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ANDRÉ PERMUTATIONS, LEXICOGRAPHIC SHELLABILITY AND THE *cd*-INDEX OF A CONVEX POLYTOPE

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ABSTRACT. The cd-index of a polytope was introduced by Fine; it is an integer valued noncommutative polynomial obtained from the flag-vector. A result of Bayer and Fine states that for any integer "flag-vector," the existence of the cd-index is equivalent to the holding of the generalized Dehn-Sommerville equations of Bayer and Billera for the flag-vector. The coefficients of the cd-index are conjectured to be nonnegative.

We show a connection between the cd-index of a polytope $\mathscr{P}$ and any CL-shelling of the lattice of faces of $\mathscr{P}$; this enables us to prove that each André polynomial of Foata and Schützenberger is the cd-index of a simplex. The combinatorial interpretation of this cd-index can be extended to cubes, simplicial polytopes, and some other classes (which implies that the cd-index has nonnegative coefficients for these polytopes). In particular, we show that any polytope of dimension five or less has a positive cd-index.

1. INTRODUCTION

The combinatorial properties of a polytope $\mathscr{P}$ are the properties of the lattice of faces of the polytope $\mathbf{L}(\mathscr{P})$; properties that are the same for all polytopes with the same lattice of faces are called *combinatorial invariants.* (For background on polytopes, see [8, 13, 14].) The *cd*-index is an important new combinatorial invariant, introduced by Fine, which is related to the flag-vector, which in turn is a generalization of the f-vector. Both the flag- and f-vectors can be defined for any ranked poset, not just the lattice of faces of a polytope. (For background on posets and lattices, see [4, 20].)

The f-vector of a n-polytope (or, in general, of any ranked poset of rank $n+1$) is defined to be $(f_{-1}, \dots, f_{n-1})$, where f_{i-1} is the number of faces of rank i (that is, in the polytope case, the number of faces of dimension $i-1$). Hence, for polytopes, $f_{-1} = 1$ (counting the empty face), f_0 is the number of vertices of $\mathscr{P}$, and f_{n-1} is the number of facets. (Notice that this notation has a subscript that is off-by-one from that often used for the f-vector of a poset.)

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A lot of work has been done on the *f*-vector of *simplicial* polytopes (which are polytopes such that each facet is a simplex). One of the first such results is the *Dehn-Sommerville equations*; these are best stated in terms of the *h*-vector $(h_0, \dots, h_n)$, which is defined by

$$\sum_{i=0}^{n} h_i x^{n-i} = \sum_{i=0}^{n} f_{i-1}(x-1)^{n-i} .$$

The Dehn-Sommerville equations then are just $h_i = h_{n-i}$ for each $0 \le i \le n$. Since each of the h_i's is a linear combination of the f_{i-1}'s, the Dehn-Sommerville equations are linear relations among the f_{i-1}, and in fact are the most general such relations that hold for the *f*-vector of every simplicial polytope (that is, they imply all other such linear relations).

Further work by many mathematicians culminated when Stanley (only if) and Billera and Lee (if) proved a characterization of the *f*-vector of any simplicial polytope (which was originally conjectured by McMullen). The result states that an integer vector $(h_0, \dots, h_n)$ is the *h*-vector of a simplicial polytope of dimension n if and only if the Dehn-Sommerville equations hold and the vector $(h_0, h_1 - h_0, \dots, h_{\lfloor n/2 \rfloor} - h_{\lfloor n/2 \rfloor - 1})$ is an M-vector, which means that it is the *f*-vector of a multicomplex (a numeric condition for this is known). The proofs of both directions of this result are quite difficult, and Stanley's direction uses techniques from commutative algebra and algebraic geometry. For details, see [3, 19].

Now, the flag-vector of a ranked poset $\mathbf{L}$ (with least element $\hat{0}$ and greatest element $\hat{1}$) of rank $n+1$ is an integer vector indexed by subsets of $[n] := \{1, \dots, n\}$, denoted $(f_S \mid S \subseteq [n])$. (In the case of a polytope, $\mathbf{L}$ is the lattice of faces of the polytope, and n is the rank of the facets, which is thus the dimension of the polytope.) Each f_S counts the number of chains in $\mathbf{L}$ of the form $\{\hat{0} < x_1 < x_2 < \cdots < x_k < \hat{1}\}$ such that $\{\rho(x_i) \mid 1 \le i \le k\} = S$.

For simplicial polytopes, the characterization of Billera, Lee, and Stanley of the *f*-vector gives a characterization of the flag-vector (since the flag-vector of each simplex of rank r, Δ^{r-1}, is known, so the value of $f_S(\mathscr{P})$ is exactly $f_{\max S}(\mathscr{P}) \cdot f_{S \setminus \max S}(\Delta^{\max S - 1})$). Not much is known about the flag-vector of arbitrary polytopes, and it seems unlikely that a conjecture analogous to the McMullen-Stanley-Billera-Lee result for simplicial *f*-vectors will be formulated soon for the flag-vectors of arbitrary polytopes, let alone proven. However, Bayer and Billera proved that certain linear equations, called the *generalized Dehn-Sommerville equations*, hold for the flag-vector of *any* convex polytope, and that these are the most general linear equations that hold for all convex polytopes. The generalized Dehn-Sommerville equations are:

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \dot\cup j}(\mathbf{P}) = (1 - (-1)^{k-i-1}) f_S(\mathbf{P})$$

whenever $S \subseteq [n]$, $i < k$, $\{i, k\} \subseteq S \dot\cup \{0, n+1\}$

$\{j | i < j < k\} \cap S = \varnothing$, where n is the dimension of the polytope in question. As with the *f*-vector, it is convenient to define the β-vector or *flag h-vector*, by writing

$$\beta_S = \sum_{T \subseteq S} (-1)^{\#(S \setminus T)} f_T$$

for each $S \subseteq [n]$. (This notation is taken from [2], as, unfortunately, h_S is used for something else in this field.) However, while the generalized Dehn-Sommerville equations imply that $\beta_S = \beta_{[n] \setminus S}$, (which is reminiscent of $h_i = h_{n-i}$), the converse is not the case, and the general statement of generalized Dehn-Sommerville is not simplified by merely restating the equations in terms of the flag *h*-vector.

However, if we form the noncommuting polynomial $\sum_{S \subseteq [n]} \beta_S w(S)$, where $w(S)$ is a word of length n in the variables a and b such that the ith letter of $w(S)$ is b if $i \in S$ and a otherwise, then Bayer and Fine showed that this sum can be rewritten in terms of $\mathsf{c} = \mathsf{a} + \mathsf{b}$ and $\mathsf{d} = \mathsf{ab} + \mathsf{ba}$ if and only if the generalized Dehn-Sommerville equations hold for the flag-vector (f_S); this rewriting is called the *cd*-index, which was first introduced by Fine. These ideas were first published by Bayer and Klapper in [2].

Fine originally made the following conjecture (which was strengthened by Bayer and Klapper in [2] to all CW-spheres—see that paper for details):

Conjecture 1.1 (Fine). *The coefficients of the cd-index of any polytope are nonnegative.*

Unfortunately, the *cd*-index as we have defined it is a very mysterious object. So in addition to attempting to prove the Bayer-Klapper conjecture, we would like to find some combinatorial interpretation of the coefficients, that is, some set of objects for each cd-word w and each polytope $\mathscr{P}$, such that the coefficient of w in the *cd*-index of $\mathscr{P}$ is the cardinality of the given set of objects. In addition to proving the conjecture, such an interpretation might provide additional insight into the *cd*-index.

We do not find such an interpretation for all polytopes, but for several classes of polytopes, including the simplicial polytopes, we do; for each word, there is a collection of blocks in a partition of the maximal chains of the lattice of faces of $\mathscr{P}$ which has cardinality equal to the coefficient of the word in the *cd*-index. This interpretation relies on the notion of a CL-shelling of the chains of a lattice, which is a way of labeling the maximal chains of the lattice with integer vectors, one integer per covering relation. This concept is due to Björner and Wachs, who showed that the lattice of faces of every polytope is CL-shellable (a result which in turn requires the famous result of Bruggesser and Mani that every polytope is shellable). Then the flag *h*-vector of a polytope can be computed from the labels of any CL-shelling of the lattice of faces of the polytope; in fact, β_S is the number of maximal chains of $\mathscr{P}$ that have descents at exactly the rank levels specified by S. Hence, each chain gives rise to a word in a and b in the sum $\sum \beta_S w(S)$, so a partition of these chains such that the chains in each block have ab-words summing to some cd-word gives the combinatorial interpretation desired.

In the case of the simplex and octahedron (in each dimension), such a partition exists, since the lattices of faces of these polytopes are isomorphic to the boolean lattice and the lattice of signed sets respectively. Hence, we can produce a CL-labeling by labeling each covering relation $A \lessdot B$ by the unique element of $B \backslash A$. This labeling is particularly nice, and the collection of labels of all chains of the boolean lattice (resp., the lattice of signed sets) is exactly the set of (signed) permutations of an n-set (where n is one more than the dimension of the simplex, or the dimension of the octahedron; in the latter case, 0 must be added to each signed permutation).

This motivates a study of (signed) permutations, in which the work of Foata and Schützenberger on André permutations is invaluable. An *André permutation* of a totally ordered set X is a permutation without double descents satisfying an additional technical property. We extend this notion to the case of signed permutations, where we must add additional even more technical properties (for instance, we require that if $m = \max X$, then $\overline{m} = -m$ appears in the signed permutation). Foata and Schützenberger studied André (signed) permutations because they are in bijection with alternating permutations of the same set (and we extend this to the signed case as well). The main fact that makes André (signed) permutations useful is that if $\alpha_1 \cdots \alpha_n$ is a permutation with $\alpha_k = \min\{\alpha_1, \ldots, \alpha_n\}$, then it is an André permutation if and only if both $\alpha_1 \cdots \alpha_k$ and $\alpha_{k+1} \cdots \alpha_n$ are André permutations. In the signed case, if

$$\alpha_k = \min\{\alpha_1, \ldots, \alpha_n\} = -\max\{|\alpha_1|, \ldots, |\alpha_n|\},$$

then $\alpha_1 \cdots \alpha_n 0$ is an André signed permutation if and only if $\alpha_1 \cdots \alpha_k$ is an André unsigned permutation and $\alpha_{k+1} \cdots \alpha_n 0$ is an André signed permutation. (This corresponds to the fact that the downward intervals in the lattice of signed sets are isomorphic to boolean lattices).

We inductively construct a partition of the maximal chains of the boolean lattice and the lattice of signed sets, labeled as above. There is one André (signed) permutation per block, and each block sums to a word in c and d that can be read off of the ascent-descent structure of that André (signed) permutation. From this fact, it follows that the *cd*-index of the simplex is exactly one of Foata and Schützenberger's André polynomials (with the identification of their s, t and the c, d of the *cd*-index). Since the simplex is self-dual and since the *cd*-index of $\mathscr{P}$ can be derived from that of $\mathscr{P}^*$ by simply reversing each word, we immediately get their result that each André polynomial is invariant under the reversal of each word.

The partition just constructed has some very nice properties. For instance, each block Π_i that corresponds to (that is, whose chains' ab-words sum to) a cd-word w_i beginning with the letter d has all of its chains pass through a single atom of the lattice of faces of $\mathscr{P}$. A similar but more complicated fact holds if w_i begins with the letter c. Since every ordering of the atoms of a simplex gives rise to a CL-labeling as before, and hence to a partition with these nice properties, we can use these properties to extend the partition to the lattice of faces of any simple polytope; since simple polytopes are dual to simplicial polytopes, simplicial polytopes also have nonnegative *cd*-indexes. Similarly, using similar properties and a recursion for the *cd*-index for polytopes due to Bayer and Klapper, we show that any polytope with dimension less than or equal to five has a nonnegative *cd*-index.

2. The *cd*-index

Definition 2.1. For any ranked poset $\mathbf{P}$ with $\hat{0}$ and $\hat{1}$ of rank $n+1$ and for each subset of $[n]$, $S = \{i_1, i_2, \dots, i_s\}$ such that $i_1 < \cdots < i_s$, let $f_S = f_S(\mathbf{P})$ be the number of chains of $\mathbf{P}$, $\hat{0} < x_{i_1} < x_{i_2} < \cdots < x_{i_s} < \hat{1}$ such that $\rho(x_{i_j}) = i_j$. The *flag-vector* of $\mathbf{P}$ is the vector $(f_S \mid S \subseteq [n])$ in $\mathbb{Z}^{2^n}$ with basis elements indexed by $(S \mid S \subseteq [n])$.

If $\mathbf{P}$ is as in the definition, and $S \subseteq [n]$, then $\mathbf{P}_S$ is the poset with base set $\{x \in \mathbf{P} \mid \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$ with the same relation; this is called the *rank selection* of $\mathbf{P}$. The chains in this definition are exactly the flags (maximal chains) of the rank selected posets $\mathbf{P}_S$, which is why (f_S) is called the flag-vector.

Following Stanley, we say a ranked poset with $\hat{0}$ and $\hat{1}$, $\mathbf{P}$, is *Eulerian* whenever its Möbius function satisfies $\mu_{\mathbf{P}}(x, y) = (-1)^{\rho(x)-\rho(y)} = (-1)^{\rho([x,y])}$ for all $x \leq y$ in $\mathbf{P}$. For any $F \leq G$ in the face lattice, the interval $[F, G]$ is the face lattice of a polytope $\mathscr{P}_{[F,G]}$ of rank $\rho(G) - \rho(F)$, and hence of dimension $\rho(G) - \rho(F) - 1$; furthermore, it is well known that $\mu_{\mathscr{P}}(F, G) = \tilde{\chi}(\mathscr{P}_{[F,G]}) = (-1)^{\rho(\mathscr{P}_{[F,G]})}$. Hence for every polytope $\mathscr{P}$, $\mathbf{L}(\mathscr{P})$ is Eulerian; this motivated the definition of Eulerian.

