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# APPLYING COORDINATE PRODUCTS TO THE TOPOLOGICAL IDENTIFICATION OF NORMED SPACES

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ABSTRACT. Using the $l^2$-products we find pre-Hilbert spaces that are absorbing sets for all Borelian classes of order $\alpha \geq 1$. We also show that the following spaces are homeomorphic to $\Sigma^\infty$, the countable product of the space $\Sigma = \{(x_n) \in R^\infty : (x_n) \text{ is bounded}\}$:

(1) every coordinate product $\prod_C H_n$ of normed spaces $H_n$ in the sense of a Banach space $C$, where each $H_n$ is an absolute $F_{\sigma\delta}$-set and infinitely many of the $H_n$'s are $Z_\sigma$-spaces,
(2) every function space $\widetilde{L}^p = \bigcap_{p'<p} L^{p'}$ with the $L^q$-topology, $0 < q < p \leq \infty$,
(3) every sequence space $\widetilde{l}^p = \bigcap_{p<p'} l^{p'}$ with the $l^q$-topology, $0 \leq p < q < \infty$.

We also note that each additive and multiplicative Borelian class of order $\alpha \geq 2$, each projective class, and the class of nonprojective spaces contain uncountably many topologically different pre-Hilbert spaces which are $Z_\sigma$-spaces.

## 1. INTRODUCTION

We are interested in the topological classification of noncomplete normed linear spaces. The main tool in this area is the method of absorbing sets discovered and applied in the $\sigma$-compact case by Anderson and Bessaga and Pełczyński (see [2]). Absorbing sets which are not necessarily $\sigma$-closed in a considered copy $s$ of $l^2$ were developed by Bestvina and Mogilski [4]. A disadvantage of the approach presented in [4] was that two homeomorphic absorbing sets in $s$ might not have been relatively homeomorphic. The difficulty was overcome in [7] due to replacing the strong universality property by its relative version (see Theorem 2.2). We construct linear subspaces $F_\alpha$, $\alpha \geq 1$ (respectively, $G_\alpha$, $\alpha \geq 2$) of $l^2$ that are absorbing sets for the additive Borelian class $\mathscr{A}_\alpha$ (respectively, the multiplicative Borelian class $\mathscr{M}_\alpha$) and such that the pair $(l^2, F_\alpha)$ (respectively, $(l^2, G_\alpha)$) is strongly $(\mathscr{M}_1, \mathscr{A}_\alpha)$-universal (respectively, $(\mathscr{M}_1, \mathscr{M}_\alpha)$-universal). Applying Theorem 2.2, $(l^2, F_\alpha)$ and $(l^2, G_\alpha)$ are homeomorphic to $(s, \Lambda_\alpha)$ and $(s, \Omega_\alpha)$, respectively, where $\Lambda_\alpha$ and $\Omega_\alpha$ are absorbing sets in $s$ constructed in [4].

One may guess that $F_\alpha$ (respectively, $G_\alpha$) is the weak $l^2$-product $\sum_{l^2} H_n$ (respectively, the $l^2$-product $\prod_{l^2} H_n$) of pre-Hilbert spaces $H_n$ that contain a

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closed copy of $\Lambda_\alpha$ (respectively, $\Omega_\alpha$). The crucial step is to show that $\sum_{l^2} H_n$ and $\prod_{l^2} H_n$ are strongly $\mathscr{A}_\alpha$- and $\mathscr{M}_\alpha$-universal, respectively. Actually, we are able to verify the strong $(\mathscr{K}, \mathscr{L})$-universality property of an arbitrary normed coordinate product pair $(\prod_C E_n, \sum_C H_n)$ provided each element of $(\mathscr{K}, \mathscr{L})$ admits a relative closed embedding into every $(E_n, H_n)$ (see Proposition 3.1). A version of 3.1 for cartesian products was earlier applied [11, 13, 12] in order to identify some function and sequence spaces that are homeomorphic to $\Omega_2 = \Sigma^\infty$. Applying 3.1 (and its variations), we show that several absolute $F_{\sigma\delta}$-spaces that underlie a "product" structure are homeomorphic to $\Omega_2$. In particular, we prove that every normed coordinate product $\prod_C H_n$, $C$ being a Banach space, is homeomorphic to $\Omega_2$ provided each $H_n \in \mathscr{M}_2$ and infinitely many of the $H_n$'s are $Z_\sigma$-spaces. Another application concerns the function space $\widetilde{L}^p = \bigcap_{p'<p} L^{p'}$ in the $L^q$-topology $(q < p)$ and the sequence space $\widetilde{l}^p = \bigcap_{p<p'} l^{p'}$ in the $l^q$-topology $(p < q)$. We prove that $\widetilde{L}^p$, $0 < q < p \leq \infty$, and $\widetilde{l}^p$, $0 \leq p < q < \infty$, are homeomorphic to $\Omega_2$. Actually, we show that the pairs $(L^q, \widetilde{L}^p)$, $(l^q, \widetilde{l}^p)$, and $(s, \Omega_2)$ are homeomorphic. The fact that the space $\widetilde{L}^p$ considered as a subspace of $L^0$ (of all measurable functions with the topology of convergence in measure) and the space $\widetilde{l}^p$ as a subspace of $R^\infty$ are homeomorphic to $\Omega_2$ was previously obtained in [13]. Let us note that dealing with these different topologies on $\widetilde{L}^p$ (same for $\widetilde{l}^p$) the natural linear map $\Psi : L^0 \to (L^0)^\infty$ is employed. In the present paper $\Psi$ is considered as a linear isomorphism of $L^1$ onto $\prod_{l^1} L^1$ with the following key property:

$$\Psi(\widetilde{L}^p) \cap \sum_{l^1} L^1 = \sum_{l^1} \widetilde{L}^p.$$

In the last section we provide some examples of pre-Hilbert spaces with rather mysterious topological structure. They all are of the form

$$Y(A) \times F_\alpha \quad \text{and} \quad Y(A) \times G_\alpha,$$

where $Y(A)$ is the linear span of a linearly independent subset $A$ in $l^2$. In particular, we show that every projective class $\mathscr{P}_n \setminus \bigcup_{k<n} \mathscr{P}_k$, $n \geq 1$, contains uncountably many nonhomeomorphic pre-Hilbert spaces. The same is true for the class of spaces which are nonprojective. We observe that the argument of Henderson and Pełczyński [2] showing that there are uncountably many $\sigma$-compact pre-Hilbert spaces applies (after a minor change) to produce uncountably many nonhomeomorphic pre-Hilbert spaces in each class $\mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$ and $\mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$ for $\alpha \geq 2$.

The results of §3 will be applied to construct absorbing sets for all projective classes in a forthcoming paper by the first named author.

The authors wish to note that J. Dijkstra and J. Mogilski have recently obtained the same results concerning $\widetilde{L}^p$- and $\widetilde{l}^p$-spaces [10].

*Convention.* All spaces considered are separable and metrizable. Maps are continuous functions.

## 2. Preliminaries

Let us recall that a closed subset $A$ of a space $X$ is a $Z$-set (respectively, a strong $Z$-set) if for every open cover $\mathscr{U}$ of $X$ there exists a $\mathscr{U}$-close to the identity map $f : X \to X$ such that $f(X) \cap A = \varnothing$ (respectively, $\mathrm{cl}(f(X)) \cap A = \varnothing$). A space which is a countable union of $Z$-sets is called a $Z_\sigma$-space. Note that every $Z_\sigma$-space is of the first category. In the case where $X$ is an absolute neighborhood retract a closed set $A$ is a $Z$-set iff given $n$ every map of the $n$-dimensional cube $I^n$ into $X$ can be approximated by maps into $X \setminus A$. Every not necessarily closed set $A$ satisfying the above condition is called locally homotopy negligible in $X$ (see [17]).

Fix a pair of spaces $(K, L)$, i.e., $L \subseteq K$. We say that a pair of spaces $(X, Y)$ is strongly $(K, L)$-universal if, for every closed subset $D$ of $K$, every map $f : K \to X$ whose restriction to $D$ is a $Z$-embedding (i.e., $f|D$ is an embedding and $f(D)$ is a $Z$-set in $X$) and satisfies the condition

$$(f|D)^{-1}(Y) = D \cap L,$$

and every open cover $\mathscr{U}$ of $X$, there exists a $Z$-embedding $g : K \to X$ which is $\mathscr{U}$-close to $f$ and satisfies the conditions

$$g|D = f|D \quad \text{and} \quad g^{-1}(Y) = L.$$

We find it convenient to formulate the following technical fact concerning the strong $(K, L)$-universality (cf. [4, Proposition 2.2]).

**Proposition 2.1.** *Let an absolute neighborhood retract $X$, its subsets $Y \subseteq Y'$, and a pair of spaces $(K, L)$ satisfy the following conditions*:

(i) *every $Z$-set in $X$ is a strong $Z$-set,*
(ii) *$X \setminus Y$ is locally homotopy negligible in $X$,*
(iii) *$Y'$ is locally homotopy negligible in $X$,*
(iv) *given open subsets $U$ of $K$ and $V$ of $X$, a map $f : K \to X$ with $f(U) \subset V \cap Y$ and $f(K \setminus U) \subset X \setminus V$, and an open cover $\mathscr{V}$ of $V$, there exists a closed embedding $g : U \to V$ which is $\mathscr{V}$-close to $f|U$ and satisfies $g(U) \subset Y'$ and $g^{-1}(V \cap Y) = L \cap U$.*

*Then for every $Z \subseteq X$ with $Z \cap Y' = Y$, the pair $(X, Z)$ is strongly $(K, L)$-universal.*

Before we give a proof of 2.1 we recall that $f : K \to X$ is closed over a set $A \subset X$ if for every $a \in A$ and every neighborhood $U$ of $f^{-1}(\{a\})$ there exists a neighborhood $V$ of $a$ such that $f^{-1}(V) \subset U$ (see [4]).

*Proof of* 2.1. Let $D$ be a closed subset of $K$ and let $\bar{f} : K \to X$ be a map such that $\bar{f}|D$ is a $Z$-embedding satisfying $(\bar{f}|D)^{-1}(Z) = D \cap L$. Since $\bar{f}(D)$ is a strong $Z$-set in $X$ and $X \setminus Y$ is locally homotopy negligible in $X$, we can apply [4, Lemma 1.1; 17, Theorem 2.4] to approximate $\bar{f}$ by $f$ such that

(1) $\bar{f}|D = f|D$,
(2) $f$ is closed over $\bar{f}(D)$,
(3) $f(K \setminus D) \subset Y \setminus f(D)$.

Set $U = K \setminus D$ and $V = X \setminus f(D)$. Let $\mathscr{U}$ be an open cover of $X$. Fix a metric $d$ on $X$ and choose an open cover $\mathscr{V}$ of $V$ which is inscribed in $\mathscr{U}$ and such that

(4) for every element $W$ of $\mathscr{V}$, $\mathrm{diam}(W) < \mathrm{dist}(W, X \setminus V)$.

By our assumption, there exists a closed embedding $g : U \to V$ which is $\mathscr{V}$-close to $f|U$ and such that $g^{-1}(V \cap Y) = U \cap L$ and $g(U) \subset Y'$. By (4), $g$

can be continuously extended by $\overline{f} = f$ over $D$ to a one-to-one map which is $\mathscr{U}$-close to $f$. Denote this extension also by $g$. We have $g^{-1}(V \cap Y) = U \cap L$ and consequently $g(L) \subset Z$. Moreover, if $g(x) \in Z$ and $x \notin D$ then $g(x) \in Z \cap Y' = Y$. This, together with $(\overline{f}|D)^{-1}(Z) = D \cap L$, yields $g^{-1}(Z) = L$. To show that $g: K \to X$ is a closed embedding, let $\{g(x_n)\}_{n=1}^{\infty}$ converge to $y \in X$. If $y \in \overline{f}(D)$ then, by (4), $\{f(x_n)\}_{n=1}^{\infty}$ converges to $y$ and consequently, by (2), $\{x_n\}_{n=1}^{\infty}$ converges to $f^{-1}(y)$. Otherwise, $y \in V$ and $\{x_n\}_{n=1}^{\infty}$ converges to $g^{-1}(y)$. Since $g(K) \subset \overline{f}(D) \cup Y'$, the union of a $Z$-set and a locally homotopy negligible set, $g(K)$, is a $Z$-set in $X$.

Let $\mathscr{K}$ and $\mathscr{L}$ be classes of spaces. We write $(K, L) \in (\mathscr{K}, \mathscr{L})$ provided $K \in \mathscr{K}$ and $L \in \mathscr{L}$. A pair of spaces $(X, Y)$ is said to be strongly $(\mathscr{K}, \mathscr{L})$-universal if $(X, Y)$ is strongly $(K, L)$-universal for every pair $(K, L) \in (\mathscr{K}, \mathscr{L})$. This concept was introduced in [6]. If the pair $(Y, Y)$ is strongly $(L, L)$-universal for every $L \in \mathscr{L}$ then, according to [4], $Y$ is strongly $\mathscr{L}$-universal.

In what follows, $\mathscr{L}$ will satisfy the following conditions:

(a) if $L$ and $L'$ are homeomorphic and $L \in \mathscr{L}$, then $L' \in \mathscr{L}$,
(b) if a space $L$ is a union of its two closed subspaces which belong to $\mathscr{L}$, then $L \in \mathscr{L}$,
(c) every closed subset of an element of $\mathscr{L}$ belongs to $\mathscr{L}$.

The following fact proved in [7] extends the uniqueness theorem for absorbing sets discovered by Anderson and Bessaga and Pełczyński (see [2]).

