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# $T1$ THEOREMS FOR BESOV AND TRIEBEL-LIZORKIN SPACES

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ABSTRACT. We give simple proofs of the $T1$ theorem in the general context of Besov spaces and (weighted) Triebel-Lizorkin spaces. Our approach yields some new results for kernels satisfying weakened regularity conditions, while also recovering previously known results.

## 1. INTRODUCTION

In recent years, there has been significant progress on the problem of proving boundedness of generalized Calderón-Zygmund operators on various function spaces. The prototypical result is the famous " $T1$ " theorem of David and Journé [DJ]. Suppose that we have a continuous, linear mapping $T: \mathscr{D} \to \mathscr{D}'$ associated with a kernel $K(x, y)$ (in the sense that

$$\langle Tf, g\rangle = \int\int g(x)K(x, y)f(y)\,dx\,dy$$

for test functions $f$ and $g$ with disjoint support). Assume that $K(x, y)$ satisfies the pointwise conditions

$$\text{(P1)} \qquad |K(x, y)| \le A|x-y|^{-n},$$

$$\text{(P2)} \qquad \begin{aligned} &|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \\ &\quad \le \omega_\infty\left(\frac{|x-x'|}{|x-y|}\right)|x-y|^{-n} \quad (\text{for } |x-y| \ge 2|x-x'|) \end{aligned}$$

where for now, say, $\omega_\infty(r) = r^\varepsilon$, $0 < \varepsilon \le 1$. Assume also that $T$ satisfies the Weak Boundedness Property

$$\text{(WBP)} \qquad \begin{aligned} &|\langle T\varphi, \psi\rangle| \le CR^n(\|\varphi\|_\infty + R\|\nabla\varphi\|_\infty)(\|\psi\|_\infty + R\|\nabla\psi\|_\infty) \\ &\qquad\qquad (\text{if diam supp } \varphi, \text{ diam supp } \psi \le R). \end{aligned}$$

The result of David and Journé states that under the above conditions, $T$ extends to a bounded operator on $L^2$ iff $T1 \in \mathrm{BMO}$ and $T^*1 \in \mathrm{BMO}$ (where $T^*$ is defined by $\langle T\varphi, \psi\rangle \equiv \langle T^*\psi, \varphi\rangle$). Yabuta [Y], showed that this result could be recovered with a weaker regularity assumption (P2), namely that $\omega_\infty$ was now permitted to be an increasing function satisfying the Dini-type condition

$$\int_0^1 (\omega_\infty(t))^{1/3}\frac{dt}{t} < \infty.$$

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The kernel conditions were further weakened by Y. Meyer [M], who replaced the pointwise estimates (P1) and (P2) by the integral estimates

(L1) $$\sup_{R>0} \int_{R \le |x-y| \le 2R} |K(x, y)| + |K(y, x)|\, dx \le C,$$

(L2)

$$\sup_{\substack{R>0 \\ |u|+|v|\le R}} \left[ \int_{2^k R \le |x-y| \le 2^{k+1} R} |K(x+u, y+v) - K(x, y)|\, dx \right.$$
$$\left. + \int_{2^k R \le |x-y| \le 2^{k+1} R} |K(x+u, y+v) - K(x, y)|\, dy \right] \equiv \delta_1(k),$$

where $k = 1, 2, 3, \dots,$ and $\delta_1 = \delta$ satisfies

(1.1) $$\sum_{k=1}^{\infty} k\delta(k) < \infty.$$

Equivalently we may take $\delta_1(k) = \omega(2^{-k})$, where

(1.1a) $$\int_0^1 \omega(t) \log \frac{1}{t} \frac{dt}{t} < \infty.$$

I.e., since $\delta_1$ is nonincreasing by definition, we can define $\omega$ as a step function. Meyer's theorem is also notable for his method of proof, which was to show that in the special case $T1 = 0 = T^*1$, $T$ is bounded on the Besov space $\dot{B}_1^{0,1}$, and then use duality and interpolation to obtain the $L^2$ bound. The restriction $T1 = 0 = T^*1$ can then be removed as in [DJ] by using the $L^2$-boundedness of the "paraproduct"

(1.2) $$\Pi_b \equiv \int_0^\infty Q_t(Q_t b) P_t \frac{dt}{t},$$

where $b$ is a BMO function (in fact one takes $b$ to be $T1$ or $T^*1$), and $Q_t$ and $P_t$ are convolution operators defined respectively by radial $C^\infty$ kernels $\psi_t(x) = t^{-n}\psi(x/t)$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$ with support in the unit ball and $\int \psi = 0$, $\int \varphi = 1$. Also, $\int_0^\infty Q_t^2 \frac{dt}{t} = I$, so $\Pi_b 1 = b$. The $L^2$ bound for $\Pi_b$ is an easy consequence of the relationship between BMO and Carleson measures. The Besov spaces $\dot{B}_p^{\alpha,q}$, $1 \le p, q \le \infty$, can be defined in general by the norm

(1.3) $$\|f\|_{\dot{B}_p^{\alpha,q}} \equiv \left( \int_0^\infty (t^{-\alpha} \|Q_t f\|_p)^q \frac{dt}{t} \right)^{1/q} < \infty,$$

where we make the obvious change if $p$ or $q = \infty$, and where $Q_t$ is defined as above. Although in the special case $\alpha = 0$ and $q = p$, Meyer's result can be extended to $\dot{B}_p^{0,p}$ by interpolation, the first $T1$ theorem for general Besov spaces was proved by Lemariè [L], who showed that $T$ is bounded on $\dot{B}_p^{\alpha,q}$, $1 \le p, q \le \infty$, if $T1 = 0$ and if the pointwise kernel conditions (P1) and (P2) hold, in this case with $\omega_\infty(r) = r^\varepsilon$, and $0 < \alpha < \varepsilon$. (In fact this result holds if $K(x, y)$ is smooth only in the first variable, although to extend the range of $\alpha$ to $-\varepsilon < \alpha < \varepsilon$, one requires smoothness in both variables and also $T^*1 = 0$.)