Theorem 2.2 (Bayer-Billera, [1]). *If* $\mathbf{P}$ *is a rank* $n+1$ *Eulerian poset, then*

(2.1)
$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \dot{\cup} j}(\mathbf{P}) = (1 - (-1)^{k-i-1}) f_S(\mathbf{P})$$
whenever $S \subseteq [n]$, $i < k$, $\{i, k\} \subseteq S \dot{\cup} \{0, n+1\}$ *and* $\{j \mid i < j < k\} \cap S = \varnothing$

Furthermore, these equations (along with $f_{\varnothing} = 1$*) imply all linear equations that hold for the flag-vectors of all n-polytopes (and hence all Eulerian posets* $\mathbf{P}$*). The dimension of the affine span of all flag-vectors of n-polytopes is* $e_n - 1$*, where* e_n *is the nth Fibonacci number (defined by* $e_0 = e_1 = 1$ *and* $e_i = e_{i-1} + e_{i-2}$*).*

Definition 2.3. The *flag h-vector* or *β-vector* of a poset $\mathbf{P}$ of rank $n+1$ is $(\beta_S \mid S \subseteq [n])$, where
$$\beta_S := \sum_{T \subseteq S} (-1)^{\#(S \setminus T)} f_T .$$

Note: the flag h-vector is to the flag-vector as the f-vector is to the h-vector; as with the h-vector, use of the flag h-vector makes stating certain things clearer. (The generalized Dehn-Sommerville equations are unfortunately not one of these things; (2.1) becomes

(2.2)
$$\sum_{T \subseteq S} \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \beta_{T \dot{\cup} j}(\mathbf{P}) = \tfrac{1}{2}(1 - (-1)^{k-i-1}) \sum_{T \subseteq S} \beta_T(\mathbf{P})$$
whenever $S \subseteq [n]$, $i < k$, $\{i, k\} \subseteq S \dot{\cup} \{0, n+1\}$, and $\{j \mid i < j < k\} \cap S = \varnothing$.

which is not really very nice. However, Proposition 2.9 gives a nice consequence of the generalized Dehn-Sommerville equations, as does Theorem 2.4.)

As with the *h*-vector and the *f*-vector, the flag *h*-vector can be converted back into the flag-vector:

$$f_S = \sum_{T \subseteq S} \beta_S .$$

(This is a simple inclusion-exclusion argument.)

To each subset of $[n]$, we can associate a *word* in the letters a and b of length n, *via* $w_0(\varnothing) := w(\varnothing) := 1$ and $w_n(S) := w(S) := w_1 w_2 \dots w_n$, where

$$w_i := \begin{cases} \mathsf{a} & \text{if } i \notin S; \\ \mathsf{b} & \text{if } i \in S. \end{cases}$$

So for instance, if $n = 5$, then $w(\{1, 3, 4\}) = \mathsf{babba}$.

Using $w(S)$, we can associate a noncommuting polynomial $ab(\mathscr{P})$ (in the (noncommuting) variables a and b) to the flag *h*-vector of $\mathscr{P}$:

$$ab(\mathscr{P}) := \sum_{S \subseteq [n]} \beta_S w(S) .$$

We call this the *ab*-index of $\mathscr{P}$.

Theorem 2.4 (Bayer-Klapper, [2]). *The generalized Dehn-Sommerville equations holding for an integer vector* $(\beta_S \mid S \subseteq [n])$ *is equivalent to the existence of a unique noncommuting polynomial* $f(\mathsf{c}, \mathsf{d})$ *in* c *and* d *with integral coefficients such that* $\sum_{S \subseteq [n]} \beta_S w(S) = f(\mathsf{a} + \mathsf{b}, \mathsf{ab} + \mathsf{ba})$.

Definition 2.5. Given the notation of Theorem 2.4, the *cd*-index of $\mathbf{P}$ is defined to be $f(\mathsf{c}, \mathsf{d})$ and will be denoted $cd(\mathbf{P})$; the *cd*-index of a polytope $\mathscr{P}$ is defined to be $cd(\mathbf{L}(\mathscr{P}))$.

Corollary 2.6. *Every polytope has a cd-index.*

For example, consider Δ^2, which is a triangle. Then $f_\varnothing = 1$, $f_{\{1\}} = f_{\{2\}} = 3$ and $f_{\{1,2\}} = 6$. Hence, $\beta_\varnothing = 1$, $\beta_{\{1\}} = \beta_{\{2\}} = 2$ and $\beta_{\{1,2\}} = 6 - 3 - 3 + 1 = 1$. The *cd*-index of Δ^2 is thus $\mathsf{c}^2 + \mathsf{d}$.

Let $\operatorname{rev}(v_1 \cdots v_n) = v_n \cdots v_1$ for any word $v_1 \cdots v_n$ and extend rev( ) linearly to sums of words. Then we have the following fact (from [2]):

Lemma 2.7. $cd(\mathscr{P}) = \operatorname{rev}(cd(\mathscr{P}^*))$.

Since the existence of the *cd*-index is equivalent to the holding of the generalized Dehn-Sommerville equations, studying the *cd*-index is a good way to study these equations (especially since the equations are so messy and unpleasant). However, the *cd*-index appears to be a very subtle invariant; it is not even easy to calculate the *cd*-index of a simplex without some sort of general theory. (In fact, the author spent several weeks trying to do this before Dr. Stanley pointed out the paper [11].) Therefore, it would be nice to get some sort of combinatorial interpretation for the *cd*-index. In connection with this, we recall the following conjecture:

Conjecture 2.8 (Fine). *The coefficients of the cd-index of any polytope are nonnegative.*

If this holds, it might be possible to find some set of combinatorial objects that the coefficients of the *cd*-index count; conversely, if we could find such a set of objects, we would have proved the conjecture.

At this point, we note the following result, which is essentially [20, Corollary 3.14.6] for Eulerian posets and due to Bayer and Billera [1] in this context:

Proposition 2.9. *If the integer vector* $(\beta_S \mid S \subseteq [n])$ *satisfies the generalized Dehn-Sommerville equations, then* $\beta_S = \beta_{[n]\setminus S}$.

Note that the equations $\beta_S = \beta_{[n]\setminus S}$ do not imply all of the generalized Dehn-Sommerville equations, as we will see. Consider the case where $n = 3$. It is straightforward to calculate that the generalized Dehn-Sommerville equations are

$$\beta_\varnothing = 1, \quad \beta_{\{1\}} = \beta_{\{2,3\}}, \quad \beta_{\{2\}} = \beta_{\{1,3\}}, \quad \beta_{\{3\}} = \beta_{\{1,2\}}$$

and, finally

$$\beta_{\{2\}} = \beta_{\{1\}} + \beta_{\{3\}} - 1$$

which does not follow from the other four.

3. Poset labelings

Björner [5] and Björner and Wachs [7] introduced notions of shellability for posets which we require. For completeness, we give the definitions and state a few results here; see the papers just referenced for details.

Let $\mathscr{E}_*(\mathbf{P})$ be the rooted edges of a poset $\mathbf{P}$, that is, $\{(\underline{r}, x, y) \mid \underline{r} = \{r_0 \lessdot \cdots \lessdot r_l = x\} \in \mathscr{M}\mathscr{C}([\hat{0}, x]),\ x \lessdot y\}$, where $\mathscr{M}\mathscr{C}(\mathbf{Q})$ denotes the set of maximal chains of any poset $\mathbf{Q}$. Then $\lambda\colon \mathscr{E}_*(\mathbf{P}) \to \mathbb{Z}$ is a *chain-edge labeling* of $\mathbf{P}$, and we write $\lambda_{\underline{r}}(x, y)$ for $\lambda(\underline{r}, x, y)$. A *rooted interval* ${}_{\underline{r}}[x, y]$ is an interval $[x, y]$ together with a maximal chain $\underline{r}$ of $[\hat{0}, x]$, which is called the *root* of the rooted interval.

A (maximal, saturated) chain of ${}_{\underline{r}}[x, y]$ is a (maximal, saturated) chain $\underline{c}$ of the interval $[x, y]$ along with the root. For a chain-edge labeling λ, and a maximal chain $\underline{c}$ of ${}_{\underline{r}}[x, y]$, we write $\lambda_{\underline{r}}(\underline{c})$ for $(\lambda_{\underline{r}}(c_0, c_1),\ \lambda_{\underline{r}\lessdot\{c_1\}}(c_1, c_2), \ldots, \lambda_{\underline{r}\lessdot\underline{c}}(c_{l-1}, c_l))$ (in the last term, we slightly abuse the notation; the root is actually $\underline{r} \lessdot \{c_0 \lessdot c_1 \lessdot \cdots \lessdot c_{l-1}\}$). We say that a rooted chain $\underline{c}$ of ${}_{\underline{r}}[x, y]$ is (λ-)*increasing* whenever $\lambda_{\underline{r}}(\underline{c})$ is, and we have the lexicographic order $<^{\lambda}_{\text{lex}}$ on the chains of $\mathbf{P}$ ($\mathbf{a} <_{\text{lex}} \mathbf{b}$ iff for some j, $a_i = b_i$ for $i < j$ and $a_j < b_j$).

Definition 3.1. A chain-edge labeling $\lambda\colon \mathscr{E}_*(\mathbf{P}) \to \mathbf{Q}$ is a *CL-labeling* of $\mathbf{P}$ (for "*c*hain-edge *l*exicographic") iff for each *rooted* interval ${}_{\underline{r}}[x, y]$ of $\mathbf{P}$, there is a unique lexicographically first maximal chain, and this chain is the unique increasing chain of ${}_{\underline{r}}[x, y]$. A poset $\mathbf{P}$ is said to be *CL-shellable* or *CL-labelable* if there is some *CL*-labeling of $\mathbf{P}$.

If $\lambda_{\underline{r}}(x, y)$ is always independent of the chain $\underline{r}$, then we say λ is an *EL*-labeling and that the poset is *EL*-shellable.

A useful reformulation of *CL*-shellability was introduced by Björner and Wachs in [7]; for this, recall that the *atom*s of $\mathbf{P}$ are $a \in \mathbf{P}$ such that $a \gtrdot \hat{0}$.

Definition 3.2. An ordering $\{a_1, a_2, \ldots, a_t\}$ of the atoms of $\mathbf{P}$ is a *recursive atom order* (and we say $\mathbf{P}$ *admits a recursive atom order*) whenever either $\mathbf{P}$ is of rank 1, or both

(1) for all $1 \le j \le t$, the interval $[a_j, \hat{1}]$ admits a recursive atom order in which those atoms of $[a_j, \hat{1}]$ which cover some a_i with $i < j$ come first, and
(2) for all $1 \le i < j \le t$, if $a_i, a_j < y \in \mathbf{P}$, then there exists a $1 \le k < j$ and $z \in \mathbf{P}$ such that $a_k, a_j \lessdot z \le y$.

A shelling order of a polytope (which we will define in a moment) is a recursive atom order of the lattice of faces of the dual polytope. This motivated the definition of recursive atom order.

Algorithm 3.3. Given a recursive atom order of a poset $\mathbf{P}$, we construct a CL-labeling of $\mathbf{P}$.

Let $\{a_1, a_2, \dots, a_t\}$ be an recursive atom order, and pick any integer labeling $\lambda_{\{\hat{0}\}}$ of the pairs $(\hat{0}, a_i)$ such that $\lambda_{\{\hat{0}\}}(\hat{0}, a_i) < \lambda_{\{\hat{0}\}}(\hat{0}, a_j)$ for all $i < j$. Now, for each atom a_j, let $F(a_j)$ be the set of all atoms of $[a_j, \hat{1}]$ that cover some a_i with $i < j$; this is the set of atoms of $[a_j, \hat{1}]$ that must come first in any recursive atom order of $[a_j, \hat{1}]$. Pick such a recursive atom order (which exists by definition), say $b_1, b_2, \dots, b_s$, and extend λ to the bottom edges $\{a_j \lessdot b_k\}$ of $[a_j, \hat{1}]$ such that $\lambda_{\{\hat{0}\lessdot a_j\}}(a_j, b_k) < \lambda_{\{\hat{0}\lessdot a_j\}}(a_j, b_l)$ for $k < l$, and

$$b_k \in F(a_j) \Rightarrow \lambda_{\{\hat{0}\lessdot a_j\}}(a_j, b_k) < \lambda_{\{\hat{0}\}}(\hat{0}, a_j),$$
$$b_k \notin F(a_j) \Rightarrow \lambda_{\{\hat{0}\lessdot a_j\}}(a_j, b_k) > \lambda_{\{\hat{0}\}}(\hat{0}, a_j);$$

clearly this is possible. Continue inductively to create the required labeling.

See [7, Theorem 3.2] for a proof that this is, indeed, a CL-labeling. The same proof demonstrates how to go from a CL-labeling to a recursive atom order.