**Theorem 2.2** [7, Theorem 2.1]. *Let $X$ be a topological copy of $l^2$ and let $Y_1$ and $Y_2$ be two subsets of $X$. Assume that both $Y = Y_1$ and $Y_2$ satisfy the following conditions*:

(i) *$X \setminus Y$ is locally homotopy negligible in $X$,*
(ii) *$Y$ is a $Z_\sigma$-space,*
(iii) *$Y$ is a countable union of closed sets that are elements of $\mathscr{L}$,*
(iv) *$(X, Y)$ is strongly $(\mathscr{M}, \mathscr{L})$-universal, where $\mathscr{M}$ is the class of completely metrizable spaces.*

*Then, for every open cover $\mathscr{U}$ of $X$, there exists a $\mathscr{U}$-close to the identity homeomorphism of $(X, Y_1)$ onto $(X, Y_2)$.*

Every subset $Y$ of $X$ which is strongly $\mathscr{L}$-universal and fulfils (i)–(iii) is called an $\mathscr{L}$-absorbing set in $X$. In [4], it was shown that two $\mathscr{L}$-absorbing sets in a copy of $l^2$ are homeomorphic. Theorem 2.2 may be rephrased in its weaker form as follows: two $\mathscr{L}$-absorbing sets in a copy of $l^2$ are relatively homeomorphic provided they are strongly $(\mathscr{M}, \mathscr{L})$-universal.

## 3. Strong universality in products

Let $C$ be a normed countable coordinate space (briefly, a normed coordinate space), i.e., $C = (C, \|\cdot\|_C)$ is a normed linear space of real sequences such that

($c_1$) for every bounded sequence $\lambda = (\lambda_n)$ and every $c = (c_n) \in C$, we have $\lambda \cdot c = (\lambda_n c_n) \in C$ and $\|\lambda \cdot c\|_C \leq \|\lambda\|_\infty \|c\|_C$, where $\|\lambda\|_\infty = \sup_{n \geq 1} |\lambda_n|$,
($c_2$) for every $\varepsilon > 0$ and every $(c_n) \in C$ there exists $k$ such that

$$\|(0, \dots, 0, c_k, c_{k+1}, \dots)\|_C < \varepsilon,$$

($c_3$) each unit vector $u_n = (\delta_n^i)$ belongs to $C$.

We took the notion of a normed coordinate space from [1] (see also [16]) where the following equivalent condition replaces ($c_3$): $C$ is contained in no hyperplane $\{(c_n) : c_k = 0\}$, $k \geq 1$. (For examples of normed coordinate spaces, see [1].) Note that $C$ contains all eventually zero sequences $C_0$. Later on we have to assume that $C \setminus C_0 \neq \varnothing$. This, of course, is the case if $C$ is a Banach space.

Let $\{(E_n, \|\cdot\|_n)\}_{n=1}^{\infty}$ be a sequence of normed linear spaces. We consider the linear spaces

$$\prod\nolimits_C E_n = \left\{ (y_n) \in \prod_{n=1}^{\infty} E_n : (\|y_n\|_n) \in C \right\}$$

and

$$\sum\nolimits_C E_n = \left\{ (y_n) \in \prod_{n=1}^{\infty} E_n : \ y_n = 0 \text{ for almost all } n \right\}$$

which are both equipped with the norm $|||(y_n)||| = \left\| (\|y_n\|_n) \right\|_C$. These spaces are called, respectively, the normed coordinate product (of the $E_n$'s in the sense of $C$) and the weak normed coordinate product (briefly, $C$-product and weak $C$-product of the $E_n$'s). For $y = (y_n) \in \prod_C E_n$ and $k \geq 1$, we write $r_k(y) = (0, \dots, y_k, y_{k+1}, \dots)$, $s_k(y) = y - r_k(y)$, and $\pi_k(y) = y_k$. Identifying $E_n$ with the natural subspace of $\prod_C E_n$, we have

(A) $|||s_k(y)||| \leq |||y|||$,
(B) $|||\pi_k(y)||| \leq |||r_k(y)||| \leq |||y|||$,
(C) $\lim |||r_k(y)||| = 0$,

for every $k \geq 1$ and $y \in \prod_C E_n$.

We now give the main result of this section.

**Proposition 3.1.** *Let $\{(E_n, H_n)\}_{n=1}^{\infty}$ be a sequence of pairs of nontrivial normed linear spaces with each $H_n$ dense in $E_n$ and let $C$ be a normed coordinate space that contains an element with infinitely many nonzero terms. Fix a pair of spaces $(K, L)$ and assume that for every $n \geq 1$ there exists a bounded closed embedding $\psi_n : K \to E_n$ with $\psi^{-1}(H_n) = L$. Then, for every $Z \subseteq E = \prod_C E_n$ with $Z \cap \sum_C E_n = \sum_C H_n$, the pair $(E, Z)$ is strongly $(K, L)$-universal.*

We shall make use of the next two lemmas.

**Lemma 3.2.** *There exists a homotopy*

$$\Phi = (\Phi_n) : \left( E \times [0, 1], \sum\nolimits_C H_n \times [0, 1] \right) \to \left( E, \sum\nolimits_C H_n \right)$$

*satisfying the following conditions:*

(i) $\Phi(\cdot, 0) = \text{id}$,
(ii) *if* $n \geq \frac{1}{t} + 2$, *then* $\Phi_n(y, t) = 0$,
(iii) *if for some sequence* $\{(y(i), t_i)\}_{i=1}^{\infty} \subset E \times [0, 1]$ *with* $\lim t_i = 0$ *there exists* $y \in E$ *such that* $\lim \Phi(y(i), t_i) = y$, *then* $\lim y(i) = y$.

**Lemma 3.3.** *There exists a one-to-one map*

$$\varphi : K \times (0, 1] \to E$$

*satisfying the following conditions*:

(iv) $\varphi^{-1}\left(\sum_C H_n\right) = L \times (0, 1]$,
(v) $|||\varphi(x, t)||| \le t$ *for all* $(x, t) \in K \times (0, 1]$,
(vi) *if* $x \in K$ *and* $\frac{1}{n+1} < t \le \frac{1}{n}$, *then* $\pi_{n+2}\varphi(x, t) \neq 0$ *while* $\pi_k\varphi(x, t) = 0$ *for all* $k < n$ *and* $k \ge n+4$,
(vii) *if the sequence* $\{\varphi(x_i, t_i)\}_{i=1}^{\infty}$ *converges in* $E$, $\{(x_i, t_i)\}_{i=1}^{\infty} \subset K \times (0, 1]$, *and* $\lim t_i = t_0 > 0$, *then* $\{x_i\}_{i=1}^{\infty}$ *converges in* $K$.

First, we derive Proposition 3.1 from Lemmas 3.2 and 3.3.

*Proof of* 3.1. We make use of 2.1 with the following data: $X = E$, $Y = \sum_C H_n$, $Y' = \sum_C E_n$, and $Z$. It is known that $Z$-sets in $E$ are strong $Z$-sets (see [5, Lemma 2.6; 13, Lemma 2.1]). It is also clear that $E \setminus \sum_C H_n$ and $\sum_C E_n$ are locally homotopy negligible in $E$ (see, e.g., [17]). Fix a map $\overline{f} : K \to E$ and open sets $U \subset K$ and $V \subset E$ such that $f = \overline{f}|U$ maps $U$ into $V \cap \sum_C H_n$ and $\overline{f}(x) \notin V$ for all $x \notin U$. Let $\mathscr{V}$ be an open cover of $V$. Pick a map $\omega : V \to (0, 1]$ such that

(1) whenever $y \in V$ and $z \in E$ satisfy $|||y - z||| < 2\omega(y)$ then there exists an element $\mathscr{V}$ containing both $y$ and $z$.

Let $\Phi$ be a homotopy of 3.2. Pick a map $\varepsilon : E \to [0, 1]$ such that

(2) $\varepsilon^{-1}(\{0\}) = E \setminus V$,
(3) $|||\Phi(y, \varepsilon(y)) - y||| < \omega(y)$ for all $y \in V$,
(4) $(\frac{1}{\varepsilon(y)} + 4)^{-1} < \omega(y)$ for all $y \in V$.

Write $\varepsilon(x) = \varepsilon(f(x))$ and $\lambda(x) = (\frac{1}{\varepsilon(x)} + 4)^{-1}$. Pick a homotopy $\varphi$ from 3.3 and define $g : U \to E$ by the formula

$$g(x) = \Phi(f(x), \varepsilon(x)) + \varphi(x, \lambda(x)).$$

Applying (3)–(4) and (v), we get

$$|||f(x) - g(x)||| < \omega(f(x)) + \lambda(x) < 2\omega(f(x))$$

for every $x \in U$. The property (1) of $\omega$ assures that the range of $g$ is $V$ and that $g$ is $\mathscr{V}$-close to $f$. Clearly, $g$ takes values in $\sum_C E_n$ and, by (iv), $g^{-1}(V \cap \sum_C H_n) = L \cap U$.

To finish the proof, it remains to show that $g : U \to V$ is a closed embedding. First we check that $g$ is one-to-one. If $\frac{1}{n+1} < \varepsilon(x) \le \frac{1}{n}$ then, by (ii), $\Phi_p(f(x), \varepsilon(x)) = 0$ for all $p \ge n+3$. Since $\frac{1}{n+5} < \lambda(x) \le \frac{1}{n+4}$ we have, by (vi), $\varphi_{n+6}(x, \lambda(x)) \neq 0$ and $\varphi_k(x, \lambda(x)) = 0$ for $k \neq n+4, n+5, n+6, n+7$. Assume that $g(x) = g(x')$ and $\varepsilon(x') \le \varepsilon(x)$. Letting $\frac{1}{n'+1} < \varepsilon(x') \le \frac{1}{n'}$, we see that $n' \ge n$ and, by (vi), $\varphi_{n'+6}(x', \lambda(x')) \neq 0$. It follows that

$$\varphi_{n'+6}(x', \lambda(x')) = g_{n'+6}(x') = g_{n'+6}(x) = \varphi_{n'+6}(x, \lambda(x)) \neq 0;$$

hence, $n' = n$ or $n+1$. Then, for every $p \ge n+4$, we have $\Phi_p(f(x'), \varepsilon(x')) = \Phi_p(f(x), \varepsilon(x)) = 0$ and consequently $\varphi_p(x, \lambda(x)) = g_p(x) = g_p(x') = \varphi_p(x', \lambda(x'))$. Since, by (vi), $\varphi_p(x, \lambda(x)) = \varphi_p(x', \lambda(x')) = 0$ for all $p \le n+3$, we conclude that

$$\varphi(x, \lambda(x)) = \varphi(x', \lambda(x')).$$

The latter yields $x = x'$ because $\varphi$ is one-to-one. Now, suppose $\{g(x_i)\}_{i=1}^{\infty}$ converges to $y = (y_n) \in V$ for some sequence $\{x_i\}_{i=1}^{\infty} \subset U$. Write $\varepsilon_i = \varepsilon(x_i)$ and $\lambda_i = \lambda(x_i)$. We can assume that $\{\varepsilon_i\}_{i=1}^{\infty}$ converges to $\varepsilon_0 \in [0, 1]$. If $\varepsilon_0 = 0$, then $\lim \lambda_i = 0$. Using (v), we get $\lim \Phi(f(x_i), \varepsilon_i) = y$. Then, by (iii), $\{f(x_i)\}_{i=1}^{\infty}$ converges to $y$. By the continuity of $\varepsilon$, we get $\varepsilon(y) = 0$ which contradicts (2). Therefore, we can assume that $\varepsilon_0 > 0$. Let

$$\varepsilon_0 = s_0 \frac{1}{n} + (1 - s_0) \frac{1}{n+1}$$

for some $0 < s_0 \leq 1$. We can further assume that

$$\varepsilon_i = s_i \frac{1}{n} + (1 - s_i) \frac{1}{n \pm 1}.$$

Then, we have $\Phi_{n+j}(x_i, \varepsilon_i) = 0$ for all $i$ and $j \geq 3$; and consequently the sequence $\{\varphi_{n+j}(x_i, \lambda_i)\}_{i=1}^{\infty} = \{g_{n+j}(x_i)\}_{i=1}^{\infty}$ converges to $y_{n+j}$ for all $j \geq 3$. For $p \neq n + j$, we have, by (vi), $\varphi_p(x_i, \lambda_i) = 0$. It follows that $\{\varphi(x_i, \lambda_i)\}_{i=1}^{\infty}$ converges in $E$. Since $\lim \lambda_i = (\frac{1}{\varepsilon_0} + 4)^{-1} > 0$, by (vii), the sequence $\{x_i\}_{i=1}^{\infty}$ is convergent in $K$. If $\lim x_i = x \in K \setminus U$, then $\lim f(x_i) = f(x) \in E \setminus V$ and $\lim \varepsilon(x_i) = \varepsilon(f(x)) = 0$, contradicting (2). We have shown that $g$ is a closed embedding.

*Proof of* 3.2. Pick a vector $e_n \in H_n$ with $|||e_n||| = 1$. Define $\Phi : E \times [0, 1] \to E$ by $\Phi(y, 0) = y$,

$$\Phi\left(y, \frac{1}{n}\right) = (y_1, \dots, y_{n-1}, 0, |||r_n(y)||| \cdot e_{n+1}, 0, 0, \dots)$$

and

$$\Phi\left(y, s\frac{1}{n} + (1-s)\frac{1}{n+1}\right) = s\Phi\left(y, \frac{1}{n}\right) + (1-s)\Phi\left(y, \frac{1}{n+1}\right)$$

for every $n \geq 1$, $0 \leq s \leq 1$, and $y = (y_n) \in E$. It is clear that $\Phi$ transforms $\sum_C H_n \times [0, 1]$ into $\sum_C H_n$, is continuous on $E \times (0, 1]$, and satisfies (i) and (ii). The continuity of $\Phi$ at the points $(y, 0)$ will follow from the auxiliary estimations.