Related to the Besov spaces are the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$, defined by the norm (for $1 \le p, q < \infty$),

$$\|f\|_{\dot{F}_p^{\alpha,q}} \equiv \left\| \left( \int_0^\infty (t^{-\alpha}|Q_t f|)^q \frac{dt}{t} \right)^{1/q} \right\|_p < \infty. \tag{1.4}$$

For example, in the special case $\alpha = 0$, $q = 2$, the inner integral is just a Littlewood-Paley $g$-function and therefore $\dot{F}_p^{0,2} = H^p$ $(0 < p < \infty$, and in particular $L^p$, $1 < p < \infty)$. Also, $\dot{B}_p^{\alpha,p} = \dot{F}_p^{\alpha,p}$, and in particular $\dot{F}_2^{0,2} = \dot{B}_2^{0,2} = L^2$. A $T1$ theorem was proved in this context by Frazier, Jawerth, Han, and Weiss [FJHW]: $T$ is bounded on $\dot{F}_p^{\alpha,q}$, $1 \le p, q \le \infty$, if $K$ satisfies (P1) and (P2) with $\omega_\infty(r) = r^\varepsilon$, $T1 = 0$, and $0 < \alpha < \varepsilon$. If $T^*1 = 0$ also, then their result extends to $-\varepsilon < \alpha < \varepsilon$. The method of proof was to use atomic and molecular decompositions of Triebel-Lizorkin spaces due to Frazier and Jawerth [FJ] and then to show that $T$ maps atoms into molecules. Recently the present authors [HH] followed the ideas of [FJ] to prove a weak molecule condition for $\dot{F}_p^{0,q}$, $1 < p, q < \infty$, which then yielded a $T1$ theorem for these spaces with the smoothness condition (P2) relaxed so that $\int_0^1 \omega_\infty(t)|\log t|^2 \frac{dt}{t} < \infty$. Obtaining a weak molecule condition was also Y. Meyer's approach in proving his $T1$ theorem for $\dot{B}_1^{0,1}$.

The point of the present paper is to give a systematic treatment that includes the above results (and some new ones) by means of a simple, direct proof without recourse to atoms, molecules or interpolation. In the context of Besov spaces, we consider a slightly weaker version of Meyer's smoothness condition

$$\tag{L$'$2} \sup_{\substack{R>0 \\ |u|+|v|\le R}} \left[ \int_{2^j R \le |x-y|} |K(x+u, y+v) - K(x,y)|\, dx + \int_{2^j R \le |x-y|} |K(x+u, y+v) - K(x,y)|\, dy \right] \equiv \gamma_1(j),$$

where

$$\sum_{j=1}^\infty \gamma_1(j) < \infty. \tag{1.5}$$

Since $\gamma_1(j) \le \sum_{k=j}^\infty \delta_1(k)$, it is easy to see that (L2) $\Rightarrow$ (L$'$2). The point of (L$'$2) is that we do not require that the supremum be taken over each annulus separately, as was the case for (L2). Of course, if $\delta_1(j)$ or $\gamma_1(j) \le C2^{-j\varepsilon}$, then (L2) and (L$'$2) are equivalent, by the elementary fact that the sum of a convergent geometric series is comparable to the first term. We prove the following:

**Theorem 1.** *Suppose that* $T1 = 0$, *that* $T$ *satisfies* WBP, *and that the kernel* $K(x,y)$ *satisfies* (L$'$2). *If the* $L^1$ *"modulus of continuity"* $\gamma_1(j)$ *satisfies* (1.5), *and if* $T^*1 = 0$, *then* $T$ *is bounded on* $\dot{B}_p^{0,q}$, $1 \le p, q \le \infty$. *If* $\delta_1(k) \le C2^{-k\varepsilon}$, *and* $T^*1$ *is arbitrary, then* $T$ *is bounded on* $\dot{B}_p^{\alpha,q}$, $1 \le p, q \le \infty$, $0 < \alpha < \varepsilon$.

*Remarks.* For $\alpha = 0$, and $p = q$, this is essentially Meyer's result. The general case is in one sense a refinement of Lemariè's result, in that his pointwise kernel estimates are replaced by integral estimates. For $-\varepsilon < \alpha < 0$, Theorem 1 holds if $T^*1 = 0$ and $T1$ is arbitrary, since the dual of $\dot{\beta}_p^{\alpha,q}$ is $\dot{\beta}_{p'}^{-\alpha,q'}$.

For $w \in A_p$, define the weighted Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}(w)$ by the norm

$$\|f\|_{\dot{F}_p^{\alpha,q}(w)} \equiv \left(\int_{\mathbf{R}^n}\left(\int_0^\infty (t^{-\alpha}|Q_t f(x)|)^q \frac{dt}{t}\right)^{p/q} w(x)\,dx\right)^{1/p} < \infty.$$

In particular, for $\alpha = 0$, $q = 2$, and $1 < p < \infty$, $\dot{F}_p^{0,2}(w) = L^p(w)$ by the weighted norm inequality for the Littlewood-Paley $g$-function. We have the following:

**Theorem 2.** *Let $T1 = 0$, and let $T$ satisfy* WBP. *Suppose the pointwise kernel estimate* (P2) *holds. Let $1 < p, q < \infty$, and let $w \in A_p$. If $\omega_\infty = \omega$ satisfies the Dini-type condition* (1.1a), *and if $T^*1 = 0$, then $T$ is bounded on $\dot{F}_p^{0,q}(w)$. If $\omega_\infty(t) = t^\varepsilon$, then $T$ is bounded on $\dot{F}_p^{\alpha,q}(w)$, $0 < \alpha < \varepsilon$.*

*Remarks.* For $w = 1$, the case $\alpha = 0$ is a sharpened version (i.e., with weaker smoothness condition for $\omega_\infty$) of the result in [HH], and the case $\alpha > 0$ is the result of [FJHW] (although we do not obtain the endpoint cases of that paper, $p, q = 1$ or $\infty$). For $w \in A_p$, the results above are new, except that for $\alpha = 0$, $q = 2$, we recover the known boundedness of $T$ on $L^p(w)$ (this could also be obtained from the unweighted $T1$ theorem and the result of Coifman-Fefferman [CF], which does not require convolution, but only the unweighted $L^2$-bound and the kernel conditions (P1) and (P2)). As was the case for Theorem 1, if $T^*1 = 0$, we can extend to negative $\alpha$ by duality.