Recall that a *polyhedral complex* is a finite set Δ of polytopes in $\mathbb{R}^n$ such that a face of any element is again an element and the intersection of any two elements is a face of both (and hence an element). One example of a polyhedral complex is any simplicial complex, since a simplex is a polytope. The example we are most concerned with is $\partial\mathscr{P}$, which is the set of all faces of $\mathscr{P}$ except $\mathscr{P}$ itself. Hence, we say that the *facet*s of a polyhedral complex Δ are the maximal elements of Δ, so the facets of $\partial\mathscr{P}$ in the polyhedral complex sense are the same as the facets of $\mathscr{P}$ in the polytope sense. If all of the facets have the same dimension n, we say Δ is a n-complex, so if $\mathscr{P}$ is a n-polytope, $\partial\mathscr{P}$ is a $(n-1)$-complex homeomorphic to a sphere, and the complex $\overline{\mathscr{P}} := \partial\mathscr{P} \cup \{\mathscr{P}\}$ is a n-complex. A simplicial complex is a n-complex whenever it is a pure simplicial complex.

An ordering of the facets $F_1, F_2, \dots, F_t$ of a n-complex is called a *shelling order* if either $n = 0$ or $\overline{F_j} \cap \bigcup_{i=1}^{j-1} \overline{F_i}$ is a nonempty $(n-1)$-complex for all $1 < j \le t$ and the following recursive condition holds: there is a shelling of each ∂F_j, $1 \le j \le t$, in which the facets of $\overline{F_j} \cap \bigcup_{i=1}^{j-1} \overline{F_i}$ come first. (This definition, due to Björner in [5], seems the most useful in this context. Note that it implies that a recursive atom order of a polytope is equivalent to a shelling order for the dual polytope, and vice versa.) We say that a n-

complex is *shellable* if there is a shelling order on the facets, and that $\mathscr{P}$ is shellable whenever $\partial\mathscr{P}$ is. We form the lattice of faces of Δ, $\mathbf{L}(\Delta)$, by taking the union of the lattice of faces of each facet of Δ and adding a $\hat{1}$; similarly, we form $\mathbf{L}^*(\Delta)$ by taking the dual to $\mathbf{L}(\Delta)$. Hence, $\mathbf{L}(\partial\mathscr{P})$ is the same as $\mathbf{L}(\mathscr{P})$.

Theorem 3.4 (Björner-Wachs, [7, Theorem 4.3]). *Let Δ be a n-complex. Then Δ is shellable iff $\mathbf{L}^*(\Delta)$ has a recursive atom order.*

Corollary 3.5. *Let Δ be a n-complex. Then Δ is shellable iff $\mathbf{L}^*(\Delta)$ is CL-shellable.*

Note that we have slightly unfortunate terminology here: an n-complex is shellable iff the *dual* to its lattice of faces is shellable, in which case we say the lattice of faces is *co CL-shellable.* So we can say that an n-complex Δ is shellable if and only if its lattice of faces is co CL-shellable.

Now we have the following important result.

Theorem 3.6 (Bruggesser-Mani, [9]). *Every polytope $\mathscr{P}$ is shellable.*

The idea of this is fairly straightforward. We imagine that we have a polytope floating in space, and consider launching a "spacecraft" off of one of the facets on a suitable straight line off to infinity. The "spacecraft" will then return from infinity along the same straight line, but on the other side of the polytope. The order in which the facets of the polytope become visible (on the way out) and invisible (on the way in) gives a shelling order for the polytope. For details and proofs, which are not so straightforward, see [9].

In a CL-labeled poset $\mathbf{P}$ of rank $n+1$, the *descent set* of a maximal chain $\underline{c} = \{\hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_{n+1} = \hat{1}\}$ of $\mathbf{P}$ is the set of all $i \in [n]$ such that $\lambda_{\underline{c}}(c_{i-1}, c_i) > \lambda_{\underline{c}}(c_i, c_{i+1})$; this set is denoted $D_\lambda(\underline{c}) = D(\underline{c})$, and each $i \in D(\underline{c})$ is called a *descent* of the chain $\underline{c}$. Recall the definition (in §2) of the map w_n from subsets of $[n]$ to words of length n in the letters a and b, and, as before, denote the set of maximal chains of a ranked poset $\mathbf{P}$ by $\mathscr{M}\mathscr{C}(\mathbf{P})$. Then we have the following, which is basically [20, Theorem 3.13.2]. (See also [6].)

Theorem 3.7. *The ab-index $\sum \beta_{\{S\}} \cdot w(S)$ of any polytope $\mathscr{P}$ with a CL-shelling λ is equal to the sum $\sum w(D(\underline{c}))$, where the sum is taken over all $\underline{c} \in \mathscr{M}\mathscr{C}(\mathbf{L}(\mathscr{P}))$.*

So we have a combinatorial interpretation of the flag h-vector (β_S) for CL-shellable posets $\mathbf{P}$, since $\beta_S = \#\{\underline{c} \in \mathscr{M}\mathscr{C}(\mathscr{P}) \mid D(\underline{c}) = S\}$. Of course, this immediately implies that each β_S is nonnegative for every CL-shellable Eulerian poset (which was well known).

Each word in $\{\mathsf{c}, \mathsf{d}\}$ can be thought of as a set of words in $\{\mathsf{a}, \mathsf{b}\}$ of the same degree by expanding $\mathsf{c} = \mathsf{a} + \mathsf{b}$ and $\mathsf{d} = \mathsf{ab} + \mathsf{ba}$, so we might hope to construct a partition of $\mathscr{M}\mathscr{C}(\mathbf{L}(\mathscr{P}))$ so that each block B of the partition has $\sum_{\underline{c} \in B} w(D(\underline{c}))$ summing to a word in c and d. This would constitute a combinatorial interpretation of the *cd*-index.

4. (Signed) permutations and EL-shellings

Recall that the n-simplex Δ^n is the convex hull of $n+1$ points in general position in $\mathbb{R}^n$. The *boolean lattice* of rank n, $\mathbf{B}_n$, is the poset of subsets of $[n]$ ordered by inclusion. It is well known that $\mathbf{L}(\Delta^n) \cong \mathbf{B}_{n+1}$. The *$n$-octahedron*, $\mathscr{O}^n$, is the convex hull of the points $\{\varepsilon_i, \overline{\varepsilon}_i\}_{i=1}^n$ in $\mathbb{R}^n$, where $\varepsilon_i = (\delta_{1,i}, \delta_{2,i}, \dots, \delta_{n,i})$, $\overline{\varepsilon}_i = -\varepsilon_i$, and $\delta_{i,j}$ is the Kronecker delta (1 if $i=j$ and 0 otherwise). The *lattice of signed sets* of rank n, $\mathbf{B}_n^{\pm}$, is the poset on base set

$$\{A = (A^+, A^-)\}\{A^+ \cap A^- = \varnothing \text{ and } A^+ \cup A^- \subseteq [n]\} \cup \{\hat{1}\},$$

ordered by $\hat{1} > (A^+, A^-)$ for all pairs A and $(A^+, A^-) \leq (B^+, B^-)$ iff $A^+ \subseteq B^+$ and $A^- \subseteq B^-$. (Note: in many other papers, such as [15], the order used for the lattice of signed sets is the dual of the one given here.) We can think of these pairs as sets where elements of A^+ have a positive sign and A^- have a negative sign; hence the name signed sets. For instance, the signed set $(\{2, 5\}, \{1, 3\})$ can be thought of as $\{-1, +2, -3, +5\}$. Once again, it is well known that $\mathbf{L}(\mathscr{O}^n) = \mathbf{B}_n^{\pm}$.

The *n-cube*, $\mathscr{C}^n$ is the dual of the n-octahedron; alternatively, we can define it to be the convex hull of the points $\{(e_1, e_2, \dots, e_n) \mid e_i = \pm 1\}$; its lattice of faces is $\mathbf{B}_n^{\pm *}$. See [10] for more details on all three of these polytopes, and [16] for more on the lattice of signed sets.

The boolean lattice and the lattice of signed sets each have a very nice EL-labeling, defined as follows: for $A, B \in \mathbf{B}_n$, obviously $A \lessdot B$ iff $B \backslash A$ is a single element. We define $\lambda(A \lessdot B)$ to be the element of $B \backslash A$ (which is in $[n] \subseteq \mathbb{Z}$). For $A = (A^+, A^-)$, $B = (B^+, B^-) \in \mathbf{B}_n^{\pm}$, we have that $A \lessdot B$ when $B \backslash A$ is a single *signed* element (either $B^+ \backslash A^+$ is a single element and $A^- = B^-$, or vice versa). We let $\lambda(A \lessdot B)$ be the element of $B \backslash A$ (considered as an integer), so that $\lambda((\{1\}, \{2\}) \lessdot (\{1\}, \{2, 3\})) = -3$. In addition, if $A^+ \dot{\cup} A^- = [n]$, then $A \lessdot \hat{1}$. In this case, we let $\lambda(A \lessdot \hat{1}) = 0$.

These labelings of $\mathbf{B}_n$ and $\mathbf{B}_n^{\pm}$ are called the *standard labelings*; any labeling of $\mathbf{L}(\Delta^n) \cong \mathbf{B}_n$ or $\mathbf{L}(\mathscr{O}^n) \cong \mathbf{B}_n^{\pm}$ that can be derived from the standard labelings by isomorphism is also called standard. See Figure 1 for an example of the latter; each element of the poset is labeled by both a face of $\mathscr{O}^2$ and the corresponding element of $\mathbf{B}_2^{\pm}$.

Given a totally ordered n-set X, the *X-standard labelings* are defined to be $\lambda_X(a, b) := \upsilon_X^{-1}(\lambda(a, b))$, where υ is the unique order preserving bijection from X to $[n]$ (extended to $X^{\pm} \to [-n, n]$ for the octahedron). Notice that all the results we prove for the standard labelings will hold for the X-standard labelings as well.

A bijection $\pi\colon [n] \to X$ (where X is any n-set) is called a *permutation* (or an *unsigned permutation*) of X. We often write π_i for $\pi(i)$ and the word $\pi_1\pi_2\pi_3\cdots\pi_n$ for π; hence, we call n the *length* of π. The set of all permutations of a set X is denoted $X!$, and $S_n := [n]!$; we have that the size of $X!$, $\#(X!) = (\#X)!$, so $\#S_n = n!$. We think of these sets primarily as sets of words; for example, $\{x, y, z\}!$ is the set of words $\{xyz, xzy, yxz, yzx, zxy, zyx\}$, and $S_3 = \{123, 132, 213, 231, 312, 321\}$.

Consider injections $\sigma\colon [n] \to \{+, -\} \times X$, and write σ_i for $\sigma(i)$, $\overline{x}$ for

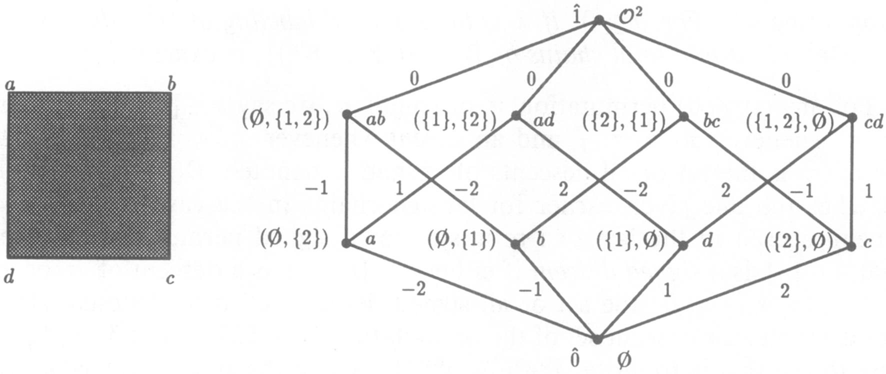

FIGURE 1. A standard labeling of $\mathscr{O}$.

$(-, x)$ and x for $(+, x)$. Define $|(\pm, x)| := x$, and $|\sigma|$ such that $|\sigma|_i := |\sigma_i|$, so $|\sigma|$ is a map from $[n]$ to X. Then we say that such a σ is a *signed permutation* whenever $|\sigma|$ is a permutation of X. In this case, we let $\sigma_{n+1} := 0$, and write $\sigma_1 \cdots \sigma_n \sigma_{n+1}$ for σ. However, we still call n the length of σ. The set of all signed permutations of a set X is denoted $\overline{X}!$, and $S_n^{\pm} := \overline{[n]}!$. Again, we consider $S_n^{\pm}$ and $\overline{X}!$ as sets of words; $S_2^{\pm} = \{120, 1\overline{2}0, \overline{1}20, \overline{1}\overline{2}0, 210, 2\overline{1}0, \overline{2}10, \overline{2}\overline{1}0\}$.

We consider only the case where X is (totally) ordered by a relation $\leq$. In this case, we extend $\leq$ to the set $X^{\pm} := (\{+, -\} \times X) \cup \{0\}$, so that

- $\overline{y} < 0 < x$ for all $x, y \in X$;
- $x < y$ iff $x < y$ in X; and
- $\overline{x} < \overline{y}$ iff $x > y$ in X.

This defines $\leq$ as a total order on $X^{\pm}$. (For instance, if $X = [2]$, we have $\overline{2} < \overline{1} < 0 < 1 < 2$.) If $X = [n]$, we identify $\overline{x}$ with $-x$, so $X^{\pm}$ is identified with $[-n, n]$. We define $\mathrm{supp}(\sigma)$ to be the image of σ in $\{+, -\} \times X$, $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, so $|\mathrm{supp}(\sigma)| = X$. For instance, $\mathrm{supp}(\overline{1}3\overline{2}0)$ is $\{\overline{1}, 2, \overline{3}\} = (\{2\}, \{1, 3\})$. Notice that σ can be thought of as a map from $[n]$ into $\mathrm{supp}(\sigma)$, and thus as a permutation of that set. Thus, a word with signs is ambiguous, in that it could be a permutation or a signed permutation; throughout this paper, the 0 at the end of the word $\sigma_1 \cdots \sigma_n \sigma_{n+1}$ will mark that the word is to be thought of as a signed permutation rather than as an unsigned permutation of $\{\sigma_i\}_{i=1}^n$. For an unsigned permutation π of X, we define $\mathrm{supp}(\pi) := X$. (Notice that this is quite different from the support of a chain.)

