphism (all orbits dense) called the (binary) adding machine.

We now define a map $l\colon I \to I$ of the unit interval $I = [0, 1]$ which restricts to an adding machine on C (l for "lightning bolt," see [N]). Let $l_1\colon [\frac{1}{3}, 1] \to [0, \frac{7}{9}]$ be defined by

$$l_1(x) = \begin{cases} \frac{-7}{3}x + \frac{14}{9}; & \frac{1}{3} \leq x \leq \frac{2}{3}, \\ x - \frac{2}{3}; & \frac{2}{3} \leq x \leq 1. \end{cases}$$

Define $l\colon I \to I$ by

$$l(x) = \begin{cases} l_1(x); & \frac{1}{3} \leq x \leq 1, \\ (\frac{1}{3^n})[l_1(3^n x) + 3^n - 1]; & \frac{1}{3^{n+1}} \leq x \leq \frac{1}{3^n}, \ n \geq 0, \\ 1; & x = 0. \end{cases}$$

See Figure 2.

We summarize several useful properties of l in the following lemma. The proofs are straightforward and are left to the reader.

Lemma 1.1. *the map l satisfies*:

(i) $l(C) = C$ *and* $l|_C = \phi \circ A \circ \phi^{-1}$;

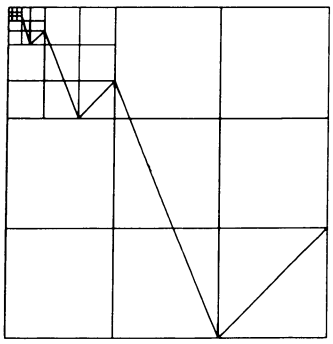

FIGURE 2. Graph of l

(ii) *for each* $n \geq 0$, l *has a single periodic orbit of period* 2^n *lying in the union of gaps,* $\bigcup\{G\langle a_1, \dots, a_n, 2\rangle : (a_1, \dots, a_n) \in \{0, 1\}^n\}$, *and* l *has no other periodic orbits*;

(iii) *for each* $n \geq 1$ *and* $(a_1, \dots, a_n) \in \{0, 1\}^n$, $l(C\langle a_1, \dots, a_n\rangle) = C\langle b_1, \dots, b_n\rangle$ *where* $(b_1, \dots, b_n) = A_n(a_1, \dots, a_n)$;

(iv) $l|_{G\langle 2\rangle}$ *is monotone decreasing, and for each* $n \geq 1$ *and* $(a_1, \dots, a_n) \in \{0, 1\}^n$, $l|_{G\langle a_1, \dots, a_n, 2\rangle}$ *is monotone with*

$$G\langle b_1, \dots, b_n, 2\rangle \subset l(G\langle a_1, \dots, a_n, 2\rangle),$$

where $(b_1, \dots, b_n) = A_n(a_1, \dots, a_n)$;

(v) $l^2(G\langle 2\rangle) = I$, *and* $l^{2^{n+1}}(G\langle a_1, \dots, a_n, 2\rangle) = C\langle a_1, \dots, a_n\rangle$, *for each* $n \geq 1$ *and* $(a_1, \dots, a_n) \in \{0, 1\}^n$.

A collection $\{C_i\}_{i=1}^n$ of nonempty open sets in a continuum X is called an *ε-chain* provided $\operatorname{diam}(C_i) < \varepsilon$ for $i = 1, \dots, n$, $C_i \cap C_j = \varnothing$ for $|i - j| \geq 2$, and $C_i \cap C_{i+1} \neq \varnothing$, for $i = 1, \dots, n-1$.

If there is an ε-chain in X that covers X, then we say that X is *ε-chainable*; X is *chainable* if X is ε-chainable for ech $\varepsilon > 0$. The reader can check that if Y is a subcontinuum of X and Y is ε-chainable (in the restricted metric), then there is an ε-chain consisting of open sets in X which covers Y.

An inverse limit space on the Cantor comb. We now construct a chainable inverse limit space, (E, f), where E is the Cantor comb, $E = \{C \times I\} \cup \{I \times \{0\}\} \subset \mathbb{R}^2$. We then show that the shift homeomorphism, $\hat{f} : (E, f) \to (E, f)$ "shuttles" Knaster continua over an adding machine and has positive topological entropy. Later $\hat{f}$ will be extended to a homeomorphism of $\mathbb{R}^3$.

Let $t : I = [0, 1] \to [0, 1]$ be the "tent map" defined by

$$t(x) = \begin{cases} 2x; & 0 \leq x \leq \frac{1}{2}, \\ 2 - 2x; & \frac{1}{2} \leq x \leq 1. \end{cases}$$

The inverse limit space (I, t) is homeomorphic with the indecomposable Knaster "bucket handle" continuum.

We now define a map $g : E \to E$ which folds parts of gaps, (components of $(I - C) \times \{0\}$) into neighboring "endhairs" of $C \times I$. The contribution of g to the map $f : E \to E$ is the mechanism by which (E, f) becomes chainable. Let $g =$ identity on $C \times I$. For each $n = 1, 2, \dots$ and on each gap $G\langle a_1, \dots, a_{n-1}, 2\rangle \times \{0\} = [\frac{m}{3^n}, \frac{m+1}{3^n}] \times \{0\}$ define

$$g(x, 0) = \begin{cases} \left(\dfrac{m}{3^n}, x - \dfrac{m}{3^n}\right); & \dfrac{m}{3^n} \leq x \leq \dfrac{6m+1}{2 \cdot 3^{n+1}}, \\ \left(\dfrac{m}{3^n}, \dfrac{3m+1}{3^{n+1}} - x\right); & \dfrac{6m+1}{2 \cdot 3^{n+1}} \leq x \leq \dfrac{3m+1}{3^{n+1}}, \\ \left(3(x - \dfrac{2m+1}{2 \cdot 3^n}) + \dfrac{2m+1}{2 \cdot 3^n}, 0\right); & \dfrac{3m+1}{3^{n+1}} \leq x \leq \dfrac{3m+2}{3^{n+1}}, \\ \left(\dfrac{m+1}{3^n}, x - \dfrac{3m+2}{3^{n+1}}\right); & \dfrac{3m+2}{3^{n+1}} \leq x \leq \dfrac{6m+5}{2 \cdot 3^{n+1}}, \\ \left(\dfrac{m+1}{3^n}, \dfrac{m+1}{3^n} - x\right); & \dfrac{6m+5}{2 \cdot 3^{n+1}} \leq x \leq \dfrac{m+1}{3^n}. \end{cases}$$

So g "folds" the two end thirds of each gap into the adjacent teeth of the comb.

Finally, define $f: E \to E$ by

$$f(x, y) = g(l(x), t(y)).$$

Theorem 1.2. *The inverse limit space (E, f) is a chainable continuum, the shift homeomorphism $\hat{f}: (E, f) \to (E, f)$ has a single periodic orbit of period 2^n for each $n \leq 0$, and no other periodic orbits, and the topological entropy of $\hat{f}$ is* $\log 2$.

Before proceeding to the proof of Theorem 1.2, we first establish the following lemma which we will use to show (E, f) is chainable.

Lemma 1.3. *Suppose the continuum X is the union of the pairwise disjoint subsets A, G, and B. Assume A and B are subcontinua of X, and that $G = \gamma(\mathbb{R})$ is the continuous image of an injection $\gamma: \mathbb{R} \to X$ such that $A = \bigcap_{T \geq 0} \mathrm{cl}(\gamma(-\infty, -T])$ and $B = \bigcap_{T \geq 0} \mathrm{cl}(\gamma[T, \infty))$. Then if A and B are ε-chainable, so is X.*

Proof. Let $\{C_i\}_{i=1}^n$ and $\{C_i'\}_{i=1}^m$ be ε-chains in X covering A and B respectively. Since A and B are disjoint closed subsets, we may assume $(\bigcup_{i=1}^n C_i) \cap (\bigcup_{i=1}^m C_i) = \varnothing$. Select T_1 and T_2 so that $\gamma(-\infty, -T_1) \subset \bigcup_{i=1}^n C_i$, $\gamma(-T_1) \in C_n$, $\gamma(T_2) \in C_1'$, and $\gamma[T_2, \infty) \subset \bigcup_{i=1}^m C_i'$. (This may require renumbering the elements of the two chains in the opposite order.) Let

$$T_1' = \inf\{t \in (-\infty, T_1): \gamma[-T, -t) \subset C_n\}$$

and let

$$T_2' = \inf\{t \in (-\infty, T_2): \gamma(t, T_2] \subset C_1'\}.$$

Let $0 < \delta < \min\{T_1 - T_1', T_2 - T_2'\}$ be small enough so that if $t_1, t_2 \in [-T_1, T_2]$ and $|t_1 - t_2| < 2\delta$, then $d(\gamma(t_1), \gamma(t_2)) < \varepsilon$. Now let $B_1, \dots, B_k$ be open intervals in $\mathbb{R}$ of radius $\leq \delta$ such that B_1 is centered at $-T_1'$ but $-T_1' \notin B_2$, B_k is centered at T_2' but $T_2' \notin B_{k-1}$, $B_i \cap B_j = \varnothing$ for $|i - j| \geq 2$, and $[-T_1', T_2'] \subset \bigcup_{i=1}^k B_i$.

