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# SYMMETRIES OF HOMOTOPY COMPLEX PROJECTIVE THREE SPACES

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ABSTRACT. We study symmetry properties of six-dimensional, smooth, closed manifolds which are homotopy equivalent to $\mathbf{C}P^3$. There are infinitely differentiably distinct such manifolds. It is known that if $m$ is an odd prime, infinitely many homotopy $\mathbf{C}P^3$'s admit $\mathbf{Z}_m$-actions whereas only the standard $\mathbf{C}P^3$ admits an action of the group $\mathbf{Z}_m \times \mathbf{Z}_m \times \mathbf{Z}_m$. We study the intermediate case of $\mathbf{Z}_m \times \mathbf{Z}_m$-actions and show that infinitely many homotopy $\mathbf{C}P^3$'s do admit $\mathbf{Z}_m \times \mathbf{Z}_m$-actions for a fixed prime $m$. The major tool involved is equivariant surgery theory. Using a transversality argument, we construct normal maps for which the relevant surgery obstructions vanish allowing the construction of $\mathbf{Z}_m \times \mathbf{Z}_m$-actions on homotopy $\mathbf{C}P^3$'s which are $\mathbf{Z}_m \times \mathbf{Z}_m$-homotopy equivalent to a specially chosen linear action on $\mathbf{C}P^3$. A key idea is to exploit an extra bit of symmetry which is built into our set-up in a way that forces the signature obstruction to vanish. By varying the parameters of our construction and calculating Pontryagin classes, we may construct actions on infinitely many differentiably distinct homotopy $\mathbf{C}P^3$'s as claimed.

## 1. INTRODUCTION

It is well known that there is a one-to-one correspondence between the integers and the set of diffeomorphism classes of six-dimensional, smooth, closed manifolds which are homotopy equivalent to $\mathbf{C}P^3$. (See [MY].) Such manifolds shall hereafter be called homotopy $\mathbf{C}P^3$'s. For every integer $k$, there is a unique homotopy $\mathbf{C}P^3$, denoted $X_k$, with first Pontryagin class $P_1(X_k) = (4+24k)x^2$, where $x \in H^2(X_k)$ is a generator. Then, $X_0$ is the standard $\mathbf{C}P^3$. In what follows, all actions shall be effective and smooth.

Some information is known about smooth finite group actions on homotopy $\mathbf{C}P^3$'s. For instance, in [H2], it is shown that if $\mathbf{D}_{2m}$ is the dihedral group of order $2m$, where $m$ is an odd prime such that the projective class group $\widetilde{K}_0(\mathbf{Z}[\mathbf{D}_{2m}])$ has 2-rank $= 0$, then there are infinitely many integers $k$ for which $X_k$ admits a $\mathbf{D}_{2m}$-action. It is also known that infinitely many homotopy $\mathbf{C}P^3$'s admit a $\mathbf{Z}_m$-action for almost every prime number $m$. (See [DM].) On the other hand, in [M1], it is shown that if $X_k$ admits a smooth, effective $\mathbf{Z}_m \times \mathbf{Z}_m \times \mathbf{Z}_m$-action, for any odd prime $m$, then $k = 0$, i.e., $X_k = \mathbf{C}P^3$. (There is a version of this result for $m = 2$ due to Masuda. Indeed, Corollary 5.2 of [M1] states

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that if $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ acts smoothly on a homotopy $\mathbf{C}P^3$, $X_k$, such that the restricted action of any order two subgroup has fixed point set consisting of precisely two components, then $X_k = X_0 = \mathbf{C}P^3$. More information about involutions on homotopy $\mathbf{C}P^3$'s can be found in [DMS], where, in particular, it is shown that every $X_k$ admits a smooth conjugation type involution, i.e., an involution with fixed point set consisting of a $\mathbf{Z}_2$-cohomology $\mathbf{R}P^3$.)

In this paper, we shall consider the intermediate case of $\mathbf{Z}_m \times \mathbf{Z}_m$-actions. We shall show that infinitely many homotopy $\mathbf{C}P^3$'s admit a $\mathbf{Z}_m \times \mathbf{Z}_m$-action. We also show how a result of Dovermann implies that for nonstandard homotopy $\mathbf{C}P^3$'s, the only possible fixed point set consists of four points. Our main result is:

**Theorem A.** *Let $m$ be a prime number. There are infinitely many integers $k$ for which $X_k$ admits a $\mathbf{Z}_m \times \mathbf{Z}_m$-action. More precisely, given relatively prime integers $p$ and $q$, each of which is $\equiv \pm 1 \bmod m$, $X_k$ admits a $\mathbf{Z}_m \times \mathbf{Z}_m$-action, where $k = (p^2 - 1)(q^2 - 1)/3$.*

The main tool used to prove the above theorems is equivariant surgery theory (see [DP and PR]). The features of this theory which are relevant to our work shall be outlined in the next section. The third section provides the proof of Theorem A and considerations on the possible fixed point sets.

## 2. Background

Let $G$ be a finite group. Equivariant ($G$-) surgery is a process for constructing $G$-manifolds which are $G$-homotopy equivalent to a given $G$-manifold $Y$. (A homotopy $F: X \times I \to Y$ is a $G$-homotopy if $F(\cdot, t)$ is a $G$-map for all $t$.) Two major steps are involved in this process.

1. We build a $G$-normal map $(X, f, b)$ with target manifold $Y$. This can be thought of as an approximation to a $G$-homotopy equivalence.
2. We must determine whether or not the obstructions to performing $G$-surgery to a $G$-homotopy equivalence vanish. The process of $G$-surgery converts $X$ to a $G$-manifold $X'$ and $f$ to a $G$-map $f': X' \to Y$ which is a $G$-homotopy equivalence.

Before we elaborate on this, we need some definitions.

**Definition 2.1.** A $G$-manifold is said to satisfy the gap hypothesis if given a nontrivial subgroup $H \subseteq G$ and a component $F$ of $X^H$, we have $2 \dim F < \dim X$.

Recall that a smooth $G$-vector bundle is a triple $(E, p, B)$, where $p: E \to B$ is an ordinary smooth vector bundle such that $E$ and $B$ support smooth $G$-actions and the projection $p$ is a $G$-map. We also require that, given $g \in G$ and $b \in B$, the map restricted to fiber $g: E_b \to E_{g(b)}$ is linear.

At this point, for simplicity, instead of defining $G$-normal maps, we choose to define a special type of $G$-normal map, namely an adjusted $G$-normal map. (See [D1]. The notion of a $G$-normal map can be found, for example, in [H1 or PR].)

**Definition 2.2.** An adjusted $G$-normal map with target $Y$ is a triple $(X, f, b)$, where

(1) $X$ is a smooth, oriented, closed $G$-manifold which satisfies the gap hypothesis and is of dimension $\geq 5$. $Y$ is a smooth, oriented, closed $G$-manifold which is simply connected and of the same dimension as $X$.