Motivated by [KW], we can also relax the kernel conditions of Theorem 2 to obtain some new results if we impose stronger assumptions on the weight $w$. We consider the conditions

$$(\mathrm{L}^r1) \qquad \sup_{R>0} R^{n/r'}\left(\int_{R\le|x-y|\le 2R} (|K(x,y)| + |K(y,x)|)^r\,dx\right)^{1/r} \le C$$

$(\mathrm{L}^r2)$

$$\sup_{\substack{R>0\\|u|+|v|\le R}} (2^kR)^{n/r'}\left[\left(\int_{2^kR\le|x-y|\le 2^{k+1}R} |K(x+u,y+v) - K(x,y)|^r\,dx\right)^{1/r} + \left(\int_{2^kR\le|x-y|\le 2^{k+1}R} |K(x+u,y+v) - K(x,y)|^r\,dy\right)^{1/r}\right] \equiv \delta_r(k).$$

**Theorem 3.** *Let $r' < \min(p, q)$ and let $w \in A_{p/r'}$. Suppose $T1 \equiv 0$, and $T$ satisfies* WBP. *If $K$ satisfies* $(\mathrm{L}^r2)$ *for $\delta_r = \delta$ satisfying* (1.1), *and if $T^*1 = 0$ then $T$ is bounded on $\dot{F}_p^{0,q}(w)$. If $\delta_r(k) \le 2^{-k\varepsilon}$, then $T$ is bounded on $\dot{F}_p^{\alpha,q}(w)$, $0 < \alpha < \varepsilon$.*

A final comment is in order about all of these results. We have assumed that $T1 = 0$ (and for $\alpha = 0$, $T^*1 = 0$). For the special case $L^p$ (or $L^p(w)$), this can be extended to $T1 \in \mathrm{BMO}$ by using the paraproduct (1.2) (that $\Pi_b$ is bounded on $L^2(w)$, $w \in A_2$, for $b \in \mathrm{BMO}$, is due to Journé [J]—the proof is similar to the case $w \equiv 1$; the boundedness on $L^p(w)$ follows by the extrapolation theorem of Rubio de Francia [GR]). In general, however, it is an open problem to determine necessary and sufficient conditions on $b$ so that $\Pi_b$ is bounded on $\dot{B}_p^{\alpha,q}$ or $\dot{F}_p^{\alpha,q}$ (see M. Meyer [MM] for a result in the special case $\dot{B}_1^{0,1}$).

## 2. Proof of Theorem 1

We will use a representation of the operator $T$ that appears in a recent paper of Coifman, David, Meyer and Semmes [CDMS]. Let $Q_t$ be as above, so in particular $\int_0^\infty Q_t^2 \frac{dt}{t} = I$. Then

$$T = \int_0^\infty \int_0^\infty Q_s^2 T Q_t^2 \frac{dt}{t}\frac{ds}{s}. \tag{2.1}$$

The authors of [CDMS] used this representation to prove a different sort of $T1$ theorem, namely that for $w \in A_1$, $T\colon L^2(w) \to L^2(\frac{1}{w})$ if $T1 \in \mathrm{BMO}_w$, $T^*1 \in \mathrm{BMO}_w$, if the kernel $K(x, y)$ satisfies certain "$w$-standard" estimates, i.e. certain size and regularity conditions involving the weight $w$, and if $T$ satisfies a "$w$-WBP." We wish to acknowledge that our approach here was motivated by their work, and the structure of our arguments follows theirs.

We also wish to thank the referee for a helpful comment concerning the exposition.

By duality, it is enough to show, with $f, g \in C_0^\infty$,

$$|\langle Tf, g\rangle| \le C\|f\|_{\dot{B}_p^{\alpha,q}}\|g\|_{\dot{B}_{p'}^{-\alpha,q'}}.$$

Using the representation (2.1), we write

$$\langle Tf, g\rangle = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} Q_s T Q_t (Q_t f)(x) Q_s g(x)\, dx \frac{dt}{t}\frac{ds}{s}. \tag{2.2}$$

We will proceed formally, but one can make the arguments rigorous by truncating the $dt$ and $ds$ integrals and obtaining bounds independent of the truncation.

Set $T_{st} \equiv Q_s T Q_t$, and let $K_{st}(x, y)$ be the associated kernel.

**Lemma 2.3.** *If $\gamma_1$ satisfies (1.5), and $T1 = 0 = T^*1$ then for $j = 1, 2, 3, \ldots$ and either $2^{-j}t \le s \le 2^{-j+1}t$ or $2^{-j}s \le t \le 2^{-j+1}s$, we have*

$$\int_{\mathbb{R}^n} |K_{st}(x, y)|\, dy + \int_{\mathbb{R}^n} |K_{st}(x, y)|\, dx \le \tilde{\gamma}(j), \tag{2.4}$$

*where*

$$\sum_{j=1}^\infty \tilde{\gamma}(j) < \infty. \tag{2.5}$$

*If $2^{-j}t \le s \le 2^{-j+1}t$, if $\delta_1(k) \le C2^{-k\varepsilon}$, and $T1 = 0$, with $T^*1$ arbitrary, then (2.4) holds with $\tilde{\gamma}(j) \le C2^{-j\varepsilon}$. (If $\varepsilon = 1$, then $\tilde{\gamma}(j) \le Cj2^{-j}$.) If $t \le s$, then the left-hand side of (2.4) is bounded by a constant.*

Let us assume Lemma 2.3 for now and deduce Theorem 1. To estimate (2.2), we integrate first over the set $\{(s, t)\colon t \ge s\}$ whose characteristic function we indicate by $\chi$ (if $\alpha = 0$, the case $s \geq t$ is dual to the one we consider). We apply Hölder's inequality to the $dx$ integral in the right side of (2.2) which is bounded by

$$\int_0^\infty \int_0^\infty \chi \|T_{st}Q_t f\|_p \|Q_s g\|_{p'} \frac{ds}{s}\frac{dt}{t} \tag{2.6}$$

plus the analogous term where $s \geq t$. Now, by Hölder,

$$\begin{aligned} &\|T_{st}Q_t f\|_p \\ &\quad \leq \left( \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |K_{st}(x, y)|\, dy \right]^{p/p'} \left[ \int_{\mathbb{R}^n} |K_{st}(x, y)|\, |Q_t f(y)|^p\, dy \right] dx \right)^{1/p}. \end{aligned} \tag{2.7}$$