The ε-chain of X can now be specified. For $i = 1, \dots, n-1$, let $U_i = C_i - \gamma[-T_1, \infty)$; let $U_n = C_n \backslash \gamma[-T_1', \infty)$; for $i = 1, \dots, k$, let $U_{n+i} = \gamma(B_i)$, let $U_{n+k+1} = C_1' - \gamma(-\infty, T_2']$; and for $i = 2, \dots, m$, let $U_{n+k+i} = C_i' - \gamma(-\infty, T_2]$. Then $\{U_i\}_{i=1}^{n+k+m}$ is an ε-chain covering X. □

Proof of Theorem 1.2. By Lemma 1.1(iii), $l^{2^n}(C\langle \mathbf{a} \rangle) = C\langle \mathbf{a} \rangle$ for each $\mathbf{a} \in \{0, 1\}^n$. Thus $\{(x, y) \in E: x \in C\langle \mathbf{a} \rangle\} = (C\langle \mathbf{a} \rangle \times I) \cap E$ is invariant under f^{2^n}. Now for each $\mathbf{a} \in \{0, 1\}^n$, let

$$\widehat{C}\langle \mathbf{a} \rangle = \{((x_0, y_0), (x_1, y_1), \dots) \in (E, f): x_{k2^n} \in C\langle \mathbf{a} \rangle \text{ for all } k \geq 0\}.$$

So $\widehat{C}\langle \mathbf{a} \rangle$ is a subcontinuum of (E, f). Next we identify an infinite collection of copies of $\mathbb{R}$ which resemble the invariant manifolds in the Bowen-Franks model [BF]. By Lemma 1.1(iv), there exists an interval, $J\langle 2 \rangle \subset \operatorname{int} G\langle 2 \rangle$ and intervals, $j\langle \mathbf{a}, 2 \rangle \subset \operatorname{int} G\langle \mathbf{a}, 2 \rangle$ for each $n \geq 1$ and $\mathbf{a} \in \{0, 1\}^n$, with these properties: $f|_{J\langle 2 \rangle \times \{0\}}$ and $f^{2^n}|_{J\langle 2 \rangle \times \{0\}}$ are injections,

$$f(J\langle 2 \rangle \times \{0\}) = G\langle 2 \rangle \times \{0\},$$
$$f^{-1}(G\langle 2 \rangle \times \{0\}) \cap (G\langle 2 \rangle \times \{0\}) = J\langle 2 \rangle \times \{0\},$$
$$f^{2^n}(J\langle \mathbf{a}, 2 \rangle \times \{0\}) = G\langle \mathbf{a}, 2 \rangle \times \{0\},$$

and

$$f^{-2^n}(G\langle \mathbf{a}, 2\rangle \times \{0\}) \cap (G\langle \mathbf{a}, 2\rangle \times \{0\}) = J\langle \mathbf{a}, 2\rangle \times \{0\}.$$

Thus for each $\mathbf{a} \in \{0, 1\}^n$,

$$\{\mathbf{z} \in (z_0, z_1, \dots) \in (E, f) \colon z_i \in G\langle 2\rangle \times \{0\}, i \geq 0\}$$

and

$$\{\mathbf{z} = (z_0, z_1, \dots) \in (E, f) \colon z_{k2^n} \in G\langle \mathbf{a}, 2\rangle \times \{0\}, k \geq 0\}$$

are arcs. So the sets

$$\widehat{G}\langle 2\rangle = \bigcup_{i=0}^{\infty} \hat{f}^i \{\mathbf{z} \in (E, f) \colon z_i \in G\langle 2\rangle \times \{0\}, \text{ all } i \geq 0\}$$

and

$$\widehat{G}\langle \mathbf{a}, 2\rangle = \bigcup_{i=0}^{\infty} \hat{f}^{i2^n} \{\mathbf{z} \in (E, f) \colon z_{k2^n} \in G\langle \mathbf{a}, 2\rangle \times 0, \text{ all } k \geq 0\},$$

for each $\mathbf{a} \in \{0, 1\}^n$, are images of continuous injective parametrizations,

$$\gamma \colon \mathbb{R} \to \widehat{G}\langle 2\rangle \quad \text{and} \quad \gamma_{\mathbf{a}} \colon \mathbb{R} \to \widehat{G}\langle \mathbf{a}, 2\rangle,$$

respectively.

We now show that the immersed lines $\widehat{G}\langle 2\rangle$ and $\widehat{G}\langle \mathbf{a}, 2\rangle$ have additional properties, so that Lemma 1.3 can be applied. Let $\pi_1, \pi_2 \colon E \to [0, 1]$ be the projections onto first and second coordinates, respectively. Notice that

$$\pi_1 \circ f(G\langle 2\rangle \times \{0\}) \subset (C\langle 1\rangle \times \{0\}) \cup (G\langle 2\rangle \times \{0\}) \cup (C\langle 0\rangle \times \{0\}),$$
$$\pi_1 \circ f(C\langle 1\rangle \times \{0\}) = C\langle 0\rangle \times \{0\},$$

and

$$\pi_1 \circ f(C\langle 0\rangle \times \{0\}) = C\langle 1\rangle \times \{0\}.$$

And for each $n \geq 1$ and $\mathbf{a} \in \{0, 1\}^n$, notice that

$$\pi_1 \circ f^{2^n}(G\langle \mathbf{a}, 2\rangle \times \{0\}) \subset (C\langle \mathbf{a}, 1\rangle \times \{0\}) \cup (G\langle \mathbf{a}, 2\rangle \times \{0\}) \cup (C\langle \mathbf{a}, 0\rangle \times \{0\}),$$
$$\pi_1 \circ f^{2^n}(C\langle \mathbf{a}, 1\rangle \times \{0\}) = C\langle \mathbf{a}, 0\rangle \times \{0\},$$

and

$$\pi_1 \circ f^{2^n}(C\langle \mathbf{a}, 0\rangle \times \{0\}) = C\langle \mathbf{a}, 1\rangle \times \{0\}.$$

Thus after first composing γ and the $\gamma_{\mathbf{a}}$ with an orientation reversing homeomorphism if necessary,

$$\bigcap_{T \geq 0} \mathrm{cl}(\gamma(-\infty, -T]) \subset \widehat{C}\langle 1\rangle,$$
$$\bigcap_{T \geq 0} \mathrm{cl}(\gamma[T, \infty)) \subset \widehat{C}\langle 0\rangle,$$
$$\bigcap_{T \geq 0} \mathrm{cl}(\gamma_{\mathbf{a}}(-\infty, -T]) \subset \widehat{C}\langle \mathbf{a}, 1\rangle,$$

and

$$\bigcap_{T \geq 0} \mathrm{cl}(\gamma_{\mathbf{a}}([T, \infty)) \subset \widehat{C}\langle \mathbf{a}, 0\rangle.$$

To apply Lemma 1.3, we need to show that these inclusions are equalities. For $\mathbf{b} \in \Sigma$, let $K_{\mathbf{b}} = \{((x_0, y_0), (x_1, y_1), \dots) \in (E, f): x_i \in \phi A^{-i}\mathbf{b}, i = 0, 1, \dots\}$. Then $\hat{\pi}_2: K_{\mathbf{b}} \to (I, t)$ defined by $\hat{\pi}_2((x_0, y_0), (x_1, y_1), \dots) = (y_0, y_1, \dots)$ is a shift-commuting homeomorphism. Let $\mathbf{a} \in \{0, 1\}^n$ and let $\mathbf{z} \in \widehat{C}\langle \mathbf{a}, 1\rangle$. Suppose first that $\mathbf{z} = ((x_0, y_0), (x_1, y_1), \dots) \in K_{\mathbf{b}}$. Then $b_i = a_i$ for $i = 1, 2, \dots, n$ and $b_{n+1} = 1$. For each N let $\mathbf{z}' = \mathbf{z}'(N) = ((x_0, y_0'), (x_1, y_1'), \dots) \in K_{\mathbf{b}}$ where $y_i' = y_i$ for $i = 0, 1, \dots, N$ and $y_{i+1}' - \frac{1}{2}y_i'$ for all $i \geq N$.

Now since $A^{2^{n+1}}$ restricted to $[a_1, a_2, \dots, a_n, 1] \subset \Sigma$ is minimal, there is a sequence $\{m_i \to \infty\}$ so that $A^{m_i 2^{n+1}}(a_1, a_2, \dots, a_n, 1, 0, \dots) \to \mathbf{b}$ as $i \to \infty$. For each N let $i = i(N)$ be large enough that $(\frac{1}{2})^{m_i 2^{n+1} - N} \leq \frac{1}{2 \cdot 3^{n+1}}$. And let $\mathbf{z}'' = \mathbf{z}''(N) = ((x_0', y_0'), (x_1', y_1'), \dots) \in \widehat{C}\langle \mathbf{a}, 1\rangle$ be the point with $x_j' = \phi(A^{-j}\mathbf{b}') = \phi(A^{m_i 2^{n+1} - j}\mathbf{a})$ for $j = 0, 1, 2, \dots$. That is, $\mathbf{z}'' \in K_{\mathbf{b}}$ where $\mathbf{b}' = A^{m_i 2^{n+1}}(a_1, a_2, \dots, a_n, 1, 0, 0, \dots)$, and $\hat{\pi}_2(z'') = \hat{\pi}_2(\mathbf{z}') = (y_0', y_1', \dots)$. Since $y_{N+k}' = (\frac{1}{2})^k y_N' \leq (\frac{1}{2})^k$ and $(\frac{1}{2})^{m_i 2^{n+1} - N} \leq \frac{1}{2 \cdot 3^{n+1}}$, we see that $y_j' \leq \frac{1}{2 \cdot 3^{n+1}}$ for $j = m_i 2^{n+1}$. For this j, $x_j' = \phi(a_1, a_2, \dots, a_n, 1, 0, 0, \dots)$. Thus $(x_j', y_j') \in f(G\langle \mathbf{a}, 2\rangle \times \{0\})$.

So there exists a point $\mathbf{z}''' = \mathbf{z}'''(N) = ((x_0'', y_0''), \dots) \in \widehat{G}\langle \mathbf{a}, 2\rangle$ with $(x_k'', y_k'') = (x_k', y_k')$ for $k = 0, \dots, m_i 2^{n+1}$. Since $\mathbf{z}'''(N) \to \mathbf{z}$ as $N \to \infty$, $\mathbf{z} \in \mathrm{cl}(\widehat{G}\langle \mathbf{a}, 2\rangle)$. Thus $K_{\mathbf{b}} \subset \mathrm{cl}(\widehat{G}\langle \mathbf{a}, 2\rangle)$ for all $\mathbf{b} = (b_1, b_2, \dots) \in \Sigma$ for which $b_i = a_i$ for $i = 1, \dots, n$ and $b_{n+1} = 1$.