(2) $f: X \to Y$ is a smooth, degree 1 $G$-map which induces a $G$-homotopy equivalence between the singular sets $X^s$ and $Y^s$. (Recall that $X^s = \{x \in X: G_x \neq 1\}$.)

(3) $b$ is a stable $G$-vector bundle isomorphism between $TX \oplus f^*(\eta_-)$ and $f^*(TY \oplus \eta_+)$, for some pair of $G$-vector bundles $\eta_\pm$. That is, there exists a $G$-representation $V$ such that $b$ is a $G$-vector bundle isomorphism between $TX \oplus f^*(\eta_-) \oplus (X \times V)$ and $f^*(TY \oplus \eta_+) \oplus (X \times V)$.

We have a further important definition.

**Definition 2.3.** Let $\eta_+$ and $\eta_-$ be $G$-vector bundles over a $G$-manifold $Y$. Assume that given $H \subseteq G$ and $y \in Y^H$, we have $\dim(\eta_+|_y)^H = \dim(\eta_-|_y)^H$. Then $\omega: \eta_+ \to \eta_-$ is a $G$-fiber homotopy equivalence if it is a proper, fiber preserving $G$-map such that, given $H \subseteq G$ and $y \in Y^H$, the map $(\omega|_y)^H: (\eta_+|_y)^H \to (\eta_-|_y)^H$ has degree 1 when extended to one point compactifications.

Using ideas found in §11 of Chapter 3 in [PR], an adjusted $G$-normal map can be constructed from a $G$-fiber homotopy equivalence over $Y$ provided that certain conditions are met. This shall be carried out in §3 of this paper.

We mention that the notion of a $G$-fiber homotopy equivalence is usually defined differently. Indeed, given two $G$-bundles $(E, p, X)$ and $(E', p', X')$ and a $G$-map $f: X \to X'$, a $G$-fiber homotopy over $f$ is a $G$-map $F: I \times E \to E'$ such that $F(t, \cdot)$ is a $G$-map over $f$ for all $t \in I = [0, 1]$. ($I$ is given the trivial $G$-action.) The maps $f_0 = F(0, \cdot)$ and $f_1 = F(1, \cdot)$ are then said to be $G$-fiber homotopic over $f$. A $G$-map $u: E \to E'$ over $\mathrm{Id}_X$ is a $G$-fiber homotopy equivalence if there is a $G$-map $v: E' \to E$ such that $vu$ and $uv$ are $G$-fiber homotopic to $\mathrm{Id}_E$ and $\mathrm{Id}_{E'}$ respectively. It is a fact [PR, §§1–13] that if $\omega$ is as in Definition 2.3, then it induces a $G$-fiber homotopy equivalence in the usual sense $\Omega: S(\eta_+ \oplus (Y \times \mathbf{R})) \to S(\eta_- \oplus (Y \times \mathbf{R}))$, where $S(\cdot)$ denotes the sphere bundle.

Once our adjusted $G$-normal map is constructed, we shall proceed to step 2, which is to determine whether surgery to a $G$-homotopy equivalence is possible.

We first mention that an equivariant map $f: X \to Y$ is a $G$-homotopy equivalence if and only if $f^H: X^H \to Y^H$ is an ordinary homotopy equivalence for all $H \subseteq G$. (See [Br1].) Therefore, given our adjusted $G$-normal map $(X, f, b)$, we must convert $X$ to a $G$-manifold $X_\eta$ and $f$ to a $G$-map $F: X_\eta \to Y$ such that $F^H$ is a homotopy equivalence for all $H \subseteq G$. There is an obstruction to obtaining such a $G$-homotopy equivalence via $G$-surgery.

**Proposition 2.4.** *Let $(X, f, b)$ be an adjusted $G$-normal map with target $Y$. There is an obstruction $\sigma_1(f, b)$ such that if $\sigma_1(f, b) = 0$, then $(X, f, b)$ is $G$-normally cobordant to an adjusted $G$-normal map $(X_\eta, F, B)$, where $F: X_\eta \to Y$ is a $G$-homotopy equivalence.*

That is, if $\sigma_1(f, b)$ vanishes, then $G$-surgery can be used to convert $X$ to $X_\eta$ and $f$ to a $G$-homotopy equivalence $F: X_\eta \to Y$. The proof of this proposition may be found in [D1]. (See Corollary 1.1 on p. 853.) Related results involving $G$-normal maps are well known and can be found in [PR and

R2]. Also, see [BQ]. We note that this surgery is done relative to the singular set $X^s$. The obstruction $\sigma_1(f, b)$ is an element of the Wall group $L_n^h(\mathbf{Z}[G], w)$, where $n = \dim Y$ and $w: G \to \mathbf{Z}_2$ is the orientation homomorphism of the $G$-action on $Y$. (See [W]).

It is often easier to deal with $L_n^s(\mathbf{Z}[G], w)$, the surgery obstruction group for simple homotopy equivalences, instead of $L_n^h(\mathbf{Z}[G], w)$. These two groups are related by the Rothenberg exact sequence [Sh]:

$$\cdots \to L_n^s(\mathbf{Z}[G], w) \to L_n^h(\mathbf{Z}[G], w) \xrightarrow{\alpha_G} H^n(\mathbf{Z}_2; \mathrm{Wh}(G)) \to \cdots,$$

where $\mathrm{Wh}(G)$ is the Whitehead group of $G$ and $\alpha_G$ is the torsion homomorphism to be considered shortly. The Tate cohomology group $H^n(\mathbf{Z}_2; \mathrm{Wh}(G))$ is defined as:

$$\{\delta \in \mathrm{Wh}(G) : \delta = (-1)^n \delta^*\}/\{\tau + (-1)^n \tau^* : \tau \in \mathrm{Wh}(G)\},$$

where $*$ denotes the conjugation involution based on the orientation homomorphism $w$.

Let us suppose that our adjusted $G$-normal map $(X, f, b)$ with target $Y$ has been constructed from a $G$-fiber homotopy equivalence $\omega: \eta_+ \to \eta_-$ over $Y$. In this situation, the work of Dovermann [D1] and Dovermann-Rothenberg [DR1] can be applied to give us information on $\alpha_G(\sigma_1(f, b)) \in H^n(\mathbf{Z}_2; \mathrm{Wh}(G))$. Given a $G$-fiber homotopy equivalence $\omega$, its generalized Whitehead torsion $\tau(\omega)$ can be defined as an element of the generalized Whitehead group $\widetilde{\mathrm{Wh}}(G) = \bigoplus \mathrm{Wh}(N_G(H)/H)$, where there is one summand for each conjugacy class of subgroups of $G$. With our set up, a formula due to Dovermann can be used to evaluate $\alpha_G(\sigma_1(f, b))$ in terms of $\tau(\omega)$ and in [DR1], a formula for $\tau(\omega)$ is given in terms of an element in the Burnside ring of $G$.