At the endpoint cases, for $p = 1$, (2.7) is simply

$$\iint |K_{st}(x, y)|\, |Q_t f(y)|\, dy\, dx.$$

If $p = \infty$, we make a slight modification in the argument, and dominate the $dx$ integral in (2.2) by $\|Q_t f\|_\infty$ times

$$\iint |K_{st}(x, y)|\, |Q_s g(x)|\, dy\, dx.$$

Write $\chi \equiv \sum_{j=1}^\infty \chi_j$, where $\chi_j(s, t)$ is the characteristic function of $\{(s, t)\colon 2^{-j}t \leq s \leq 2^{-j+1}t\}$. Then on the support of $\chi_j$, by Lemma 2.3 (2.7) is bounded by

$$\tilde{\gamma}(j)\|Q_t f\|_p, \quad \text{if } \alpha = 0 \tag{2.8}$$

or

$$C2^{-j\varepsilon}\|Q_t f\|_p, \qquad 0 < \alpha < \varepsilon \text{ and } \delta_1(k) \leq 2^{-k\varepsilon}. \tag{2.9}$$

If $p = \infty$, we make the obvious adjustment, since the corresponding integral can be treated in exactly the same way. In the case $\alpha = 0$, substitute the upper bound (2.8) into (2.6) which is then at most

$$\begin{aligned} &\sum_{j=1}^\infty \int_0^\infty \int_0^\infty \chi_j(s, t)\tilde{\gamma}(j)\|Q_t f\|_p \|Q_s g\|_{p'} \frac{ds}{s}\frac{dt}{t} \\ &\quad \leq \left( \sum_j \int_0^\infty \int_0^\infty \chi_j(s, t)\tilde{\gamma}(j)\|Q_t f\|_p^q \frac{ds}{s}\frac{dt}{t} \right)^{1/q} \\ &\quad\quad \cdot \left( \sum_j \int_0^\infty \int_0^\infty \chi_j(s, t)\tilde{\gamma}(j)\|Q_s g\|_{p'}^{q'} \frac{dt}{t}\frac{ds}{s} \right)^{1/q'}. \end{aligned}$$

If $q = 1$ or $\infty$, we again make the obvious adjustment, pulling out an appropriate sup norm in $s$ or $t$. Since $\int \chi_j \frac{ds}{s} = \log 2 = \int \chi_j \frac{dt}{t}$, by (2.5) the last expression is in turn less than or equal to

$$C\|f\|_{\dot{B}_p^{0,q}} \|g\|_{\dot{B}_{p'}^{0,q'}}.$$

The case $\alpha \neq 0$ is similar. We bound (2.6) by

$$C \sum_j \int_0^\infty \int_0^\infty \chi_j(s, t) 2^{-j\varepsilon} s^{-\alpha} \|Q_t f\|_p s^\alpha \|Q_s g\|_{p'} \frac{ds}{s} \frac{dt}{t},$$

which by Hölder's inequality and the fact that $2^{-j} \approx \frac{s}{t}$ on $\operatorname{supp} \chi_j$ is no larger than a constant times

$$(2.10) \qquad \left( \int_0^\infty \int_0^t s^{-\alpha q} \left(\frac{s}{t}\right)^{\varepsilon \beta q} \|Q_t f\|_p^q \frac{ds}{s} \frac{dt}{t} \right)^{1/q} \cdot \left( \int_0^\infty \int_s^\infty \left(\frac{s}{t}\right)^{\varepsilon(1-\beta) q'} s^{\alpha q'} \|Q_s g\|_{p'}^{q'} \frac{dt}{t} \frac{ds}{s} \right)^{1/q'}.$$

The endpoint cases are slightly simpler. For example if $q = \infty$, we have in place of (2.10) the bound

$$C \sup_{t>0} (t^{-\alpha} \|Q_t f\|_p) \int_0^\infty \int_s^\infty \left(\frac{s}{t}\right)^{\varepsilon - \alpha} s^\alpha \|Q_s g\|_{p'} \frac{dt}{t} \frac{ds}{s}.$$

The case $q = 1$ may be handled in a similar fashion. In (2.10), $\beta$ has been chosen close enough to 1 so that $\varepsilon\beta - \alpha > 0$. Note that in the case $\varepsilon = 1$, we have (instead of $(\frac{s}{t})^\varepsilon$) $\frac{s}{t} \log \frac{t}{s}$ from Lemma 2.3, but we can always choose (for $\alpha < 1$) an $\varepsilon'$ so that $\alpha < \varepsilon' < 1$ and $\frac{s}{t} \log \frac{t}{s} \leq C (\frac{s}{t})^{\varepsilon'}$. Straightforward integration shows that (2.10) (or its endpoint analogue) is dominated by $C \|f\|_{\dot{B}_p^{\alpha, q}} \|g\|_{\dot{B}_{p'}^{-\alpha, q'}}$.

Consider now the case $t \leq s$. As mentioned above, if $\alpha = 0$, this case follows immediately from the previous one by duality, since our assumptions about $T$ and $T^*$ are symmetric. If $\alpha > 0$, this symmetry no longer holds, so we require an additional argument. We now estimate (2.6) with $\chi$ replaced by $1 - \chi$. By Lemma 2.3, for $t \leq s$, (2.7) is now bounded by a constant times $\|Q_t f\|_p$. Thus, the analogue of (2.6) is bounded by

$$C \int_0^\infty \int_0^\infty (1 - \chi) s^{-\alpha + \beta} t^{-\beta} \|Q_t f\|_p s^{\alpha - \beta} t^\beta \|Q_s g\|_{p'} \frac{ds}{s} \frac{dt}{t},$$

where $\beta$ is now chosen so that $0 < \beta < \alpha$. We apply Hölder's inequality as above, and the required estimate follows easily since $t \leq s$.