If $\mathbf{z} = (z_0, z_1, \dots) \in \widehat{C}\langle \mathbf{a}, 1\rangle$ and $\mathbf{z} \notin K_{\mathbf{b}}$ for any $K_{\mathbf{b}}$, then $z_{j \cdot 2^{n+1}} \in C\langle \mathbf{a}, 1\rangle \times \{0\}$ for all sufficiently large j and $\mathbf{z} \in \mathrm{cl}\,\widehat{G}\langle \mathbf{a}, 2\rangle$ since $f^{2^{n+1}}(G\langle \mathbf{a}, 2\rangle \times \{0\})$ contains $C\langle \mathbf{a}, 1\rangle \times \{0\}$. Therefore $\widehat{C}\langle \mathbf{a}, 1\rangle \subset \mathrm{cl}\,\widehat{G}\langle \mathbf{a}, 2\rangle$. Similarly $\widehat{C}\langle \mathbf{a}, 0\rangle \subset \mathrm{cl}\,\widehat{G}\langle \mathbf{a}, 2\rangle$ and $\widehat{C}\langle 1\rangle \cup \widehat{C}\langle 0\rangle \subset \mathrm{cl}\,\widehat{G}\langle 2\rangle$.

Before continuing with the proof of Theorem 1.2 we first prove an additional lemma. In the discussion to follow we will use the metric on (E, f) given by

$$d(((x_0, y_0), (x_1, y_1), \dots), ((x_0', y_0'), (x_1', y_1'), \dots)) = \sum_{i=0}^{\infty} \frac{|x_i - x_i'|}{2^i} + \sum_{i=0}^{\infty} \frac{|y_i - y_i'|}{2^i}.$$

Lemma 1.4. *Given $\varepsilon > 0$, there exists $n = n(\varepsilon)$ such that $\widehat{C}\langle \mathbf{a}\rangle$ is ε-chainable for all $\mathbf{a} \in \{0, 1\}^n$.*

Proof. Let $f(x, y) = (f_1(x, y), f_2(x, y))$. Then $|f_2(x_1, y_1) - f_2(x_2, y_2)| \leq \max\{2|y_1 - y_2|, \frac{1}{2 \cdot 3^{n+1}}\}$ for any pair $(x_1, y_1), (x_2, y_2) \in E$ where $x_1, x_2 \in C\langle \mathbf{a}\rangle$ and $\mathbf{a} \in \{0, 1\}^n$. Thus for any such pair, if $|y_1 - y_2| < \delta$, then $|(f^k)_2(x_1, y_1) - (f^k)_2(x_2, y_2)| \leq \max\{2^k\delta, \frac{k}{2 \cdot 3^{n+1}}\}$. Now let $\varepsilon > 0$ be given, and let n be large enough that $\frac{2}{3^n} < \frac{\varepsilon}{8}$ and $\frac{3 - \log_2 \varepsilon}{2 \cdot 3^{n+1}} < \frac{\varepsilon}{8}$. Let $m = [|3 - \log_2 \varepsilon|]$ so that $\frac{m}{2 \cdot 3^{n+1}} < \frac{\varepsilon}{8}$ and $\frac{1}{2^{m-1}} \leq \frac{\varepsilon}{2}$. Now for $\mathbf{a} \in \{0, 1\}^n$, we will show that $\widehat{C}\langle \mathbf{a}\rangle$ is ε-chainable. Let $B_1, \dots, B_k$ be (relatively) open intervals in $[0, 1]$ of diameter $< \delta$ with $\bigcup_{i=1}^k b_i = [0, 1]$ and $B_i \cap B_j = \varnothing$ for $|i - j| \geq 2$. Here $\delta > 0$ is small enough that $2^m\delta < \frac{\varepsilon}{8}$. For $i = 1, 2, \dots, k$, let $C_i = \{((x_0, y_0), (x_1, y_1), \dots)\} \in \widehat{C}\langle \mathbf{a}\rangle: y_m \in B_i\}$. Then C_i is open for each i and $C_i \cap C_j = \varnothing$ for $|i - j| \geq 2$.

Moreover, if $\mathbf{z} = ((x_0, y_0), (x_1, y_1), \dots)$ and $\mathbf{z}' = ((x_0', y_0'), (x_1', y_1'), \dots)$ are elements of C_i, then

$$\begin{aligned} d(\mathbf{z}, \mathbf{z}') &\le \sum_{j=0}^{m} \frac{|x_i - x_j'|}{2^j} + \sum_{j=0}^{m} \frac{|y_i - y_i'|}{2^j} + \frac{1}{2^{m-1}} \\ &\le \frac{2}{3^n} + \sum_{j=0}^{m} \frac{\max\{2^{m-j}\delta, \frac{m-j}{2\cdot 3^{n+1}}\}}{2^j} + \frac{1}{2^{m-1}} \\ &< 2\left(\frac{\varepsilon}{8}\right) + 2\left(\frac{\varepsilon}{8}\right) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $\operatorname{diam}(C_i) < \varepsilon$, and $\{C_i\}_{i=1}^k$ is an ε-chain covering $\widehat{C}\langle \mathbf{a} \rangle$. □

We now complete the proof of Theorem 1.2. To show that (E, f) is chainable let $\varepsilon > 0$ and let n be sufficiently large so that $\widehat{C}\langle \mathbf{a} \rangle$ is ε-chainable for all $\mathbf{a} \in \{0, 1\}^n$ (Lemma 1.4). Then $\widehat{C}\langle \mathbf{b} \rangle = \widehat{C}\langle \mathbf{b}, 1 \rangle \cup \widehat{G}\langle \mathbf{b}, 2 \rangle \cup \widehat{C}\langle \mathbf{b}, 0 \rangle$ is ε-chainable for all $\mathbf{b} \in \{0, 1\}^{n-1}$ by Lemma 1.3. Continuing in this way, we find that $\widehat{C}\langle 1 \rangle$ and $\widehat{C}\langle 0 \rangle$ are ε-chainable. So again by Lemma 1.3, $\{E, f\} = \widehat{C}\langle 1 \rangle \cup \widehat{G}\langle 2 \rangle \cup \widehat{C}\langle 0 \rangle$ is ε-chainable.

Since the adding machine has no periodic orbits, $\operatorname{per}(f) \subset I \times \{0\}$. These are the periodic orbits of $(x, 0) \to g(l(x), 0)$. It follows from Lemma 1.1 and the construction of g, that f and hence $\hat{f}$ have a single periodic orbit of period 2^n for each $n \ge 0$ and no other periodic orbits.

That the topological entropy of $\hat{f}$, $h(\hat{f})$, is $\log 2$ can be deduced as follows (see, for example [Bo]):

$$h(\hat{f}) = h(f) = h\left(f|_{C \times [0,1]}\right) = h(A) + h(t) = 0 + \log 2. \quad \square$$

Since (E, f) is chainable, there is an embedding of (E, f) into $\mathbb{R}^2$. As a consequence of Theorem 2.1, however, under no such embedding can $\hat{f}$ be extended to a homeomorphism of the plane.

Theorem 1.5. *There is an embedding $e: (E, f) \to \mathbb{R}^3$ and a homeomorphism $F: \mathbb{R}^3 \to \mathbb{R}^3$ so that $F \circ e(E, f) = e(E, f)$ and $F = e \circ \hat{f} \circ e^{-1}$ on $e(E, f)$.*

Proof. We will use the following consequence of a theorem due to M. Brown [Br]: if $F: X \to X$ is a near homeomorphism (that is, can be uniformly approximated by homeomorphisms) of the compact metric space X, then the inverse limit space (X, F) is homeomorphic with X. We will show that the map, $f \times \mathrm{id}: E \times \{0\} \to E \times \{0\}$, can be extended to a near homeomorphism $F: S^3 \to S^3$ where S^3 is the one-point compactification $\mathbb{R}^3 \cup \{\infty\}$ of $\mathbb{R}^3$, and where $F^{-1}\{\infty\} = \{\infty\}$. The map $\hat{i}: (E, f) \to (S^3, f)$ given by $\hat{i}((x_0, y_0), (x_1, y_1), \dots) = ((x_0, y_0, 0), (x_1, y_1, 0), \dots)$ is then an embedding and the shift homeomorphism, $\widehat{F}: (\mathbb{R}^3, F) \to (\mathbb{R}^3, F)$ is a homeomorphism of (a space homeomorphic to) $\mathbb{R}^3$ that extends $\hat{i} \circ \hat{f} \circ \hat{i}^{-1}$.

To produce such an F, we will extend the "lightning bold," l, the tent map, t, and the "folding map," g, each to near homeomorphisms, then compose these extensions. Let $L: S^3 \to S^3$ be a homeomorphism that extends the embedding $(x, y, 0) \mapsto (l(x), y, x)$ of $E \times \{0\}$ into S^3 with the property that $L(\infty) = 0$. Let $H_s^1: S^3 \to S^3$, $s \in [0, 1]$ be a continuous family of maps such that each H_s^1 is a homeomorphism for $s > 0$, $(H_s^1)^{-1}(\infty) = \{\infty\}$ for all s, and

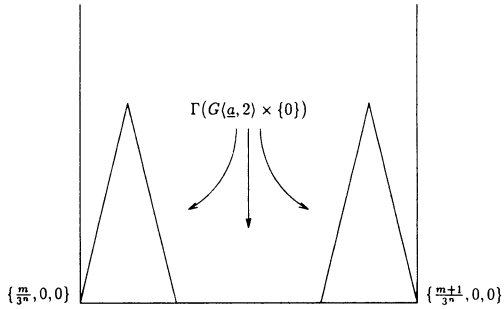

FIGURE 3. Γ on the gap, $G\langle \mathbf{a}, 2\rangle \times \{0\}$

$H_0^1(x, y, z) = (x, y, 0)$ for all $0 \le x, y, z \le 1$. Then $H_0^1 \circ L$ is a near homeomorphism of S^3 with $H_0^1 \circ L|_{E\times\{0\}} = l \times \mathrm{id} \times \mathrm{id}$, and $(H_0^1 \circ L)^{-1}(\infty) = \{\infty\}$.

