

INVERSE SCATTERING FOR SINGULAR POTENTIALS IN TWO DIMENSIONS

ZIQI SUN AND GUNTHER UHLMANN

ABSTRACT. We consider the Schrödinger equation for a compactly supported potential having jump type singularities at a subdomain of \mathbb{R}^2 . We prove that knowledge of the scattering amplitude at a fixed energy, determines the location of the singularity as well as the jump across the curve of discontinuity. This result follows from a similar result for the Dirichlet to Neumann map associated to the Schrödinger equation for a compactly supported potential with the same type of singularities.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this paper we consider the Schrödinger equation for a compactly supported potential, q , having jump type singularities at a subdomain of \mathbb{R}^2 . We prove that knowledge of the scattering amplitude at a fixed energy, λ_0 , determines the location of the singularity as well as the jump across the curve of discontinuity.

This problem is reduced to the study of the Dirichlet to Neumann map for the Schrödinger operator $-\Delta + q - \lambda_0^2$ in a bounded domain of \mathbb{R}^2 . (For the application considered here it is enough to consider the domain to be a ball containing the support of q .) We prove that in dimension two the Dirichlet to Neumann map for the Schrödinger operator $-\Delta + q - \lambda_0^2$ determines uniquely the location of the singularity of q as well as its jump across the curve of discontinuity.

The scattering amplitude of a potential $q \in L^\infty(\mathbb{R}^n)$ with compact support is defined via the outgoing eigenfunctions. Namely, $\forall \lambda \in \mathbb{R} \setminus 0$, $\omega \in S^{n-1}$, there exists $\psi_+(\lambda, x, \omega)$, solution of

$$(1.1) \quad (-\Delta + q - \lambda^2)\psi_+ = 0$$

satisfying

$$(1.2) \quad \psi_+ = e^{i\lambda x \cdot \omega} + \frac{a_q(\lambda, \theta, \omega)e^{i\lambda|x|}}{|x|^{(n-1)/2}} + O(|x|^{-((n-1)/2+1)})$$

with $\theta = x/|x|$. The scattering amplitude, $a_q(\lambda, \theta, \omega)$, measures the effect of the potential q on plane waves of the form $e^{i\lambda x \cdot \omega}$.

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The inverse scattering problem at a fixed frequency $\lambda_0 \in \mathbb{R} \setminus 0$ is to study the map

$$(1.3) \quad q \xrightarrow{\mathcal{A}_{\lambda_0}} a_{q, \lambda_0}$$

where

$$(1.4) \quad a_{q, \lambda_0}(\theta, \omega) = a_q(\lambda_0, \theta, \omega).$$

In dimensions $n \geq 3$ the map \mathcal{A}_{λ_0} is formally overdetermined in the sense that we want to determine a function of n -variables from a function depending on $2(n-1)$ variables. Injectivity of \mathcal{A}_{λ_0} was proved by Novikov [No] in this case. This result can also be proven, as we shall indicate below, as a consequence of the global uniqueness result proven by Sylvester and Uhlmann for the Dirichlet to Neumann map $\Lambda_{q-\lambda_0^2}$ [S-U, I]. See §2 for more details.

However, injectivity of \mathcal{A}_{λ_0} is an open question in the 2-dimensional case. The difficulty is that this is a formally determined problem in this case. Injectivity of \mathcal{A}_{λ_0} for q close to zero was proven by Novikov [No]. Again this result follows from the corresponding result for the Dirichlet to Neumann map, $\Lambda_{q-\lambda_0^2}$, proven by Sylvester and Uhlmann [S-U, II]. Sun and Uhlmann [Su-U, I] used the results of [Su-U, II] for the Dirichlet to Neumann to prove that \mathcal{A}_{λ_0} is locally injective near most potentials (an open and dense set of potentials in the $W^{1, \infty}$ topology) and globally injective for pairs of potentials in an open and dense set in the $W^{1, \infty} \times W^{1, \infty}$ topology.

In this paper we consider potentials $q_j \in L^\infty(\mathbb{R}^2)$, $j = 1, 2$, with compact support, having jump type singularities across the boundary of a bounded smooth domain $\Omega_0^j \subset \mathbb{R}^2$, $j = 1, 2$. More precisely,

Theorem 1. *Let $q_j \in C^1(\overline{\Omega_0^j})$, $q_j \in C^1(\mathbb{R}^2 \setminus \Omega_0^j)$, $j = 1, 2$. Also assume that $(q_j^+ - q_j^-)|_{\partial\Omega_j} \neq 0$, $j = 1, 2$. If $\mathcal{A}_{\lambda_0}(q_1) = \mathcal{A}_{\lambda_0}(q_2)$, then $\Omega_0^1 = \Omega_0^2$ and $(q_1^+ - q_1^-)|_{\partial\Omega_0^1} = (q_2^+ - q_2^-)|_{\partial\Omega_0^2}$ where q_j^+ (resp. q_j^-) denotes the interior (resp. exterior) limit of q_j on $\partial\Omega_0^j$.*