We turn now to the proof of Lemma (2.3), and consider first the case $t \geq s$. Formally,

$$K_{st}(x, y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi_s(x - z) K(z, u) \psi_t(u - y) \, du \, dz.$$

We suppose first that $|x - y| \geq 4t$. Here, the integral above is well defined, because $|z - u| > 2t$. Since $\int \psi_s = 0$, we can write

$$K_{st}(x, y) = \iint \psi_s(x - z) [K(z, u) - K(x, u)] \psi_t(u - y) \, du \, dz.$$

Thus, since $\psi_t$ is supported in the ball of radius $t$, we have

$$\begin{aligned}
&\int_{|x-y|\geq 4t} |K_{st}(x, y)|\,dy \\
&\quad\leq \int |\psi_s(x-z)| \int_{|x-u|\geq 2t} |K(z, u) - K(x, u)| \int |\psi_t(u-y)|\,dy\,du\,dz \\
&\quad\leq C \int |\psi_s(x-z)| \int_{2\frac{t}{s}s\leq|x-u|} |K(z, u) - K(x, u)|\,du\,dz.
\end{aligned}$$

But, for $2^{-j}t \leq s \leq 2^{-j+1}t$, the inner integral is bounded by $\gamma_1(j)$, and this yields Lemma 2.3 in the present case. To handle $\int_{|x-y|\geq 4t} |K_{st}(x, y)|\,dx$, we make a translation $z \to z + x$ and then use Fubini's Theorem to obtain the bound

$$\int |\psi_s(-z)| \int |\psi_t(u-y)| \int_{|x-u|\geq 2t} |K(x+z, u) - K(x, u)|\,dx\,du\,dz,$$

which is then handled exactly like the previous $(dy)$ integral.

We now consider the case $|x-y| \leq 4t$. Following [CDMS], we choose a smooth cut-off function $\eta \in C_0^\infty(-3, 3)$, $\eta \equiv 1$ on $[-2, 2]$. Using $T1 = 0$, we write (formally)

(2.11)

$$\begin{aligned}
K_{st}(x, y) &= \iint \psi_s(x-z)K(z, u)[\psi_t(u-y) - \psi_t(x-y)]\,du\,dz \\
&= \iint \psi_s(x-z)K(z, u)[\psi_t(u-y) - \psi_t(x-y)]\eta\left(\frac{|x-u|}{s}\right)du\,dz \\
&\quad+ \iint \psi_s(x-z)K(z, u)[\psi_t(u-y) - \psi_t(x-y)]\left[1-\eta\left(\frac{|x-u|}{s}\right)\right]du\,dz \\
&\equiv A + B.
\end{aligned}$$

The integral representing $B$ is well defined since $|z-u| > s$. We can interpret $A$ in a distributional sense as $\langle\varphi, T\tilde{\varphi}\rangle$ where $\varphi(z) \equiv \psi_s(x-z)$ and

$$\tilde{\varphi}(u) \equiv [\psi_t(u-y) - \psi_t(x-y)]\eta\left(\frac{|x-u|}{s}\right).$$

The reader may routinely verify that WBP implies $|A| \leq Cs/t^{n+1}$. Integrating over the set $|x-y| \leq 4t$ in either the $x$ or $y$ variable gives a bound of $C\frac{s}{t}$, which certainly satisfies any of the estimates claimed in Lemma 2.3.

To handle $B$, we use the fact that $\int \psi_s = 0$, so

$$\begin{aligned}
B = \iint &\psi_s(x-z)[K(z, u) - K(x, u)] \\
&\times [\psi_t(u-y) - \psi_t(x-y)]\left[1-\eta\left(\frac{|x-u|}{s}\right)\right]du\,dz.
\end{aligned}$$

We split $B$ so that

$$B = \iint_{2s\leq|x-u|\leq 2t} du\,dz + \iint_{2t\leq|x-u|} du\,dz \equiv B_1 + B_2.$$

Since

$$|\psi_t(u-y) - \psi_t(x-y)| \leq C\min\left(\frac{|x-u|}{t^{n+1}}, \frac{1}{t^n}\right),$$

we have

$$|B_2| \le \frac{1}{t^n} \int |\psi_s(x-z)| \int_{2t \le |x-u|} |K(z,u) - K(x,u)| \, du \, dz.$$

The inner integral in the last expression is handled exactly like the analogous integral in the case $|x-y| \ge 4t$ above. Thus, $|B_2| \le t^{-n}\gamma_1(j)$, so integrating over the set $|x-y| \le 4t$ in either $dx$ or $dy$, we obtain the desired estimates of Lemma 2.3.

Finally, to conclude the case $t \ge s$, we need only estimate $|B_1|$, which is no larger than

$$\frac{1}{t^{n+1}} \int |\psi_s(x-z)| \int_{2s \le |x-u| \le 2t} |K(z,u) - K(x,u)| \, |x-u| \, du \, dz.$$

For $2^{-j}t \le s \le 2^{-j+1}t$, the inner integral is bounded by

$$\begin{aligned} &\sum_{k=1}^{j} \int_{2^k s \le |x-u| \le 2^{k+1}s} |x-u| \, |K(z,u) - K(x,u)| \, du \\ &\quad \le Cs \sum_{k=1}^{j} 2^k \gamma_1(k) \approx t \sum_{k=1}^{j} 2^{k-j}\gamma_1(k) \equiv t\gamma_0(j). \end{aligned} \tag{2.12}$$

Thus $|B_1| \le Ct^{-n}\gamma_0(j)$, and again we obtain (2.4) by integrating over $|x-y| \le 4t$.

By summing in "$j$" and interchanging the order of summation, the reader may easily verify that the conclusions of Lemma 2.3 hold for $\gamma_0$, and thus (2.4) holds with $\tilde{\gamma}(j) \equiv \gamma_0(j) + \gamma_1(j) + 2^{-j}$.

We now treat the case $t \le s$, which for $\alpha = 0$ is dual to the previous one. It is therefore enough to consider $\alpha > 0$. For $|x-y| \ge 4s$, using the fact that $Q_t 1 = 0$, we write $K_{st}(x,y) = \iint \psi_s(x-z)[K(z,u) - K(z,y)]\psi_t(u-y)\,du\,dz$.