Note that if $\#X = n$, we have order-preserving bijections $\upsilon_X \colon X \to [n]$ and $\upsilon_X^{\pm} \colon X^{\pm} \to [-n, n]$, which can be extended to bijections $\mathrm{can}_X(\pi) := \upsilon_X \circ \pi$ from $X!$ to S_n, and $\mathrm{can}_X^{\pm}(\sigma) := \upsilon_X^{\pm} \circ \sigma$ from $\overline{X}!$ to $S_n^{\pm}$. For any (signed) permutation π (resp. σ), we define $\mathrm{can}(\pi) = \mathrm{can}_{\mathrm{supp}(\pi)}(\pi)$ (resp. $\mathrm{can}^{\pm}(\sigma) := \mathrm{can}^{\pm}_{|\mathrm{supp}(\sigma)|}(\sigma)$); $\mathrm{can}(\pi)$ and $\mathrm{can}^{\pm}(\sigma)$ are called the *canonical equivalents* of π and σ. For example, if $X = \{x, y, z\}$ ordered alphabetically $(x < y < z)$, then $\mathrm{can}(xzy) = 132$ and $\mathrm{can}^{\pm}(x\overline{z}\overline{y}0) = 1\overline{3}\overline{2}0$.

Proposition 4.1. *For $n \geq 0$, if λ is the standard labeling of $\mathbf{B}_n$, then the set of labels of all maximal chains of $\mathbf{B}_n$, $\lambda(\mathscr{M}\mathscr{C}(\mathbf{B}_n))$, is exactly S_n.*

Proposition 4.2. *For $n \geq 0$, if λ is the standard labeling of $\mathbf{B}_n^{\pm}$, then the set of all labels of all maximal chains of $\mathbf{B}_n^{\pm}$, $\lambda(\mathscr{M}\mathscr{C}(\mathbf{B}_n^{\pm}))$, is exactly $S_n^{\pm}$.*

For any (signed) permutation π of length n, we say $i \in [n-1]$ is a *descent* of π whenever $\pi_i > \pi_{i+1}$ and an *ascent* whenever $\pi_i < \pi_{i+1}$. The *descent set* of π is the set of all descents of π and is denoted $D(\pi)$ (this definition matches the one given earlier for labeled chains in the case of the standard labelings used in the last two results). For a signed permutation σ, we say that $i \in [n]$ is a *signed descent* if either $i \in [n-1]$ is a descent of σ or $i = n$ and $\sigma_n > \sigma_{n+1} = 0$; the set of all signed descents of σ is denoted $D^{\pm}(\sigma)$.[1] For example, the descent set of the permutation 7462531 is $\{1, 3, 5, 6\}$. One way to see this is to write $7^{\mathsf{b}}4^{\mathsf{a}}6^{\mathsf{b}}2^{\mathsf{a}}5^{\mathsf{b}}3^{\mathsf{b}}1$, where the superscript is a for an ascent and b for a descent. The word bababb is $w(D(7462531))$, where w is the function defined in §2. By using Theorem 3.7, we see that

Proposition 4.3. *$cd(\Delta^n) = \sum_{\pi \in S_{n+1}} w(D(\pi))$, and $cd(\mathscr{O}^n) = \sum_{\sigma \in S_n^{\pm}} w(D^{\pm}(\sigma))$.*

5. André (signed) permutations

From here until the end of the section, we follow Foata and Schützenberger's paper [11] (the first part of which was published as [12]), adding the corresponding signed concepts which were not considered there. So all of the results in this section for permutations originally appeared in [11]. If $i-1$ and i are both (signed) descents, we call i a *double (signed) descent*. Similarly, we can define double (signed) ascents, (signed) peaks (ascent, descent) and (signed) valleys (descent, ascent).

For any permutation π of length n, we define the *restriction* of π to an interval $[i, j]$ of $[n]$ to be the permutation $\pi|_{[i,j]}$ of $\{\pi_i, \pi_{i+1}, \dots, \pi_j\}$ of length $j - i + 1$ defined by $\pi|_{[i,j]}(k) = \pi(k + i - 1)$. In other words, the restriction is the word $\pi_i \pi_{i+1} \cdots \pi_j$. For any signed permutation σ of an n-set X and $[i, j] \subseteq [n]$ we define $\sigma|_{[i,j]}$ in the same way, so it is an unsigned permutation $\sigma_i \sigma_{i+1} \cdots \sigma_j$ of the set $\{\sigma_i, \sigma_{i+1}, \dots, \sigma_j\}$, not a signed permutation, (because there is no 0 at the end—the reason for this definition is because of our application to the standard labelings of $\mathbf{B}_n^{\pm}$). However, we define $\sigma|_{[i,n+1]}$ to be a signed permutation of length $n - i + 1$ because we have $\sigma|_{[i,n+1]}(n - i + 2) = \sigma_{n-i+2+i-1} = \sigma_{n+1} = 0$, so $\sigma|_{[i,n+1]}$ is the word $\sigma_i \sigma_{i+1} \cdots \sigma_n \sigma_{n+1}$, which ends with 0.

A permutation α of an n-set X is an *André permutation* whenever α has no double descents and α satisfies condition A_n:

(A_n): for all $1 < j < j' \leq n$, if

$$\alpha_{j-1} = \max\{\alpha_{j-1}, \alpha_j, \alpha_{j'-1}, \alpha_{j'}\} \text{ and}$$
$$\alpha_{j'} = \min\{\alpha_{j-1}, \alpha_j, \alpha_{j'-1}, \alpha_{j'}\},$$

there exists a j'', with $j < j'' < j'$, such that $\alpha_{j''} < \alpha_{j'}$.

This is rather confusing; see Figure 2 which shows the condition on a graph of α thought of as a function. Notice that, in fact, (A_n) implies that there are no double descents; it can be thought of as a generalization of the requirement

[1] This is nonstandard; we are adding the 0 to match the labels of $\mathbf{B}_n^{\pm}$.

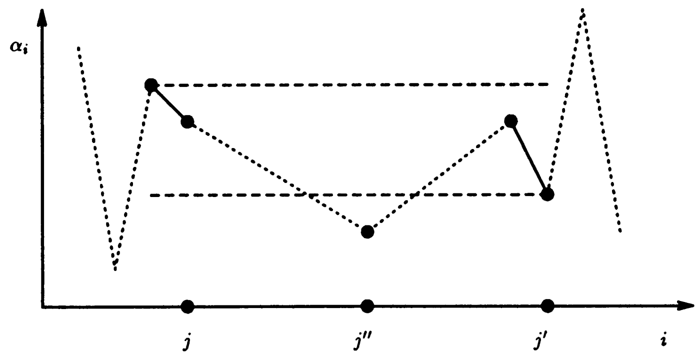

FIGURE 2. Condition (A_n): the existence of $j < j'$ such that $\alpha_{j-1} > \alpha_j$, $\alpha_{j'-1} > \alpha_{j'}$, $\alpha_{j'} < \alpha_j$, and $\alpha_{j-1} > \alpha_{j'-1}$ implies the existence of $j < j'' < j'$ so that $\alpha_{j''} < \alpha_{j'}$.

that there be no double descents. The set of all André permutations of $[n]$ is denoted D_n^*.

Examples.

$$D_1^* = \{1\}, \quad D_2^* = \{12, 21\}, \quad D_3^* = \{123, 132, 213, 231, 312\},$$
$$D_4^* = \{1234, 1243, 1324, 1342, 1423, 2134, 2143, 2314, 2341, 2413, 3124, 3142, 3241, 3412, 4123, 4132\}.$$

Proposition 5.1. *The restriction of an André permutation is an André permutation.*

Proof. We just note that if α has no double descents and satisfies condition (A_n), then any restriction of it of length m has no double descents and satisfies (A_m). □

We say that an André permutation α of an n-set X is *augmented* whenever $\alpha_n = \max \operatorname{supp}(\alpha)$; we denote the augmented André permutations of $[n]$ by A_n.

Examples.

$$A_1 = \{1\}, \quad A_2 = \{12\}, \quad A_3 = \{123, 213\},$$
$$A_4 = \{1234, 1324, 2134, 2314, 3124\}.$$

Proposition 5.2. *For $n \geq 0$, restriction to $[n]$ gives a bijection $A_{n+1} \to D_n^*$ and hence, $\#A_{n+1} = \#D_n^*$.*

A signed permutation α of an n-set X is an *André signed permutation* if α has no double signed descents, α satisfies condition A_{n+1} (which means that the 0 at the end must be considered), and for $x = \max X$, $\overline{x} \in \operatorname{supp}(\alpha)$ and $\alpha|_{[\alpha^{-1}(\overline{x})+1, n+1]}$ is also an André signed permutation, where we take $0 \in S_0^{\pm}$ to be an André signed permutation. (These extra conditions are needed to make various results that hold for André unsigned permutations also hold for André signed permutations; for instance, the following.)

Proposition 5.3. *The restriction of an André signed permutation is an André permutation or an André signed permutation.*

Proof. For a signed permutation α and its restriction $\alpha|_{[i,j]}$ with $j \leq n$, the proof is as in Proposition 5.1 (and since we have removed the 0 at the end, the result is an unsigned permutation). Otherwise, we have $\beta = \alpha|_{[i,n+1]}$, and let $x = \max X$; either $i \leq \alpha^{-1}(\overline{x})$, in which case $x = \max|\mathrm{supp}(\beta)|$ (so $\overline{x} \in \mathrm{supp}(\beta)$) and $\beta|_{[\beta^{-1}(\overline{x})+1,n+1]}$ is the same as $\alpha|_{[\alpha^{-1}(\overline{x})+1,n+1]}$, which is an André signed permutation; or $i > \alpha^{-1}(\overline{x})$, in which case $\gamma = \alpha|_{[\alpha^{-1}(\overline{x})+1,n+1]}$ is an André signed permutation, so by induction on the length of α, the signed permutation β is again André, since it is a restriction of the shorter André signed permutation γ. □

The following was pointed out by Bayer (in a personal communication):

Corollary 5.4. *If* α *is an André signed permutation of* $[n]$, *then* $\alpha_n < 0$.

Proof. By the proposition, $\alpha|_{\{n,n+1\}}$ is an André signed permutation; Hence, $-|\alpha_n| \in \{\alpha_n\}$ by one of the conditions on André signed permutation, so $\alpha_n = -|\alpha_n| < 0$. □

Because of the last result, all André signed permutations end with a descent, like augmented André (signed) permutations; we shall see that the André signed permutations correspond to augmented André (signed) permutations in other respects as well. Hence, we say that all André signed permutations are augmented. (It would be nice to find a better definition for André signed permutations, so that they would properly contain the augmented André signed permutations, as with the unsigned permutations. It would be especially nice if an analog of Proposition 5.2 existed.) The set of all (augmented) André signed permutations of $[n]$ is denoted $A_n^{\pm}$.

Examples.

$$A_0^{\pm} = \{0\}, \quad A_1^{\pm} = \{\overline{1}0\}, \quad A_2^{\pm} = \{1\overline{2}0, \overline{12}0, \overline{21}0\},$$
$$A_3^{\pm} = \{\overline{3}1\overline{2}0, \overline{312}0, \overline{321}0, 1\overline{32}0, 2\overline{31}0,$$
$$\overline{132}0, \overline{231}0, 12\overline{3}0, \overline{12}\overline{3}0, \overline{2}1\overline{3}0, \overline{213}0\}.$$

The *variation* of a (signed) permutation π of length n is $V(\pi) := w_{n-1}(D(\pi))$, where w is the function introduced in §2. Similarly, we have the *signed variation* $V^{\pm}(\sigma) := w_n(D^{\pm}(\sigma))$ of a signed permutation σ. For instance, $V^{\pm}(\overline{3}1\overline{2}0) = w(\{2\}) = \mathsf{aba}$.

For an augmented André permutation α of the usual n-set X, we define the *reduced variation* $U(\alpha)$ to be the word obtained from $V(\alpha)$ by replacing each ba with a d, and each remaining a by a c.[2] Since α has no double descents and ends with an ascent (so $V(\alpha)$ ends with an a), there will be no extra b's, so $U(\alpha)$ is a word in c and d. Similarly, we have the *reduced signed variation* of an augmented André signed permutation α, $U^{\pm}(\alpha)$ which comes from $V^{\pm}(\alpha)$, both in the same way as $U(\alpha)$ comes from $V(\alpha)$.

[2] In [11], the original notation used was s for c and t for d. The fact that the letters in question sound similar is apparently a coincidence (see Theorem 6.1 for why we use c and d).

Examples. We have that $V(2134) = \mathsf{baa}$, so $U(2134) = \mathsf{dc}$. Similarly, $V^{\pm}(\bar{3}1\bar{2}0) = \mathsf{aba}$, so $U^{\pm}(\bar{2}1\bar{3}0) = \mathsf{cd}$.