Indeed, in our proof of Theorem A, we shall use an adjusted $G$-normal map constructed in such a way that the generalized Whitehead torsion of $f^s$ vanishes. In this case, Dovermann's formula for $\alpha_G(\sigma_1(f, b))$ reduces to a particularly simple form, namely, $\alpha_G(\sigma_1(f, b)) = [T\tau(\omega)]$, where $T$ is conjugation on $\widetilde{\mathrm{Wh}}(G)$ (i.e., the ordinary conjugation involution on each summand) and $[\cdot]$ denotes the cohomology class as indicated above. (We note that in [D1], the right-hand side of the equation actually appears as $[T\tau(\varphi)]$, where $\varphi$ is a $G$-fiber homotopy equivalence closely related to our $\omega$. Indeed, there exists a complex $G$-vector bundle $F$ such that $\varphi$ is obtained by adding $\mathrm{id}: F \to F$ to $\omega: \eta_+ \to \eta_-$. However, the addition formula Corollary 8.15 of [DR1] implies that $\tau(\varphi) = \tau(\omega)$. We further note that the results of [D1 and DR1] are written in terms of sphere bundles. The Whitney sum corresponds to fiberwise join. This is not a restriction for us. See §§1–13 of [PR].)

**Lemma 2.5.** *Let $G$ be a finite abelian group and $Y$ an even dimensional $G$-manifold on which $G$ preserves orientation. Let $(X, f, b)$ be an adjusted $G$-normal map with target $Y$ constructed as above from a $G$-fiber homotopy equivalence $\widetilde{\omega}$ such that the generalized Whitehead torsion of $f^s$ vanishes. Suppose that $\widetilde{\omega} = \omega \oplus \omega: \eta_+ \oplus \eta_+ \to \eta_- \oplus \eta_-$, where $\omega: \eta_+ \to \eta_-$ is a $G$-fiber homotopy equivalence over $Y$. Then $\alpha_G(\sigma_1(f, b)) = 0$.*

*Proof.* As mentioned above, [DR1] provides a formula for the generalized Whitehead torsion of $\widetilde{\omega}$, $\tau(\widetilde{\omega})$. The addition formula, Corollary 8.15 of that paper, implies that with our set-up, $\tau(\tilde{\omega})$ is twice an element of $\widetilde{\mathrm{Wh}}(G)$. Therefore, $T\tau(\tilde{\omega})$ is also a "multiple of 2." At this point, we note that from our geometric set-up, the only nonzero coordinate of $\tau(\tilde{\omega})$ lies in $\mathrm{Wh}(G)$. Now, it is known (see [B1] or [Ba]), that if $G$ is finite abelian and preserves orientation (i.e., the orientation homomorphism $w$ is trivial), then the conjugation involution $*$ is trivial on $\mathrm{Wh}(G)$. Then, since $n = \dim Y$ is even, we have that $H^n(\mathbf{Z}_2; \mathrm{Wh}(G)) = \mathrm{Wh}(G)/2\,\mathrm{Wh}(G)$. Since 2 divides $\tau(\tilde{\omega})$, it divides $[T\tau(\tilde{\omega})]$ implying that $\alpha_G(\sigma_1(f, b)) = 0$ as claimed. Q.E.D.

Our purpose for introducing the Rothenberg sequence is to show that $\sigma_1(f, b) \in L_n^h(\mathbf{Z}[G], w)$ comes from an element $\sigma_1^s(f, b) \in L_n^s(\mathbf{Z}[G], w)$, which will be shown to vanish, thereby guaranteeing that $\sigma_1(f, b) = 0$, and that surgery to a $G$-homotopy equivalence is possible. Clearly, $\sigma_1(f, b)$ will come from some $\sigma_1^s(f, b)$ if $\alpha_G(\sigma_1(f, b)) = 0$.

## 3. Proof of Theorem A

In this section, we shall give the proof of

**Theorem A.** *Let $m$ be a prime number. There are infinitely many integers $k$ for which $X_k$ admits a $\mathbf{Z}_m \times \mathbf{Z}_m$-action. More precisely, given relatively prime integers $p$ and $q$, each of which is $\equiv \pm 1 \bmod m$, $X_k$ admits a $\mathbf{Z}_m \times \mathbf{Z}_m$-action, where $k = (p^2 - 1)(q^2 - 1)/3$.*

Our proof will depend upon an appropriate choice of a model $Y$ on which to base our surgery constructions. Given $p$ and $q$ as above, we will construct a $\mathbf{Z}_m \times \mathbf{Z}_m$-fiber homotopy equivalence over $Y$, and from it, an adjusted $\mathbf{Z}_m \times \mathbf{Z}_m$-normal map. A key point will be the use of a particular involution on $Y$ to kill the signature obstruction. Then, we will show that our set-up is such that all obstructions to surgery vanish.

Our model $Y$ and $\mathbf{Z}_m \times \mathbf{Z}_m$-fiber homotopy equivalence will be constructed so as to satisfy an important technical condition called the Transversality Condition (Definition 3.1) which will allow us to build from them an adjusted $\mathbf{Z}_m \times \mathbf{Z}_m$-normal map.

First, we set up some notation. Let $G$ be finite. Given any irreducible, real $G$-representation $\psi$, we define $m_\psi \colon RO(G) \to \mathbf{Z}$ by setting $m_\psi(V)$ equal to the multiplicity of $\psi$ in the virtual representation $V$. ($RO(G)$ denotes the real representation ring of $G$.) Let $d_\psi$ denote the dimension of the real division algebra of $\mathbf{R}$-linear $G$-endomorphisms of $\psi$, $\mathrm{Hom}_{\mathbf{R}}^G(\psi, \psi)$. Finally, let $1_{\mathbf{R}}$ denote the real one-dimensional trivial $G$-representation.

**Definition 3.1** (Transversality Condition). (See [P2].) Let $\omega\colon \eta_+ \to \eta_-$ be a $G$-fiber homotopy equivalence over the smooth $G$-manifold $Y$. We say that the transversality condition is satisfied if for each $H \in \mathrm{Iso}(Y) = \{G_y : y \in Y\}$ and each component $Y_g^H \subseteq Y^H$ the following holds. Let $y \in Y_\alpha^H$. For each real $H$-representation $\psi$ with $m_\psi(\eta_-|_y) \neq 0$ we have

$$\dim Y_\alpha^H = m_{1_{\mathbf{R}}}(TY|_y) \leq d_\psi m_\psi(TY + \eta_+ - \eta_-|_y) + d_\psi - 1 .$$

If the transversality condition is met, Petrie's Transversality Lemma tells us that there are no obstructions to moving $\omega$ by a proper $G$-homotopy to a smooth $G$-map $h$ which is transverse to $Y$, the zero-section of $\eta_-$. We then set $X = h^{-1}(Y)$, $f = h|_X$, and $b$ is constructed using the $G$-vector bundles $\eta_\pm$. More precisely, for $H \subseteq G$, we set $X^H = (f^H)^{-1}(Y^H)$. Note that if a path component $X^H_\alpha$ lies in $(f^H)^{-1}(Y^H_\beta)$, for some component $Y^H_\beta \subseteq Y^H$, then $\dim X^H_\alpha = \dim Y^H_\beta$. Since $\omega$ is a $G$-fiber homotopy equivalence, we can choose the orientation of $X$ so that the $G$-map $f$ will be of degree $1$.