We now actually require only that the smoothness condition in the second variable of the kernel $K$ be a Hörmander condition (i.e. (L′2) holds with $u = 0$ merely for $j = 1$ with $\gamma_1(1) < \infty$). The integral of $|K_{st}(x,y)|$ in either $dx$ or $dy$ is then bounded by a constant, as desired. The argument is similar to the one for the analogous case $t \ge s$, $|x-y| \ge 4t$, and the details are left to the reader.

Suppose now that $|x-y| \le 4s$. We again proceed in an anology with the case $t \ge s$, $|x-y| \le 4t$, except that we no longer impose any restriction on $T^*1$. Formally,

$$\begin{aligned} K_{st}(x,y) = &\iint \psi_s(x-z)K(z,u)\psi_t(u-y)\eta\left(\frac{|y-z|}{t}\right) du\,dz \\ &+ \iint \psi_s(x-z)K(z,u)\psi_t(u-y)\left[1 - \eta\left(\frac{|y-z|}{t}\right)\right] du\,dz, \end{aligned}$$

where $\eta$ is defined as above. By WBP, the first term, again interpreted in the sense of distributions, is no larger than $Cs^{-n}$. The second term is bounded by

$$\frac{1}{s^n} \int |\psi_t(u-y)| \int_{|z-y| \ge 2t} |K(z,u) - K(z,y)| \, dz \, du,$$

which is also no larger than $Cs^{-n}$, and again we only require a Hörmander condition in the second variable. Integrating over the set $|x-y| \le 4s$ gives the desired estimate. This concludes the proof of Lemma 2.3, and hence also that of Theorem 1. We remark that in the case $\alpha > 0$, we required that the kernel $K(x, y)$ satisfy the full smoothness assumption of the theorem only in the first variable, and that a Hörmander condition in the second variable was sufficient.

## 3. Proof of Theorem 2

Again, we use the representation (2.1), and we show that

$$\left| \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} T_{st} Q_t f(x) Q_s g(x)\, dx \frac{ds}{s}\frac{dt}{t} \right| \le C \|f\|_{\dot{F}_p^{\alpha,q}(w)} \|g\|_{\dot{F}_{p'}^{\alpha,q'}(v)}, \tag{3.1}$$

where $w \in A_p$ and $v \equiv w^{-p'/p} \in A_{p'}$. Under the pointwise kernel condition (P2), we follow [CDMS] to obtain a pointwise version of Lemma 2.3 (for $\alpha = 0$ we only state the case $t \ge s$: the other case is dual, as in the previous section):

**Lemma 3.2.** *Let $K_{st}(x, y)$ be defined as in the previous section let $t \ge s$, and now suppose that $K(x, y)$ satisfies the pointwise condition* (P2). *If $\omega_\infty$ satisfies* (1.1a), *then there exist $\tilde{\omega}$ satisfying $\int_0^1 \tilde{\omega}(t)\frac{dt}{t} < \infty$ such that*

$$\begin{aligned} |K_{st}(x, y)| \le{}& \omega_\infty\left(\frac{s}{t + |x-y|}\right)(|x-y| + t)^{-n} \\ &+ \tilde{\omega}\left(\frac{s}{t}\right) t^{-n} \chi(|x-y| \le 4t). \end{aligned} \tag{3.3}$$

*If $\omega_\infty(r) = r^\varepsilon$, then*

$$|K_{st}(x, y)| \le \frac{s^\varepsilon}{(t + |x-y|)^{n+\varepsilon}} \tag{3.4}$$

(*as in the previous section, if $\varepsilon = 1$, we choose an $\varepsilon'$ so that $\alpha < \varepsilon' < 1$*).

For $\alpha > 0$, $s \ge t$, if $K(x, y)$ merely satisfies a Dini condition in the "$y$" variable, then

$$|K_{st}(x, y)| \le s^{-n}\chi(|x-y| \le 4s) + \omega_\infty\left(\frac{t}{s + |x-y|}\right)(|x-y| + s)^{-n}.$$

*Proof of Lemma* 3.2. Suppose $t \ge s$. The case $|x-y| \le 4t$ does not require the pointwise kernel estimate (P2), but only the weaker integral estimate (L′2). Thus, (3.3) and (3.4) in this case are immediate from the pointwise estimates obtained for $A$, $B_1$, and $B_2$ in the proof of Lemma 2.3 in the previous section, where in the present situation we interpret $\tilde{\gamma}(j) \equiv \tilde{\omega}(\frac{s}{t})$, if $2^{-j}t \le s \le 2^{-j+1}t$. It is therefore enough to consider the case $|x-y| \ge 4t$.

Using $\int \psi_s = 0$ and (P2), we write

$$\begin{aligned} |K_{st}(x, y)| &\le \iint |\psi_s(x-z)|\,|K(z, u) - K(x, u)|\,|\psi_t(u-y)|\, du\, dz \\ &\le \iint |\psi_s(x-z)| \omega_\infty\left(\frac{|x-z|}{|x-u|}\right) |x-u|^{-n} |\psi_t(u-y)|\, du\, dz. \end{aligned}$$

But now we observe that $|x-z| \le s$, and also $|x-u| \approx |x-y| \approx |x-y| + t$, for $|x-y| \ge 4t$. We then easily obtain (3.3) and (3.4).

Now assume $s \ge t$. The case $|x-y| \le 4s$ again follows immediately from the pointwise estimates for the analogous term from the proof of Lemma 2.3, and requires only a Hörmander condition in the second variable. If $|x-y|\geq 4s$, then

$$|K_{st}(x,y)|\leq \iint |\psi_s(x-z)|\,|K(z,u)-K(z,y)|\,|\psi_t(u-y)|\,du\,dz,$$

which can then be handled as above.