We now define the noncommutative André polynomial of Foata and Schützenberger as follows: $U(A_n) := \sum_{\alpha \in A_n} U(\alpha)$ is the *nth noncommutative André polynomial*. In addition, we define the *nth signed noncommutative André polynomial* to be $U^{\pm}(A_n^{\pm}) := \sum_{\alpha \in A_n^{\pm}} U^{\pm}(\alpha)$.

Examples.

$$\begin{aligned}
U(A_1) &= 1, \quad U(A_2) = \mathsf{c}, \quad U(A_3) = \mathsf{c}^2 + \mathsf{d}, \\
U(A_4) &= \mathsf{c}^3 + 2\mathsf{cd} + 2\mathsf{dc}, \\
U(A_5) &= \mathsf{c}^4 + 3\mathsf{c}^2\mathsf{d} + 5\mathsf{cdc} + 3\mathsf{dc}^2 + 4\mathsf{d}^2, \\
U^{\pm}(A_0^{\pm}) &= 1, \quad U^{\pm}(A_1^{\pm}) = \mathsf{c}, \\
U^{\pm}(A_2^{\pm}) &= \mathsf{c}^2 + 2\mathsf{d}, \quad U^{\pm}(A_3^{\pm}) = \mathsf{c}^3 + 4\mathsf{dc} + 6\mathsf{cd}.
\end{aligned}$$

By letting c and d commute, we get the *commutative André polynomial* of each type.

Proposition 5.5. *For any permutation α of an (ordered) n-set, α is an André permutation iff for $\alpha_m = \min \operatorname{supp}(\alpha)$, both $\alpha|_{[m]}$ and $\alpha|_{[m+1,n]}$ are André permutations.*

Proof. The 'only if' is Proposition 5.1. So suppose $\alpha|_{[m]}$ and $\alpha|_{[m+1,n]}$ are André permutations. Since m is a valley of α, we need only check condition A_n for $j < m < j'$, but recall that $\alpha_m = \min \operatorname{supp}(\alpha)$, so it is smaller than $\alpha_{j'}$, so we take $j'' = m$. □

Corollary 5.6. *For any permutation α of an (ordered) n-set, α is an augmented André permutation iff for $\alpha_m = \min \operatorname{supp}(\alpha)$, we have that $\alpha|_{[m-1]}$ is an augmented André permutation, $\alpha|_{[m+1,n]}$ is an augmented André permutation and $\max \operatorname{supp}(\alpha) \in \operatorname{supp}(\alpha|_{[m+1,n]})$.*

Proposition 5.7. *Let $\bar{\alpha}$ be an augmented André permutation of an n-set with $n \geq 2$. Let $\bar{\alpha}_m = \min \operatorname{supp}(\bar{\alpha})$, $\bar{\alpha}^{(1)} = \bar{\alpha}|_{[m-1]}$ and $\bar{\alpha}' = \bar{\alpha}|_{[m+1,n]}$. Then*

$$U(\bar{\alpha}) = \begin{cases} \mathsf{c}U(\bar{\alpha}') & \text{if } m = 1, \\ U(\bar{\alpha}^{(1)})\mathsf{d}U(\bar{\alpha}') & \text{otherwise.} \end{cases}$$

Corollary 5.8.

$$U(A_{n+2}) = \mathsf{c}U(A_{n+1}) + \sum_{j=1}^{n} \binom{n}{j} U(A_j)\mathsf{d}U(A_{n+1-j}), \qquad n \geq 0.$$

Proposition 5.9. *For any signed permutation α of an (ordered) n-set X with $x = \max X$, α is an André signed permutation iff $\alpha_m = \bar{x} \in \operatorname{supp}(\alpha)$, $\alpha|_{[m]}$ is an André permutation and $\alpha|_{[m+1,n+1]}$ is an André signed permutation.*

Proof. By Proposition 5.3, the 'only if' part is clear. Conversely, there are no double signed descents and (A_{n+1}) holds just as the corresponding facts were true in the proof of Proposition 5.5. The other two necessary conditions ($\bar{x} \in \operatorname{supp}(\alpha)$ and $\alpha|_{[m+1,n+1]}$ being André) are true by assumption. □

Corollary 5.10. *There is a bijection*

$$A^{\pm}_{n+1} \to \bigoplus_{m=1}^{n+1} \left(\binom{[n]}{m-1}^{\pm} \times A_{m-1} \times A^{\pm}_{n-m+1} \right)$$

(*where* $\binom{X}{k}^{\pm}$ *is the set of all signed sets,* (A^+, A^-), *such that* $A^+ \dot\cup A^- \subseteq X$ *and* $\#A^+ + \#A^- = k$).

Proposition 5.11. *For all* $\alpha \in A^{\pm}_n$, *if* $\alpha_m = \overline{n}$, *then*

$$U^{\pm}(\alpha) = \begin{cases} \mathsf{c}U^{\pm}(\alpha|_{[m+1,n+1]}) & \text{if } m = 1, \\ U(\alpha|_{[m-1]})\mathsf{d}U^{\pm}(\alpha|_{[m+1,n+1]}) & \text{otherwise.} \end{cases}$$

Corollary 5.12.

$$U^{\pm}(A^{\pm}_{n+1}) = \mathsf{c}U^{\pm}(A^{\pm}_n) + \sum_{j=1}^{n} 2^j \binom{n}{j} U(A_j)\mathsf{d}U^{\pm}(A^{\pm}_{n-j}).$$

6. Connection with the cd-index

Theorem 6.1. *For all* $n > 0$,

$$cd(\Delta^n) = U(A_{n+1}), \tag{6.1}$$

$$cd(\mathcal{O}^n) = U^{\pm}(A^{\pm}_n). \tag{6.2}$$

Proof. By Proposition 4.3, we have that $cd(\Delta^n) = \sum_{\pi \in S_{n+1}} w(D(\pi))$ and $cd(\mathcal{O}^n) = \sum_{\sigma \in S^{\pm}_n} w(D^{\pm}(\sigma))$. We will define a map $Q\colon A_{n+1} \to \mathbf{2}^{S_{n+1}}$ so that $\{Q(\alpha) \mid \alpha \in A_{n+1}\}$ is a partition of S_{n+1}; for all $\alpha, \beta \in A_{n+1}$, $\alpha \in Q(\beta)$ iff $\alpha = \beta$; and $\sum_{\pi \in Q(\alpha)} w(D(\pi)) = U(\alpha)$; this will prove (6.1). Similarly, we will define $Q^{\pm}\colon A^{\pm}_n \to \mathbf{2}^{S^{\pm}_n}$ so that $\{Q^{\pm}(\alpha) \mid \alpha \in A^{\pm}_n\}$ partitions $S^{\pm}_n$; for all $\alpha, \beta \in A^{\pm}_n$, $\alpha \in Q^{\pm}(\beta)$ iff $\alpha = \beta$; and $\sum_{\pi \in Q^{\pm}(\alpha)} w(D(\pi)) = U^{\pm}(\alpha)$ to prove (6.2). For both of these, we will proceed by induction on n.

For $n = 1$, $S_2 = \{12, 21\}$, $A_2 = \{12\}$, and we set $Q(12) = \{12, 21\}$. Similarly, $S^{\pm}_1 = \{10, \overline{1}0\}$, $A^{\pm}_1 = \{\overline{1}0\}$ and so we set $Q^{\pm}(\overline{1}0) = \{10, \overline{1}0\}$. Note that again the facts required of Q and $Q^{\pm}$ hold.

To define Q and $Q^{\pm}$ for $n > 1$, it is convenient to define

$$\overline{Q}(\alpha) = \operatorname{can}^{-1}_{\operatorname{supp}(\alpha)}(Q(\operatorname{can}(\alpha)))$$

for α *any* augmented André permutation, and $\overline{Q}^{\pm}$ similarly. This means that we will have partitions of $X!$ or $\overline{X}!$ once we have partitions of the corresponding $S_{\#X}$ and $S^{\pm}_{\#X}$. For instance, $\overline{Q}(23) = \{23, 32\}$ because $\operatorname{can}(23) = 12$, $Q(12) = \{12, 21\}$, and $\operatorname{can}^{-1}_{\{2,3\}}(\{12, 21\}) = \{23, 32\}$.

For α an André signed permutation of $[n]$, let $\alpha_m = \overline{n}$, $\bar{\alpha}^{(1)} = \alpha|_{[1,m-1]}$ (an unsigned permutation), $\bar{\alpha}^{(2)} = \alpha|_{[m+1,n+1]}$ (a signed permutation) and $X_i = \operatorname{supp}(\bar{\alpha}^{(i)})$. Then we set

$$Q^{\pm}(\alpha) = \{\pi n \sigma,\ \pi \overline{n} \sigma \mid \pi \in \bar{Q}(\bar{\alpha}^{(1)}),\ \sigma \in \bar{Q}^{\pm}(\bar{\alpha}^{(2)})\}.$$

For example, we have $A^{\pm}_2 = \{1\overline{2}0, \overline{12}0, \overline{21}0\}$. For $1\overline{2}0$, $m = 2$ and so $Q(1\overline{2}0) = \{\pi\overline{2}\sigma, \pi 2\sigma \mid \pi \in \overline{Q}(1) = \{1\}, \sigma \in \overline{Q}^{+}(0) = \{0\}\} = \{1\overline{2}0, 120\}$.

Similarly, $Q^{\pm}(\overline{1}\overline{2}0) = \{\overline{1}\overline{2}0, \overline{1}20\}$ and $Q^{\pm}(\overline{2}\overline{1}0) = \{\overline{2}\sigma, 2\sigma \mid \sigma \in \overline{Q}^{\pm}(\overline{1}0) = \{\overline{1}0, 10\}\} = \{\overline{2}\overline{1}0, \overline{2}10, 2\overline{1}0, 210\}$ (in the latter case, $m = 1$).

For α an augmented André permutation of $[n+1]$, let $\alpha_m = 1$, $\bar{\alpha}^{(1)} = \alpha|_{[1,m-1]}$, $\bar{\alpha}^{(2)} = \alpha|_{[m+1,n+1]}$, $X_i = \text{supp}(\bar{\alpha}^{(i)})$ (so $n+1 \in X_2$). Then we set

$$Q(\alpha) = \{\pi^{(1)}1\pi^{(2)}, \pi^{(1)}(n+1)\downarrow(\pi^{(2)}) | \pi^{(i)} \in \overline{Q}(\bar{\alpha}^{(i)})\},$$

where $\downarrow(\pi) = \text{can}^{-1}_{\text{supp}(\pi)\cup\{1\}\setminus\{n+1\}}(\text{can}(\pi))$.

For example, $A_2 = \{12\}$, and $Q(12) = \{1\pi, 2\downarrow(\pi) \mid \pi \in \overline{Q}(2)\}$, that is, $\{12, 2\downarrow(2)\} = \{12, 21\}$. Then, we have $A_3 = \{123, 213\}$; for $\alpha = 123$, $m = 1$, and so $Q(123) = \{1\pi, 3\downarrow(\pi) \mid \pi \in \overline{Q}(23)\}$, since $\overline{Q}(\varnothing) = \varnothing$. We know that $\overline{Q}(23) = \{23, 32\}$, so $Q(123) = \{123, 132, 312, 321\}$, because $\downarrow(23) = 12$ and $\downarrow(32) = 21$. Note that $\sum_{\pi\in Q(123)} w(D(\pi)) = \mathsf{a}^2 + \mathsf{ab} + \mathsf{ba} + \mathsf{b}^2 = \mathsf{c}^2 = U(123)$. Similarly, $Q(213) = \{213, 231\}$, and $\sum_{\pi\in Q(213)} w(D(\pi)) = \mathsf{ab} + \mathsf{ba} = \mathsf{d} = U(213)$.

We will now show

(a) $Q(\alpha) \cap Q(\beta) \neq \varnothing$ implies $\alpha = \beta$;
$Q^{\pm}(\alpha) \cap Q^{\pm}(\beta) \neq \varnothing$ implies $\alpha = \beta$.
(b) For all $\pi \in S_{n+1}$, there exists $\alpha \in A_{n+1}$ such that $\pi \in Q(\alpha)$;
for all $\pi \in S_n^{\pm}$, there exists $\alpha \in A_n^{\pm}$ such that $\pi \in Q^{\pm}(\alpha)$.
(c) $\sum_{\pi\in Q(\alpha)} w(D(\pi)) = U(\alpha)$;
$\sum_{\sigma\in Q^{\pm}(\alpha)} w(D^{\pm}(\sigma)) = U^{\pm}(\alpha)$.

The first two show that we have a partition, and the third shows that the sum is right. Note that all three are true for $n = 1$ (from the complete description of Q and $Q^{\pm}$ for this case given above). So we may proceed by induction.

(a) Note that for all $\pi \in Q(\alpha)$, if $\alpha_k = 1$, then by construction $\pi_k \in \{1, n+1\}$. So if $\pi \in Q(\alpha) \cap Q(\beta)$, then $\alpha_k = \beta_k = 1$. Hence, we have that $\pi = \pi^{(1)}x\pi^{(2)}$, where $x = \pi_k \in \{1, n+1\}$. Suppose $x = 1$ (the $x = n+1$ case is similar). By definition of $\overline{Q}(\ )$, we have that $\pi^{(i)} \in \overline{Q}(\bar{\alpha}^{(i)}) \cap \overline{Q}(\bar{\beta}^{(i)})$, so by induction $\bar{\alpha}^{(i)} = \bar{\beta}^{(i)}$. Hence $\alpha = \bar{\alpha}^{(1)}1\bar{\alpha}^{(2)} = \bar{\beta}^{(1)}1\bar{\beta}^{(2)} = \beta$. For the signed case, we do the same thing with $\{n, \overline{n}\}$ instead of $\{1, n+1\}$; note that we must use the unsigned case in the induction step.