Next, let $T\colon S^3 \to S^3$ be a homeomorphism with $T(\infty) = \infty$ that extends the embedding $(x, y, 0) \to (x, t(y), y)$ of $E \times \{0\}$ into S^3. Then $H_0^1 \circ T$ is a near homeomorphism of S^3 with $H_0^1 \circ T|_{E\times 0} = \mathrm{id} \times t \times \mathrm{id}$ and $(H_0^1 \circ T)^{-1}(\infty) = \{\infty\}$. Finally, to extend the "folding map" g to a near homeomorphism, we first push the folds up by a homeomorphism Γ, then squeeze them onto the "end hairs" by a near homeomorphism. Let $\Gamma\colon S^3 \to S^3$ be a homeomorphism that extends the embedding $(x, y, 0) \to (x, g_2(x, y), 0)$ of $E \times \{0\}$ into S^3 with $\Gamma(\infty) = \infty$ (see Figure 3). Here $g(x, y) = (g_1(x, y), g_2(x, y))$. Now let H_s^2, $s \in [0, 1]$, be a continuous family of maps of S^3 such that: $(H_s^2)^{-1}(\infty) = \infty$ for all s; H_s^2 is a homeomorphism for $s > 0$; and, for each $n \ge 1$ and $\mathbf{a} \in \{0, 1\}^{n-1}$, H_0^2 on $G\langle \mathbf{a}, 2\rangle \times [0, 1] \times \{0\} = [\frac{m}{3^n}, \frac{m+1}{3^n}] \times \{0\} \times \{0\}$ is given by

$$(x, y, 0) \mapsto \begin{cases} \left(\dfrac{m}{3^n}, y, 0\right); & \dfrac{m}{3^n} \le x \le \dfrac{3m+1}{3^{n+1}}, \\ \left(3\left(x - \dfrac{2m+1}{2\cdot 3^n}\right) + \dfrac{2m+1}{2\cdot 3^n}, y, 0\right); & \dfrac{3m+1}{3^{n+1}} \le x \le \dfrac{3m+2}{3^{n+1}}, \\ \left(\dfrac{m+1}{3^n}, y, 0\right); & \dfrac{3m+2}{3^{n+1}} \le x \le \dfrac{m+1}{3^n}. \end{cases}$$

Then $H_0^2 \circ \Gamma$ is a near homeomorphism of S^3 satisfying $(H_0^2 \circ \Gamma)^{-1}(\infty) = \{\infty\}$ and $H_0^2 \circ \Gamma|_{E\times\{0\}} = g \times \mathrm{id}$. Thus $F = H_0^2 \circ \Gamma \circ H_0^1 \circ T \circ H_0^1 \circ L\colon S^3 \to S^3$ is a near homeomorphism with $F^{-1}(\infty) = \{\infty\}$ and $F|_{E\times\{0\}} = f \times \mathrm{id}$, as desired. □

Remark. A circularly chainable invariant set of a three-space homeomorphism, F, which supports positive topological entropy where $\operatorname{per} F$ is empty, can be similarly constructed. One starts with a circular Cantor comb with spine map, the Denjoy circle homeomorphism.

2. NONEXISTENCE OF PLANAR HOMEOMORPHISMS EXTENDING CERTAIN ACTIONS OVER THE ADDING MACHINE

We will consider continua K that have the following property:

(2.1) there is a unique point $p \in K$ such that the arc component of p in K is a continuous one-to-one image of $\mathbb{R}^+ \cup \{0\}$ with p the image of 0.

Clearly any self-homeomorphism of such a K must fix p and leave invariant the arc component of p. Examples of such continua are the topologists sine curve, and of more interest in this paper, the Knaster bucket handle continuum that appears as the fiber over the Cantor set in Theorem 1.2.

For a continuum K satisfying (2.1), assume that $h: \Sigma \times K \to \mathbb{R}^2$ is an embedding of the Cantor set Σ crossed with K into the plane, and let Λ denote the image of h. Suppose further that $f: \Lambda \to \Lambda$ is a homeomorphism that factors over the adding machine homeomorphism on Σ. The goal of this section is to prove that no such f can be extended to a homeomorphism of the plane. As a consequence we will see that, under no embedding of the chainable continuum of Theorem 1.2 into the plane, does the homeomorphism of that continuum constructed in §1 extend to a homeomorphism of the plane.

The main result of this section holds for a more general class of continua (that includes, for example, the interval) than those satisfying (2.1). We will remark on this following the proof of Theorem 2.2. As before, let Σ denote the Cantor set

$$\Sigma = \{0, 1\}^{\mathbb{N}} = \{\mathbf{a} = (a_1, a_2, a_3, \dots) | a_i \in \{0, 1\},\ i = 1, 2, \dots\}$$

and let $A: \Sigma \to \Sigma$ be the adding machine homeomorphism.

Theorem 2.2. *Suppose that K is a continuum that satisfies property* (2.1) *and that $h: \Sigma \times K \to \mathbb{R}^2$ is an embedding. If $\Lambda = h(\Sigma \times K)$ and if $f: \Lambda \to \Lambda$ is a homeomorphism such that*

$$\pi_1 \circ h^{-1} \circ f \circ h = A \circ \pi_1 : \Sigma \times K \to \Sigma$$

then there is no homeomorphism $F: \mathbb{R}^2 \to \mathbb{R}^2$ such that $F|_\Lambda = f$.

Before proceeding to the proof of Theorem 2.2 we will introduce some notation and briefly describe the idea of the proof. Given $\mathbf{a} = (a_1, a_2, a_3, \dots, a_n) \in \{0, 1\}^{\mathbb{N}}$, $[\mathbf{a}] = [a_1, a_2, a_3, \dots, a_n]$ will denote the "n-block" $[\mathbf{a}] = \{\mathbf{b} \in \Sigma | b_i = a_i \text{ for } i = 1, \dots, n\}$ of Σ. Again $h: \Sigma \times K \to \mathbb{R}^2$ is an embedding with image Λ. For $\mathbf{a} \in \Sigma$, let $\Lambda(\mathbf{a})$ denote the fiber $h(\{\mathbf{a}\} \times K)$ and, for $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$ let $\Lambda[\mathbf{a}] = h([\mathbf{a}] \times K)$. We will show that the topology of the plane may be used to define an ordering of the fibers in each "n-block" $\Lambda[\mathbf{a}]$ for n sufficiently large. The ordering will be such that if F is an orientation preserving homeomorphism (otherwise, consider F^2) of the plane extending f, then F must preserve the order, at least on sufficiently close fibers. From this we will derive the existence of positive integers m and k with the property that if $\Lambda[\mathbf{a}]$ is an m-block then $\Lambda[\mathbf{a}] < F^k(\Lambda[\mathbf{a}]) < F^{2k}(\Lambda[\mathbf{a}]) < \cdots$. But then $\{F^{ik}(\Lambda[\mathbf{a}]) | i = 0, 1, 2, \dots\}$ would be a disjoint collection of m-blocks, in contradiction to the fact that there are only 2^m disjoint m-blocks in Λ.

For a continuum K that satisfies (2.1), let $K^{\mathbb{R}}$ denote the arc component of K with endpoint p. As $K^{\mathbb{R}}$ is the continuous one-to-one image of $\mathbb{R}^+ \cup \{0\}$, we may use the order on the nonnegative real numbers to order the points of $K^{\mathbb{R}}$. If $x, y \in K^{\mathbb{R}}$ with $x < y$ we will let $[x, y]$ denote the arc in $K^{\mathbb{R}}$ with endpoints x and y. If J is any such arc in $K^{\mathbb{R}}$, $\mathbf{a} \in \Sigma$, and $\mathbf{b} \in \{0, 1\}^{\mathbb{N}}$, we will denote $h(\{\mathbf{a}\} \times J)$ by $\Lambda^J(\mathbf{a})$ and $h([\mathbf{b}] \times J)$ by $\Lambda^J[\mathbf{b}]$.

Now let $q \in K^{\mathbb{R}}$, $q \neq p$, be chosen and let I be the arc $[p, q]$ in $K^{\mathbb{R}}$. According to a theorem of E. Moise [M], there is a homeomorphism of the plane that takes the Cantor sets $h(\Sigma \times \{p\})$ and $h(\Sigma \times \{q\})$ into the lines $\mathbb{R} \times \{-1\}$ and

$\mathbb{R}\times\{1\}$ respectively. By composing such a homeomorphism with the embedding h we may assume, as we will from now on, that $h(\Sigma \times \{p\}) \subset \mathbb{R} \times \{-1\}$ and $h(\Sigma \times \{q\}) \subset \mathbb{R} \times \{1\}$.

Suppose now that $\mathbf{a}, \mathbf{b} \in \Sigma$ and that γ_{-1} and γ_1 are arcs in the plane with the properties: γ_{-1} has endpoints $h(\mathbf{a}, p)$ and $h(\mathbf{b}, p)$ and is otherwise disjoint from $\Lambda^I(\mathbf{a}) \cup \Lambda^I(\mathbf{b})$; γ_1 has endpoints $h(\mathbf{a}, q)$ and $h(\mathbf{b}, q)$ and is otherwise disjoint from $\Lambda^I(\mathbf{a}) \cup \Lambda^I(\mathbf{b})$; and $(\gamma_{-1} \cup \gamma_1) \cap (\mathbb{R} \times \{0\}) = \varnothing$. Such arcs, γ_{-1} and γ_1, will be called *admissible arcs* joining $\Lambda^I(\mathbf{a})$ and $\Lambda^I(\mathbf{b})$.

Definition of the order. Given $\mathbf{a}, \mathbf{b} \in \Sigma$, $\mathbf{a} \neq \mathbf{b}$, we will write $\Lambda(\mathbf{a}) < \Lambda(\mathbf{b})$ in the case there are admissible arcs γ_{-1} and γ_1 joining $\Lambda^I(\mathbf{a})$ and $\Lambda^I(\mathbf{b})$, as above, and the orientation $\gamma_{-1} \to \Lambda^I(\mathbf{b}) \to \gamma_1 \to \Lambda^I(\mathbf{a})$ is the positive (counterclockwise) orientation on the simple closed curve $\gamma_{-1} \cup \Lambda^I(\mathbf{b}) \cup \gamma_1 \cup \Lambda^I(\mathbf{a})$.