Given $y = (y_n) \in E$, we have

$$\left|\left|\left|y - \Phi\left(y, \frac{1}{n}\right)\right|\right|\right| \leq |||y - s_n(y)||| + \left|\left|\left|s_n(y) - \Phi\left(y, \frac{1}{n}\right)\right|\right|\right|$$
$$= |||r_n(y)||| + ||| \, |||r_n(y)||| \cdot e_{n+1} ||| = 2\,|||r_n(y)||| .$$

For $t = s\frac{1}{n} + (1-s)\frac{1}{n+1}$, $0 \leq s \leq 1$, and $y \in E$, we have

(1) $|||y - \Phi(y, t)||| = |||s\,(y - \Phi\,(y, \frac{1}{n})) + (1-s)(y - \Phi(y, \frac{1}{n+1}))||| \leq s|||y - \Phi(y, \frac{1}{n})||| + (1-s)|||y - \Phi(y, \frac{1}{n+1})||| \leq 2s|||r_n(y)||| + 2(1-s)|||r_{n+1}(y)||| \leq 2|||r_n(y)|||$.

Let $\{(y(i), t_i)\}_{i=1}^{\infty}$ be a sequence of $E \times (0, 1]$ that is convergent to $(y, 0) \in E \times \{0\}$ and let

(2) $t_i = s_i\frac{1}{n_i} + (1-s_i)\frac{1}{n_i+1}$ for some $0 \leq s_i \leq 1$ and $n_i \to \infty$.

Using (1), we get

(3) $|||y - \Phi(y(i), t_i)||| \leq |||y - y(i)||| + |||y(i) - \Phi(y(i), t_i)||| \leq |||y - y(i)||| + 2|||r_{n_i}(y(i))|||$.

On the other hand, applying (B), we obtain

(4) $|||r_n(y(i))||| \le |||r_n(y)||| + |||r_n(y - y(i))||| \le |||r_n(y)||| + |||y - y(i)|||$.

Combining (3) and (4), we get

$$|||y - \Phi(y(i), t_i)||| \le 3|||y - y(i)||| + 2|||r_{n_i}(y)|||.$$

The latter inequality together with (C) yields the continuity of $\Phi$ at $(y, 0)$.

Now, let $\{(y(i), t_i)\}_{i=1}^{\infty} \subset E \times (0, 1]$ be such that $\lim \Phi(y(i), t_i) = y \in E$ and $\lim t_i = 0$. Express $t_i$ in the form of (2). We have

$$|||y - y(i)||| \le |||s_{n_i}(y) - s_{n_i}(y(i))||| + |||r_{n_i}(y)||| + |||r_{n_i}(y(i))|||.$$

We see that $s_{n_i}(y(i)) = s_{n_i}(\Phi(y(i), t_i))$. Therefore, after using (A), we get

$$|||s_{n_i}(y) - s_{n_i}(y(i))||| = |||s_{n_i}(y - \Phi(y(i), t_i))||| \le |||y - \Phi(y(i), t_i)|||.$$

This implies

$$|||y - y(i)||| \le |||y - \Phi(y(i), t_i)||| + |||r_{n_i}(y)||| + |||r_{n_i}(y(i))|||.$$

Note that the first two terms tend to $0$ if $i \to \infty$. To show (iii), it remains to verify that the last term also tends to $0$. It is clear that it is enough to consider the case where $s_i \ge \frac{1}{2}$ for all $i$ and the case where $s_i < \frac{1}{2}$ for all $i$. In the first case, we apply (B) to the $(n_i + 1)$-coordinate and get

$$|||y - \Phi(y(i), t_i)||| \ge |||y_{n_i+1} - s_i|||r_{n_i}(y(i))||| \cdot e_{n_i+1}|||.$$

Then, since $|||e_{n_i+1}||| = 1$, we estimate

$$\begin{aligned} |||r_{n_i}(y(i))||| &\le \frac{1}{s_i}(|||y - \Phi(y(i), t_i)||| + |||y_{n_i+1}|||) \\ &\le 2(|||y - \Phi(y(i), t_i)||| + |||y_{n_i+1}|||). \end{aligned}$$

Finally, according to (B) and (C), $\lim_{i\to\infty} r_{n_i}(y(i)) = 0$. In the case where $s_i < \frac{1}{2}$, we apply (B) to the $n_i$-coordinate and obtain

$$|||y - \Phi(y(i), t_i)||| \ge |||y_{n_i} - (1 - s_i)y_{n_i}(i)|||$$

and hence

$$\begin{aligned} |||y_{n_i}(i)||| &\le \frac{1}{1 - s_i}(|||y - \Phi(y(i), t_i)||| + |||y_{n_i}|||) \\ &< 2(|||y - \Phi(y(i), t_i)||| + |||y_{n_i}|||). \end{aligned}$$

The same argument applied to the $(n_i + 2)$-coordinate yields

$$|||y - \Phi(y(i), t_i)||| \ge |||y_{n_i+2} - (1 - s_i)|||r_{n_i+1}(y(i))||| \cdot e_{n_i+2}|||.$$

As before, we get

$$\begin{aligned} |||r_{n_i+1}(y(i))||| &\le \frac{1}{1 - s_i}(|||y - \Phi(y(i), t_i)||| + |||y_{n_i+2}|||) \\ &\le 2(|||y - \Phi(y(i), t_i)||| + |||y_{n_i+2}|||). \end{aligned}$$

The latter, in turn, implies

$$\begin{aligned} |||r_{n_i}(y(i))||| &\le |||y_{n_i}(i)||| + |||r_{n_i+1}(y(i))||| \\ &\le 4|||y - \Phi(y(i), t_i)||| + 2(|||y_{n_i}||| + |||y_{n_i+2}|||). \end{aligned}$$

Finally, according to (B) and (C), the last two terms of the above inequality tend to $0$ if $i \to \infty$.

*Note.* Condition (iii) is equivalent to the fact that the map $(y, t) \to (\Phi(y, t), t)$ from $E \times [0, 1]$ into $E \times [0, 1]$ is closed over $E \times \{0\}$.

*Proof of* 3.3. By our assumption there exists a closed embedding $\psi_n : K \to E_n$ such that

$$\psi^{-1}(H_n) = L \quad \text{and} \quad |||\psi_n(x)||| \leq \frac{1}{2n}.$$

Pick a vector $e_n \in H_n$ with $|||e_n||| = \frac{1}{2n}$. Define $\varphi = (\varphi_p)$ as follows:

$$\varphi_p(x, \frac{1}{n}) = \begin{cases} 0 & \text{if } p \neq n, n+2, \\ \psi_n(x) & \text{if } p = n, \\ e_{n+2} & \text{if } p = n+2. \end{cases}$$

and

$$\varphi\left(x, s\frac{1}{n} + (1-s)\frac{1}{n+1}\right) = s\varphi\left(x, \frac{1}{n}\right) + (1-s)\varphi\left(x, \frac{1}{n+1}\right)$$

for $n \geq 1$ and $0 \leq s \leq 1$. It is clear that $\varphi$ is continuous and satisfies (vi) and (iv). We have

$$\left|\left|\left|\varphi\left(x, \frac{1}{n}\right)\right|\right|\right| \leq |||\psi_n(x)||| + |||e_{n+2}||| \leq \frac{1}{n}.$$

Consequently, we estimate

$$\begin{aligned} |||\varphi(x, t)||| &\leq s\left|\left|\left|\varphi\left(x, \frac{1}{n}\right)\right|\right|\right| + (1-s)\left|\left|\left|\varphi\left(x, \frac{1}{n+1}\right)\right|\right|\right| \\ &\leq s\frac{1}{n} + (1-s)\frac{1}{n+1} = t. \end{aligned}$$

To show that $\varphi$ is one-to-one, let $\varphi(x, t) = \varphi(x', t')$ for some $(x, t), (x', t') \in K \times (0, 1]$. If $t = s\frac{1}{n} + (1-s)\frac{1}{n+1}$ with $n \geq 1$ and $0 \leq s < 1$ (respectively, $t = 1$), then the last nonvanishing coordinate of $\varphi(x, t)$ is the $(n+3)$-coordinate (respectively, the third coordinate) and it equals $(1-s)e_{n+3}$ (respectively, $e_3$). This shows that $t = t'$. Clearly, we have $\varphi_{n+1}(x, t) = (1-s)\psi_{n+1}(x)$ (respectively, $\varphi_1(x, t) = \psi_1(x)$). Since $\psi_n$ (respectively, $\psi_1$) is an embedding, we get $x = x'$.

Let $\{(x_i, t_i)\}_{i=1}^{\infty}$ be a sequence of $K \times (0, 1]$ such that $\{\varphi(x_i, t_i)\}_{i=1}^{\infty}$ converges in $E$ and $\lim t_i = t_0 > 0$. Assume that $t_0 = s_0\frac{1}{n} + (1-s_0)\frac{1}{n+1}$ for some $n \geq 1$ and $0 < s_0 < 1$ (the case where $t_0 = \frac{1}{n}$, $n \geq 1$, can be treated similarly). We may suppose that $t_i = s_i\frac{1}{n} + (1-s_i)\frac{1}{n+1}$ for all $i$, where $0 < s_i < 1$ and $\lim s_i = s_0$. Since $\varphi_n(x_i, t_i) = s_i\psi_n(x_i)$, $\{\psi_n(x_i)\}_{i=1}^{\infty}$ converges in $E_n$. Finally, $\{x_i\}_{i=1}^{\infty}$ converges in $K$ because $\psi_n$ is a closed embedding.

In §§4 and 5, we will employ the following variation of Proposition 3.1.

**Proposition 3.4.** *Let $\{(E_n, H_n)\}_{n=1}^{\infty}$ be a sequence of pairs of nontrivial normed linear spaces with each $H_n$ dense in $E_n$ and let $C$ be a normed coordinate space. Fix a pair of spaces $(K, L)$. Assume there are pairwise disjoint infinite subsets $N_1, N_2, \ldots$ of the set of integers $N$ such that $N_k \cap \{1, 2, \ldots, k-1\} = \varnothing$ and, writing*

$$C_k = \{(c_p)_{p \in N_k} : \exists_{c \in C} \forall_{p \in N_k}\ \pi_p(c) = c_p\}$$

*and identifying* $C_k$ *with the natural subspace of* $C$, *each* $C_k$ *contains an element with infinitely many nonzero terms and there exists a bounded closed embedding* $\psi_k : K \to \prod_{C_k} E_p$ *with* $\psi_k^{-1}(\prod_{C_k} H_p) = L$ *for* $k \geq 1$. *Then, the pair* $(\prod_C E_n, \prod_C H_n)$ *is strongly* $(K, L)$*-universal.*

A proof of 3.4 will be omitted. Let us indicate that to get it one has to follow the proof of 3.1 and replace 3.3 by the lemma below.

**Lemma 3.5.** *There exists a one-to-one map*

$$\varphi : K \times (0, 1] \to \prod\nolimits_C E_n$$

*satisfying conditions* (v) *and* (vii) *of* 3.3 *together with*

(iv′) $\varphi^{-1}(\prod_C H_n) = L \times (0, 1]$,

(vi′) *given* $n \geq 1$ *there exists an integer* $k_n > k_{n-1}$ $(k_1 \geq 1)$ *such that, if* $x \in K$ *and* $\frac{1}{n+1} < t \leq \frac{1}{n}$ *then* $\pi_{k_n}\varphi(x, t) \neq 0$ *while* $\pi_k\varphi(x, t) = 0$ *for all* $k \in N \setminus (N_{n+1} \cup N_{n+2} \cup \{k_n, k_{n+1}\})$.

*Proof.* Pick $k_n \in N_1$ with $k_n > k_{n-1}$ ( $k_1 \geq 1$ ). By our assumption, there exists a closed embedding $\psi_n : K \to \prod_{C_{n+1}} E_p$, $n \geq 1$, such that

$$\psi_n^{-1}\left(\prod\nolimits_{C_{n+1}} H_p\right) = L \quad \text{and} \quad |||\psi_n(x)||| \leq \frac{1}{n}$$

for all $x \in K$. Pick a vector $e_{k_n} \in H_{k_n}$ with $|||e_{k_n}||| = \frac{1}{2n}$. Define $\varphi = (\varphi_k)$ as

$$\varphi_k\left(x, \frac{1}{n}\right) = \begin{cases} 0 & \text{if } k \in N \setminus (N_{n+1} \cup \{k_n\}), \\ e_{k_n} & \text{if } k = k_n, \\ \pi_k \psi_n(x) & \text{if } k \neq k_n, \end{cases}$$

and

$$\varphi\left(x, s\frac{1}{n} + (1-s)\frac{1}{n+1}\right) = s\varphi\left(x, \frac{1}{n}\right) + (1-s)\varphi\left(x, \frac{1}{n+1}\right)$$

for $n \geq 1$ and $0 \leq s \leq 1$. Conditions (iv′) and (vi′) follow easily. To verify (v) and (vii), repeat a reasoning of the proof of 3.3.

The next result is a counterpart of Proposition 3.1 for cartesian products and can be viewed as a relative version of [4, Proposition 2.5]. We need to recall that by the weak product of $X_i$'s with the basepoints $*_i \in X_i$ we mean

$$W(X_i, *_i) = \left\{(x_i) \in \prod_{i=1}^{\infty} X_i : \ x_i = *_i \ \text{for almost all } i\right\}$$

(endowed with the subspace topology).

**Proposition 3.6.** *Let* $X_i$ *be a noncompact absolute retract and let* $Y_i$ *be a subset of* $X_i$ *such that* $X_i \setminus Y_i$ *is locally homotopy negligible in* $X_i$ *for* $i = 1, 2, \ldots$. *Fix a pair of spaces* $(K, L)$ *and assume that for every* $i \geq 1$ *there exists a closed embedding*

$$h_i : K \to X_i \quad \textit{with} \quad h_i^{-1}(Y_i) = L.$$

*Then, for every choice of basepoints* $*_i \in Y_i$ *and every set* $Z \subseteq X = \prod_{i=1}^{\infty} X_i$ *with* $Z \cap W(X_i, *_i) = W(Y_i, *_i)$, *the pair* $(X, Z)$ *is strongly* $(K, L)$*-universal.*

*Proof.* We apply Proposition 2.1 with the following data: $X$, $Y = W(Y_i, *_i)$, $Y' = W(X_i, *_i)$, and $Z$. It is clear that $X$ is an absolute retract and both $X \setminus Y$

and $Y'$ are locally homotopy negligible in $X$. By [13, Lemma 2.2] $Z$-sets are strong $Z$-sets in $X$. Let $\overline{f}: K \to X$ be such that $f = (f_i) = \overline{f}|U$ maps $U$ into $V \cap Y$ and $\overline{f}(x) \notin V$ for all $x \notin U$, where $U \subset K$ and $V \subset X$ are open sets. Let $\mathscr{V}$ be an open cover of $V$. We pick a map $\mu : X_i \times X_i \times [0, 1] \to X_i$ such that for all $i \geq 1$

(1) $\mu_i(x, y, 0) = x$ and $\mu_i(x, y, 1) = y$ for every $x, y \in X_i$,
(2) $\mu_i(Y_i \times Y_i \times [0, 1]) \subset Y_i$.