We now apply Lemma 3.2 to obtain

**Lemma 3.5.** *Under the assumptions of Lemma* 3.2, *if* $\omega_\infty$ *satisfies* (1.1a), *then there exists* $\tilde{\omega}$ *satisfying* $\int_0^1 \tilde{\omega}(t)\frac{dt}{t}<\infty$ *such that, for* $s\leq t$,

$$\int_{\mathbb{R}^n} |K_{st}(x,y)|\,|h(y)|\,dy\leq \tilde{\omega}\left(\frac{s}{t}\right)Mh(x), \tag{3.6}$$

*where* $M$ *is the Hardy-Littlewood maximal function. If* $\omega_\infty(r)=r^\varepsilon$, *then*

$$\int_{\mathbb{R}^n} |K_{st}(x,y)|\,|h(y)|\,dy\leq \left(\frac{s}{t}\right)^\varepsilon Mh(x). \tag{3.7}$$

For $\alpha>0$, $t\leq s$, then (3.6) holds with $\tilde{\omega}\equiv 1$. The proof is easy. Write (in the case $t\geq s$),

$$\int_{\mathbb{R}^n} |K_{st}(x,y)|\,|h(y)|\,dy=\int_{|x-y|\leq 4t}+\sum_{k=2}^{\infty}\int_{2^k t\leq |x-y|\leq 2^{k+1}t}\equiv \mathrm{I}+\mathrm{II}.$$

The estimates for I are immediate from Lemma 3.2. We also have

$$\mathrm{II}\leq\sum_{k=2}^{\infty}\omega_\infty\left(2^{-k}\frac{s}{t}\right)(2^k t)^{-n}\int_{|x-y|\leq 2^{k+1}t}|h(y)|\,dy.$$

Lemma 3.5 then follows, in this case.

If $s\geq t$, and $\alpha>0$, the obvious analog of the above works, and here we merely require that $\omega_\infty$ satisfy $\int_0^1\omega_\infty(t)\frac{dt}{t}<\infty$.

We now deduce Theorem 2 from Lemma 3.5. Let $\chi$ be the characteristic function of $\{(s,t)\colon t\geq s\}$.

Apply Lemma 3.5 to $T_{st}Q_tf(x)=\int K_{st}(x,y)Q_tf(y)\,dy$ so that the left side of (3.1) is bounded by

$$\int_0^\infty\int_0^\infty\int_{\mathbb{R}^n}\chi\omega\left(\frac{s}{t}\right)M(Q_tf)(x)|Q_sg(x)|\,dx\frac{ds}{s}\frac{dt}{t}, \tag{3.8}$$

plus the analogous term with $s\geq t$, where $\omega(r)=\tilde{\omega}(r)$ $(\alpha=0)$ or $\omega(r)=r^\varepsilon$ $(0<\alpha<\varepsilon)$, and $f\in\dot{F}_p^{\alpha,q}(w)$. Also, for $s\geq t$, $\alpha>0$, the analog of (3.8) has $\omega\equiv 1$.

The case $\alpha=0$ is now easy. We apply Hölder's inequality to the $\frac{ds}{s}\frac{dt}{t}$ integration so that (3.8) is at most

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(\int_0^\infty\int_0^t\tilde{\omega}\left(\frac{s}{t}\right)[M(Q_tf)(x)]^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q}\\&\qquad\cdot\left(\int_0^\infty\int_s^\infty\tilde{\omega}\left(\frac{s}{t}\right)|Q_sg(x)|^{q'}\frac{dt}{t}\frac{ds}{s}\right)^{1/q'}dx\\&\quad\leq C\int_{\mathbb{R}^n}\left(\int_0^\infty[M(Q_tf)(x)]^q\frac{dt}{t}\right)^{1/q}\left(\int_0^\infty|Q_sg(x)|^{q'}\frac{ds}{s}\right)^{1/q'}dx,\end{aligned}$$

where in the last inequality we have used that $\tilde{\omega}$ satisfies a Dini condition. Now multiply and divide by $w^{1/p}$, and apply Hölder again to obtain the upper bounds

$$C\left(\int_{\mathbb{R}^n}\left(\int_0^\infty [M(Q_tf)(x)]^q\frac{dt}{t}\right)^{p/q} w(x)\,dx\right)^{1/p}$$
$$\cdot\left(\int_{\mathbb{R}^n}\left(\int_0^\infty |Q_sg(x)|^{q'}\frac{ds}{s}\right)^{p'/q'} v(x)\,dx\right)^{1/p'},$$

where $v = w^{-p'/p}$. The second factor is by definition $\|g\|_{\dot{F}^{0,q'}_{p'}(v)}$. The first factor is bounded by $\|f\|_{\dot{F}^{0,q}_{p}(w)}$, if we apply the weighted version of the Fefferman-Stein vector valued maximal inequality from Andersen and John [AJ].

We now consider (3.8) in the case that $\omega(r) = r^\varepsilon$, $t \geq s$, and $0 < \alpha < \varepsilon$. Multiply and divide by $s^{-\alpha}$, and choose $\beta$ close enough to 1 so that $\varepsilon\beta - \alpha > 0$. Then (3.8) is bounded by

$$\int_{\mathbb{R}^n}\left(\int_0^\infty\int_0^t\left(\frac{s}{t}\right)^{\beta q\varepsilon} s^{-\alpha q}[M(Q_tf(x))]^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q}$$
$$\cdot\left(\int_0^\infty\int_s^\infty\left(\frac{s}{t}\right)^{(1-\beta)q'\varepsilon} s^{\alpha q'}|Q_sg(x)|^{q'}\frac{dt}{t}\frac{ds}{s}\right)^{1/q'} dx$$
$$\leq C\int_{\mathbb{R}^n}\left(\int_0^\infty (M(t^{-\alpha}|Q_tf|)(x))^q\frac{dt}{t}\right)^{1/q}\left(\int_0^\infty (s^\alpha|Q_sg(x)|)^{q'}\frac{ds}{s}\right)^{1/q'} dx.$$

For $t \geq s$, the rest of the proof is then exactly like the case $\alpha = 0$. If $\alpha > 0$ and $s \geq t$, we treat the analog of (3.8) as in the proof of Theorem 1. We multiply and divide by $s^{-\alpha+\beta}t^{-\beta}$, with $0 < \beta < \alpha$, and again the theorem follows from Hölder's inequality.

## 4. Proof of Theorem 3

Since the reader is by now familiar with the procedure, we shall be brief. We first prove an analogue of Lemmas 2.3 and 3.2.