To see that this order is antisymmetric, suppose that γ'_{-1} and γ'_1 is another pair of admissible arcs joining $\Lambda^I(\mathbf{a})$ and $\Lambda^I(\mathbf{b})$. Then the interiors of γ_{-1} and γ'_{-1} lie in the same component of $\mathbb{R}^2\backslash(\Lambda^I(\mathbf{a}) \cup \Lambda^I(\mathbf{b}) \cup (\mathbb{R} \times \{0\}))$, which is simply connected, and the interiors of γ_1 and γ'_1 lie in the same component of $\mathbb{R}^2\backslash(\Lambda^I(\mathbf{a}) \cup \Lambda^I(\mathbf{b}) \cup (\mathbb{R} \times \{0\}))$, which is also simply connected. Thus there is an isotopy of $\mathbb{R}^2$, rel. $\Lambda^I(\mathbf{a}) \cup \Lambda^I(\mathbf{b})$, that takes γ'_{-1} to γ_{-1} and γ'_1 to γ_1 and we see that $\gamma_{-1} \to \Lambda^I(\mathbf{b}) \to \gamma_1 \to \Lambda^I(\mathbf{a})$ and $\gamma'_{-1} \to \Lambda^I(\mathbf{b}) \to \gamma'_1 \to \Lambda^I(\mathbf{a})$ are either both positive or both negative orientations of the corresponding simple closed curves.

Lemma 2.3. *Given $J = [p, z] \subset K^{\mathbb{R}}$ and an $\varepsilon > 0$, there is an integer N_0 so that if $\mathbf{a}$ and $\mathbf{b}$ are in the same N_0-block $[\mathbf{c}] \in \{0, 1\}^{N_0}$, then there are arcs γ_p and γ_z in the plane such that: γ_p has endpoints $h(\mathbf{a}, p)$ and $h(\mathbf{b}, p)$ and is otherwise disjoint from $\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})$; γ_z has endpoints $h(\mathbf{a}, z)$ and $h(\mathbf{b}, z)$ and is otherwise disjoint from $\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})$; diameter $(\gamma_p) < \varepsilon$; and diameter $(\gamma_z) < \varepsilon$.*

Proof of Lemma 2.3. Let $\varepsilon > 0$ and J be given and let $\delta_1 > 0$ be small enough so that if $\mathbf{a} \in \Sigma$ and $x, y \in J$ with $x < y$ and with $\operatorname{diam}(\{\mathbf{a}\} \times [x, y]) < \delta_1$, then $\operatorname{diam}(h(\{\mathbf{a}\} \times [x, y])) < \varepsilon/3$. Let $\delta_2 > 0$ be small enough so that if $\mathbf{a} \in \Sigma$ and $x, y \in J$ with $x < y$, and $d(h(\mathbf{a}, x), h(\mathbf{a}, y)) < \delta_2$ then $\operatorname{diam}(\{\mathbf{a}\} \times [x, y]) < \delta_1$. Now let N_0 be large enough so that if $\mathbf{a}, \mathbf{b} \in [\mathbf{c}]$, $\mathbf{c}$ any element of $\{0, 1\}^{N_0}$, and $x \in J$, then the straight line segment joining $h(\mathbf{a}, x)$ and $h(\mathbf{b}, x)$ has diameter less than $\min\{\delta_2, \varepsilon/3\}$.

For such N_0 let $\mathbf{c} \in \{0, 1\}^{N_0}$ and let $\mathbf{a}, \mathbf{b} \in [\mathbf{c}]$. Let η be the straight line segment joining $h(\mathbf{a}, z)$ and $h(\mathbf{b}, z)$ so that diameter $(\eta) < \min\{\delta_2, \varepsilon/3\}$. Let γ_2 be a subarc of η with one endpoint on $\Lambda^J(\mathbf{a})$, and the other endpoint on $\Lambda^J(\mathbf{b})$, and otherwise disjoint from $\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})$. Let $h(\mathbf{a}, x)$ and $h(\mathbf{b}, y)$ be the endpoints of γ_2. Then $d(h(\mathbf{a}, z), h(\mathbf{a}, x)) < \delta_2$ and $d(h(\mathbf{b}, z), h(\mathbf{b}, y)) < \delta_2$ so that $\operatorname{diam}(\{\mathbf{a}\} \times [x, z]) < \delta_1$, and $\operatorname{diam}(\{\mathbf{b}\} \times [y, z]) < \delta_1$. Consequently $\operatorname{diam}(h(\{\mathbf{a}\} \times [x, z])) < \varepsilon/3$, and $\operatorname{diam}(h(\{\mathbf{b}\} \times [y, z])) < \varepsilon/3$. There are then arcs γ_1 and γ_3, approximating $h(\{\mathbf{a}\} \times [x, z])$ and $h(\{\mathbf{b}\} \times [y, z])$ respectively, such that: γ_1 has one endpoint $h(\mathbf{a}, z)$, the other endpoint on γ_2, $\gamma_1 \cap (\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})) = h(\mathbf{a}, z)$, and $\operatorname{diam}(\gamma_1) < \varepsilon/3$; γ_3 has one endpoint $h(\mathbf{b}, z)$, the other endpoint on γ_2, $\gamma_3 \cap (\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})) = h(\mathbf{b}, z)$, and $\operatorname{diam}(\gamma_3) < \varepsilon/3$. Let γ_z be the subarc of $\gamma_1 \cup \gamma_2 \cup \gamma_3$ with endpoints $h(\mathbf{a}, z)$ and $h(\mathbf{b}, z)$. The arc γ_p is similarly constructed. $\square$

In particular, if $J = I = [p, q]$ and $\varepsilon = 1$, there is a corresponding integer N_0 so that if $\mathbf{a}$ and $\mathbf{b}$ are distinct elements of the same N_0-block in Σ, then either $\Lambda(\mathbf{a}) < \Lambda(\mathbf{b})$ or $\Lambda(\mathbf{b}) < \Lambda(\mathbf{a})$. We will need the following improvement of Lemma 2.3.

Lemma 2.4. *Let x, y, z be elements of $K^{\mathbb{R}}$ with $p < x < y < z$ and let $\varepsilon > 0$ be given. Let $J = [p, z]$, then there exist an integer N and $\delta > 0$ so that if $\mathbf{a} \neq \mathbf{b}$ are in the same N-block of Σ and $x', y' \in [x, y]$ with $d(x', y') < \delta$ then there are arcs γ_p, γ_z, and η in $\mathbb{R}^2$ with these properties:*

(i) *γ_p, γ_z, and η have diameter less than ε;*

(ii) *γ_p has endpoints $h(\mathbf{a}, p)$ and $h(\mathbf{b}, p)$ and is otherwise disjoint from $\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})$;*

(iii) *γ_z has endpoints $h(\mathbf{a}, z)$ and $h(\mathbf{b}, z)$ and is otherwise disjoint from $\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})$;*

(iv) *η has endpoints $h(\mathbf{a}, x')$ and $h(\mathbf{b}, y')$ and is otherwise disjoint from $\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})$; and*

(v) *η is contained in the closed topological disk in $\mathbb{R}^2$ bounded by $\gamma_p \cup \Lambda^J(\mathbf{b}) \cup \gamma_z \cup \Lambda^J(\mathbf{a})$.*

Proof of Lemma 2.4. Fix $\mathbf{a} \in \Sigma$ and let $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism such that $\psi(h(\mathbf{a}, p)) = (0, -1)$, $\psi(h(\mathbf{a}, z)) = (0, 1)$, and $\psi(\Lambda^J(\mathbf{a})) = \{0\} \times [-1, 1]$. Let $\varepsilon' > 0$ be such that if $u, v \in \mathbb{R}^2$ with $d(u, v) < \varepsilon'$, then $d(\psi^{-1}(u), \psi^{-1}(v)) < \varepsilon$. Let $\delta' > 0$ be small enough so that if $x', x'' \in K$ and $d(x', x'') < \delta$; then $\operatorname{diam}(\psi(h(\{\mathbf{b}\} \times [x', x'']))) < \varepsilon'/4$ for all $\mathbf{b} \in \Sigma$. Now take $r_1 > 0$ so that if $d(\psi(h(\mathbf{b}', x'')), \psi(h(\mathbf{b}'', y''))) < r_1$ then $d(x'', y'') < \delta'/2$. Let

$$r_2 = \frac{1}{2} \min \left\{ r_1, \frac{\varepsilon'}{2}, \frac{1}{2} d(\psi(h(\mathbf{a}, x)), (0, -1)), \frac{1}{2} d(\psi(h(\mathbf{a}, y)), (0, 1)) \right\}.$$

Thus $\psi(h(\{\mathbf{a}\} \times [x, y]))$ is disjoint from the closed balls D_{-1} and D_1 centered at $(0, -1)$ and $(0, 1)$, respectively, of radius r_2. Let $\mathcal{N}$ denote the r_2-neighborhood $\mathring{D}_{-1} \cup ((-r_2, r_2) \times [-1, 1]) \cup \mathring{D}_1$ of $\{0\} \times [-1, 1]$. Now take N large enough so that if $\mathbf{b}'$, $\mathbf{b}''$, and $\mathbf{a}$ are all in the same N-block of Σ then:

(i) $\psi(\Lambda^J(\mathbf{b}')) \subset \mathcal{N}$, $\Lambda^J(\mathbf{b}') \backslash (\mathring{D}_{-1} \cup \mathring{D}_1)$ has exactly one component that meets both D_{-1} and D_1, and all other components of $\Lambda^J(\mathbf{b}') \backslash (\mathring{D}_{-1} \cup \mathring{D}_1)$ are disjoint from $\mathbb{R} \times [-1 + 2r_2, 1 - 2r_2]$;

(ii) $\psi(h(\{\mathbf{b}'\} \times [x, y])) \subset \mathbb{R} \times (-1 + 3r_2, 1 - 3r_2)$;

(iii) There are arcs γ_p' and γ_z' of diameter less than $r_2/2$, joining $\psi(h(\mathbf{b}', p))$ to $\psi(h(\mathbf{b}'', p))$ and $\psi(h(\mathbf{b}', z))$ and $\psi(h(\mathbf{b}'', z))$, respectively, that are otherwise disjoint from $\psi(\Lambda^J(\mathbf{b}')) \cup \psi(\Lambda^J(\mathbf{b}''))$.

Finally, let $\delta > 0$ be small enough so that $\delta \leq \delta'/2$ and if $x', y' \in J$ with $d(x', y') < \delta$, then $\operatorname{diam}(\psi(h(\{\mathbf{b}\} \times [x', y']))) < \varepsilon'/4$ for all $\mathbf{b} \in \Sigma$.