To construct $\mu_i$, choose any map $\lambda_i : X_i \times X_i \times [0, 1] \to X_i$ satisfying (1) and a homotopy $(\varphi_t^i) : X_i \times [0, 1] \to X_i$ with $\varphi_0^i = \mathrm{id}_{X_i}$ and $\varphi_t^i(X_i) \subseteq Y_i$ for all $t > 0$ and define

$$\mu_i(x, y, t) = \varphi_{t(1-t)}^i\big(\lambda_i(x, y, t)\big).$$

To produce $\varphi_t^i$, use the fact that $X_i \setminus Y_i$ is locally homotopy negligible in $X_i$ and apply [17, Theorem 2.4]. The same property implies that $Y_i$ is an absolute retract [17, Theorem 3.1]; moreover, since $X_i$ is noncompact, $Y_i$ is nontrivial. As a consequence, there exists an embedding $\alpha_i : [0, 1] \to Y_i$ with

(3) $\alpha_i(0) = *_i$ for $i \geq 1$.

Fix $t = s\frac{1}{n} + (1-s)\frac{1}{n+1}$, $n \geq 1$ and $0 \leq s \leq 1$, and let $\Phi(x, t) = (y_i)$, where

(4) $y_i = f_i(x)$ for $i \leq n$,
(5) $y_i = *_i$ for $i = n+6$ and $i \geq n+9$,
(6) $y_{n+1} = \mu_{n+1}\big(f_{n+1}(x), *_{n+1}, s\big)$,
(7) $y_{n+2} = \mu_{n+2}\big(*_{n+2}, h_{n+2}(x), s\big)$,
(8) $y_{n+i} = h_{n+i}(x)$ for $i = 3$ and $4$,
(9) $y_{n+5} = \mu_{n+5}\big(h_{n+5}(x), *_{n+5}, s\big)$,
(10) $y_{n+7} = \alpha_{n+7}(s)$ and $y_{n+8} = \alpha_{n+8}(1-s)$.

Letting $\Phi(x, 0) = f(x)$, we easily check that $\Phi : K \times [0, 1] \to X$ is well defined and continuous. Notice that, by (2),

(11) $\Phi(K \times (0, 1]) \subset W(X_i, *_i)$,
(12) $\Phi^{-1}\big(W(Y_i, *_i)\big) \cap \big(K \times (0, 1]\big) = L \times (0, 1]$.

We claim that $\Phi|K \times (0, 1]$ is one-to-one. In fact, let $(x, t)$ and $(x', t')$ be such that $\Phi(x, t) = \Phi(x', t')$. If $t = s\frac{1}{n} + (1-s)\frac{1}{n+1}$, $n \geq 1$ and $0 \leq s < 1$ (respectively, $t = 1$), the last $p$th coordinate of $\Phi(x, t)$, different from $*_p$, occurs when $p = n+8$ (respectively, $p = 8$) and it is equal to $\alpha_{n+8}(1-s)$ (respectively, $\alpha_8(1)$). Since $\alpha$ is an embedding, $\Phi(x, t)$ determines $t$ and hence $t = t'$. According to (8), $\Phi_{n+3}(x, t) = h_{n+3}(x)$ (respectively, $\Phi_4(x, t) = h_4(x)$). Therefore, $h_{n+3}(x) = h_{n+3}(x')$ (respectively, $h_4(x) = h_4(x')$) and consequently we get $x = x'$.

Choose a map $\varepsilon : X \to [0, 1]$ such that

(13) $\varepsilon^{-1}(\{0\}) = X \setminus V$,
(14) whenever $y \in V$ and $y' \in X$ satisfy $d(y, y') < \varepsilon(y)$ then there is an element of $\mathscr{V}$ containing both $y$ and $y'$,

where $d$ is a metric on $X = \prod_{i=1}^{\infty} X_i$ chosen so that $d(y, y') < \frac{1}{n+1}$ if $y$ and $y'$ agree on the first $n$ coordinates. By the choice of $d$ and (4), we get

(15) $d\big(\Phi(x, \varepsilon(f(x))), f(x)\big) < \varepsilon(f(x))$.

To see (15), observe that if $\frac{1}{n+1} < \varepsilon(f(x)) \leq \frac{1}{n}$, then $d\big(\Phi(x, \varepsilon(f(x))), f(x)\big) \leq$

$\frac{1}{n+1} < \varepsilon(f(x))$. Define $g : U \to X$ by

$$g(x) = \Phi(x, \varepsilon(f(x))).$$

By (14) and (15), $g$ is $\mathscr{V}$-close to $f$ and takes values in $V$. In turn, (5) and (12) imply that $g$ takes values in $W(X_i, *_i)$ and satisfies $g^{-1}(V \cap W(Y_i, *_i)) = L \cap U$.

It remains to verify that $g: U \to V$ is a closed embedding. For some sequence $\{x_k\}_{k=1}^{\infty} \subset U$ let $\lim g(x_k) = y = (y_i) \in V$. We may assume that $\{\varepsilon(f(x_k))\}_{k=1}^{\infty}$ converges to some $\varepsilon_0 \in [0, 1]$. If $\varepsilon_0 = 0$ then, by (15), $\lim f(x_k) = y$, contradicting the fact that $\varepsilon(y) > 0$. If $\varepsilon_0 > 0$, then we may assume that that $\varepsilon(f(x_k)) \in (\frac{1}{n+1}, \frac{1}{n-1})$ for some $n$ and all $k$. According to (8), $g_{n+3}(x_k) = h_{n+3}(x_k)$ and consequently $\lim h_{n+3}(x_k) = y_{n+3}$. Since $h_{n+3}$ is a closed embedding, $\{x_k\}_{k=1}^{\infty}$ converges in $K$. If $\lim x_k = x \in K \setminus U$, then $\lim f(x_k) = f(x) \notin V$ and, by (13), $\lim \varepsilon(f(x_k)) = \varepsilon(f(x)) = 0$, a contradiction.

*Note* 3.7. Our proof of Proposition 3.6 requires that at least infinitely many of the $X_n$'s are noncompact. Otherwise, it may happen that not all $Z$-sets are strong $Z$-sets in $X$ (see [13]). However, the proof (after minor modifications) still works if one assumes that all the $X_n$'s are nontrivial local compacta.

## 4. Borelian absorbing sets can be linearly represented in $l^2$

For every countable ordinal $\alpha \geq 0$, by $\mathscr{A}_\alpha$ and $\mathscr{M}_\alpha$ we denote the additive and multiplicative classes of all absolute Borelian sets of order $\alpha$, respectively. To be more specific, $\mathscr{M}_0$ consists of all compacta, $\mathscr{A}_1$ consists of all $\sigma$-compact spaces, $\mathscr{M}_1 = \mathscr{M}$ consists of all completely metrizable spaces, and $\mathscr{M}_2$ consists of all absolute $F_{\sigma\delta}$-sets. By $\mathscr{P}_n$, $n \geq 1$, we denote the class of all projective sets of order $n$; $\mathscr{P}_0 = \bigcup_\alpha \mathscr{A}_\alpha$. A set that does not belong to $\bigcup_{n=1}^{\infty} \mathscr{P}_n$ is called nonprojective.

The aim of this section is to find a linear representation of an $\mathscr{A}_\alpha$-absorbing set $F_\alpha$ (respectively, $\mathscr{M}_\alpha$-absorbing set $G_\alpha$) in $l^2$. To perform this, we will make use of $\mathscr{A}_\alpha$-absorbing sets $\Lambda_\alpha$ and $\mathscr{M}_\alpha$-absorbing sets $\Omega_\alpha$ constructed in copies $s$ of $l^2$ in [4]. By the uniqueness theorem for absorbing sets [4], $F_\alpha$ is homeomorphic to $\Lambda_\alpha$ and $G_\alpha$ is homeomorphic to $\Omega_\alpha$. Actually, we show that the pairs $(l^2, F_\alpha)$ and $(s, \Lambda_\alpha)$, $\alpha \geq 1$ (respectively, $(l^2, G_\alpha)$ and $(s, \Omega_\alpha)$, $\alpha \geq 2$), are homeomorphic. The last is achieved by proving the strong $(\mathscr{M}, \mathscr{A}_\alpha)$- and $(\mathscr{M}, \mathscr{M}_\alpha)$-universality of suitable pairs. The multiplicative case of order $\alpha = 1$ differs from the others and is treated separately in [8]; we include a description of $G_1$ in our text in order to formulate the result in full generality.

We briefly recall the definition of $\Lambda_\alpha$ and $\Omega_\alpha$. Set $\Lambda_1 = \Sigma \subset R^\infty = s$ and $\Omega_1 = W(R^\infty, 0) \subset (R^\infty)^\infty = s$. Inductively, if $\alpha = \beta + 1$ let $\Omega_\alpha = \Lambda_\beta^\infty \subset s_\beta^\infty = s$, where $\Lambda_\beta$ is represented in $s_\beta$. If $\alpha$ is a limit ordinal let $\Omega_\alpha = \prod_{\xi<\alpha} \Lambda_\xi^\infty \subset \prod_{\xi<\alpha} s_\xi^\infty = s$, where $\Lambda_\xi$ is represented in $s_\xi$. Finally, let $\Lambda_\alpha = W(s_\alpha \setminus \Omega_\alpha, *) \subset s_\alpha^\infty = s$, where $\Omega_\alpha$ is represented in $s_\alpha$ and $*$ is an arbitrary basepoint of $s_\alpha \setminus \Omega_\alpha$. By the Kadec-Anderson theorem [2], the spaces $s$ (in which $\Lambda_\alpha$ and $\Omega_\alpha$ are represented) are copies of $l^2$.

**Proposition 4.1** (cf. [4]). *For every $\alpha \geq 2$, the pairs $(s, \Lambda_\alpha)$ and $(s, \Omega_\alpha)$ are strongly $(\mathscr{M}, \mathscr{A}_\alpha)$- and $(\mathscr{M}, \mathscr{M}_\alpha)$-universal, respectively. The pair $(s, \Lambda_1) = (R^\infty, \Sigma)$ is strongly $(\mathscr{M}, \mathscr{A}_1)$-universal.*

*Proof*. We only present a proof of the multiplicative case. We repeat the argument of [4, Lemma 6.3] and use Proposition 3.6.

To show that $(R^\infty, \Sigma)$ is strongly $(\mathcal{M}, \mathcal{A}_1)$-universal and fix a pair $(K, L) \in (\mathcal{M}, \mathcal{A}_1)$. There exists a closed embedding $h : K \to R^\infty$ with $h^{-1}(\Sigma) = L$. (Take any closed embedding of $K$ into $R^\infty$ and compose it with a homeomorphism of $R^\infty$ sending $h(L) \cup \Sigma$ onto $\Sigma$; see [2].) Now, by 3.6, $((R^\infty)^\infty, W(\Sigma, 0))$ is strongly $(\mathcal{M}, \mathcal{A}_1)$-universal. Since the latter pair is homeomorphic to $(R^\infty, \Sigma)$ [2, p. 275], the strong $(\mathcal{M}, \mathcal{A}_1)$-universality of $(s, \Lambda_1)$ follows.

We assume that $\alpha \geq 2$ and $\alpha = \beta + 1$ (the case of a limit ordinal is analogous). Given $(K, L) \in (\mathcal{M}, \mathcal{M}_\alpha)$ there exists $L_i \subset K$, $L_i \in \mathcal{A}_\beta$ $(i \geq 1)$, such that $L = \bigcap_{i=1}^\infty L_i$. By the inductive assumption, we find a closed embedding $h_i : K \to s_\beta$ with $h^{-1}(\Lambda_\beta) = L_i$. Writing $h = (h_i)$, we see that $h$ is a closed embedding of $K$ into $s_\beta^\infty = s$ with $h^{-1}(\Omega_\alpha) = \bigcap_{i=1}^\infty L_i = L$. Finally, Proposition 3.6 yields the strong $(\mathcal{M}, \mathcal{M}_\alpha)$-universality of $(s_\beta^\infty, \Lambda_\beta^\infty) = (s, \Omega_\alpha)$.

**Theorem 4.2.** *For every $\alpha \geq 1$, there exists a linear subspace $F_\alpha$ of $l^2$ which is an $\mathcal{A}_\alpha$-absorbing set and such that the pair $(l^2, F_\alpha)$ is strongly $(\mathcal{M}, \mathcal{A}_\alpha)$-universal. In particular, $(l^2, F_\alpha)$ and $(s, \Lambda_\alpha)$ are homeomorphic.*

*Proof. Construction of $F_\alpha$*. Let $(A, B)$ be a copy of $(s, \Lambda_\alpha)$. Consider a closed embedding $h$ of $A$ onto a linearly independent subset of the unit sphere in $l^2$ satisfying the following condition:

(∗) *for every $a \in A$ and every closed subset $F \subset A$ with $a \notin F$ there exists a continuous linear functional $x^* : l^2 \to R$ such that $x^*(h(F)) \subseteq \{0\}$ while $x^*(h(a)) \neq 0$.*

Condition (∗) is taken from [3] where it was checked that the embedding described by Bessaga and Pełczyński [2, p. 193] fulfils (∗). Denote by $H$ the linear span of $h(B)$ in $l^2$ and by $\overline{H}$ the closure of $H$. Since $B$ is dense in $A$, $\overline{H}$ contains $h(A)$ as a (closed) subset. Write $(E_n, H_n) = (\overline{H}, H)$ and set

$$E = \prod\nolimits_{l^2} E_n \quad \text{and} \quad F_\alpha = \sum\nolimits_{l^2} H_n.$$

Clearly, $E$ is isomorphic to $l^2$ and $F_\alpha$ is dense in $E$.