At this point, provided that a few other conditions are met, the triple $(X, f, b)$ will be an adjusted $G$-normal map. In the proof of Theorem A, we shall consider these conditions in detail and show how an adjusted $\mathbf{Z}_m \times \mathbf{Z}_m$-normal map can be constructed from a particular $\mathbf{Z}_m \times \mathbf{Z}_m$-fiber homotopy equivalence satisfying the transversality condition.

A further preliminary to the proof of Theorem A is to provide a list of conditions which guarantee the vanishing of the surgery obstruction associated to an adjusted $G$-normal map $(X, f, b)$. In our case,

$$\sigma_1(f, b) \in L^h_6(\mathbf{Z}[\mathbf{Z}_m \times \mathbf{Z}_m], 1).$$

The first condition is that $\alpha_G(\sigma_1(f, b)) = 0$. Then, $\sigma_1(f, b)$ comes from a unique element $\sigma^s_1(f, b) \in L^s_6(\mathbf{Z}[\mathbf{Z}_m \times \mathbf{Z}_m], 1)$. (Uniqueness does not hold in general, but it does in our case by applying the Rothenberg sequence and using the fact that $H^{\mathrm{odd}}(\mathrm{Wh}(\mathbf{Z}_m \times \mathbf{Z}_m); \mathbf{Z}_2) = 0$. Indeed, $\mathrm{Wh}(\mathbf{Z}_m \times \mathbf{Z}_m)$ is torsion free, in particular, having no two-torsion. See [La].) But, according to [B2], $\sigma^s_1(f, b) = 0 \Leftrightarrow \mathrm{Sign}(\sigma^s_1(f, b)) = 0$ and $c(\sigma^s_1(f, b)) = 0$, where

$$\mathrm{Sign}\colon L^s_6(\mathbf{Z}[\mathbf{Z}_m \times \mathbf{Z}_m], 1) \to R(\mathbf{Z}_m \times \mathbf{Z}_m)$$

is the multisignature and

$$c\colon L^s_6(\mathbf{Z}[\mathbf{Z}_m \times \mathbf{Z}_m], 1) \to \mathbf{Z}_2$$

is the Kervaire-Arf invariant of classical surgery theory. (Here $R(f\mathbf{Z}_m \times \mathbf{Z}_m)$ denotes the complex representation ring.)

There is an interesting $S^1$-map due to Ted Petrie (see [MeP, p. 74]) which will be used in our constructions. Given a pair of relatively prime integers $p$ and $q$, take integers $a$ and $b$ such that $-ap + bq = 1$ and let $t^i$ denote the one-dimensional complex $S^1$-representation, where $t \in S^1$ acts on $\mathbf{C}$ by $t \cdot z = t^i z$ (complex multiplication). Define $f\colon t^{-2p} + t^{-2q} = V_+ \to t^{-2} + t^{-2pq} = V_-$ by $f(z_0, z_1) = (\overline{z}^a_0 z^b_1, z^q_0 + z^p_1)$. It can be shown that $f$ is a proper $S^1$-map such that $\deg f^+ = 1$, where $f^+$ is the extension of $f$ to one point compactifications.

We are now ready to handle the proof of Theorem A.

*Proof of Theorem* A. For $m = 2$, the result is a special case of Theorem 3 in [H1]. So, we let $m$ be an odd prime and let $G = \mathbf{Z}_m \times \mathbf{Z}_m$. We choose our model $Y$ to be $\mathbf{C}P^3$ with the following $\mathbf{Z}_m \times \mathbf{Z}_m$-action.

Let $g$ denote the one-dimensional complex $G$-representation which associates the $1 \times 1$ matrix $(\xi^i)$ to the element $(\xi^i, \xi^j) \in G$. (Here, $\xi = e^{2\pi i/m}$ and we let the generators of $G$ correspond to $(\xi, 1)$ and $(1, \xi)$.) Likewise, $h$ shall denote the $G$-representation which associates $(\xi^j)$ to $(\xi^i, \xi^j)$. Then, $A = g + h + g^{-1} + h^{-1}$ is a $G$-representation and we shall take our model $Y$ to be $p(A)$, the space of complex lines in $A$. $P(A)$ can be thought of as the orbit space $S(A \otimes t)/S^1$, where $t$ denotes the $S^1$-representation $t^1$ using the above notation and $S(\cdot)$ denotes the unit sphere of the indicated representation with respect to a $G \times S^1$-invariant metric.

Let us look more closely at this action on $P(A)$. Notice that $G$ has $m+1$ nontrivial proper subgroups, each of which is, of course, isomorphic to $\mathbf{Z}_m$. We have $H_{01} = 1 \times \mathbf{Z}_m$, $H_{10} = \mathbf{Z}_m \times 1$, and $\{H_{1i} = \langle(\xi, \xi^i)\rangle\}_{i=1}^{m-1}$. Now, if $i \neq 1$, $m-1$, then it is easy to see, using the fact that $m$ is prime, that $P(A)^{H_{1i}} = P(A)^G = \{p_i\}_{i=0}^3$, where $p_0 = [1:0:0:0]$, $p_1 = [0:1:0:0]$, et cetera. Let $Y_{01} = \{[z_0 : z_1 : 0 : 0] | z_0, z_1 \in \mathbf{C}\} = \mathbf{C}P^1$ and let $Y_{ij}$, $i, j = 0, \ldots, 3$ be similarly defined. Then it is not difficult to see that $\mathrm{Iso}(P(A)) = \{1, H_{10}, H_{01}, H_{11}, H_{1m-1}, G\}$ and that $P(A)^{H_{10}} = p_0 \amalg p_2 \amalg Y_{13}$, $P(A)^{H_{01}} = Y_{02} \amalg p_1 \amalg p_3$, $P(A)^{H_{11}} = Y_{01} \amalg Y_{23}$, and $P(A)^{H_{1m-1}} = Y_{03} \amalg Y_{12}$. Clearly, our model $Y = P(A)$ satisfies the gap hypothesis.