Suppose now that $\mathbf{b}'$ and $\mathbf{b}''$ are in the same N-block of Σ as $\mathbf{a}$ and that $x', y' \in [x, y]$ with $d(x', y') < \delta$. Let l be the horizontal line in $\mathbb{R}^2$ through $\psi(h(\mathbf{b}', x'))$. Then by (ii) above, l separates D_{-1} from D_1 and by (i) and (iii), l separates γ_p' from γ_z' in the disk D bounded by $\gamma_{-1}' \cup \psi(\Lambda^J(\mathbf{b}'')) \cup \gamma_1' \cup \psi(\Lambda^J(\mathbf{b}'))$. There is thus an arc η_2' that lies on l, has one endpoint on $\psi(\Lambda^J(\mathbf{b}'))$, the other endpoint on $\psi(\Lambda^J(\mathbf{b}''))$, and otherwise lies in the interior of D. Let $\psi(h(\mathbf{b}', x''))$ and $\psi(h(\mathbf{b}'', y''))$ be the endpoints of η_2'. Since

$\psi(\Lambda^J(\mathbf{b}'))$ and $\psi(\Lambda^J(\mathbf{b}''))$ are contained in $\mathscr{N}$, whose horizontal diameter is $2r_2$, $d(\psi(h(\mathbf{b}', x')), \psi(h(\mathbf{b}', x''))) < 2r_2 \leq r_1$, and $\operatorname{diam}(\eta_2') < 2r_2 \leq \varepsilon'/2$. It follows from the choice of r_1 that $d(x', x'') < \delta'$ so that

$$\operatorname{diam}(\psi(h(\{\mathbf{b}'\} \times [x', x'']))) < \varepsilon'/4.$$

Furthermore, $d(\psi(h(\mathbf{b}', x')), \psi(h(\mathbf{b}'', y''))) < 2r_2 \leq r_1$, so that $d(x', y'') < \delta'/2$. Since $d(x', y') < \delta \leq \delta'/2$ we have $d(y', y'') < \delta'$ and

$$\operatorname{diam}(\psi(h(\{\mathbf{b}''\} \times [y', y'']))) < \varepsilon'/4.$$

There is thus an arc η', obtained by perturbing $\psi(h(\{\mathbf{b}'\} \times [x', x''])) \cup \eta_2' \cup \psi(h(\{\mathbf{b}''\} \times [y', y'']))$ slightly into the interior of D, except at its endpoints, of diameter less than ε, with endpoints $\psi(h(\mathbf{b}', x'))$ and $\psi(h(\mathbf{b}'', y'))$, and otherwise lying in the interior of D. Letting $\gamma_p = \psi^{-1}(\gamma_p')$, $\gamma_z = \psi^{-1}(\gamma_z')$, and $\eta = \psi^{-1}(\eta')$, we see that the conclusions of the lemma hold throughout the N-block containing $\mathbf{a}$. A compactness argument removes the dependence of N and δ on $\mathbf{a}$. □

Lemma 2.5. *Suppose that* $\Lambda^I(\mathbf{a}) < \Lambda^I(\mathbf{b})$. *Then if* $\mathbf{a}'$ *is sufficiently near* $\mathbf{a}$ *and* $\mathbf{b}'$ *is sufficiently near* $\mathbf{b}$, $\Lambda^I(\mathbf{a}') < \Lambda^I(\mathbf{b}')$.

Proof of Lemma 2.5. Let U and V be open topological disks in $\mathbb{R}^2$ with $\Lambda^I(\mathbf{a}) \subset U$, $\Lambda^I(\mathbf{b}) \subset V$, and $U \cap V = \varnothing$. If $\mathbf{a}'$ is sufficiently near $\mathbf{a}$ and $\mathbf{b}'$ is sufficiently near $\mathbf{b}$, then $\Lambda^I(\mathbf{a}') \subset U$ and $\Lambda^I(\mathbf{b}') \subset V$, and there are admissible arcs α_1 joining $h(\mathbf{a}', q)$ to $h(\mathbf{a}, q)$, α_{-1} joining $h(\mathbf{a}', p)$ to $h(\mathbf{a}, p)$, β_1 joining $h(\mathbf{b}', q)$ to $h(\mathbf{b}, q)$, and β_{-1} joining $h(\mathbf{b}', p)$ to $h(\mathbf{b}, p)$. Moreover, if $\mathbf{a}'$ and $\mathbf{b}'$ are close enough to $\mathbf{a}$ and $\mathbf{b}$, respectively, it follows from Lemma 2.3 that α_1, α_{-1}, β_1, and β_{-1} can be chosen to lie in $U \cup V$.

Let γ_1 and γ_{-1} be admissible arcs joining $h(\mathbf{a}, q)$ to $h(\mathbf{b}, q)$ and $h(\mathbf{a}, p)$ to $h(\mathbf{b}, p)$, respectively, and let γ_1' and γ_{-1}' be closures of components of $\gamma_1 \backslash (\alpha_1 \cup \Lambda^I(\mathbf{a}') \cup \beta_1 \cup \Lambda^I(\mathbf{b}'))$ and $\gamma_{-1} \backslash (\alpha_{-1} \cup \Lambda^I(\mathbf{a}') \cup \beta_{-1} \cup \Lambda^I(\mathbf{b}'))$ respectively, with the property that γ_1' meets each of $\alpha_1 \cup \Lambda^I(\mathbf{a}')$ and $\beta_1 \cup \Lambda^I(\mathbf{b}')$ and γ_{-1}^1 meets each of $\alpha_{-1} \cup \Lambda^I(\mathbf{a}')$ and $\beta_{-1} \cup \Lambda^I(\mathbf{b}')$. Now if $\mathbf{a}'$ and $\mathbf{b}'$ are sufficiently close to $\mathbf{a}$ and $\mathbf{b}$, respectively, the subarcs

α_1' of $\alpha_1 \cup \Lambda^I(\mathbf{a}')$ from $h(\mathbf{a}, q)$ to $\gamma_1' \cup (\alpha_1 \cup \Lambda^I(\mathbf{a}'))$,
β_1' of $\beta_1 \cup \Lambda^I(\mathbf{b}')$ from $h(\mathbf{b}, q)$ to $\gamma_1' \cap (\beta_1 \cup \Lambda^I(\mathbf{b}'))$,
α_{-1}' of $\alpha_{-1} \cup \Lambda^I(\mathbf{a}')$ from $h(\mathbf{a}, p)$ to $\gamma_{-1}' \cap (\alpha_{-1} \cup \Lambda^I(\mathbf{a}'))$, and
β_{-1}' of $\beta_{-1} \cup \Lambda^I(\mathbf{b}')$ from $h(\mathbf{b}, p)$ to $\gamma_{-1}' \cap (\beta_{-1} \cup \Lambda^I(\mathbf{b}'))$,

are all disjoint from $\mathbb{R} \times \{0\}$. Thus the arcs $\alpha_1' \cup \gamma_1' \cup \beta_1'$ from $h(\mathbf{a}, q)$ to $h(\mathbf{b}, q)$ and $\alpha_{-1}' \cup \gamma_{-1}' \cup \beta_{-1}'$ from $h(\mathbf{a}, p)$ to $h(\mathbf{b}, p)$ are admissible. Let D_1 be the closed disk bounded by $\alpha_{-1} \cup \Lambda^I(\mathbf{a}) \cup \alpha_1 \cup \Lambda^I(\mathbf{a}')$ and let D_2 be the closed disk bounded by $\beta_{-1} \cup \Lambda^I(\mathbf{b}) \cup \beta_1 \cup \Lambda^I(\mathbf{b}')$. Since $\alpha_1' \cup \gamma_1' \cup \beta_1'$ and $\alpha_{-1}' \cup \gamma_{-1}' \cup \beta_{-1}'$ are disjoint from $(\mathring{D}_{-1} \cup \mathring{D}_1)$, we may perturb these arcs slightly into $\mathbb{R}^2 \backslash (D_1 \cup D_2)$, except at their endpoints, to obtain admissible arcs γ_1'' from $h(\mathbf{a}, q)$ to $h(\mathbf{b}, q)$ and γ_{-1}'' from $h(\mathbf{a}, p)$ to $h(\mathbf{b}, p)$ with the property that $(\gamma_1'' \cup \gamma_{-1}'') \cap (D_1 \cup D_2) = \{h(\mathbf{a}, p), h(\mathbf{a}, q), h(\mathbf{b}, p), h(\mathbf{b}, q)\}$. We now have that $\alpha_1 \cup \gamma_1'' \cup \beta_1$ and $\alpha_{-1} \cup \gamma_{-1}'' \cup \beta_{-1}$ are admissible from $h(\mathbf{a}', q)$ to $h(\mathbf{b}', q)$ and from $h(\mathbf{a}', p)$ to $h(\mathbf{b}', p)$ respectively. Now suppose that $\Lambda^I(\mathbf{a}) < \Lambda^I(\mathbf{b})$. Then $\gamma_{-1}'' \to \Lambda^I(\mathbf{b}) \to \gamma_1'' \to \Lambda^I(\mathbf{a})$ is positively oriented. Let H_t be a homotopy of the plane, rel. $\gamma_{-1}'' \cup \Lambda^I(\mathbf{b}) \cup \gamma_1'' \cup \Lambda^I(\mathbf{a})$ such that H_0 is the identity and H_1 collapses:

D_1 to $\Lambda^I(\mathbf{a})$; D_2 to $\Lambda^I(\mathbf{b})$; α_1 to $h(\mathbf{a}, q)$; α_{-1} to $h(\mathbf{a}, p)$; β_1 to $h(\mathbf{b}, q)$; and β_{-1} to $h(\mathbf{b}, p)$. Then H_1 takes $\alpha_{-1} \cup \gamma''_{-1} \cup \beta_{-1} \cup \Lambda^I(\mathbf{b}) \cup \alpha_1 \cup \gamma''_1 \cup \beta_1 \cup \Lambda^I(\mathbf{a})$ to $\gamma''_{-1} \cup \Lambda^I(\mathbf{b}) \cup \gamma''_1 \cup \Lambda^I(\mathbf{a})$ and takes the orientation $\alpha_{-1} \cup \gamma''_{-1} \cup \beta_{-1} \to \Lambda^I(\mathbf{b}) \to \alpha_1 \cup \gamma''_1 \cup \beta_1 \to \Lambda^I(\mathbf{a})$ to the orientation $\gamma''_{-1} \to \Lambda^I(\mathbf{b}) \to \gamma''_1 \to \Lambda^I(\mathbf{a})$. Thus $\alpha_{-1} \cup \gamma''_{-1} \cup \beta_{-1} \to \Lambda^I(\mathbf{b}) \to \alpha_1 \cup \gamma''_1 \cup \beta_1 \to \Lambda^I(\mathbf{a})$ is a positive orientation and $\Lambda^I(\mathbf{a}') < \Lambda^I(\mathbf{b}')$. □

Proof of Theorem 2.2. Suppose that F is an orientation preserving homeomorphism of the plane that extends f. Let $0 < \varepsilon < 1$, be small enough so that if γ is an arc of diameter less than ε that meets Λ, then $\operatorname{diam}(F(\gamma)) < 1$. Let x, y, z be element of $K^{\mathbb{R}}$ with $p < x < q < y < z$ and such that $f(h(\Sigma \times [x, y])) \supset h(\Sigma \times \{q\})$. For these x, y, z and the above ε, let $N_1 = N$ and δ be as in Lemma 2.4. Let N_2 be large enough so that if $\mathbf{a}$ and $\mathbf{b}$ are in the same N_2-block of Σ, $f(h(\mathbf{a}, x')) = h(A(\mathbf{a}), q)$, and $f(h(\mathbf{b}, y')) = h(A(\mathbf{b}), q)$, then $d(x', y') < \delta$. Let $N_3 = \max\{N_1, N_2\}$ and suppose that $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \neq \mathbf{b}$, are in the same N_3-block of Σ. It follows from Lemma 2.4 that $\Lambda(\mathbf{a})$ and $\Lambda(\mathbf{b})$ are comparable, say $\Lambda(\mathbf{a}) < \Lambda(\mathbf{b})$.

Claim. $F(\Lambda(\mathbf{a})) < F(\Lambda(\mathbf{b}))$.

To see that this is the case, let $x', y' \in [x, y]$ be such that $f(h(\mathbf{a}, x')) = h(A(\mathbf{a}), q)$ and $f(h(\mathbf{b}, y')) = h(A(\mathbf{b}), q)$. Let γ_p and γ_q be admissible arcs joining $\Lambda^I(\mathbf{a})$ to $\Lambda^I(\mathbf{b})$ of diameter less that ε and, using Lemma 2.4 again, let γ_z and η be arcs of diameter less than ε such that γ_z has endpoints $h(\mathbf{a}, z)$ and $h(\mathbf{b}, z)$, and is otherwise disjoint from $\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})$, and η has endpoints $h(\mathbf{a}, x')$ and $h(\mathbf{b}, y')$, and is otherwise disjoint from $\Lambda^J(\mathbf{a}) \cup \Lambda^J(\mathbf{b})$. According to Lemma 2.4 we can choose these arcs so that η and γ_q lie in the disk D bounded by $\gamma_p \cup \Lambda^J(\mathbf{b}) \cup \gamma_z \cup \Lambda^J(\mathbf{a})$.

Since $\Lambda(\mathbf{a}) < \Lambda(\mathbf{b})$, the orientation $\gamma_p \to \Lambda^I(\mathbf{b}) \to \gamma_z \to \Lambda^I(\mathbf{a})$ is positive. Thus the orientation $\gamma_p \to h(\{\mathbf{b}\} \times [p, y']) \to \eta \to h(\{\mathbf{a}\} \times [p, x'])$ is also positive. Now $\operatorname{diam}(F(\eta)) < 1$ and $\operatorname{diam}(F(\gamma_p)) < 1$ so the arcs $F(\gamma_p)$ and $F(\eta)$ are admissible arcs joining $\Lambda^I(A(\mathbf{a}))$ to $\Lambda^I(A(\mathbf{b}))$. Since F is orientation preserving, the orientation $F(\gamma_p) \to F(h(\{\mathbf{b}\} \times [p, y'])) = \Lambda^I(A(\mathbf{b})) \to F(\eta) \to F(h(\{\mathbf{a}\} \times [p, x'])) = \Lambda^I(A(\mathbf{a}))$ is positive. Thus $F(\Lambda^I(\mathbf{a})) < F(\Lambda^I(\mathbf{b}))$ as claimed.

Now let $\mathbf{a}$ and $\mathbf{b}$ be as above and let $m > N_3$ be large enough so that if $\mathbf{a}'$ is in the same m-block, call it $[c]$, of Σ and $\mathbf{b}'$ is in the same m-block, $[c']$, of Σ as $\mathbf{b}$, then $\Lambda^I(\mathbf{a}') < \Lambda^I(\mathbf{b}')$ (Lemma 2.5). That is $\Lambda[c] < \Lambda[c']$. It follows from the above claim that $F^n(\Lambda[c]) < F^n(\Lambda[c'])$ for all $n \geq 0$. Since A permutes the m-blocks of Σ cyclically, there is a $k > 0$ such that $A^k(\Lambda[c]) = [c']$. Then $F^k(\Lambda[c]) = \Lambda[c']$ so that $\Lambda[c] < F^k(\Lambda[c]) < F^{2k}(\Lambda[c]) \cdots$. But $A^{2^m}(\Lambda[c]) = [c]$ so we have $F^{k \cdot 2^m}(\Lambda[c]) = \Lambda[c]$, and thus the impossibility $\Lambda[c] < \Lambda[c]$. Consequently there is no orientation preserving homeomorphism F which extends f. In case F is orientation reversing, F^2 is orientation preserving. If F extends f, F^2 extends f^2 in $\Lambda[0] = h(\{(a_1, a_2, \ldots) \in \Sigma | a_1 = 0\} \times K)$. Since A^2 restricted to the one-block $[0]$ is conjugate to A on Σ, there can be no such F^2 by the argument given above for the orientation preserving case. □

Remark. With minor adjustments in the argument given above, one can prove Theorem 2.2 with K replaced by an arc or by any continuum that satisfies

(2.1) modified to allow for finitely many points of the nature of p. Also, if the adding machine is replaced by any "generalized adding machine" (see §3), the theorem remains valid. In fact, we know of no examples for which the theorem fails for any nontrivial continuum K.

3. Fibers over generalized adding machines

The C^1 Denjoy circle diffeomorphism [Den] can be used to construct a planar diffeomorphism which shuttles arcs, with diameters bounded above zero, over a minimal action on a Cantor set. Let $F(r, \theta) = (r, q(\theta))$ when g is the Denjoy diffeomorphism. If C_g is the g-invariant minimal Cantor set, then F on $\{\{r, \theta): r \in [1, 2]\}: \theta \in C_g\}$ is a fiber bundle map which covers g.

The Denjoy homeomorphism restricted to C_g differs from the adding machine, in this context, in one key respect: there do not exist finite collections of subblocks of C_g having arbitrarily small diameters which are cyclically permuted by g. The main result of this section is the construction of a C^∞ planar diffeomorphism which shuttles compact connected fibers (continua), *having diameters above zero*, over a "generalized adding machine" having this cyclic block-permuting structure. The collection of fibers do not comprise a fiber bundle, but collapsing the fibers defines a continuous map onto the Cantor set.

Let $\Sigma = \prod_{k=1}^{\infty}\{0, 1, 2, \dots, n_k - 1\}$ with the metric topology induced by $d(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{k=1}^{\infty} \frac{|\alpha_k - \beta_k|}{2^{(n_k)}}$. Addition on Σ consists of adding corresponding components mod n_k and carrying to the right. The *generalized adding machine*, $A: \Sigma \to \Sigma$ is defined by $A(\alpha_1, \alpha_2, \dots) = (\alpha_1, \alpha_2, \dots) + (1, 0, \dots)$. So when $n_k = 2$ for all k, A is a two-symbol adding machine of the previous sections.

Remark. Theorem 2.2 may be generalized to include planar homeomorphisms which cover generalized adding machines for any collection $\{n_k \in \mathbb{N}\}$. This is because A^2 is itself a generalized adding machine on Σ (or a subset of Σ under some assignment of symbols), and $A^{n_1 n_2 \cdots n_m}[\alpha_1, \alpha_2, \dots, \alpha_m] = [\alpha_1, \alpha_2, \dots, \alpha_m]$ for all $\boldsymbol{\alpha} \in \Sigma$.

Example. We now construct a C^∞-diffeomorphism, F, of the disk which has an invariant union of fibers, Λ, having diameters bounded above zero; $F|_\Lambda$ will cover a generalized adding machine. F is constructed as the limit of a sequence of C^∞ diffeomorphisms, $\{F_n\}$ which are uniformly Cauchy in each C^k norm, $\|\cdot\|_k$; simultaneously $\{F_n^{-1}\}$ will be uniformly C^k Cauchy and thus must converge to a C^∞ disk map, the inverse of F.

Let $\{\varepsilon_1, \varepsilon_2, \dots\}$ be an infinite sequence of positive real numbers such that $\sum_{k=1}^{\infty} \varepsilon_k < 1/4$. Let $D = \{(r, \theta) | r \le 1\}$. For $\delta > 0$, define $b_\delta: D \to D$ by

$$b_\delta(r, \theta) = \begin{cases} (r.\theta); & r \le 1 - 2\delta, \\ \left(r, \theta + \dfrac{2\pi}{\delta}(r + 2\delta - 1)\right); & 1 - 2\delta \le r \le 1 - \delta, \\ (r, \theta + 2\pi); & r \ge 1 - \delta. \end{cases}$$

Set $T_\delta = b_\delta\{(r, \theta): 1 - 2\delta \le r \le 1 - \delta,\ \theta = 0\}$. For $n \in \mathbb{N}$, let

$$R(\delta, n, r, \theta) = \begin{cases} \left(r, \theta + \dfrac{2\pi}{n}\right); & r \le 1 - \delta, \\ (r, \theta + g_{\delta, n}(r)); & r \ge 1 - \delta, \end{cases}$$

where $g_{\delta,n}\colon [1-\delta, 1] \to [0, \frac{2\pi}{n}]$ is a C^∞ bump function such that $g_{\delta,n}(1-\delta) = \frac{2\pi}{n}$ and $g_{\delta,n}(1) = 0$. When $S \subset D$ denote the set $\{R(\delta, n, r, \theta)\colon (r, \theta) \in S\}$ by $R(\delta, n, S)$. For each δ and n let $D_\delta(n) \subset \operatorname{int} D$ be a closed disk with smooth boundary such that $T_\delta \subset \operatorname{int} D_\delta(n)$ and such that the disks $R^l(\delta, n, D_\delta(n))$, $l = 0, 1, \dots, n-1$, are pairwise disjoint.