According to 4.1 and the fact that $\Lambda_\alpha$ is an $\mathcal{A}_\alpha$-absorbing set [4], it suffices to prove that the pair $(E, F_\alpha)$ fulfils the requirements (i)–(iv) of 2.2 for the class $\mathcal{L} = \mathcal{A}_\alpha$. Condition (i) is a consequence of the fact that $F_\alpha$ is linear and dense in $E$ (see, e.g., [17, Remark 2.9]). Since each set $A_k = \{(y_n) \in F_\alpha : y_i = 0 \text{ for } i \geq k+1\}$ is a $Z$-set in $F_\alpha$ and $F_\alpha = \bigcup_{k=1}^\infty A_k$, $F_\alpha$ is a $Z_\sigma$-space. Condition (iv) follows directly from 3.1 and 4.1 because $A$ is closed in $E_n$. The remaining condition (iii) can be concluded from (ii) and the lemma below.

**Lemma 4.3.** *The space $H$ is in $\mathcal{A}_\alpha$.*

*Proof*. We shall make use of the cross-section argument due to Klee [2, p. 271]. The $n$-fold product $B^n$ admits a $\sigma$-closed cross-section, i.e., there exists a subset $F$ of $B^n$ that is a countable union of closed sets $F_k$ such that

(1) if $(b_1, b_2, \ldots, b_n) \in F$ then $b_i \neq b_j$ for $i \neq j$,
(2) whenever $\{y_i\}_{i=1}^n$ are $n$ distinct points of $B$ then there exists exactly one permutation of $y_1, y_2, \ldots, y_n$ that belongs to $F$.

By (1) and (2) the linear combination map $\chi$ given by

$$((b_1, b_2, \dots, b_n), (\lambda_1, \lambda_2, \dots, \lambda_n)) \to \lambda_1 h(b_1) + \lambda_2 h(b_2) + \cdots + \lambda_n h(b_n)$$

transforms in a one-to-one way the product $F_k \times D_k^p$ onto $N_k^p \subset H$, where

$$D_k^p = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_n) : \frac{1}{p} \le |\lambda_i| \le p \text{ for all } i \right\}$$

and $p = 1, 2, \dots$. Employing, as in [3, Lemma 3.3], the condition $(*)$ one shows that $\chi | F_k \times D_k^p$ is a homeomorphism. It shows that each $N_k^p$, and hence $H^n = \bigcup_{k,p=1}^{\infty} N_k^p$, belongs to $\mathscr{A}_\alpha$. Since $H = \bigcup_{n=1}^{\infty} H^n$, we get $H \in \mathscr{A}_\alpha$.

**Theorem 4.4.** *For every $\alpha \ge 1$, there exists a linear subspace $G_\alpha$ of $l^2$ which is an $\mathscr{M}_\alpha$-absorbing set and such that the pair $(l^2, G_\alpha)$ is strongly $(\mathscr{M}, \mathscr{M}_\alpha)$-universal. In particular, $(l^2, G_\alpha)$ and $(s, \Omega_\alpha)$ are homeomorphic for $\alpha \ge 2$.*

*Proof.* The case for $\alpha = 1$ differs from the others. In [8] it was shown that the space

$$G_1 = \left\{ (x_n) \in l^2 ;\ \sum_{n=1}^{\infty} |x_n| < \infty \text{ and } \sum_{n=1}^{\infty} x_n = 0 \right\}$$

is an $\mathscr{M}$-absorbing set and, moreover, the pair $(l^2, G_1)$ is strongly $(\mathscr{M}, \mathscr{M})$-universal.

*Construction of $G_\alpha$.* Let $\alpha \ge 2$. If $\alpha$ is a limit ordinal, then choose an increasing sequence of ordinals $\{\beta_n\}_{n=1}^{\infty}$ convergent to $\alpha$; otherwise, $\alpha = \beta + 1$ and let $\beta_n = \beta$. Pick, by 4.2, a pair $(E_n, F_n) = (l^2, F_{\beta_n})$ which is $(\mathscr{M}, \mathscr{A}_{\beta_n})$-universal. Set

$$E = \prod\nolimits_{l^2} E_n \quad \text{and} \quad G_\alpha = \prod\nolimits_{l^2} F_n.$$

The space $E$ is isomorphic to $l^2$ and $G_\alpha$ is its linear dense subspace.

Using 4.1 and the fact that $\Omega_\alpha$ is an $\mathscr{M}_\alpha$-absorbing set [4], it is enough to verify conditions (i)–(iv) of 2.2 for the pair $(E, G_\alpha)$ and $\mathscr{L} = \mathscr{M}_\alpha$. Condition (i) follows as in the proof of 4.2. A simple argument shows that $G_\alpha \in \mathscr{M}_\alpha$. Since $F_1$ is a $Z_\sigma$-space, $G_\alpha$ is also a $Z_\sigma$-space. It remains to verify that $(E, G_\alpha)$ is strongly $(\mathscr{M}, \mathscr{M}_\alpha)$-universal. We will apply 3.4. Let $N_1, N_2, \dots$ be any decomposition of the set of integers $N$ into pairwise disjoint infinite sets. Then, the space $C_k$ defined in 3.4 is $l^2(N_k)$, the space of all square summable sequences indexed by the integers of $N_k$. As a result $(\prod_{C_k} E_p, \prod_{C_k} H_p) = (\prod_{l^2(N_k)} E_p, \prod_{l^2(N_k)} F_p)$. By the choice of $\beta_n$, it is clear that the following lemma will finish the proof of 4.4.

**Lemma 4.5.** *For every $(K, L) \in (\mathscr{M}, \mathscr{M}_\alpha)$ there exists a bounded closed embedding $\psi : K \to E$ with $\psi^{-1}(G_\alpha) = L$.*

*Proof.* Let $L = \bigcap_{k=1}^{\infty} L_k$, where $L_k \subset K$, $L_{k+1} \subset L_k$, and $L_k \in \mathscr{A}_{\beta_{n_k}}$ for some $n_k$ with $n_{k+1} > n_k$. Write $\beta(k) = \beta_{n_k}$. Since $(E_{n_k}, F_{n_k})$ is strongly $(\mathscr{M}, \mathscr{A}_{\beta(k)})$-universal, there exists a closed embedding $\psi_{n_k} : K \to E_{n_k}$ such that

(1) $\|\psi_{n_k}(x)\| \le (\frac{1}{2})^{n_k}$ for all $x \in K$,
(2) $\psi_{n_k}^{-1}(F_{n_k}) = L_k$.

Write $\psi_k \equiv 0$ for all $n \neq n_k$ $(k \geq 1)$ and set $\psi = (\psi_n)$. By (1), $\psi$ is continuous and bounded. It is easy to see that $\psi : K \to E$ is a closed embedding with $\psi^{-1}(G_\alpha) = \bigcap_{k=1}^{\infty} L_k = L$.

*Remark* 4.6. As pointed out in [8], the pair $(s, \Omega_1) = ((R^\infty)^\infty, W(R^\infty, 0))$ is not strongly $(\mathscr{M}, \mathscr{M})$-universal. (If it were, then by 2.2, $(s, \Omega_1)$ and $(l^2, G_1)$ would be homeomorphic; consequently, $G_1$ would be $\sigma$-closed in $l^2$, which contradicts a result of [15].)

*Remark* 4.7. The spaces $\Lambda_\alpha$ and $\Omega_\alpha$ can be realized as linear subspaces in other normed coordinate products $\prod_C E_n$. The only restriction is the condition $(*)$.

*Remark* 4.8. The result of 4.1 can be readily generalized to the triple case. Representing $\Lambda_\alpha$ $(\alpha \geq 1)$ and $\Omega_\alpha$ $(\alpha \geq 2)$ in $R^\infty$, we could consider the triples

$$(\overline{R}^\infty, R^\infty, \Lambda_\alpha) \quad \text{and} \quad (\overline{R}^\infty, R^\infty, \Omega_\alpha),$$

where $\overline{R} = [-\infty, +\infty]$. These triples are strongly $(\mathscr{M}_0, \mathscr{M}, \mathscr{A}_\alpha)$- and $(\mathscr{M}_0, \mathscr{M}, \mathscr{M}_\alpha)$-universal, respectively (with an obvious meaning of the triple strong universality).

In §6 we shall need the following fact concerning the complements of $F_\alpha$ and $G_\alpha$.

**Corollary 4.9.** *The space $l^2 \setminus F_\alpha$ (respectively, $l^2 \setminus G_\alpha$) has the following properties*:

(i) $l^2 \setminus F_\alpha$ *(respectively, $l^2 \setminus G_\alpha$) is a Baire space,*
(ii) $l^2 \setminus F_\alpha \in \mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$ *(respectively, $l^2 \setminus G_\alpha \in \mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$),*
(iii) $l^2 \setminus F_\alpha$ *(respectively, $l^2 \setminus G_\alpha$) is homogeneous,*
(iv) *for every (closed) ball $B \subset l^2$, $B \setminus F_\alpha$ (respectively, $B \setminus G_\alpha$) is an absolute retract.*

*Proof*. We will only deal with the $F_\alpha$-case, the $G_\alpha$-case is analogous. Conditions (i) and (ii) follow from the fact that $F_\alpha$ is of the first category and that $F_\alpha \in \mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$. To show (iii), we produce a homeomorphism of $l^2$ that preserves $F_\alpha$ and carries $y \in l^2 \setminus F_\alpha$ onto $y' \in l^2 \setminus F_\alpha$. Let $h$ be any homeomorphism of $l^2$ with $h(y) = y'$, e.g., $h$ is the translation. Then, the pairs $(l^2, h(F_\alpha))$ and $(l^2, F_\alpha)$ are strongly $(\mathscr{M}, \mathscr{A}_\alpha)$-universal. The proof of [7, Theorem 2.1] can be easily modified to achieve a homeomorphism $k$ of $l^2$ that carries $h(F_\alpha)$ onto $F_\alpha$ and preserves $\{y'\}$ (set $X_0 = Y_0 = \{y'\} \subset X_1 \cap Y_1$). We see that $k \circ h$ preserves $F_\alpha$ and sends $y$ onto $y'$. Condition (iv) is a consequence of the fact that $B \cap F_\alpha$ is locally homotopy negligible in $B$ and [17, Theorem 3.1]. Assume $0 \in \operatorname{int} B$ and pick $y_0 \in B \setminus F_\alpha$. Since the homotopy $f_t(y) = (1-t)y + ty_0$ $(0 \leq t \leq 1)$ takes its values in $B \setminus F_\alpha$ for $t > 0$ and $f_0 = \mathrm{id}$, the local homotopy negligibility of $B \cap F_\alpha$ in $B$ follows.

## 5. Application to $F_{\sigma\delta}$-spaces

In this section we identify various absolute $F_{\sigma\delta}$-sets carrying product structures to be homeomorphic to $\Omega_2 = \Sigma^\infty$. The spaces we deal with will be considered with both normed and cartesian product topologies. We start with a direct application of Proposition 3.4 to coordinate products of normed $F_{\sigma\delta}$-spaces. A counterpart of 5.1 for cartesian products was previously obtained in [13].

**Theorem 5.1.** *Let $\prod_C H_n$ be a normed coordinate product of absolute $F_{\sigma\delta}$-spaces $H_n$ in the sense of a Banach space $C$. Assume that infinitely many of the $H_n$'s are $Z_\sigma$-spaces. Then $\prod_C H_n$ is homeomorphic to $\Omega_2$. Moreover, writing $E_n$ for the linear completion of $H_n$, the pairs $(\prod_C E_n, \prod_C H_n)$ and $(s, \Omega_2)$ are homeomorphic.*

We shall make use of the two lemmas below. A proof of the first one is implicitly contained in [12, Lemma 5.4] and therefore it will be omitted.