We shall now construct a $G$-fiber homotopy equivalence over $P(A)$ using Petrie's $S^1$-map described above. Let $p$ and $q$ be a pair of relatively prime integers each of which are $\equiv \pm 1 \bmod m$ and let $f\colon V_+ \to V_-$ be Petrie's map. By taking twisted products we can form vector bundles associated to the $S^1$-principal bundle $S(A \otimes t) \to P(A)$; namely, $\eta_+ = S(A \otimes t) \times_{S^1} V_+$ and $\eta_- = S(A \otimes t) \times_{S^1} V_-$. The $G$-action on $S(A \otimes t) \times V_\pm$ given by the representation $A$ in the first coordinate and the identity in the second coordinate induces a $G$-vector bundle structure on the $S^1$ orbit spaces $\eta_\pm$. Upon passing to orbit spaces, the map $\mathrm{id} \times f\colon S(A \otimes t) \times V_+ \to S(A \otimes t) \times V_-$ descends to a $G$-map $\omega\colon \eta_+ \to \eta_-$. Since $p$ and $q$ are prime to $|G|$, it can be shown that $\omega$ is actually a $G$-fiber homotopy equivalence. We provide some of the details of the verification of this. Take $H \in \mathrm{Iso}(P(A))$ and suppose that $\mathrm{res}_H A = n_0\psi_0 + \cdots + n_3\psi_3$, where $n_i \in \{0, 1, 2\}$, $\sum_{i=0}^3 n_i = 4$, $\mathrm{res}_H$ denotes restriction to $H$, and the $\psi_i$'s are complex one-dimensional $H$-representations. Then $P(A)^H = \coprod_i P(n_i\psi_i) = \coprod_i \mathbf{C}P^{n_i - 1}$, where $\mathbf{C}P^{-1}$ denotes the empty set. If $x \in P(n_i\psi_i)$, then $\eta_+|_x = \psi_i^{-2p} + \psi_i^{-2q}$ and $\eta_-|_x = \psi_i^{-2} + \psi_i^{-2pq}$ as $H$-representations. (Indeed, suppose $a \in S(A \otimes t)$ lies over $x$. Then $g \cdot [a, v] = [g \cdot a, v] = [\psi_i(g)a, v] = [a, \psi_i(g)v]$ for $[a, v] \in S(A \otimes t) \times_{S^1} V_\pm$.) Since $p$ and $q$ are prime to $|G|$, these $H$-representations shall be free off the origin so long as $\psi_i \neq 1_H$, the trivial complex $H$-representation. So, for $H = G$, $H_{11}$, or $H_{1m-1}$, we have $\eta_\pm^H = P(A)^H$, thought of as part of the zero section. For $H = H_{10}$ or $H_{01}$, $\eta_\pm^H = \eta_\pm|_{\mathbf{C}P^1} \amalg$ (two points), and, of course, for $H = 1$, $\eta_\pm^H = \eta_\pm$. In every case, the fact that $\deg f^+ = 1$ implies that $\deg(\omega|_x)^H = 1$ (when extended to one-point compactifications) as desired.

In order to ensure that the equivariant surgery obstructions which arise vanish, we shall put a bit more symmetry into this $G$-fiber homotopy equivalence before we construct a normal map from it. Specifically, we shall equip $\eta_\pm$ with $G$-equivariant involutions in such a way that $\omega$ becomes a $G \times \mathbf{Z}_2$-fiber homotopy equivalence. Equivariant transversality shall provide a $G \times \mathbf{Z}_2$-manifold and from this, we will construct an adjusted $G$-normal map for which the signature obstruction vanishes.

Notice that $P(A)$ admits a $G$-equivariant, orientation reversing involution $\phi$. Indeed, given $z = [z_0 : z_1 : z_2 : z_3] \in P(A)$, let $\phi(z) = [-\overline{z_2} : -\overline{z_3} : \overline{z_0} : \overline{z_1}]$. Define $\phi'\colon S(A \otimes t) \to S(A \otimes t)$ as $\phi'(a_0, a_1, a_2, a_3) = (-\overline{a_2}, -\overline{a_3}, \overline{a_0}, \overline{a_1})$ and note that $\phi'$ is a $G$-map covering $\phi$. Also notice that $\phi'$ induces a $\mathbf{Z}_4$-action on $S(A \otimes t)$. When passing to the $S^1$ quotient, this action is noneffective and results in the $\mathbf{Z}_2$-action generated by $\phi$. Then $\tilde{\phi}_\pm\colon \eta_\pm \to \eta_\pm$ defined by $\tilde{\phi}_\pm[a, v] = [\phi'(a), v]$ are involutions which cover $\phi$ and make $\eta_\pm$ into $G\times\mathbf{Z}_2$-vector bundles. It is easy to check that $\omega \circ \tilde{\phi}_+ = \tilde{\phi}_- \circ \omega$ and therefore, we see that $\phi$ lifts to $\eta_\pm$ in such a way that $\omega$ becomes a $G \times \mathbf{Z}_2$-fiber homotopy equivalence.

Now, let $\widetilde{\omega} = \omega\oplus\omega : \widetilde{\eta}_+ \to \widetilde{\eta}_-$, where $\widetilde{\eta}_\pm = \eta_\pm \oplus \eta_\pm$. This is the $G\times\mathbf{Z}_2$-fiber homotopy equivalence which we shall work with.

Our next step is to show that the Transversality Condition is satisfied and that we can construct a $G\times\mathbf{Z}_2$-manifold $X$ and a $G\times\mathbf{Z}_2$-map $f\colon X \to P(A)$. First of all, notice that since $\mathbf{Z}_2$ acts freely on $P(A)$, for al $y \in P(A)$, the isotropy group $(G\times\mathbf{Z}_2)_y$ is just $G_y$. Now, the Transversality Condition (Definition 3.1) may be easily verified using the following computations given for each of the isotropy subgroups $H \in \operatorname{Iso}(P(A))$.

If $H = G$, we have isolated fixed points and it is easily checked that $\widetilde{\eta}_+|_{p_i}$ and $\widetilde{\eta}_-|_{p_i}$ are equivalent as $G$-representations for all $i$.

If $H = H_{10}$, then $P(\operatorname{res}_{H_{10}} A) = P(\rho + 1 + \rho^{-1} + 1)$, where $\rho$ denotes the one-dimensional complex $\mathbf{Z}_m$-representation which sends a generator to multiplication by $e^{2\pi i/m}$. Therefore, we compute

$$(TP(A) + \widetilde{\eta}_+)|_{p_0} = \rho^{-1} + \rho^{-2} + \rho^{-1} + 4\rho^{-2}, \qquad \widetilde{\eta}_-|_{p_0} = 4\rho^{-2},$$
$$(TP(A) + \widetilde{\eta}_+)|_y = 1_H + \rho + \rho^{-1} + 4\cdot 1_H, \qquad \widetilde{\eta}_-|_y = 4\cdot 1_H,$$

where $y$ is any point in $Y_{13}$. For the computation of the isotropy representations over $P(A)$, see Proposition 2.3 in [H1]. The representations over the point $p_2$ are conjugate to those over $p_0$. The computations for $H = H_{01}$ are analogous.