**Lemma 4.1.** *With $K_{st}$ as above, $2^{-j}t \leq s \leq 2^{-j+1}t$, if $K(x, y)$ satisfies* (L$^r$2) *with $\delta_r$ satisfying* (1.1) *or $\delta_r(k) \leq 2^{-k\varepsilon}$, then for $k = 2, 3, \ldots$*

$$\left(\int_{2^kt\leq|x-y|\leq 2^{k+1}t}|K_{st}(x, y)|^r\,dy\right)^{1/r} \leq (2^kt)^{-n/r'}\sum_{l=-2}^{2}\delta_r(k+j+l). \tag{4.2}$$

*Also, if $2^{-j}s \leq t \leq 2^{-j+1}s$, and $\alpha = 0$ then*

$$\left[\int_{2^ks\leq|x-y|\leq 2^{k+1}s}|K_{st}(x, y)|^r\,dy\right]^{1/r} \leq [2^ks]^{-n/r'}\sum_{l=-2}^{2}\delta_r(k+j+l). \tag{4.2a}$$

*Proof*. The left side of (4.2) equals

$$\left(\int_{2^kt\leq|x-y|\leq 2^{k+1}t}\left|\iint \psi_s(x-z)[K(z, u) - K(x, u)]\psi_t(u-y)\,du\,dz\right|^r dy\right)^{1/r}$$
$$= \left(\int_{2^kt\leq|x-y|\leq 2^{k+1}t}\left|\iint \psi_s(x-z)[K(z, u+y) - K(x, u+y)]\psi_t(u)\,du\,dz\right|^r dy\right)^{1/r}.$$

Now (4.2) follows from Minkowski's inequality and (L$'$2). For $\alpha = 0$, the case $s \geq t$ is handled like the first case, except that we subtract $K(z, y)$ rather than $K(x, u)$.

We now prove a version of Lemma 3.5.

**Lemma 4.3.** *Under the assumptions of Lemma* (4.1),

$$\int_{\mathbf{R}^n} |K_{st}(x, y)|\,|h(y)|\,dy \leq \tilde{\delta}(j)(M(|h|^{r'})(x))^{1/r'} \tag{4.4}$$

*where* $\sum_j \tilde{\delta}(j) < \infty$, *or* $\tilde{\delta}(j) \leq C2^{-j\varepsilon}$, *as appropriate.*

*Proof*. Write (with $t \geq s$) the right side of (4.4) as

$$\sum_{k=2}^{\infty} \int_{2^k t \leq |x-y| \leq 2^{k+1} t} |K_{st}(x, y)|\,|h(y)|\,dy + \int_{|x-y| \leq 4t} |K_{st}(x, y)|\,|h(y)|\,dy.$$

The first term is handled by using Hölder's inequality and Lemma 4.1. The second term is bounded by the sharper estimate $\tilde{\delta}(j)Mh(x)$, because in particular $K$ satisfies the weak condition (L2) so on $\{|x-y| \leq 4t\}$, $|K_{st}(x, y)|$ satisfies the pointwise bounds for $|A| + |B_1| + |B_2|$ from the proof of Lemma 2.3. For $\alpha = 0$, $s \geq t$, we make the obvious modification. We now deduce Theorem 3 in the case $\alpha = 0$.

We again will obtain estimate (3.1), which, by Lemma 4.3 applied to $T_{st}Q_t f$, and for $\chi_j(r)$ the characteristic function of $\{2^{-j} \leq r \leq 2^{-j+1}\}$, is bounded by

$$\int_{\mathbf{R}^n} \int_0^{\infty} \int_0^{\infty} \sum_{j=1}^{\infty} \chi_j\left(\frac{s}{t}\right) \tilde{\delta}(j)[M(|Q_t f|^{r'})(x)]^{1/r'} |Q_s g(x)| \frac{dt}{t}\frac{ds}{s}\,dx \tag{4.5}$$

plus the same expression with $\chi_j(\frac{t}{s})$ instead of $\chi_j(\frac{s}{t})$. As in §3 (see (3.8)), we apply Hölder's inequality, multiply and divide by $w^{1/p}$, and apply Hölder again. For example, one factor is

$$\left(\int_{\mathbf{R}^n} \left[\int_0^{\infty} [M(|Q_t f|^{r'})(x)]^{q/r'} \int_0^{\infty} \sum_j \chi_j\left(\frac{s}{t}\right) \tilde{\delta}(j) \frac{ds}{s}\frac{dt}{t}\right]^{(r'/q)(p/r')} w(x)\,dx\right)^{1/p}.$$

The inner ($\frac{ds}{s}$) integral is a finite constant. The term with $s \geq t$ is handled in exactly the same way. The conclusion then follows by [AJ], if $p, q > r'$ and $w \in A_{p/r'}$.

For $\alpha > 0$, and $2^{-j}t \leq s \leq 2^{-j+1}t$, we have $2^{-j\varepsilon} \approx (\frac{s}{t})^{\varepsilon}$, so we can apply Lemma 4.3 and follow the argument at the end of §3. The details are left to the reader. If $t \leq s$, one can show that, in analogy with Lemma (3.5).

$$|T_{st}Q_t f| \leq M(|Q_t f|^{r'})^{1/r'},$$

even if $\delta_r(j)$ merely satisfies $\sum \delta_r(j) < \infty$. The proof is then finished as in §3. Again we leave the details to the reader.

In conclusion, we wish to add some remarks about recent results which were obtained after the submission of this manuscript. E. T. Sawyer and one of the present authors (Han), have observed that, in the case $\alpha > 0$, a refinement of the above arguments permits one to dispense with any smoothness assumption in the $y$ variable of $K(x, y)$. The idea is to modify the proof given for the case $t \le s$. They state their result for the unweighted version of the case $\alpha > 0$ of Theorem 2, but the same observation carries over to all the results of the present paper for positive $\alpha$ (at least those for $1 < p, q < \infty$). We refer the reader to [HS]. A different proof of the Han-Sawyer theorem was subsequently given by the other author (Hofmann), via the formulation of a weaker molecule condition for $\dot{F}_p^{\alpha, q}$, $\alpha > 0$, $1 < p, q < \infty$, than that given in [FJ]. See [H].

*Note added in proof.* Yet another proof of this last result was given by Han, Jawerth, Taibleson and Weiss [HJTW].

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