(b) For $\sigma \in S_n^{\pm}$, we can write $\sigma = \pi x \sigma'$, where $x \in \{n, \overline{n}\}$. By induction, the permutation π is in some $\overline{Q}(\bar{\alpha}^{(1)})$ (for $\bar{\alpha}^{(1)}$ some augmented André permutation of supp(π)) and the signed permutation σ' is in some $\overline{Q}^{\pm}(\bar{\alpha}^{(2)})$ (for $\bar{\alpha}^{(2)}$ some André signed permutation of $|\text{supp}(\sigma')|$). The signed permutation $\alpha = \bar{\alpha}^{(1)}\overline{n}\bar{\alpha}^{(2)}$ is an André signed permutation by Proposition 5.9 and the fact that $\bar{\alpha}^{(1)}\overline{n}$ is André, and $\sigma \in Q^{\pm}(\alpha)$ by definition of $Q^{\pm}(\ )$.

For any $\pi \in S_{n+1}$, we write $\pi = \pi^{(1)}x\pi^{(2)}$, where $x \in \{1, n+1\}$, and $\{1, n+1\} \cap \text{supp}(\pi^{(1)}) = \varnothing$. (So x is the first occurrence of either 1 or $n+1$.) Then proceed in the same way.

(c) Let $X(\alpha) := \sum_{\pi\in Q(\alpha)} w(D(\pi))$ and $X^{\pm}(\alpha) := \sum_{\sigma\in Q^{\pm}(\alpha)} w(D^{\pm}(\sigma))$. These have the same initial conditions as $U(\ )$ and $U^{\pm}(\ )$, so if we show they satisfy the same recursion, we are done. This consists of showing that $X(\alpha) = \mathsf{c}X(\bar{\alpha}^{(2)})$ and $X^{\pm}(\alpha) = \mathsf{c}X^{\pm}(\bar{\alpha}^{(2)})$ if $\bar{\alpha}^{(1)} \in A_0$, and that $X(\alpha) = X(\bar{\alpha}^{(1)})\mathsf{d}X(\bar{\alpha}^{(2)})$

and $X^{\pm}(\alpha) = X(\bar{\alpha}^{(1)})\mathsf{d}X^{\pm}(\bar{\alpha}^{(2)})$ otherwise. We will show the first and last of the four statements; the other two are very similar.

Let $\alpha \in A_{n+1}$, and $\bar{\alpha}^{(1)} \in A_0$. Hence, $\alpha = 1\alpha_2\alpha_3\cdots\alpha_{n+1}$. Then let $Q(\alpha) = Q^1 \dot{\cup} Q^{n+1}$, where $Q^i := \{\pi \in Q(\alpha) \mid \pi_1 = i\}$ for $i \in \{1, n+1\}$. Then

$$\begin{aligned} X(\alpha) &= \sum_{\pi \in Q^1} w(D(\pi)) + \sum_{\pi \in Q^{n+1}} w(D(\pi)) \\ &= \sum_{\pi^{(2)} \in \overline{Q}(\bar{\alpha}^{(2)})} \mathsf{a}w(D(\pi^{(2)})) + \sum_{\pi^{(2)} \in \overline{Q}(\bar{\alpha}^{(2)})} \mathsf{b}w(D(\pi^{(2)})) \\ &= \sum_{\pi \in \overline{Q}(\bar{\alpha}^{(2)})} \mathsf{c}w(D(\pi)) = \mathsf{c}X(\bar{\alpha}^{(2)}) . \end{aligned}$$

Let $\alpha \in A_n^{\pm}$, with $\alpha_1 \notin A_0^{\pm}$, and let $k = \alpha^{-1}(\overline{n})$. Then let $\overline{Q}^{\pm}(\alpha) = Q^n \cup Q^{\overline{n}}$, where $Q^i := \{\sigma \in Q^{\pm}(\alpha) \mid \sigma_k = i\}$ for $i \in \{n, \overline{n}\}$. Then

$$\begin{aligned} X^{\pm}(\alpha) &= \sum_{\sigma \in Q^n} w(D^{\pm}(\sigma)) + \sum_{\sigma \in Q^{\overline{n}}} w(D^{\pm}(\sigma)) \\ &= \sum_{\substack{\pi \in \overline{Q}(\bar{\alpha}^{(1)}) \\ \sigma \in \overline{Q}^{\pm}(\bar{\alpha}^{(2)})}} (w(D(\pi))\mathsf{ab}w(D^{\pm}(\sigma))) + \sum_{\substack{\pi \in \overline{Q}(\bar{\alpha}^{(1)}) \\ \sigma \in \overline{Q}^{\pm}(\bar{\alpha}^{(2)})}} (w(D(\pi))\mathsf{ba}w(D^{\pm}(\sigma))) \\ &= \sum_{\substack{\pi \in \overline{Q}(\bar{\alpha}^{(1)}) \\ \sigma \in \overline{Q}^{\pm}(\bar{\alpha}^{(2)})}} (w(D(\pi))\mathsf{d}w(D^{\pm}(\sigma))) \\ &= \Bigg(\sum_{\pi \in \overline{Q}(\bar{\alpha}^{(1)})} w(D(\pi))\Bigg)\mathsf{d}\Bigg(\sum_{\sigma \in \overline{Q}^{\pm}(\bar{\alpha}^{(2)})} w(D^{\pm}(\sigma))\Bigg) = X(\bar{\alpha}^{(1)})\mathsf{d}X^{\pm}(\bar{\alpha}^{(2)}) . \end{aligned}$$

□

We now have a combinatorial interpretation of coefficients of the *cd*-index of the simplex, octahedron, and cube, as counting certain classes of André (signed) permutation:

$$\begin{aligned} \mathbb{C}_w\, cd(\Delta^n) &= \#\{\alpha \in A_{n+1} \mid U(\alpha) = w\} , \\ \mathbb{C}_w\, cd(\mathscr{O}^n) &= \#\{\alpha \in A_n^{\pm} \mid U(\alpha) = w\} , \\ \mathbb{C}_w\, cd(\mathscr{Q}^n) &= \#\{\alpha \in A_n^{\pm} \mid U(\alpha) = \mathrm{rev}(w)\} \end{aligned}$$

(where $\mathbb{C}_w f(\mathsf{c}, \mathsf{d})$ is the coefficient of the cd-word w in $f(\mathsf{c}, \mathsf{d})$).

Since the simplex is self-dual in every dimension, we immediately get (from the above and Lemma 2.7) that

$$U(A_{n+1}) = cd(\Delta^n) = \mathrm{rev}(cd(\Delta^n)) = \mathrm{rev}(U(A_{n+1})) ;$$

this is the duality theorem of Foata and Schützenberger (which takes up a whole section of [11]); we get it for free.

Furthermore, since the proof of Theorem 6.1 gives us a bijection between augmented André (signed) permutations and blocks of a partition of S_n ($S_n^{\pm}$), we can prove the following result originally due (in unsigned form) to Foata and Schützenberger. (A *porism* is a corollary of the *proof* of a result.)

Porism 6.2.

$$\#A_n = \#\{\pi \in S_n \, D(\pi) = \{1, 3, 5, \dots, 2\lfloor \tfrac{n}{2} \rfloor - 1\}\}, \tag{6.3}$$

$$\#A_n^{\pm} = \#\{\sigma \in S_n^{\pm} \, D(\sigma) = \{1, 3, 5, \dots, 2\lfloor \tfrac{n+1}{2} \rfloor - 1\}\}. \tag{6.4}$$

Proof. Note that for each cd-word w, there is exactly one ab-word in its expansion of the form ababab$\cdots$. Hence, in each block of the partition $\{Q(\alpha) \mid \alpha \in A_n\}$ there is exactly one (signed) permutation of the right-hand side of (6.3). □

The permutations of the right-hand side of (6.3) are called *alternating permutations*, and their cardinality is called the nth *Euler number* E_n. (It was interest in these numbers that motivated the paper [11].) We will define the signed permutations on the right-hand side of (6.4) to be the *alternating signed permutations* and their cardinality to be the nth *signed Euler number*, $E_n^{\pm}$.

Notice that by Corollary 5.6, Corollary 5.10 and Porism 6.2, we have the recurrence relations

$$E_{n+2} = \sum_{j=0}^{n} \binom{n}{j} E_j E_{n-j+1} \quad \text{and} \quad E_{n+1}^{\pm} = \sum_{j=0}^{n} 2^j \binom{n}{j} E_j E_{n-j}^{\pm}$$

for $n \geq 1$, and $E_0 = E_1 = E_2 = E_0^{\pm} = E_1^{\pm} = 1$. If we let $F(x) = \sum_{n\geq 0} E_n x^n/n!$ and $G(x) = \sum_{n\geq 0} E_n^{\pm} x^n/n!$, then we have $F(0) = E_0 = 1$, $F'(0) = E_1 = 1$, and

$$\begin{aligned} F''(x) &= \sum_{n\geq 0} E_{n+2} x^n/n! = 1 + \sum_{n\geq 1} \sum_{j=0}^{n} \binom{n}{j} E_j E_{n-j+1} x^n/n! \\ &= \sum_{n\geq 0} E_n x^n/n! \cdot \sum_{n\geq 0} E_{n+1} x^n/n! = F(x)F'(x). \end{aligned}$$

Integrating both sides of the last equation gives the more familiar form $2F'(x) = F(x)^2 + 1$; either way, the solution is well known to be $F(x) = \sec x + \tan x$.

For $G(x)$, we get $G(0) = E_0^{\pm} = 1$ and

$$\begin{aligned} G'(x) &= \sum_{n\geq 0} E_{n+1}^{\pm} x^n/n! = 1 + \sum_{n\geq 1} \sum_{j=0}^{n} \binom{n}{j} (2^j E_j) E_{n-j}^{\pm} \\ &= \sum_{n\geq 0} 2^n E_n x^n/n! \cdot \sum_{n\geq 0} E_n^{\pm} x^n/n! = F(2x)G(x). \end{aligned}$$

Hence,

$$\ln G(x) = \int_0^x F(2t)\,dt, \qquad G(x) = \sec(2x)(\sin(x) + \cos(x)).$$

In [18], Shanks studies two matrices of numbers denoted $c_{a,n}$ and $d_{a,n}$, which he calls the *generalized Euler and class numbers*. The definition is rather complicated and noncombinatorial, but recursions are found for $c_{a,n}$ and for $d_{a,n}$ for each a. Furthermore, it is observed that

$$E_n = \begin{cases} c_{1,k} & \text{if } n = 2k, \\ d_{1,k} & \text{if } n = 2k+1. \end{cases}$$

Shanks notes the combinatorial interpretation this gives for $c_{1,k}$ and $d_{1,k}$, and wonders if there are combinatorial interpretations for higher values of a.

For $a = 2$, we have the following:

Proposition 6.3.

$$E_n^{\pm} = \begin{cases} c_{2,k} & \textit{if } n = 2k, \\ d_{2,k} & \textit{if } n = 2k+1. \end{cases}$$

Proof. From [18], we have the recursions

$$\sum_{j=0}^{n} (-4)^j \binom{2n}{2j} c_{2,n-j} = (-1)^n ,$$

$$\sum_{j=0}^{n-1} (-4)^j \binom{2n-1}{2j} d_{2,n-j} = (-1)^{n-1} .$$

If we let $\Gamma(x) := \sum_{n\geq 0} c_{2,n} x^{2n}/(2n)!$ and $\Delta(x) := \sum_{n\geq 1} d_{2,n} x^{2n-1}/(2n-1)!$, then we have

$$\begin{aligned} \sum_{n\geq 0} (-1)^n x^{2n}/(2n)! &= \sum_{n\geq 0}\sum_{j=0}^{n} (-4)^j \binom{2n}{2j} c_{2,n-j} x^{2n}/(2n)! \\ &= \sum_{n\geq 0}\sum_{j=0}^{n} (2\sqrt{-1})^{2j} \binom{2n}{2j} c_{2,n-j} x^{2n}/(2n)! \\ &= \left(\sum_{n\geq 0} (\sqrt{-1})^{2n} (2x)^{2n}/(2n)!\right)\left(\sum_{n\geq 0} c_{2,n} x^{2n}/(2n)!\right) ; \\ \cos(x) &= \cos(2x)\Gamma(x) , \end{aligned}$$

and, similarly,

$$\begin{aligned} \sum_{n\geq 1} (-1)^{n-1} x^{2n-1}/(2n-1)! &= \sum_{n\geq 1}\sum_{j=0}^{n-1} (2\sqrt{-1})^{2j} \binom{2n-1}{2j} d_{2,n-j} x^{2n-1}/(2n-1)! \\ &= \left(\sum_{n\geq 0} (\sqrt{-1})^{2n} (2x)^{2n}/(2n)!\right)\left(\sum_{n\geq 1} d_{2,n} x^{2n-1}/(2n-1)!\right) ; \end{aligned}$$

$$\sin(x) = \cos(2x)\Delta(x) .$$

Hence,

$$\Gamma(x) + \Delta(x) = \sec(2x)(\sin(x) + \cos(x)) = G(x) ,$$

which proves the result. □

To abstract the results of this section for Δ^n and $\mathscr{O}^n$, we say that any polytope $\mathscr{P}$ satisfying the generalized Dehn-Sommerville equations has a *combinatorial cd-index* whenever there is a CL-labeling λ and a partition $\Pi = \{\Pi_1, \ldots, \Pi_t\}$ of $\mathscr{M}\mathscr{C}(\mathbf{L}(\mathscr{P}))$ such that for each block Π_i of the partition,