If $H = H_{11}$, then $P(\operatorname{res}_H A) = P(2\rho + 2\rho^{-1})$ and we compute

$$(TP(A) + \widetilde{\eta}_+)|_y = 1_H + 2\rho^{-2} + 4\rho^{-2}, \qquad \widetilde{\eta}_-|_y = 4\rho^{-2},$$

where $y$ is any point in $Y_{01}$, while the representations over $Y_{23}$ are conjugate to those over $Y_{01}$. The computations for $H = H_{1m-1}$ are analogous.

As the verification for $H = 1$ is trivial, we see that the transversality condition holds. We can therefore construct the $G \times \mathbf{Z}_2$-manifold $X$, the degree $1 G \times \mathbf{Z}_2$-map $f\colon X \to Y$, and the $G \times \mathbf{Z}_2$-vector bundle isomorphism $b$ as indicated after Definition 3.1.

We now proceed to construct an adjusted $G$-normal map from the triple $(X, f, b)$. There are two technical matters which must be considered.

First of all, we claim that $\operatorname{Iso}(\widetilde{\eta}_+) \subseteq \operatorname{Iso}(P(A))$. Indeed, consider the subgroups $H_{1i}$ of $G$, where $i = 2, \dots, m-2$. Given $y \in P(A)^{H_{1i}}$, $\widetilde{\eta}_+|_y$ contains no trivial $H_{1i}$-representations, so $\widetilde{\eta}_+^{H_{1i}}$ lies in the zero section. Thus, $\widetilde{\eta}_+^{H_{1i}} = \widetilde{\eta}_+^G = \{p_0, p_1, p_2, p_3\}$ and our claim holds. This condition implies that $\operatorname{Iso}(X) = \operatorname{Iso}(P(A))$ and it can be shown that $X$ satisfies the gap hypothesis since $P(A)$ does. (See [PR] for details.)

It remains to show that all fixed point set components can be made simply connected and that $f^s\colon X^s \to P(A)^s$ can be made into a $G$-homotopy equivalence. First notice that it follows from our construction that for $H = G$, $H_{11}$, or $H_{1m-1}$, $X^H$ is $G/H$-diffeomorphic to $P(A)^H$. This is a result of the fact that for these $H$, $1_H$ is not a subrepresentation of $\widetilde{\eta}_\pm|_y$ for any $y \in P(A)^H$.

Next, let us consider the cases of $H = H_{10}$ and $H = H_{01}$. Set $X_{13} = (h^{H_{10}})^{-1}(Y_{13})$ and $X_{02} = (h^{H_{01}})^{-1}(Y_{02})$. First of all, it is not hard to see that every component of $X_{13}$ (resp. $X_{02}$) contains a point with isotropy group equal to $H_{10}$ (resp. $H_{01}$). Indeed, if this were not the case for one of these components, then it would also be a component of $X^G$, which consists of four isolated fixed points. However, this cannot happen since each component of $X_{13}$ and $X_{02}$ is two-dimensional by construction. By applying zero-dimensional $G/H_{10} \times \mathbf{Z}_2$ (resp. $G/H_{01} \times \mathbf{Z}_2$) surgery, we can make $X_{13}$ (resp. $X_{02}$) into a connected $G/H_{10} \times \mathbf{Z}_2$ (resp. $G/H_{01} \times \mathbf{Z}_2$) manifold. (To see why $X_{13}$ and $X_{02}$ admit $G$-equivariant involutions, note that the involution $\phi: P(A) \to P(A)$ restricts to give involutions on $Y_{13}$ and $Y_{02}$ which lift to $\widetilde{\eta}_{\pm}|_{Y_{13}}$ and $\widetilde{\eta}_{\pm}|_{Y_{02}}$ respectively (commuting with the $G$-action).) Then, we perform zero- and one-dimensional $G \times \mathbf{Z}_2$-surgery on $X$ to render it connected and simply connected. (This can be done relative to all fixed point set components.) There are no obstructions to these surgeries. (See §9 of [DP] and Chapter 3 of [PR].) It is important to note that the involution on $X$ reverses orientation. That this holds can be seen using the $\mathbf{Z}_2$-equivariance of $f$.

More care is needed in carrying out one-dimensional $G$-surgeries on $X_{13}$ and $X_{02}$. (Note that we are no longer performing $G \times \mathbf{Z}_2$-surgery.) We consider $X_{13}$ in detail, the case of $X_{02}$ being similar. We need to show that the surgery kernel $\operatorname{Ker}\{f_*: H_1(X_{13}) \to H_1(Y_{13})\}$ can be killed by subtracting $G$-handles. Notice that since $Y_{13}$ is simply connected, this kernel is simply $H_1(X_{13})$, a direct sum of $2n$ copies of $\mathbf{Z}$, where $n$ is the genus of $X_{13}$. Since $\mathbf{Z}_m$ acts on $X_{13}$, we have that $n$ is a multiple of $m$. (See, for instance, Theorem 3 of [Y].)

At this point, we appeal to the famous result of Jacob Nielsen that an orientation preserving action of a cyclic group is determined up to equivalence by its collection of isotropy representations [N]. Notice that $X_{13}$ has two fixed points, namely, $x_i = f^{-1}(p_i)$, for $i = 1$ or $3$. The stable $G$-vector bundle isomorphism $b$ allows us to identify the isotropy representations above these points. Indeed, we have, $TX_{13}|_{x_1} = TY_{13}|_{p_1} = g^{-2}$ and $TX_{13}|_{x_3} = TY_{13}|_{p_3} = g^2$. Nielsen's result then tells us that the action of $\mathbf{Z}_m$ on $X_{13}$ is equivalent to the standard "rotational" action on a genus $n$ surface. By this, we mean the action obtained by considering this surface as a two-sphere with $n$ handles attached symmetrically about the equator so that $\mathbf{Z}_m$ acts by rotation about the axis through the north and south poles. (Recall that $n$ is a multiple of $m$.) These poles are left fixed, of course, and are the two fixed points of the action.

However, for this action, the $\mathbf{Z}[\mathbf{Z}_m]$-module structure of $H_1(X_{13})$ is easily seen and it is also easy to see how to kill $H_1(X_{13})$ by removing $\mathbf{Z}_m$-handles. It must further be checked that the stable bundle isomorphism $b$ extends to the new $G$-manifold resulting from these surgeries. That is, we need to have the appropriate bundle data to give us a new normal map on which to continue our surgery constructions. Notice that the $\mathbf{Z}_m$-handles mentioned above consist of $S^1 \times D^1$'s (thickened appropriately via the equivariant normal bundle). Upon removal, $D^2 \times S^0$'s are glued in (again appropriately thickened). The question is whether or not the bundles over the $S^1 \times S^0$'s extend over the $D^2 \times S^0$'s. A priori, this need not be the case (consider the Lie framing of the torus). However, in our set-up the bundles extend. This can be seen by noting that $X$ is at this point simply connected. So, the bundle restricted to each $S^1$ already extends over a disk inside $X$. We can use these extensions to define the desired extension when the $D^2 \times S^0$'s are attached to $X$ minus the $\mathbf{Z}_m$-handles. (Notice that we are not doing ambient surgery as it would introduce points on the surface not fixed by the subgroup $H$. Rather, we use the fact that the bundles extend over disks as a way to extend the bundles over the new disks which we attach.) Therefore, one-dimensional $G$-surgery is possible on $X_{13}$ and $X_{02}$ converting them into one-connected $G$-surfaces.