**Lemma 5.2.** *Let $X \in \mathscr{M}$ be an absolute retract and $Y \subset X$ be a $Z_\sigma$-space such that $X \setminus Y$ is locally homotopy negligible in $X$. Then, for every $L \in \mathscr{A}_1$ of the Hilbert cube $I^\infty$, there exists a map $\varphi : I^\infty \to X$ with $\varphi^{-1}(Y) = L$.*

**Lemma 5.3.** *Let $(H_n, \|\cdot\|_{H_n})$ be a normed linear space that is noncompactly embedded into a Banach space $(E_n, \|\cdot\|_{E_n})$, i.e., $H_n \subseteq E_n$, $\|\cdot\|_{E_n} \le \|\cdot\|_{H_n}$, and the $E_n$-closures of $H_n$-balls are noncompact. Then, for every coordinate Banach space $C$ and every $K \in \mathscr{M}$, there exists a bounded closed embedding $\psi : K \to \prod_C E_n$ such that $\psi(K) \subset \prod_C H_n$.*

*Proof.* Since $C$ is complete, there is $c = (c_n) \in C$ with all the $c_n$'s strictly positive. It is enough to construct a closed embedding $\psi = (\psi_n) : K \to \prod_C E_n$ with $\psi_n(x) \in H_n$ and $\|\psi_n(x)\|_{H_n} \le c_n$ for all $x \in K$ and $n \ge 1$. Fix a metric $d$ on the Hilbert cube $I^\infty = [0, 1]^\infty$. Embed $K$ into $I^\infty$ and write $I^\infty \setminus K = \bigcup_{n=1}^\infty F_n$, where each $F_n$ is a closed subset of $I^\infty$. Define $d_n(q) = \operatorname{dist}_d(q, F_n)$, $q \in I^\infty$. We claim that there exists a closed embedding $\alpha_n : [0, \infty) \to E_n$ such that $\alpha_n(t) \in H_n$ and $\|\alpha_n(t)\|_{H_n} \le 1$ for all $t \ge 0$ and $n \ge 1$. This follows from the noncompactness of the $E_n$-closure of the $H_n$-unit ball and Klee's result [14] that every noncompact closed convex subset $F$ of $E_n$ contains a copy of $[0, \infty)$. (The piecewise linear embedding $\alpha$ constructed by Klee can be improved to get the nodes of $\alpha$ contained in any dense linear subset of $F$.) Pick a vector $e_{2n} \in H_{2n}$ with $\|e_{2n}\|_{H_{2n}} = 1$. Define $\psi = (\psi_n)$ by

$$\psi_{2n}(q) = c_{2n} q_n e_{2n} \quad \text{and} \quad \psi_{2n-1}(q) = c_{2n-1}\alpha_{2n-1}\left((d_n(q))^{-1}\right)$$

for $q = (q_n) \in K$. It is clear that $\psi : K \to \prod_C E_n$ is one-to-one and $\|\psi_n(q)\|_{H_n} \le c_n$, $n \ge 1$, $q \in K$. By [16, Lemma 1.4], $\psi$ is continuous. If $\{\psi(q(i))\}_{i=1}^\infty$ is convergent in $\prod_C E_n$, then there exists $q_0 \in I^\infty$ with $\lim q(i) = q_0$. Assume that $q_0 \in F_k$ for some $k$. Then $\lim_{i\to\infty} d_k(q(i)) = 0$, contradicting the fact that $\{(d_k(q(i)))^{-1}\}_{i=1}^\infty$ converges. The latter is a consequence of the facts that the sequence $\{\alpha_{2k-1}((d_k(q(i)))^{-1})\}_{i=1}^\infty$ is convergent in $E_k$ and that $\alpha$ is a closed embedding.

*Proof of* 5.1. We show that the pair $(E, H) = (\prod_C E_n, \prod_C H_n)$ satisfies (i)–(iv) of 2.2 with $\mathscr{L} = \mathscr{M}_2$. By the Kadec-Anderson theorem [2], $E$ is a copy of $l^2$. A standard argument yields (i)–(iii). The strong $(\mathscr{M}, \mathscr{M}_2)$-universality of $(E, H)$ will be derived from Proposition 3.4. Fix a pair $(K, L) \in (\mathscr{M}, \mathscr{M}_2)$. Find pairwise disjoint infinite subsets $N_1, N_2, \ldots$ of $N$ so that $H_p$ is a $Z_\sigma$-space for every $p \in \bigcup_{k=1}^\infty N_k$. Let $C_k$ be the subspace of $C$ that corresponds to $N_k$ (see 3.4). Write $E^k = \prod_{C_k} E_p$ and $H^k = \prod_{C_k} H_p$. To fulfil the hypothesis of 3.4 we have to produce a bounded closed embedding $\psi_k : K \to E^k$ with $\psi_k^{-1}(H^k) = L$. To this end we split $C_k = C_k^1 \oplus C_k^2$ into two coordinate spaces and find a bounded closed embedding $\psi_k^1 : K \to \prod_{C_k^1} E_p$ with $\psi_k^1(K) \subset \prod_{C_k^1} H_p$

and a bounded map $\psi_k^2 : K \to \prod_{C_k^2} E_p$ with $(\psi_k^2)^{-1}(\prod_{C_k^2} H_p) = L$. Finally, letting $\psi_k = (\psi_k^1, \psi_k^2)$ we get a required embedding (we identify $\prod_{C_k} E_p$ with $\prod_{C_k^1} E_p \oplus \prod_{C_k^1} E_p)$.

To find $\psi_k^1$ we apply Lemma 5.3. In this case $H_p$ is a genuine subspace of an infinite-dimensional Banach space $E_p$; hence the balls in $E_p$ are noncompact. Let $c = (c_p) \in C_k^2$ be such that all $c_p$ are strictly positive. Embed $K$ into $I^\infty$ and represent $L = \bigcap_p L_p$ so that each $L_p$ is $\sigma$-compact and $\{L_p\}$ is descending. Use Lemma 5.2 with $X = B(c_p)$, the closed ball in $E_p$ centered at 0 with radius $c_p$, $Y = B(c_p) \cap H_p$, and $L = L_p$. We get maps $\varphi_p : I^\infty \to B(c_p)$ with $\varphi_p^{-1}(Y) = L_p$. Finally, we set $\psi_k^2(x) = (\varphi_p(x))$, $x \in K$. The continuity of $\psi_k^2$ follows from [16, Lemma 4.1]. To verify the hypothesis of 5.2, notice that $Y$ is convex and dense in $X$. Let $H_p = \bigcup_{m=1}^\infty A_m$, where each $A_m$ is a $Z$-set in $H_p$. Then $\operatorname{int}(B(c_p)) \cap A_m$ is a $Z$-set in $\operatorname{int}(B(c_p)) \cap H_p$. Since $\operatorname{int}(B(c_p)) \cap H_p$ is convex and dense in $B(c_p) \cap H_p$, $B(c_p) \cap A_m$ is a $Z$-set in $B(c_p) \cap H_p$. It shows that $Y$ is a $Z_\sigma$-space.

A direct consequence of 5.1 is

*Note* 5.4. The simplest pre-Hilbert space representation of $\Omega_2$ is the space $\prod_{l^2} l_f^2$, where $l_f^2 = \{(x_i) \in l^2 : \ x_i = 0 \text{ for almost all } i\}$. Moreover, the pairs $(\prod_{l^2} l^2, \prod_{l^2} l_f^2)$ and $(s, \Omega_2)$ are homeomorphic.

Consider the set $\prod_C H_n = H$ as a subspace of the cartesian product $\prod_{n=1}^\infty E_n = E$. By the Kadec-Anderson theorem [2], $E$ is a copy of $l^2$. Easily, $E \setminus H$ is locally homotopy negligible in $E$. We claim that $C$ is an $F_{\sigma\delta}$-subset of $R^\infty$. This is a consequence of the equality

$$C = \left\{ (x_n) \in R^\infty : \ \forall_{\varepsilon>0} \exists_k \forall_{m>k} \left\| \sum_{i=k}^m x_i u_i \right\|_C \le \varepsilon \right\}$$

($u_i$ is the $i$th unit vector). Consider the map $f(x) = (\|x_n\|)$, $x = (x_n) \in E$, and notice that $f^{-1}(C) = \prod_C E_n$. This shows that $\prod_C E_n \in \mathscr{M}_2$. Since $\prod_C H_n = \prod_C E_n \cap \prod_{n=1}^\infty H_n$, $H$ is an absolute $F_{\sigma\delta}$-set. Repeating (with obvious changes) the remaining part of the proof of 5.1, we get the following generalization of a result [13].

**Theorem 5.5.** *Let $\{H_n\}_{n=1}^\infty$ be a sequence of normed linear spaces such that each $H_n$ is an absolute $F_{\sigma\delta}$-set and infinitely many of the $H_n$'s are $Z_\sigma$-spaces. Then, for every coordinate Banach space $C$, the space $\prod_C H_n$ considered in the product topology, is homeomorphic to $\Omega_2$. Moreover, if $E_n$ is the linear completion of $H_n$ then the pairs $\left(\prod_{n=1}^\infty E_n, \prod_C H_n\right)$ and $(s, \Omega_2)$ are homeomorphic.*

*Remark* 5.6. The hypothesis that infinitely many of the $H_n$'s are $Z_\sigma$-spaces is essential. Consider the coordinate space $c_0 = \{(x_i) \in R^\infty : \ \lim x_i = 0 \}$ with the $\|\cdot\|_\infty$-norm. Note that $c_0 \subset \bigcup_{k=1}^\infty B_\infty(k)$, where $B_\infty(k) = \{x \in R^\infty : \|x\|_\infty \le k \}$. This shows that $c_0$ is contained in a $\sigma$-compact subset of $R^\infty$. On the other hand $\Omega_2$ contains a copy of $R^\infty$ closed in $s$. This shows that $(R^\infty, \prod_{c_0} R)$ and $(s, \Omega_2)$ are not homeomorphic, contrary to the expectation expressed in [13]. Proposition 3.6 and Lemma 5.2 yield the strong $(\mathscr{M}_0, \mathscr{M}_2)$-universality of the pair $(R^\infty, c_0)$.

For the $c_0$-products we have the following generalization of [13, Theorem 4.2].

**Theorem 5.7.** *Let, for* $n \geq 1$, $X_n$ *be a subset of a Banach space* $(E_n, \|\cdot\|_n)$ *with* $0 \in X_n$ *so that* $\inf_{n\geq 1} \operatorname{diam}(X_n) = \alpha > 0$. *Assume that each* $X_n$ *is an absolute retract that is an absolute* $F_{\sigma\delta}$*-set. Then the space*

$$X = \prod_{c_0} X_n = \left\{ (x_n) \in \prod_{n=1}^{\infty} X_n ;\ \|x_n\|_n \to 0 \right\}$$

(*endowed with the product topology*) *is homeomorphic to* $\Omega_2$.

*Proof.* We will show that $X$ is an $\mathscr{M}_2$-absorbing set in some copy of $l^2$, i.e., we will verify conditions (i)–(iii) of 2.2 and the strong $\mathscr{M}_2$-universality of $X$. Then, the uniqueness theorem for absorbing sets [4] yields our assertion. Since $\prod_{n=1}^{\infty} X_n$ is an absolute retract and $\prod_{n=1}^{\infty} X_n \setminus X$ is locally homotopy negligible, $X$ is also an absolute retract [17, Theorem 3.1]. Decompose the set of integers $N$ into pairwise disjoint infinite sets $N_1, N_2, \ldots$. Write $X^i = \prod_{p\in N_i} X_p$ and $Y^i = \{(x_p) \in X^i : \|x_p\|_p \to 0\}$ and let $\Psi : \prod_{n=1}^{\infty} X_n \to \prod_{i=1}^{\infty} X^i$ be the natural isomorphism. We have

(1) $\Psi(X) \supset W(Y^i, 0)$.

Note that each $Y^i$ is noncompact. (If it were compact then, because $Y^i$ is dense in $X^i$, we would get $X^i = Y^i$, contradicting the fact that $\alpha > 0$.) Now, each $Y^i$ has a completion $Z^i$ with $Z^i \setminus Y^i$ locally homotopy negligible in $Z^i$ [17, Proposition 4.1]. Since $Y^i$ is noncompact, we can assume that $Z^i$ is also noncompact (if it were compact take $Z^i \setminus \{*\}$, where $* \in Z^i \setminus Y^i$). Then the product $s = \prod_{i=1}^{\infty} Z^i$ is a copy of $l^2$ [18, Theorem 5.1] with $s \setminus X$ locally homotopy negligible in $s$; this shows (i). An argument preceding Theorem 5.5 applies to show that $X$ is an absolute $F_{\sigma\delta}$-set. Writing $A_m = \{(x_n) \in X : \|x_j\|_j \leq \frac{\alpha}{2}$ for all $j \geq m+1\}$, we see that $\bigcup_{m=1}^{\infty} A_m = X$ and that each $A_m$ is a $Z$-set in $X$. In proving the strong $\mathscr{M}_2$-universality of $X$ we employ 3.6 with $Y_i = Y^i$, $K = L \in \mathscr{M}_2$, and $Z = \Psi(X)$. To produce a closed embedding of $L$ into $Y^i$, we may assume that $Y^i = X$.

Write $B^i(\varepsilon) = \{(x_p) \in X^i : \|x_p\|_p \leq \varepsilon$ for all $p \in N_i\}$ and notice that

(2) $\prod_{i=1}^{\infty} Y^i \cap B^i(2^{-i})$ is a closed subset of $\Psi(X)$.

By (2) and the fact that a countable product of $Z_\sigma$-spaces that are absolute retracts contains a closed copy of $\Omega_2$ [13, Corollary 2.5], it suffices to check that each $Y^i \cap B^i(2^{-i})$ contains a closed $Z_\sigma$-space that is an absolute retract. Assuming $\alpha \geq 2^{-i}$, we choose in each $X_p$ an arc $T_p$ joining 0 with some $x_p$ with $\|x_p\|_p = 2^{-i}$. Write $T^i = Y^i \cap \prod_{p\in N_i} T_p$. Then $T^i$ is a closed subset of $Y^i$. The argument showing that $X$ is a $Z_\sigma$-space applies also to verify that $T^i$ is a $Z_\sigma$-space. The proof is completed.

Let us note a relative version of [13, Corollary 2.7].

*Remark 5.8.* Let $X_n \in \mathscr{M}$ be a noncompact absolute retract and let $Y_n$ be a subset of $X_n$ such that $X_n \setminus Y_n$ is locally homotopy negligible in $X_n$, $n = 1, 2, \ldots$. If each $Y_n$ is an absolute $F_{\sigma\delta}$-set and infinitely many of the $Y_n$'s are $Z_\sigma$-spaces, then the pairs $\left(\prod_{n=1}^{\infty} X_n, \prod_{n=1}^{\infty} Y_n\right)$ and $(s, \Omega_2)$ are homeomorphic. Apply 2.2 together with 3.6. To produce a closed embedding $h : K \to \prod_{n=1}^{\infty} X_n$

with $h^{-1}\left(\prod_{n=1}^{\infty} Y_n\right) = L$ employ Lemma 5.2 and the fact that $\prod_{n=1}^{\infty} X_n$ contains a closed copy of $[0, \infty)$ that lives in $\prod_{n=1}^{\infty} Y_n$. Moreover, adopting Theorem 2.2 and Lemma 5.2 to the triple case one can get a homeomorphism of suitable triples (see [9]); 4.8.