Let $\delta_1 = \varepsilon_1$. Choose n_1 sufficiently large that

$$\|R_1 - \mathrm{id}\|_0 < 1 \quad \text{and} \quad \|R_1^{-1} - \mathrm{id}\|_0 < 1,$$

where $R_1(\cdot, \cdot) = R(\delta_1, n_1, \cdot, \cdot)$. Then specify $F_0 = R_1$, and call $D_1 = D_{\delta_1}(n)$, $T_1 = T_{\delta_1}$.

Let $\phi_1\colon D_1 \to D$ be any diffeomorphism. Notice that for all m, $\operatorname{diam}(F_0^m T_1) > 2 - 4\varepsilon_1$. Thus $\operatorname{diam}(F_0^m D_1) > 2 - 4\varepsilon_1$. By construction, for all $p \in \partial D$, l, n, and δ, there exists $q \in R^l(\delta, n, T_\delta)$ such that $d(p, q) < 2\delta$. It follows that, there exists δ_2 sufficiently small so that for all m, l, and n,

$$\operatorname{diam}(F_0^m \phi_1^{-1} R(\delta_2, n, T_2)) > 2 - 4\varepsilon_1 - 4\varepsilon_2,$$

where $T_2 = T_{\delta_2}$. Now choose n_2 large enough that $\|\phi_1^{-1} R_2 \phi_1 - \mathrm{id}\|_1 < 1/2$ and $\|\phi_1^{-1} R_2 \phi_1 - \mathrm{id}\|_1 < 1/2$ where $R_2(\cdot, \cdot) = R(\delta_2, n_2, \cdot, \cdot)$. Specify $F_1\colon D \to D$ by

$$F_1 = \begin{cases} F_0 & \text{off } D_1, \\ F_0 \circ (\phi_1^{-1} R_2 \phi_1) & \text{on } D_1. \end{cases}$$

F_1 is a diffeomorphism because $R_2 = \mathrm{id}$ on ∂D. Notice $\|F_1 - F_0\|_1 < 1/2$, and $\|F_1^{-1} - F_0^{-1}\|_1 < 1/2$. Set $D_2 = \phi_1^{-1} D_{\delta_2}(n_2)$. So for all m, $\operatorname{diam}(F_1^m D_2) > 2 - 4\varepsilon_1 - 4\varepsilon_2$.

The remainder of the disk diffeomorphisms are defined recursively. Assume that n_{k+1}, F_k, ϕ_k, δ_{k+1}, and $D_{k+1} \subset D_k$ have been defined such that

$$F_k = \begin{cases} F_{k-1} & \text{off } D_k, \\ F_{k-1} \circ (\phi_k^{-1} R_{k+1} \phi_k) & \text{on } D_k, \end{cases}$$

where $R_{k+1}(\cdot, \cdot) = R(\delta_{k+1}, n_{k+1}, \cdot, \cdot)$. Further, assume $\|F_k - F_{k-1}\|_k < \frac{1}{2^k}$, $\|F_k^{-1} - F_{k-1}^{-1}\|_k < \frac{1}{2^k}$ and $\operatorname{diam}(F_k^m D_{k+1}) < 2 - 4\sum_{i=1}^{k+1} \varepsilon_i$ for all m where $D_{k+1} = \phi_k^{-1} D_{\delta_{k+1}}(n_{k+1})$.

To initiate the recursive step, let $\phi_{k+1}\colon D_{k+1} \to D$ be any C^∞ diffeomorphism. Then there exists $\delta_{k+2} > 0$ sufficiently small that

$$\operatorname{diam}(F_k^m \phi_{k+1}^{-1} R^l(\delta_{k+2}, n, T_{k+2})) > 2 - 4\sum_{i=1}^{k+2} \varepsilon_i$$

for all m, l, and n, where $T_{k+2} = T_{\delta_{k+2}}$. Now choose n_{k+2} sufficiently large that $\|\phi_{k+1}^{-1} R_{k+2} \phi_{k+1}^{-1} - \mathrm{id}\|_{k+1} < \frac{1}{2^{k+1}}$, and $\|\phi_{k+1}^{-1} R_{k+2}^{-1} \phi_{k+1}^{-1} - \mathrm{id}\|_{k+1} < \frac{1}{2^{k+1}}$ where $R_{k+2}(\cdot, \cdot) = R(\delta_{k+2}, n_{k+2}, \cdot, \cdot)$. The next C^∞ disk diffeomorphism is

$$F_{k+1} = \begin{cases} F_k & \text{off } D_{k+1}, \\ F_k \circ (\phi_{k+1}^{-1} R_{k+2} \phi_{k+1}) & \text{on } D_{k+1}. \end{cases}$$

Again, F_{k+1} is a C^∞ diffeomorphism because $R_{k+2} = \mathrm{id}$ on ∂D. And we have $\|F_{k+1} - F_k\|_{k+1} < \frac{1}{2^{k+1}}$ and $\|F_{k+1}^{-1} - F_k^{-1}\|_{k+1} < \frac{1}{2^{k+1}}$. To complete the recursive

step, set $D_{k+2} = \phi_{k+1}^{-1} D_{\delta_{k+2}}(n_{k+2})$. By choice of δ_{k+2}, $\operatorname{diam}(F_{k+1}^m D_{k+1}) > 2 - 4\sum_{i=1}^{k+2} \varepsilon_1$, for all m.

For each k, and for all $J > k$, $\|F_{j_1} - F_{j_2}\|_k < \sum_{i=J}^{\infty} \frac{1}{2^i}$, for all $j_1, j_2 > J$. Thus for each k, $\{F_j\}$ is a C^k uniformly Cauchy sequence. Because the space of C^∞ maps on D is complete in the C^∞ norm, $\{F_j\}$ converges to a C^∞ map, F. But similarly, $\{F_j^{-1}\}$ converges to a C^∞ map which must by F^{-1}.

Recall that the $\varepsilon_i > 0$ have been chosen small enough so that $\sum_{i=1}^{\infty} \varepsilon_i < 1/4$. Thus it is guaranteed that $\operatorname{diam}(F^m(D_k)) = \operatorname{diam}(F_k^m(D_k)) > 1$ for all m and k.

We now turn to the action of F on the invariant set

$$\Lambda = \bigcap_{k=1}^{\infty} \left(\bigcup_{i=0}^{m_k} F^i(D_k) \right) \quad \text{where } m_k = \prod_{i=1}^{k} (n_i - 1).$$

Note that for each $k \geq 1$, $F^{n_1 \cdot n_2 \cdots n_k}(D_k) = D_k$ and $F^j(D_k) \cap D_k = \varnothing$ for $0 < j < n_1 \cdot n_2 \cdots n_k$. Let $\Sigma = \prod_{k=1}^{\infty} \{0, \dots, n_k - 1\}$ and define $P\colon \Lambda \to \Sigma$ by $P(x) = (\alpha_1, \alpha_2, \dots)$ provided

$$x \in F^{\alpha_1}(D_1) \cup F^{\alpha_1 + \alpha_2 n_1}(D_2) \cup \cdots = \bigcup_{k=1}^{\infty} F^{\gamma_k}(D_k),$$

where

$$\gamma_k = \sum_{i=1}^{k} \alpha_i \left(\prod_{j=1}^{i-1} n_j \right).$$

It is clear that P is a continuous surjection and that the fibers $\Lambda(\boldsymbol{\alpha}) = P^{-1}(\boldsymbol{\alpha})$, $\boldsymbol{\alpha} \in \Sigma$, are the connected components of Λ. Moreover, $F(\Lambda(\boldsymbol{\alpha})) = \Lambda(A(\boldsymbol{\alpha}))$, A being the generalized adding machine on Σ. That is, $F|_\Lambda$ is semiconjugate to the generalized adding machine. Notice, finally, that since the disks $F^m(D_k)$ have diameter larger than 1 for all m and k, $\operatorname{diam}(\Lambda(\boldsymbol{\alpha})) \geq 1$ for all $\boldsymbol{\alpha} \in \Sigma$. Properties of this example are summarized in the following theorem.

Theorem 3.1. *There exists a C^∞ diffeomorphism $F\colon D \to D$ of the two-dimensional disk D, a compact F-invariant set*

$$\Lambda = \bigcup_{\boldsymbol{\alpha} \in \Sigma} \Lambda(\boldsymbol{\alpha}),$$

and a continuous surjection $P\colon \Lambda \to \Sigma$ of Λ onto the Cantor set Σ, with $P^{-1}(\boldsymbol{\alpha}) = \Lambda(\boldsymbol{\alpha})$, such that:

(i) *$\Lambda(\boldsymbol{\alpha})$ is a continuum for each $\boldsymbol{\alpha} \in \Sigma$;*

(ii) *$\inf_{\boldsymbol{\alpha} \in \Sigma} \operatorname{diam}(\Lambda(\boldsymbol{\alpha})) > 0$; and*

(iii) *$PF(\Lambda(\boldsymbol{\alpha})) = A(\boldsymbol{\alpha})$ where $A\colon \Sigma \to \Sigma$ is a generalized adding machine.*

Remarks. By choosing the collection, $\{\phi_k\colon D_k \to D\}$ more carefully, the fiber

$$\Lambda(0, 0, \dots) = \bigcap_{k=1}^{\infty} D_k$$

can be made an imbedded Knaster continuum (K_2). And simultaneously, another fiber can be forced to have a different topological type. Since A is a minimal on Σ, Λ will then contain arbitrarily close fibers of large diameter which are not homeomorphic.

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