Now, since $X_{13}$ and $X_{02}$ are simply connected, closed surfaces, they must be spheres which, by construction, are $G/H$-homotopy equivalent to $Y_{13}$ and $Y_{02}$ respectively (where $H = H_{10}$ and $H_{01}$ respectively). Indeed, $f|_{X_{13}}\colon X_{13} \to Y_{13}$ is a $G/H_{10}$-map of degree 1 between two-spheres and is hence a homotopy equivalence. Further, it is a $G/H_{10}$-homotopy equivalence since $(f|_{X_{13}})^{G/H_{10}}$ is trivially an ordinary homotopy equivalence, being a map taking two points onto two points. Similar considerations apply to $X_{02}$.

At this point, we have an adjusted $G$-normal map, which we shall denote by $(X', f', b')$. Note that $X'$ need not admit an involution, but it is $G$-cobordant to $X$ which does admit a $G$-equivariant, orientation reversing involution. We are then brought to step 2 of the surgery process; i.e., the consideration of the surgery obstruction $\sigma_1(f', b')$.

According to the criterion for the vanishing of $\sigma_1(f', b')$ set out previously, there are three things to be demonstrated.

First of all, we must show that $\alpha_G(\sigma_1(f', b')) = 0$. According to Lemma 2.5, this torsion invariant will vanish provided that the generalized Whitehead torsion of $(f')^s$, $\tau((f')^s)$, vanishes. Notice that $(X')^s$ and $Y^s$ are connected, consisting of a union of six two-spheres. (Think of the complete graph on four vertices (with the edges having pairwise disjoint interiors) with edges corresponding to two-spheres and vertices corresponding to points.) We have seen that $(f')^G$, $(f')^{H_{11}}$, and $(f')^{H_{1m-1}}$ are $G$-diffeomorphisms. There is a Mayer-Vietoris formula for generalized Whitehead torsion found on p. 67 of [DR1]; namely, say that $f_i\colon A_i \to B_i$, $i = 1, 2$ are $G$-maps and that $f_1$ and $f_2$ coincide on $A = A_1 \cap A_2$. Then $\tau(f_1 \cup f_2) = (j_1)_*\tau(f_1) + (j_2)_*\tau(f_2) - j_*\tau(f_1 \cap f_2)$, where $j_i\colon B_i \to B_1 \cup B_2$ and $j\colon B_1 \cap B_2 \to B$ are inclusions. Using this, we see that it suffices to show that $\tau(f'|_{X'_{02}}\colon X'_{02} \to Y_{02}) = 0 = \tau(f'|_{X'_{13}}\colon X'_{13} \to Y_{13})$. However, this fact follows from our set-up. Indeed, in both cases we have a $\mathbf{Z}_m$-homotopy equivalence between two-spheres. (Note that $G = \mathbf{Z}_m \times \mathbf{Z}_m$ is not acting effectively, so we only need consider the effect $\mathbf{Z}_m$-action.) Therefore, the vanishing of $\alpha_G(\sigma_1(f', b'))$ follows from the following claim.

*Claim* 3.2. The generalized Whitehead torsion of a $\mathbf{Z}_m$-homotopy equivalence $h\colon S^2 \to S^2$ vanishes.

*Proof*. Note that in this proof, we are actually computing the torsion of $h$ in the equivariant Whitehead group $\mathrm{Wh}(S^2)$, rather than its image in the generalized Whitehead group $\widetilde{\mathrm{Wh}}(G)$ which is what appears in Dovermann's formula. (See [I] or [Lü] for the equivariant Whitehead group.) Now, it is well known that any compact Lie group action on $S^2$ is smoothly equivalent to a linear action. (This is a classical result due to Brouwer [B], Kerekjarto [K], and Eilenberg [E].) So, by composing $h$ on the left and right by $G$-diffeomorphisms, we may assume that we have $h\colon S(V) \to S(W)$, where $V$ and $W$ are real three-dimensional $\mathbf{Z}_m$-representations. (Note that this new $h$ has vanishing torsion if and only if the original $h$ does. Indeed, $\tau(f \circ h \circ g) = f_*(\tau(h))$, where $f$ and $g$ are assumed to be $G$-diffeomorphisms. This uses the composition formula found, for instance, on p. 64 of [Lü] and the fact that the torsion of a $G$-diffeomorphism vanishes. Further note that $f_*$, the homomorphism induced on the equivariant Whitehead groups, is an isomorphism.) Furthermore, since $S(V)$ and $S(W)$ are $\mathbf{Z}_m$-homotopy equivalent $S^2$'s, we must have that $V = W = 1_{\mathbf{R}} + (\rho^a)_{\mathbf{R}}$, where $\rho$ is as in the preceeding discussion and $a$ is an integer prime to $m$. For example, it is not hard to see that $Y_{02} = S(1_{\mathbf{R}} + (g^{-2})_{\mathbf{R}})$. So, we need to show that the generalized Whitehead torsion of a $\mathbf{Z}_m$-homotopy equivalence $h: S(V) \to S(V)$ vanishes. Now, $\deg h = \pm 1$, i.e., we have $\deg h^H = \deg 1^H_{S(V)}$ or $\deg h^H = \deg a^H$, for $H = 1$ or $\mathbf{Z}_m$, where $1_{S(V)}$ is the identity map and $a$ is the antipodal map. (Either of which is a $G$-diffeomorphism and therefore having vanishing torsion.) Then, according to Proposition 3.1 on p. 288 of [T], which is essentially an equivariant Hopf Theorem for linear $G$-spheres, $h$ is $G$-homotopic to either $1_{S(V)}$ or $a$. However, the generalized torsion of a $G$-homotopy equivalence is a $G$-homotopy invariant (see Theorem 4.8 of [Lü]). This establishes our claim. I thank the referee for pointing out the fact that $\mathbf{Z}_m$-homotopy equivalences between spheres of the same representation can be considered as units in the Burnside ring and that in this case the units are $\{\pm 1\}$, both of which may be represented by $\mathbf{Z}_m$-diffeomorphisms. Q.E.D.