Let us recall that by $L^p[a, b]$ we denote the space of equivalence classes of Lebesgue measurable functions $x : [a, b] \to R$ with

$$\|x\|_p = \left(\int_a^b |x(t)|^p\, dt\right)^{\min(1, \frac{1}{p})} < \infty$$

with the topology induced by the $F$-norm $\|\cdot\|_p$, $0 < p < \infty$. Write $\widetilde{L}^p[a, b] = \bigcap_{p'<p} L_{p'}[a, b]$, $0 < p \leq \infty$, and by $\widetilde{L}^p_q[a, b]$ denote the set $\widetilde{L}^p[a, b]$ with the $\|\cdot\|_q$-topology, $q < p$. Note that $\widetilde{L}^p_q[a, b]$ is dense in $L^q[a, b]$. We skip the symbol $[a, b]$ if $[a, b] = [0, 1]$.

**Theorem 5.9.** *The pairs* $(L^q, \widetilde{L}^p_q)$ *and* $(s, \Omega_2)$ *are homeomorphic for* $0 < q < p \leq \infty$.

*Proof.* Mazur's homeomorphism [2, p. 207] of $L^1$ onto $L^q$ transforms $\widetilde{L}^p$ onto $\widetilde{L}^{pq}$. Therefore, it sufficies to consider the case of $q = 1$ (and arbitrary $p > 1$).

We write $\widetilde{L}^p = \widetilde{L}^p_1$. Since $L^1$ is a copy of $l^2$ [2], it is enough to verify conditions (i)–(iv) of 2.2. The local homotopy negligibility of $L^1 \setminus \widetilde{L}^p$ follows in a standard way. Note that each $L^p$ is an $F_\sigma$-subspace of $L^q$ for $p > q$. (This is a consequence of the facts that $L^p$ is an $F_\sigma$-subspace of $L^0$, the space of measurable functions with the convergence in measure topology (see [13]), and that the $L^0$-topology is weaker than the $\|\cdot\|_q$-topology.) Select an increasing sequence $\{p_n\}_{n=1}^{\infty} \subset (1, p)$ that converges to $p$. Since $\widetilde{L}^p = \bigcap_{n=1}^{\infty} L^{p_n}$, we get $\widetilde{L}^p \in \mathscr{M}_2$.

To prove that $\widetilde{L}^p$ is a $Z_\sigma$-space, we choose $1 < p' < p$ and write

$$B_{p'}(\varepsilon) = \{x \in L^{p'} : \|x\|_{p'} \leq \varepsilon\}.$$

Since $\widetilde{L}^p = \bigcup_{k=1}^{\infty} B_{p'}(k) \cap \widetilde{L}^p$ it suffices to check that each $A = B_{p'}(k) \cap \widetilde{L}^p$ is a $Z$-set in $\widetilde{L}^p$. First of all, note that $B_{p'}(k)$ is a $Z$-set in $L^1$ because it is a closed subset of a locally homotopy negligible set $L^{p'}$ in $L^1$. Then, using the fact that $L^1 \setminus \widetilde{L}^p$ is locally homotopy negligible in $L^1$, we infer that $A$ is a $Z$-set in $\widetilde{L}^p$ (see [5, Lemma 2.6]).

We make use of 3.1 to verify the strong $(\mathscr{M}, \mathscr{M}_2)$-universality of $(L^1, \widetilde{L}^p)$. The map $\Psi$ given by

$$\Psi(x) = (x|[2^{-n}, 2^{-n+1}])_{n=1}^{\infty},$$

$x \in L^1$, is a linear isomorphism of $L^1$ onto $\prod_{l^1} L^1[2^{-n}, 2^{-n+1}]$. Writing $Z = \Psi(\widetilde{L}^p)$, we have

$$Z \cap \sum\nolimits_{l^1} L^1[2^{-n}, 2^{-n+1}] = \sum\nolimits_{l^1} \widetilde{L}^p[2^{-n}, 2^{-n+1}].$$

Since the pair $(E_n, H_n) = (L^1[2^{-n}, 2^{-n+1}], \widetilde{L}^p[2^{-n}, 2^{-n+1}])$ is (naturally) isomorphic to $(L^1, \widetilde{L}^p)$, the lemma below verifies the hypothesis of 3.1 and thus finishes the proof of 5.9.

**Lemma 5.10.** *Let* $(K, L) \in (\mathscr{M}, \mathscr{M}_2)$. *There exists a bounded closed embedding* $\psi : K \to L^1$ *with* $\psi^{-1}(\widetilde{L}^p) = L$.

*Proof.* We repeat a reasoning from the proof of 5.1. First, we find a bounded closed embedding $\psi^1 : K \to L^1[\frac{1}{2}, 1]$ with $\psi^1(K) \subset \widetilde{L}^p$. Then, we produce a bounded map $\psi^2 : K \to L^1[0, \frac{1}{2}]$ with $(\psi^2)^{-1}(\widetilde{L}^p[0, \frac{1}{2}]) = L$. Finally, we let $\psi = (\psi^1, \psi^2)$. To get $\psi^1$, we apply 5.3 with $H_n = L^p[\frac{1}{2}, 1]$, $E_n = L^1[\frac{1}{2}, 1]$, and $C = l^1$. It is clear that $H_n$ is noncompactly embedded in $E_n$. Consequently there exists a bounded closed embedding $\psi^1 : K \to \prod_{l^1} L^1[\frac{1}{2}, 1] = L^1[\frac{1}{2}, 1]$ such that $\psi^1(K) \subset \prod_{l^1} L^p[\frac{1}{2}, 1] \subset \prod_{l^p} L^p[\frac{1}{2}, 1] = L^p[\frac{1}{2}, 1] \subset \widetilde{L}^p[\frac{1}{2}, 1]$. To obtain $\psi^2$, embed $K$ into $I^\infty$ and represent $L = \bigcap_{n=2}^\infty L_n$ with each $L_n$ $\sigma$-compact and $L_{n+1} \subset L_n$ for $n \geq 2$. Recall that $\{p_n\} \subset (1, p)$ converges to $p$. If we find maps $\varphi_n : K \to L^{p_n}[2^{-n}, 2^{-n+1}]$ such that $\varphi_n^{-1}(\widetilde{L}^p[2^{-n}, 2^{-n+1}]) = L_n$ and $\|\varphi_n(x)\|_{p_n} \leq 2^{-n}$ for all $x \in K$ and $n \geq 2$, then $\psi^2$ defined by

$$\psi^2(x)|[2^{-n}, 2^{-n+1}] = \varphi_n(x),$$

$x \in K$, $n \geq 2$, is as required.

To produce $\varphi_n$, we apply 5.2 for $(X, Y) = (B_{p_n}(2^{-n}), B_{p_n}(2^{-n}) \cap \widetilde{L}^p)$. Since $Y$ is convex and dense in $X$, $X \setminus Y$ is locally homotopy negligible in $X$. Pick $p_n < p' < p$. We have

$$Y = B_{p_n}(2^{-n}) \cap \widetilde{L}^p = \bigcup_{k=1}^{\infty} B_{p'}(k) \cap B_{p_n}(2^{-n}) \cap \widetilde{L}^p.$$

We claim that each $A = B_{p'}(k) \cap B_{p_n}(2^{-n}) \cap \widetilde{L}^p$ is a $Z$-set in $Y$. Since $B_{p'}(k)$ is a $Z$-set in $L^{p_n}$, it easily follows that $B_{p'}(k) \cap B_{p_n}(2^{-n})$ is a $Z$-set in $B_{p_n}(2^{-n})$. Now the local homotopy negligibility of $X \setminus Y$ in $X$ implies, via [5, Lemma 2.6], that $A$ is a $Z$-set in $Y$. This finishes the proof.

*Remark* 5.11. One could likely elaborate an abstract scheme of identifying some normed coordinate products that are homeomorphic to $\Omega_2$, as done for cartesian products in [13]. Due to replacing the convex structure by a suitable equiconnected structure on $L^0([0, 1], G)$, the space of measurable $G$-valued functions on $[0, 1]$, it was proved in [13] that $\widetilde{L}^p([0, 1], G)$ (with the $L^0$-topology) is homeomorphic to $\Omega_2$, provided $G$ is a closed unbounded subset of a Banach space. Using 3.6 and 5.2, one can show that the pair $(L^0([0, 1], G), \widetilde{L}^p([0, 1], G))$ is homeomorphic to $(s, \Omega_2)$ for $0 < p \leq \infty$. To produce a closed embedding of $R^\infty$ in $L^0([0, 1], G)$ with values in $B = \{x \in L^1 : |x(t)| \leq \varepsilon$ almost everywhere $\}$, we use the argument of 5.3 and the fact that $B \cap L^0([0, 1], G)$ is a copy of $l^2$ [2]. It is likely that the pairs $(L^q([0, 1], G), \widetilde{L}^p_q([0, 1], G))$ and $(s, \Omega_2)$ are also homeomorphic.

By $l^p$ we denote the space of real-valued sequences $x = (x_n)$ such that

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\min(1, \frac{1}{p})} < \infty$$

with the topology induced by the $F$-norm $\|\cdot\|_p$, $0 < p < \infty$. Write $\widetilde{l}^p =$

$\bigcap_{p'>p} l^{p'}$, $0 \le p < \infty$, and denote by $\widetilde{l^p_q}$ the space $\widetilde{l^p}$ with the $\|\cdot\|_q$-topology, $q > p$. Note that $\widetilde{l^p_q}$ is a dense linear subspace of $l^q$.

**Theorem 5.12.** *The pairs* $(l^q, \widetilde{l^p_q})$ *and* $(s, \Omega_2)$ *are homeomorphic for* $0 \le p < q < \infty$.

*Proof.* As in the proof of 5.9, we only need to check that $(l^1, \widetilde{l^p})$, $0 < p < 1$, fulfils conditions (i)–(iv) of 2.2; we write $\widetilde{l^p} = \widetilde{l^p_1}$. A verification of (i) and (iii) is almost the same as in 5.9 and uses the observation that

$$B_{p'}(\varepsilon) = \{x \in l^{p'} : \|x\| \le \varepsilon\}$$

is closed in the $\|\cdot\|_p$-topology ($p > p'$). Also, every set $B_{p'}(k) \cap \widetilde{l^p}$ is a $Z$-set in $\widetilde{l^p}$, yielding (ii). To verify (iv) we make use of 3.1. Decompose $N$ into pairwise disjoint infinite sets $N_1, N_2, \ldots$. Consider the linear isomorphism $\Psi : l^1 \to \prod_{l^1} l^1(N_n)$, where $l^1(N_n)$ is an isomorphic copy of $l^1$ of sequences indexed by integers of $N_n$, given by

$$\Psi(x) = ((x_k)_{k\in N_1}, (x_k)_{k\in N_2}, \ldots),$$

$x \in l^1$. Writing $Z = \Psi(\widetilde{l^p})$, we see that

$$Z \cap \sum\nolimits_{l^1} l^1(N_n) = \sum\nolimits_{l^1} \widetilde{l^p}(N_n).$$

The following lemma enables us to apply 3.1 and hence to finish the proof of 5.12.

**Lemma 5.13.** *Let* $(K, L) \in (\mathscr{M}, \mathscr{M}_2)$. *There exists a bounded closed embedding* $\psi : K \to l^1$ *with* $\psi^{-1}(\widetilde{l^p}) = L$.

*Proof.* We follow the proof of 5.10. As in 5.10 we embed $K$ into $I^\infty$, represent $L = \bigcap_{n=1}^\infty L_n$, and pick a sequence $(p_n) \subset (p, 1)$ convergent to $p$. A bounded closed embedding $\psi^1 : K \to l^1(N_1)$ with $\psi^1(K) \subset \widetilde{l^p}$ is obtained via Lemma 5.3. We take $H_n = (l^p(N_1), \|\cdot\|_p)$, $E_n = l^1(N_1)$, and $C = l^1$. (Formally, we are not eligible to apply 5.3 because $H_n$ is not a normed space. This assumption was only used to construct the closed embedding of $[0, \infty)$. In our case the unit closed ball $B$ in $l^p(N_1)$ is homeomorphic, via Mazur's homeomorphism [2, p. 207], to the closed unit ball in the Hilbert space which, in turn, is homeomorphic to $R^\infty$. Therefore $B$ being closed in $l^1(N_1)$ admits a required embedding.) Hence, we get a bounded closed embedding $\psi^1 : K \to \prod_{l^1} l^1(N_1) = l^1(N_1)$ with $\psi^1(K) \subset \prod_{l^1} l^p = l^p \subset \widetilde{l^p}$. To produce $\psi^2$, we apply 5.2 to the pair $(X, Y) = (B_{p_n}(2^{-n}), B_{p_n}(2^{-n}) \cap \widetilde{l^p})$ and find maps $\varphi_n : K \to l^{p_n}(N_n)$, $n \ge 2$, with $\varphi_n^{-1}(\widetilde{l^p}(N_n)) = L_n$ and $\|\varphi_n(x)\|_{p_n} \le 2^{-n}$ for all $x \in K$. It is easy to see that the map $\psi^2(x) = (\varphi_n(x))_{n=2}^\infty$ satisfies $(\psi^2)^{-1}(\widetilde{l^p}(N \setminus N_1)) = L$. We let $\psi = (\psi^1, \psi^2)$.

Let us formulate a more specific result concerning $\widetilde{l^p}$-products whose proof is a modification of the proof of 5.7 (and therefore will be omitted).