$\sum_{\underline{c}\in\Pi_i} w(D(\underline{c}))$ sums to a cd-word; the labeling λ is called a *combinatorial labeling.* In other words, if there is a partition (called a *combinatorial partition*) like $\{Q(\alpha) \mid \alpha \in A_n\}$ for some λ. As above, the existence of this partition and labeling implies that the coefficients of the *cd*-index are nonnegative. The converse is also true, since if the coefficients of the *cd*-index are nonnegative, then for any CL-labeling λ (which exists by Theorem 3.6), we can associate each chain with an ab-word in the *ab*-index. Then the expansion of the *cd*-index gives us a partition as required. However, when we say that $\mathscr{P}$ has a combinatorial *cd*-index, we would like there to be some combinatorial way of producing the partition, as in the results of this section. Notice that we have shown that Δ^n and $\mathscr{O}^n$ have combinatorial *cd*-index. Since $\mathscr{Q}^n$ is the dual of $\mathscr{O}^n$, we could say that it has a *dually combinatorial cd-index.*

We will need the following observation later:

Proposition 6.4. *For $\mathscr{P} \in \{\Delta^n, \mathscr{O}^n\}_{n\geq 1}$ and λ a standard labeling, consider a block $Q(\alpha)$ of the partition of the chains, and suppose the chains in $Q(\alpha)$ sum to the* cd*-word w. Then if w begins with a* d*, all the chains of $Q(\alpha)$ pass through a common atom of $\mathbf{L}(\mathscr{P})$, and if w begins with a* c*, then all the chains of $Q(\alpha)$ pass through one of two atoms $a^{(\mathsf{a})}, a^{(\mathsf{b})} \in Q(\alpha)$, those chains whose* ab*-word begin with* a *(resp.* b*) pass through $a^{(\mathsf{a})}$ (resp. $a^{(\mathsf{b})}$), and λ satisfies $\lambda_{\hat{0}}(\hat{0}, a^{(\mathsf{a})}) < \lambda_{\hat{0}}(\hat{0}, a^{(\mathsf{b})})$.*

Proof. We proceed by induction. Write $\alpha = \bar{\alpha}^{(1)}x\bar{\alpha}^{(2)}$, where x is 1 or $\overline{n}$, depending on whether $\mathscr{P}$ is Δ^n or $\mathscr{O}^n$. If $\bar{\alpha}^{(1)} \notin A_0 \cup A_1$, then $w(\bar{\alpha}^{(1)}) \neq 1$, and so the result holds by induction. (This follows from the structure of λ: all elements of $Q(\overline{\alpha}^{(1)})$ will label chains in $[\hat{0}, e]$, where e is the bottom of the edge labeled x in the chain labeled α.)

Otherwise, if w begins with c, then $\bar{\alpha}^{(1)} \in A_0$ and $\alpha = x\bar{\alpha}^{(2)}$, and all the chains $Q(\alpha)$ have labels starting with $\{1, n\}$ or $\{n, \overline{n}\}$, depending on $\mathscr{P}$. All of those chains whose labels begin with 1 or $\overline{n}$ pass through a single atom of $\mathbf{L}(\mathscr{P})$ (since no two atoms have the same label $\lambda(\hat{0} \lessdot a)$) and they all start with an ascent since all other labels are larger than 1 or $\overline{n}$, whichever it is. Similarly, all the chains whose labels begin with either $n+1$ or n pass through a single atom and start with a descent.

So suppose w starts with d; then $\bar{\alpha}^{(1)} \in A_1$. Hence all the chains pass through a single atom (the one such that $\lambda(\hat{0} \lessdot a) = \bar{\alpha}^{(1)}$), as required. □

We will say that any polytope that has some labeling which satisfies Proposition 6.4 for some partition $\Pi = \{Q(\alpha)\}$ has a *strongly combinatorial cd-index* and that the labeling is a *strongly combinatorial labeling.* The standard (and X-standard) labelings of the simplex and octahedron are strongly combinatorial CL-labelings, and the standard labeling of the octahedron is a strongly combinatorial co CL-labeling of the cube.

7. Extending strongly combinatorial labelings

The results of the previous section only apply to a few polytopes. In this section, we will show how the strongly combinatorial labelings can be extended

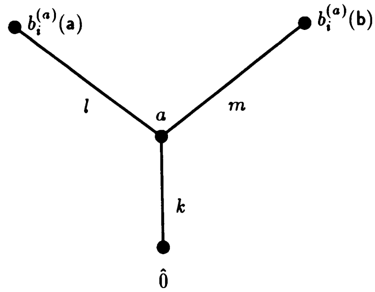

FIGURE 3. When the word begins with a c.

to other classes of polytopes. Then, we will use the fact that any simplex can always be strongly combinatorially labeled starting from any values of the bottom labels to get some additional specific results.

Theorem 7.1. *Let* $\mathbf{P}$ *be an Eulerian poset. If* λ *is a CL-labeling of* $\mathbf{P}$ *such that for each atom* a, λ *restricted to* $[a, \hat{1}]$ *is a strongly combinatorial CL-labeling, then* λ *is a combinatorial CL-labeling of* $\mathbf{P}$, *so the cd-index of* $\mathbf{P}$ *has nonnegative coefficients.*

Proof. For each atom $a \in \mathbf{P}$, we have a partition $\Pi^{(a)} := \{\Pi_1^{(a)}, \dots, \Pi_t^{(a)}\}$ of the maximal chains of $[a, \hat{1}]$ such that $\sum_{\underline{c}\in\Pi_i^{(a)}} w(\underline{c}) = w_i^{(a)}$ is a cd-word (where $w(\underline{c}) = w(D(\underline{c}))$). Furthermore, if $w_i^{(a)}$ begins with a d, there is an atom $b_i^{(a)}$ of $[a, \hat{1}]$ that all the chains of $\Pi_i^{(a)}$ pass through; and if $w_i^{(a)}$ begins with a c, then there are two atoms $b_i^{(a)}(\mathsf{a})$ and $b_i^{(a)}(\mathsf{b})$ through which all chains of $\Pi_i^{(a)}$ pass, such that those chains beginning with ascents (resp. descents) pass through $b_i^{(a)}(\mathsf{a})$ (resp. $b_i^{(a)}(\mathsf{b})$), and the label $\lambda_{\{\hat{0}\lessdot a\}}(a, b_i^{(a)}(\mathsf{a})) = l$ is less than the label $\lambda_{\{\hat{0}\lessdot a\}}(a, b_i^{(a)}(\mathsf{b})) = m$. (This is just the definition of a strongly combinatorial labeling.)

Now, let $\mathrm{T}_i^{(a)}$ be the set of chains in $\{\{\hat{0}\} \lessdot \underline{c} \mid \underline{c} \in \Pi_i^{(a)}\}$. Clearly $\{\mathrm{T}_i^{(a)} \mid a \in \mathbf{P},\ \rho(a) = 1\}$ is a partition of the maximal chains of $\mathbf{P}$. If $w_i^{(a)}$ (which is still the sum of the ab-words of the chains in $\Pi_i^{(a)}$) begins with a d, then each chain of $\mathrm{T}_i^{(a)}$ begins $\{\hat{0} \lessdot a \lessdot b_i^{(a)}\}$, and hence all either start with an ascent or all start with a descent, since the first two labels are the same for them all. Hence, $\sum_{\underline{c}\in\mathrm{T}_i^{(a)}} w(\underline{c}) = xw_i^{(a)}$, where $x \in \{\mathsf{a}, \mathsf{b}\}$. We let $\mathscr{A}_w$ (resp. $\mathscr{B}_w$) be the collection of blocks $\mathrm{T}_i^{(a)}$ such that $\sum_{\underline{c}\in\mathrm{T}_i^{(a)}} w(\underline{c}) = \mathsf{a}w$ (resp. $\mathsf{b}w$) for each w.

On the other hand, if $w_i^{(a)}$ begins with a c, we have the situation shown in Figure 3: $b_i^{(a)}(\mathsf{a})$ and $b_i^{(a)}(\mathsf{b})$ each cover a and the labels of these (rooted) covering relations are l and m respectively, and $a \gtrdot \hat{0}$ with label k. We have that $m > l$, so there are three cases: $k > m > l$, $m > k > l$ and $m > l > k$. In the first case, all chains of $\mathrm{T}_i^{(a)}$ begin with an ascent and in the last case, with a descent, so we form $\mathscr{A}_w$ and $\mathscr{B}_w$ as before. In the remaining case, we have that each chain of $\mathrm{T}_i^{(a)}$ passing through $b_i^{(a)}(\mathsf{a})$ begins with a descent, since $k > l$; following this is an ascent by definition of $b_i^{(a)}(\mathsf{a})$. Similarly, each chain passing through $b_i^{(a)}(\mathsf{b})$ begins with an ascent followed by a descent. Since $\sum_{\underline{c}\in\Pi_i^{(a)}} w(\underline{c}) = w_i^{(a)} = \mathsf{c}w'$, we have that $\sum_{\underline{c}\in\mathrm{T}_i^{(a)}} w(\underline{c}) = \mathsf{d}w'$. Let $\mathscr{D}_{w'}$ be the collection of blocks $\mathrm{T}_i^{(a)}$ so that $\sum_{\underline{c}\in\Pi_i^{(a)}} w(\underline{c}) = \mathsf{d}w'$.

Now, every block $\mathrm{T}_i^{(a)}$ is in one of $\mathscr{A}_w$, $\mathscr{B}_w$ or $\mathscr{D}_w$ for some w, and every maximal chain is in exactly one block. Hence we have that

$$cd(\mathbf{P}) = \sum_w \#\mathscr{A}_w\, \mathsf{a}w + \sum_w \#\mathscr{B}_w\, \mathsf{b}w + \sum_w \#\mathscr{D}_w\, \mathsf{d}w\,. \tag{7.1}$$

Now, since $\mathbf{P}$ has a *cd*-index, the right-hand side of (7.1) can be written in terms of c and d. The last term already is in this form, so we have that

$$\sum_w \#\mathscr{A}_w\, \mathsf{a}w + \sum_w \#\mathscr{B}_w\, \mathsf{b}w$$

can be written in terms of c and d. This implies that it is invariant under the map switching a and b, but so are c and d, and thus so is a word w in c and d, so we have

$$\sum_w \#\mathscr{A}_w\, \mathsf{a}w + \sum_w \#\mathscr{B}_w\, \mathsf{b}w = \sum_w \#\mathscr{A}_w\, \mathsf{b}w + \sum_w \#\mathscr{B}_w\, \mathsf{a}w\ ,$$

$$\sum_w (\#\mathscr{A}_w\, \mathsf{a}w + \#\mathscr{B}_w\, \mathsf{b}w) = \sum_w (\#\mathscr{A}_w\, \mathsf{b}w + \#\mathscr{B}_w\, \mathsf{a}w)\ ,$$

$$\sum_w (\#\mathscr{A}_w - \#\mathscr{B}_w)(\mathsf{a}-\mathsf{b})w = 0\ ;$$

considering this as a noncommuting polynomial in $\{\mathsf{c}, \mathsf{d}\}$ with coefficients in $\mathbb{Z}[\mathsf{a}-\mathsf{b}]$ implies that $\#\mathscr{A}_w = \#\mathscr{B}_w$. Hence, there exists a bijection $\phi_w = \phi$ from $\mathscr{A}_w$ to $\mathscr{B}_w$ for each w, and we can define $\mathscr{C}_w$ to be $\{\mathrm{T}_i^{(a)} \cup \phi\mathrm{T}_i^{(a)} \mid \mathrm{T}_i^{(a)} \in \mathscr{A}_w\}$, and then $\{\mathscr{C}_w, \mathscr{D}_w\}$ is a combinatorial partition as required. □

Note that we do not have that λ is a *strongly* combinatorial CL-labeling because although those chains in the block $\mathrm{T}_i^{(a)} \cup \phi\mathrm{T}_i^{(a)}$ that begin with ascents pass through one atom (a) and those that begin with a descent pass through another atom $(\phi(a))$, it could well be that $\lambda_{\{\hat{0}\}}(\hat{0}, a) < \lambda_{\{\hat{0}\}}(\hat{0}, \phi(a))$. We will describe such Eulerian posets, labelings and partitions as *almost strongly combinatorial* Eulerian posets, labelings and partitions when they satisfy all of the conditions for a strong combinatorial labeling except this one. It would be very nice to find a combinatorial way for finding the bijection ϕ_w in this proof, especially if we could find one that would have $\lambda_{\{\hat{0}\}}(\hat{0}, a) > \lambda_{\{\hat{0}\}}(\hat{0}, \phi(a))$ (since if we could do this in general, it would prove the nonnegativity of the *cd*-index in general).

Porism 7.2. *Under the same hypothesis as Theorem* 7.1, λ *is almost strongly combinatorial.*

A polytope is said to be *simplicial* whenever each of its facets is a simplex. A polytope is said to be *simple* whenever each of its vertex figures is a simplex, or, equivalently, whenever it is the dual of a simplicial polytope. Similar definitions

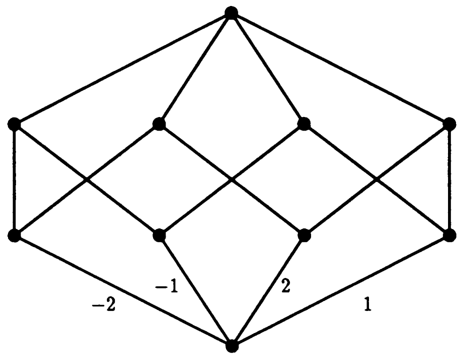

FIGURE 4. This cannot be completed to a standard labeling.

can be stated for Eulerian posets and lattices; for instance, an Eulerian poset is *simple* if every interval $[a, \hat{1}]$ is isomorphic to a boolean algebra for all $a \neq \hat{0}$.