It now remains to show the vanishing of the associated obstruction $\sigma_1^s(f', b') \in L_6^s(\mathbf{Z}[G], 1)$. The first of the two required steps is to show that the signature $\text{Sign}(\sigma_1^s(f', b')) = 0$. Here, a useful formula comes into play, namely, $\text{Sign}(\sigma_1^s(f', b')) = \text{Sign}(G, X') - \text{Sign}(G, P(A))$, where $\text{Sign}(G, \cdot)$ denotes the $G$-signature of Atiyah and Singer. (See [AS and P1].) Now, by definition, it is clear that $\text{Sign}(G, P(A)) = 0$ as $P(A)$ has no middle dimensional cohomology. To show that $\text{Sign}(G, X') = 0$ we need two facts. The first is that if a $G$-manifold $M$ admits a $G$-equivariant, orientation reversing diffeomorphism, then $\text{Sign}(G, M) = 0$. Secondly, we need the fact that $\text{Sign}(G, \cdot)$ is a $G$-cobordism invariant. Our $X'$ may not be equipped with such a diffeomorphism, but it is $G$-cobordant to $X$ which does admit such a diffeomorphism, namely, the involution that was built into it. Therefore, $\text{Sign}(G, X') = 0$ as desired.

The second and final step in showing that $\sigma_1^s(f', b') = 0$ is to show that the Kervaire-Arf invariant $c(\sigma_1^s(f', b'))$ vanishes. This invariant depends only on the initial (nonequivariant) fiber homotopy equivalence used in the construction of our normal map. It is known that the Kervaire-Arf invariant of twice a fiber homotopy equivalence vanishes. This follows from the fact that the Kervaire-Arf invariant can be expressed in terms of the Kervaire-Sullivan classes via Sullivan's characteristic variety formula (p. 152 of [BM]). The primitivity of these classes implies that a normal map obtained from twice a fiber homotopy equivalence will have vanishing Kervaire-Arf invariant. (Also of relevance is the formula due to Masuda [M2] and independently to Schultz [S1] which gives the Kervaire-Arf invariant for certain fiber homotopy equivalences closely related to ours.) As we are working with $\tilde{\omega} = \omega \oplus \omega$, we do have $c(\sigma_1^s(f', b')) = 0$. Together with the preceeding paragraph this shows that $\sigma_1^s(f', b') = 0$.

Therefore, we have that $\sigma_1(f', b')$ vanishes and according to Proposition 2.4, $G$-surgery provides us with an adjusted $G$-normal map $(X_{\tilde{\eta}}, F, B)$, where $F: X_{\tilde{\eta}} \to P(A) = \mathbf{C}P^3$ is a $G$-homotopy equivalence.

The stable bundle isomorphism $B$ between $TX_{\tilde{\eta}}$ and $F^*(TY + \tilde{\eta}_+ - \tilde{\eta}_-)$ allows us to compute the Pontryagin class of the smooth manifold $X_{\tilde{\eta}}$. In particular, the first Pontryagin class is given by

$$p_1(X_{\tilde{\eta}}) = (4 + 8(p^2 - 1)(q^2 - 1))x^2,$$

where $x \in H^2(X_{\tilde{\eta}})$ is a generator. (See [H1, §§6 and 7] for more details on this calculation.) So, $X_{\tilde{\eta}} = X_k$, where $k = (p^2 - 1)(q^2 - 1)/3$ and by varying $p$ and $q$ (within the constraints that $p$ and $q$ are relatively prime and are each $\equiv \pm 1 \bmod m$) we can construct $\mathbf{Z}_m \times \mathbf{Z}_m$-actions for infinitely many $k$. Q.E.D.

*Remark.* Notice that our $\mathbf{Z}_m \times \mathbf{Z}_m$-actions on $X_k$ are such that the fixed point sets consist of four isolated fixed points. It follows from a theorem due to Dovermann that, for $m$ odds, this is the only fixed point set that can arise.

**Theorem** (Dovermann [D2]). *If $\mathbf{Z}_m$ ($m$ an odd prime) acts on $X_k$ with fixed point set $F \amalg$ point, where $F$ is connected, then $k = 0$. (Note that $F$ must have dimension 4 by Bredon's Theorem* [Br2, Chapter 7].)

Actually, his statement is more general than this. We have rephrased his result for our purposes. A similar result for $m = 2$ is due to Masuda [M1]. Also see [DMSu].

**Corollary 3.3.** *If $\mathbf{Z}_m \times \mathbf{Z}_m$ ($m$ an odd prime) acts effectively on $X_k$ with $k \neq 0$, then $(X_k)^{\mathbf{Z}_m \times \mathbf{Z}_m}$ consists of four isolated fixed points.*

*Proof*. Suppose that $G = \mathbf{Z}_m \times \mathbf{Z}_m$ acts on $X_k$ with $k \neq 0$. First of all, $X_k^G$ cannot be empty since it must have Euler characteristic $\equiv 4 \bmod m$. (This follows from results in Chapter 3 of [Br2].) Next, note that since $\dim X_k$ is even and the order of $G$ is odd, each component of $X_k^G$ must have even dimension. (Indeed, $\nu(X_k^G, X_k)$ is a sum of irreducible nontrivial $G$-representations and is hence even dimensional.) Now, according to Chapter VI, §3 of [Hs] every component of $X_k^G$ must be a mod $m$ cohomology $\mathbf{C}P^r$, with $r = 0, 1$, or $2$. If any mod $m$ cohomology $\mathbf{C}P^2$ occurs, then Dovermann's result implies that $k = 0$. So, we may suppose that this does not happen. Next, suppose that $X_k^G$ contains a mod $m$ cohomology $\mathbf{C}P^1$. In [Hs], we learn that there is a linear action of $\mathbf{Z}_m \times \mathbf{Z}_m$ on $\mathbf{C}P^3$ which provides a model, up to mod $m$ cohomology type, of the orbit structure of our action on $X_k$. Now, any linear $G$-action on $\mathbf{C}P^3$ leaving a $\mathbf{C}P^1$ fixed would have to look like $P(\psi + \psi + \psi_1 + \psi_2)$, where $\psi$, $\psi_1$, and $\psi_2$ are irreducible complex $G$-representations. Note that $\psi \neq \psi_i$ for $i = 1, 2$, but we may have $\psi_1 = \psi_2$. The irreducible character $\psi \circ \psi_1^{-1} : G \to S^1$ must have kernel of rank 1. Therefore, there is a subgroup $K$, isomorphic to $\mathbf{Z}_m$, for which $\mathrm{res}_K\, \psi = \mathrm{res}_K\, \psi_1$ as $K$-representations. Then, we must have a group isomorphic to $\mathbf{Z}_m$ acting on $X_k$ and fixing a mod $m$ cohomology $\mathbf{C}P^2$ or all of $X_k$. This forces $k = 0$ which is contrary to assumption. Thus, we have no mod $m$ cohomology $\mathbf{C}P^1$ fixed point set components and the fixed point set consists of isolated fixed points (being a closed submanifold of $X_k$). The above-mentioned linear model shows that there are four of them. Q.E.D.

So, for instance, we could not have constructed $\mathbf{Z}_m \times \mathbf{Z}_m$-actions on non-standard homotopy $\mathbf{C}P^3$'s by basing our surgery constructions on a model like $P(2g + 2h)$, which has fixed point set consisting of two copies of $\mathbf{C}P^1$.

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