**Theorem 5.14.** *Let, for* $n \ge 1$, $X_n$ *be a subset of a Banach space* $(E_n, \|\cdot\|_n)$ *with* $0 \in X_n$ *so that* $\inf_{n\ge1} \operatorname{diam}(X_n) = \alpha > 0$. *Assume that each* $X_n$ *is an* *absolute retract that is an absolute $F_{\sigma\delta}$-set. Then the space*

$$\widetilde{l}^p(X_n) = \left\{ (x_n) \in \prod_{n=1}^{\infty} X_n : \forall_{p'>p} \sum_{n=1}^{\infty} \|x_n\|_n^{p'} < \infty \right\}$$

*(as a subspace of $\prod_{n=1}^{\infty} X_n$) is homeomorphic to $\Omega_2$ for every $0 \le p < \infty$.*

*Remark* 5.15. Assume that each $X_n \in \mathscr{M}$. We may ask whether $(\prod_{n=1}^{\infty} X_n, \widetilde{l}^p(X_n))$ is homeomorphic to $(s, \Omega_2)$. This, in general, is not necessarily the case. The space $\widetilde{l}^p(R) = \widetilde{l}^p$ is contained in a $\sigma$-compact subset of $R^\infty$ (cf. Remark 5.6). Let us notice that the pair $(R^\infty, \widetilde{l}^p)$ is strongly $(\mathscr{M}_0, \mathscr{M}_2)$-universal. The assertion of Theorem 2.2 holds if one replaces $\mathscr{M}$ by $\mathscr{M}_0$ and add in the hypothesis that both $Y_1$ and $Y_2$ are contained in $\sigma$-compact subsets of $X$. As a consequence, the pairs $(R^\infty, c_0)$ and $(R^\infty, \widetilde{l}^p)$ are homeomorphic for $0 \le p < \infty$. This shows that two $\mathscr{L}$-absorbing sets $Y_1$ and $Y_2$ can be relatively homeomorphic in a copy $X$ of $l^2$ while none of the pairs $(X, Y_1)$ and $(X, Y_2)$ are strongly $(\mathscr{M}, \mathscr{L})$-universal.

## 6. The spaces $F_\alpha$ and $G_\alpha$ as factors of exotic pre-Hilbert spaces

In this section we present some examples concerning the topological classification of pre-Hilbert spaces. Examples we deal with are of the form $Y(A) \times F_\alpha$ and $Y(A) \times G_\alpha$, where $Y(A) =$ span$(A)$ and $A$ is a linearly independent subset of $l^2$.

Fix a linearly independent arc $T = [0, 1]$ in $l^2$ such that $Y(A)$ is dense in $l^2$ for every infinite set $A \subseteq T$ (see [2, p. 267]). Since $Y(A)$ is contained in a $\sigma$-compact subspace of $l^2$, $Y(A)$ is a $Z_\sigma$-space provided it is infinite-dimensional (i.e., $A$ is infinite).

**Proposition 6.1.** *Let $A$ be any subset of $T$ and $\alpha \ge 1$. Then:*

(a) *$Y(A) \times F_\alpha$ (respectively, $Y(A) \times G_\alpha$) contains no closed copy of $l^2 \setminus G_{\alpha+1}$ (respectively, $l^2 \setminus F_{\alpha+1}$),*

(b) *$Y(A) \times F_\alpha$ (respectively, $Y(A) \times G_\alpha$) contains no closed copy of $l^2 \setminus F_\alpha$ (respectively, $l^2 \setminus G_\alpha$).*

First, in full detail, we consider the following particular case of part (b) with $\alpha = 1$ (as $l^2 \setminus F_1$ is a copy of $l^2$, see [2]).

**Lemma 6.2.** *For every subset $A$ of $T$, the space $Y(A) \times \Sigma$ contains no closed copy of $l^2$. In particular, $Y(A) \times \Sigma$ is homeomorphic neither to $\Lambda_\alpha$, $\alpha \ge 2$, nor to $\Omega_\alpha$, $\alpha \ge 1$.*

*Proof.* We apply the cross-section argument described in 4.3. For $k$ and $p \ge 1$, we write

$$C_k^p = \left\{ (t_1, t_2, \dots, t_k) \in T^k : t_1 \le t_2 \le \dots \le t_k, \ \|t_i - t_j\| \ge \frac{1}{p} \right\}$$

and

$$D_k^p = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_k) \in R^k : \frac{1}{p} \le |\lambda_i| \le p \ \text{ for all } i \right\}.$$

(The union $\bigcup_{k,p=1}^{\infty} C_k^p$ is a particular $\sigma$-compact cross-section for $T^k$.) The map $\chi_k$ given by $\chi_k((t_1, t_2, \dots, t_k), (\lambda_1, \lambda_2, \dots, \lambda_k)) = \lambda_1 t_1 + \lambda_2 t_2 + \dots + \lambda_k t_k$

is a homeomorphism of $C_k^p \times D_k^p$ onto $M_k^p \subset Y(T)$. Clearly,

$$M_k^p \cap Y(A) = \chi_k((C_k^p \cap A^k) \times D_k^p) = N_k^p$$

is a closed subset in $Y(A)$. Since $Y(A) = \{0\} \cup \bigcup_{k,p=1}^{\infty} N_k^p$, we get

$$Y(A) \times \Sigma = (\{0\} \times \Sigma) \cup \bigcup_{k,p=1}^{\infty} (N_k^p \times \Sigma).$$

Assume that $X$ being a copy of $l^2$ is contained as a closed subset of $Y(A) \times \Sigma$. Using a Baire category argument and the fact that no open set in $l^2$ is $\sigma$-compact we find indices $k$ and $p$ such that $N_k^p \times \Sigma$ contains an open subset $U$ of $X$. It follows that a copy $B$ of a closed ball in $l^2$ inscribed in $U$ is closed in $N_k^p \times \Sigma$. It is easy to see that there exists a connected set $K \subset A^k$ such that

$$B \subset \chi_k((K \cap C_k^p) \times D_k^p) \times \Sigma.$$

Each connected subset $K$ of $A^k$ is of the form $I_1 \times I_2 \times \cdots \times I_k$, where every $I_j$ is a connected component of $A$ (i.e., $I_j$ is an interval). Hence, $K \cap C_k^p$ is locally compact and so is $\chi_k\left((K \cap C_k^p) \times D_k^p\right)$. Finally, $B$, being a closed subset of a $\sigma$-compact space $\chi_k\left((K \cap C_k^p) \times D_k^p\right) \times \Sigma$, is itself $\sigma$-compact, a contradiction.

*Proof of* 6.1. Assume $Y(A) \times F_\alpha$ contains a closed copy $X$ of $l^2 \setminus G_{\alpha+1}$. Using the notation of the proof of 6.2, we have

$$Y(A) \times F_\alpha = (\{0\} \times F_\alpha) \cup \bigcup_{k,p=1}^{\infty} (N_k^p \times F_\alpha).$$

Since $X$ is a Baire space (see 4.9), there exist $k$ and $p$ and a closed set $P \subset N_k^p \times F_\alpha$ such that $P$ has nonempty interior in $X$ and $P$ is a copy of $B \setminus G_{\alpha+1}$ for some closed ball in $l^2$. According to 4.9, $P$ is connected. As in the proof of 6.2, we get

$$P \subset \chi_k((K \cap C_k^p) \times D_k^p) \times F_\alpha,$$

where $K \cap C_k^p$ is locally compact. Now, it follows that $\chi_k\left((K \cap C_k^p) \times D_k^p\right) \times F_\alpha \in \mathscr{A}_\alpha$; consequently $P \in \mathscr{A}_\alpha$. Since $X$ is homogeneous (see 4.9) and the interior of $P$ in $X$ is nonempty, $X$ is locally in the class $\mathscr{A}_\alpha$. The latter yields $X \in \mathscr{A}_\alpha$, contradicting $G_{\alpha+1} \in \mathscr{M}_{\alpha+1} \setminus \mathscr{A}_{\alpha+1}$.

All the remaining cases can be proved in the same way. (A minor change is needed for $G_1$; namely, $G_1$ must be represented as a countable union of complete metrizable spaces.)

**Corollary 6.3.** *For every subset $A$ of $T$ the spaces $Y(A) \times F_\alpha$ and $Y(A) \times G_\alpha$, $\alpha \geq 1$, are topologically distinct.*

*Proof.* By 6.1, $Y(A) \times F_\alpha$ contains no closed copy of $l^2 \setminus F_\alpha \in \mathscr{M}_\alpha$. Since $\mathscr{M}_\alpha \subset \mathscr{A}_\beta \cap \mathscr{M}_\beta$ for $\beta > \alpha$, the spaces $Y(A) \times F_\beta$ and $Y(A) \times G_\beta$ do contain such a copy. Also $Y(A) \times G_\alpha$ contains a closed copy of $l^2 \setminus F_\alpha$. As a consequence, we conclude that $Y(A) \times F_\alpha$ is not homeomorphic to $Y(A) \times F_\beta$ for $\alpha \neq \beta$ and that $Y(A) \times F_\alpha$ is not homeomorphic to $Y(A) \times G_\beta$ for $\beta \geq \alpha$. Analogously, we prove that $Y(A) \times G_\alpha$ is not homeomorphic to $Y(A) \times G_\beta$ for $\beta \neq \alpha$ and that $Y(A) \times G_\beta$ is not homeomorphic to $Y(A) \times F_\alpha$ for $\beta \leq \alpha$.

The same argument applies in the following

**Corollary 6.4.** *For every subset $A$ of $T$, the spaces $Y(A)\times F_\alpha$ and $Y(A)\times G_\alpha$ are homeomorphic neither to $F_\beta$ nor to $G_\beta$ for $\beta\neq\alpha$.*

**Corollary 6.5.** *We have:*

(a) *if $A\in\mathscr{A}_\alpha\setminus\mathscr{M}_\alpha$, then the spaces $F_\alpha$, $Y(A)\times F_\beta$, and $Y(A)\times G_\beta$ belong to $\mathscr{A}_\alpha\setminus\mathscr{M}_\alpha$ and are topologically distinct for $\beta<\alpha$,*
(b) *if $A\in\mathscr{M}_\alpha\setminus\mathscr{A}_\alpha$, then the spaces $G_\alpha$, $Y(A)\times G_\beta$, and $Y(A)\times F_\beta$ belong to $\mathscr{M}_\alpha\setminus\mathscr{A}_\alpha$ and are topologically distinct for $\beta<\alpha$,*
(c) *if $A\in\mathscr{P}_n\setminus\bigcup_{k<n}\mathscr{P}_k$, then the spaces $Y(A)\times F_\alpha$ and $Y(A)\times G_\alpha$ belong to $\mathscr{P}_n\setminus\bigcup_{k<n}\mathscr{P}_k$ and are topologically distinct,*
(d) *if $A\notin\bigcup_{n=1}^\infty\mathscr{P}_n$, then the spaces $Y(A)\times F_\alpha$ and $Y(A)\times G_\alpha$ do not belong to $\bigcup_{n=1}^\infty\mathscr{P}_n$ and are topologically distinct.*

Corollary 6.5 is a direct consequence of 6.3 and 6.4 and the following fact, which seems to be well known; however we could not find it formulated in such a generality in literature.

**Lemma 6.6.** *We have:*

(a) *if $A\in\mathscr{A}_\alpha\setminus\mathscr{M}_\alpha$, then $Y(A)\in\mathscr{A}_\alpha\setminus\mathscr{M}_\alpha$, $\alpha\geq 1$,*
(b) *if $A\in\mathscr{M}_\alpha\setminus\mathscr{A}_\alpha$, then $Y(A)\in\mathscr{M}_\alpha\setminus\mathscr{A}_\alpha$, $\alpha\geq 2$,*
(c) *if $A\in\mathscr{P}_n\setminus\bigcup_{k<n}\mathscr{P}_k$, then $Y(A)\in\mathscr{P}_n\setminus\bigcup_{k<n}\mathscr{P}_k$, $n\geq 1$,*
(d) *if $A\notin\bigcup_{n=1}^\infty\mathscr{P}_n$, then $Y(A)\notin\bigcup_{n=1}^\infty\mathscr{P}_n$.*

*Proof*. Since $A$ is closed in $Y(A)$, $A\notin\mathscr{L}$ implies $Y(A)\notin\mathscr{L}$ provided $\mathscr{L}$ is closed with respect to closed subsets. Therefore, it suffices to show that $A\in\mathscr{L}$ implies $Y(A)\in\mathscr{L}$, where $\mathscr{L}=\mathscr{A}_\alpha, \mathscr{M}_\alpha$ and $\bigcup_{k<n}\mathscr{P}_k$. The case $\mathscr{A}_\alpha$ and $\bigcup_{k<n}\mathscr{P}_n$ is a result of Klee [2, p. 272]. Let $A\in\mathscr{M}_\alpha$ and $\alpha\geq 2$. Represent $A=\bigcap_{n=1}^\infty A_n$, $A_n\in\mathscr{A}_{\beta_n}$ for $\beta_n<\alpha$, and employ $Y(A_n)\in\mathscr{A}_{\beta_n}$ to conclude that $Y(A)=\bigcap_{n=1}^\infty Y(A_n)\in\mathscr{M}_\alpha$.

*Remark* 6.7. Corollary 6.5(a) and (b) (see also 6.2) provide a negative answer to the question of whether a pre-Hilbert space that contains a Hilbert cube and is of the exact Borelian class of order $\alpha$ must be homeomorphic to either $F_\alpha$ or $G_\alpha$, $\alpha\geq 2$. The answer to this question is "yes" for $\mathscr{A}_1$.

*Remark* 6.8. From 6.5(c) it follows that each class $\mathscr{P}_n\setminus\bigcup_{k<n}\mathscr{P}_k$ contains uncountably many topologically distinct pre-Hilbert spaces that are $Z_\sigma$-spaces.

*Remark* 6.9. Each class $\mathscr{A}_\alpha\setminus\mathscr{M}_\alpha$, $\alpha\geq 1$, and $\mathscr{M}_\alpha\setminus\mathscr{A}_\alpha$, $\alpha\geq 2$, contains uncountably many topologically distinct pre-Hilbert spaces that are $Z_\sigma$-spaces. To show this, take $A\in\mathscr{A}_\alpha\setminus\mathscr{M}_\alpha$ (respectively, $A\in\mathscr{M}_\alpha\setminus\mathscr{A}_\alpha$) and repeat Henderson and Pełczyński's argument to the spaces $Y(A)\times X$, $X\in\mathscr{X}$, where $\mathscr{X}$ is that of [2, p. 282].

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