For the lattice of faces $\mathbf{L} = \mathbf{L}(\mathscr{P})$ of any simple polytope $\mathscr{P}$ of dimension n (or any other simple, CL-labelable Eulerian poset $\mathbf{L}$ of rank $n+1$) we can create a CL-labeling that satisfies the hypothesis of Theorem 7.1. If $n \leq 0$, there is nothing to do; otherwise, order the facets of $\mathscr{P}^*$ in a shelling order (Theorem 3.6) or some other recursive atom order for $\mathbf{L}$ (which exists because $\mathbf{L}$ is CL-shellable). By definition of dual shelling order (recursive atom order), we have dual shelling orders (recursive atom orders) of each $[a, \hat{1}]$, each of which is isomorphic to $\mathbf{B}_n$. Now, define the first two levels of λ as in Algorithm 3.3. Then continue so that each $[a, \hat{1}]$ is labeled in the X-standard labeling, where X is the set of labels of the form $\lambda_{\{\hat{0} \lessdot a\}}(a, b)$; this set has the right cardinality by the choice of labels in Algorithm 3.3. As in that result, the resulting λ is a CL-labeling. Now, by our previous results, the X-standard labelings are strongly combinatorial, and by Corollary 2.6, $\mathbf{L}$ has a *cd*-index. Hence we have:

Corollary 7.3. *Any CL-shellable simple Eulerian poset has a combinatorial cd-index, and hence has nonnegative cd-index.*

Corollary 7.4. *Any CL-shellable simplicial Eulerian poset has a dually combinatorial cd-index, and hence has nonnegative cd-index.*

Unfortunately, we do not have similar results for cubic and dual-cubic polytopes, because it is *not* true that any ordering of the facets of a cube gives rise to a standard labeling of its dual—see Figure 4.

If we weaken the hypotheses of Theorem 7.1 to only require that each interval $[a, \hat{1}]$ is *almost* strongly combinatorially labeled, then the proof will not go through, since there will be the additional case of $k < l < m$, which will give a term of the form $(\mathsf{a}^2 + \mathsf{b}^2)w = (\mathsf{c}^2 - \mathsf{d})w$. However, since this term will come from a block having all of its chains pass through a single atom (a), we can state the following:

Proposition 7.5. *Every Eulerian poset's cd-index can be written as a nonnegative noncommuting polynomial in the variables* c, d *and* $\mathsf{e} := \mathsf{c}^2 - \mathsf{d}$.

Of course, such a noncommuting polynomial need not be unique, even if all possible $\mathsf{d} + \mathsf{e}$'s are replaced by c^2's: $\mathsf{c}^2\mathsf{d} + \mathsf{d}\mathsf{c}^2 - \mathsf{d}^2 = \mathsf{e}\mathsf{d} + \mathsf{d}\mathsf{c}^2 = \mathsf{d}\mathsf{e} + \mathsf{c}^2\mathsf{d}$.

However, in [2], Bayer and Klapper produce a recursion for the *cd*-index of a polytope, which in our notation is

$$2cd(\mathscr{P}) = \sum_{j=1}^{t} cd(F_j)\mathsf{c} + cd(E_j)(2\mathsf{d} - \mathsf{c}^2) \ , \tag{7.2}$$

where $F_1, \ldots, F_t$ is a line shelling of the polytope $\mathscr{P}$ and E_j is the boundary of $F_j \cap \bigcup_{i<j} F_i$ in ∂F_j. It is relatively easy to see that E_j is a polyhedral complex isomorphic (as a polyhedral complex) to a polytope, which we also call E_j. (Since the lattice of faces of the two E_j's is the same, no confusion will result.) The facets of E_j are faces of $\mathscr{P}$ of codimension 3.

As a result of (7.2), we can inductively show that the *cd*-index of any polytope can be written as a nonnegative noncommuting polynomial of the variables c, d and $\mathsf{e}' := 2\mathsf{d} - \mathsf{c}^2$. This is not enough to show that the *cd*-index is positive, even with Proposition 7.5, as can be seen from the fact that $\mathsf{c}^4 + \mathsf{d}(2\mathsf{d} - \mathsf{c}^2) = \mathsf{c}^4 - \mathsf{d}\mathsf{c}^2 + 2\mathsf{d}^2 = 2\mathsf{d}^2 + (\mathsf{c}^2 - \mathsf{d})\mathsf{c}^2$.

However, we can use this recursion to get the following result:

Proposition 7.6. *Let $\mathscr{P}$ be a polytope such that the CL-labeling λ of $\mathbf{L} := \mathbf{L}(\mathscr{P})$ coming from a line shelling of the dual polytope $\mathscr{P}^*$ is such that for each element $b \in \mathbf{L}$ of rank 2, λ restricted to $[b, \hat{1}]$ is a strongly combinatorial labeling. Suppose further that in the line shelling of $\mathscr{P}^*$ that gives rise to λ, each E_j has a nonnegative cd-index. Then λ is a combinatorial labeling of $\mathbf{L}$.*

Proof. By Porism 7.2, for each atom a, the interval $[a, \hat{1}]$ is almost strongly combinatorially labeled by λ. So we can mimic the proof of Theorem 7.1 to get

$$cd(\mathscr{P}) = \sum_{w} \#\mathscr{C}_w \, \mathsf{c}w + \sum_{w} \left(\#\mathscr{D}_w \, \mathsf{d}w + \#\mathscr{E}_w \, (\mathsf{c}^2 - \mathsf{d})w \right) \ , \tag{7.3}$$

where each $\Pi \in \mathscr{C}_w$ sums to $\mathsf{c}w$, each $\Pi \in \mathscr{D}_w$ sums to $\mathsf{d}w$ and each $\Pi \in \mathscr{E}_w$ sums to $(\mathsf{c}^2 - \mathsf{d})w$.

From (the dual formulation of) the recursion (7.2), we have

$$cd(\mathscr{P}) = \tfrac{1}{2} \sum_{i=1}^{t} \mathsf{c} \cdot cd([a_i, \hat{1}]) + (2\mathsf{d} - \mathsf{c}^2) cd(E_i^*) \ , \tag{7.4}$$

where E_i^* is the dual polytope to E_i. Its lattice of faces can be constructed by taking the union of certain intervals of the form $[x, \hat{1}]$, where $\rho(x) = 3$, and adding a $\hat{0}$. By assumption (and Lemma 2.7), $cd(E_j^*)$ is nonnegative. Furthermore, each polytope F_i satisfies Theorem 7.1, and hence, each F_i has nonnegative *cd*-index.

Hence, the only negative terms of the right-hand side of (7.4) are of the form $\mathsf{c}^2 w$, while the only negative terms of the right-hand side of (7.3) are of the form $\mathsf{d}w$. Since the left-hand sides of the two equations are the same, the right-hand sides must match, and hence neither one has any negative terms, that is, the negative terms must be cancelled out by positive terms.

In the case of (7.3), this implies that for each w, $\#\mathscr{D}_w \geq \#\mathscr{E}_w$. So we may pair each partition block in $\mathscr{E}_w$ with a partition block in $\mathscr{D}_w$ to form new blocks which, along with the other blocks of $\mathscr{D}_w$ and the blocks of $\mathscr{C}_w$,

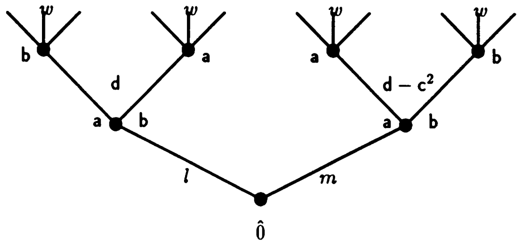

FIGURE 5. A union of blocks as formed in the proof of Proposition 7.6.

form our combinatorial partition of $\mathcal{MC}(\mathbf{L})$. However, this partition is not even almost strongly combinatorial, since the new blocks that are formed by the union of a block from $\mathcal{D}_w$ and a block from $\mathcal{E}_w$ are as shown in 7: although all chains pass through one of two atoms, chains starting with both ascents and descents pass through both atoms. (However, these are the only blocks that do not satisfy the almost combinatorial condition.) □

Corollary 7.7. *Let $\mathcal{P}$ be a polytope such that each vertex figure of $\mathcal{P}$ is simple. Then $\mathcal{P}$ has a combinatorial cd-index.*

Proof. Note that there exists a line shelling for $\mathcal{P}^*$. In this shelling, each E_j is simplicial (since each facet of $\mathcal{P}$ is simplicial) and hence has nonnegative *cd*-index. Hence, Proposition 7.6 proves the result. □

Corollary 7.8. *Let $\mathcal{P}$ be a polytope such that each facet of $\mathcal{P}$ is simplicial. Then $\mathcal{P}$ has dually combinatorial cd-index.*

Proposition 7.9. *Let $\mathcal{P}$ be a polytope with face-lattice labeled by a CL-labeling λ coming from a line-shelling of $\mathcal{P}^*$. If each E_j^* in this line shelling has a nonnegative cd-index, and each vertex figure of $\mathcal{P}$ satisfies the conditions of Proposition 7.6, then $\mathcal{P}$ has a combinatorial cd-index.*

Proof. By the proof of Proposition 7.6, each vertex figure $[a, \hat{1}]$ is almost combinatorially labeled by λ, except for blocks of the sort shown in Figure 5. We can proceed as in the proof of Proposition 7.6 to get

$$cd(\mathcal{P}) = \sum_w \#\mathcal{C}_w\, \mathsf{c}w + \sum_w \#\mathcal{D}_w\, \mathsf{d}w + \sum_w \#\mathcal{E}_w\, (\mathsf{c}^2 - \mathsf{d})w$$
$$+ \text{(blocks as in Figure 5)}\,.$$

Consider such a block, and assume the label going to the $\hat{0}$ of Figure 5 (which is an atom of $\mathcal{P}$) is k. Then we have four cases: $k < l, m$; $k > l, m$; $l > k > m$ or $l < k < m$. The first two produce elements of $\mathcal{A}_{\mathsf{c}^2 w}$ and $\mathcal{B}_{\mathsf{c}^2 w}$ and match up as usual. For the other two cases, we get

$$l > k > m: \qquad (\mathsf{a}(\mathsf{ab} + \mathsf{ba}) + \mathsf{b}(\mathsf{a}^2 + \mathsf{b}^2))w\,,$$
$$l < k < m: \qquad (\mathsf{b}(\mathsf{ab} + \mathsf{ba}) + \mathsf{a}(\mathsf{a}^2 + \mathsf{b}^2))w\,.$$

Neither of these can be written in terms of c and d (so they will not interfere with the $\mathcal{D}_w$ and $\mathcal{E}_w$ terms, which match as in Proposition 7.6). However, they sum to $\mathsf{cd} + \mathsf{c}(\mathsf{c}^2 - \mathsf{d}) = \mathsf{c}^3$, and in fact it is easy to see that the existence of the *cd*-index implies that they match up just as blocks in $\mathscr{A}_w$ and $\mathscr{B}_w$ did before. Hence, the *cd*-index is again combinatorial. □

Corollary 7.10. *For any polytope such that every interval of the form* $[b, \hat{1}]$, $\rho(b) = 3$, *is a simplex, the cd-index is combinatorial.*

Proof. The vertex figures of such a polytope satisfy Proposition 7.6 as in Corollary 7.7, and the E_j^*'s will have simplicial facets, so by Corollary 7.8, they have nonnegative *cd*-indexes. □

Since every 1-polytope is a simplex, this implies that every polytope of dimension 4 or less has a combinatorial *cd*-index. However, we can do one better:

Proposition 7.11. *If* $\mathscr{P}$ *is a polytope of dimension* 5 *or less, it has a combinatorial cd-index.*

Proof. For dimensions less than 5, the result follow from the remarks just made. For dimension 5, each E_j^* is of dimension 3, and hence has nonnegative *cd*-index. Furthermore, let $\mathscr{V}$ be a vertex figure of $\mathscr{P}$; its E_j's are of dimension 2 and thus have nonnegative *cd*-indexes. Then $\mathscr{V}$ is of dimension 4, and if $b \in \mathbf{L}(\mathscr{V})$ is of rank 2, then b is of rank 3 in $\mathbf{L}(\mathscr{P})$, and the interval $[b, \hat{1}]$ is of rank 3 (in both) and hence is the lattice of faces of a 2-polytope, that is, an n-gon. But it is easy to see that any line shelling of an n-gon gives rise to a strongly combinatorial labeling, so $\mathscr{V}$ satisfies Proposition 7.6. Hence, by Proposition 7.9, the result follows. □

It seems that it might be possible to extend this technique further, but for the moment we will stop here.

8. Conclusions

We have given a combinatorial interpretation of the *cd*-index for several types of polytopes, most notably simple and simplicial polytopes. This implies that the *cd*-indexes of these polytopes have nonnegative coefficients. It would be most interesting if these techniques could be extended to other classes of polytopes, especially to all polytopes. The remarks after Theorem 7.1 may provide a clue as to how to proceed in this direction.

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Note added in proof. Richard Stanley has recently proven Conjecture 1.1.

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