We now define the Dirichlet to Neumann map. Assume that $q \in L^\infty(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded smooth domain. Assume also that $\lambda_0^2 \in \mathbb{R} \setminus 0$ is not a Dirichlet eigenvalue for $-\Delta + q$. Then for every $f \in H^{1/2}(\partial\Omega)$ there is a unique solution $u \in H^1(\Omega)$ of the Dirichlet problem

$$(1.5) \quad (-\Delta + q - \lambda_0^2)u = 0, \quad u|_{\partial\Omega} = f.$$

Then we define

$$(1.6) \quad \Lambda_{q-\lambda_0^2}(f) = \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega}$$

with u a solution of (1.5) and ν the outer unit normal to $\partial\Omega$. The inverse problem is then to study the map

$$(1.7) \quad q \xrightarrow{\Lambda} \Lambda_{q-\lambda_0^2}.$$

Injectivity of Λ was proven in [S-U, I] in the case $n \geq 3$ for $q \in C^\infty(\Omega)$. This was extended to $q \in L^\infty(\Omega)$ in [N-S-U]. This problem is formally overdetermined in this case. For the formally determined 2-dimensional case injectivity of Λ was proved in a $W^{1, \infty}(\Omega)$ neighborhood of $q = 0$ [S-U, II]. This

was extended by Sun to potentials close to constant [Su, I]. Recently, Sun and Uhlmann [Su-U, II] proved that Λ is injective near most potentials (an open and dense set in $W^{1,\infty}(\Omega)$). In this paper we prove

Theorem 2. *Let $q_j \in L^\infty(\Omega)$ where Ω is a bounded smooth domain in \mathbb{R}^2 and $q_j \in C^1(\overline{\Omega}_0^j)$, $q_j \in C^1(\Omega \setminus \Omega_0^j)$ where Ω_0^j is a bounded smooth domain such that $\overline{\Omega}_0^j \subset \Omega$, $j = 1, 2$. Assume that $\lambda_0^2 \in \mathbb{R}$ is not an eigenvalue for $-\Delta + q_j$, and $(q_j^+ - q_j^-)|_{\partial\Omega_0^j} \neq 0$, $j = 1, 2$. If*

$$(1.8) \quad \Lambda_{q_1 - \lambda_0^2} = \Lambda_{q_2 - \lambda_0^2},$$

then

$$\Omega_0^1 = \Omega_0^2 \quad \text{and} \quad (q_1^+ - q_1^-)|_{\partial\Omega_0^1} = (q_2^+ - q_2^-)|_{\partial\Omega_0^2}$$

with q_j^\pm defined as in Theorem 1.

In §2 we shall show that Theorem 2 implies Theorem 1. The proof of Theorem 2 consists in showing that the hypothesis (1.8) implies that $q_1 - q_2 \in C^0(\Omega)$. This is done by using an extension of the special solutions constructed in [S-U, II] to potentials with jump type singularities across a submanifold.

Isakov [I] has considered earlier the inverse transmission problem and obtained several interesting uniqueness results. However, his methods do not apply here.

The plan of this paper is as follows: In §2 we shall prove Theorem 1 and Theorem 2, and in §3 we shall prove a technical proposition needed in §2.

2. PROOFS OF THE THEOREMS

First we show how to prove Theorem 1 using Theorem 2. It is well known by now how to relate the Dirichlet to Neumann map $\Lambda_{q - \lambda_0^2}$ with the fixed energy scattering amplitude $a(\lambda_0, \theta, \omega)$ in any dimension. We sketch the procedure below. We assume $q \in L^\infty(\mathbb{R}^n)$, $\text{supp } q \subseteq B(0, R)$. Let $G_q(x, y, \lambda_0)$ be the outgoing Green's kernel for $-\Delta + q - \lambda_0^2$. The single-layer operator, which is an invertible operator from $H^{1/2}(\partial\Omega)$ to $H^{3/2}(\partial\Omega)$, is defined by

$$(2.1) \quad \mathcal{S}_{q - \lambda_0^2} f(x) = \int_{\partial B(0, R)} G_q(x, y, \lambda_0) f(y) dS$$

where dS denotes surface measure.

It was proven in [N] (see Theorem 1.6; the proof is also valid in two dimensions) that

$$(2.2) \quad \Lambda_{q - \lambda_0^2} \rightarrow \mathcal{S}_{q - \lambda_0^2}$$

is injective. More precisely (see (1.40) in [N])

$$(2.3) \quad \Lambda_{q - \lambda_0^2} = \Lambda_{-\lambda_0^2} + \mathcal{S}_{q - \lambda_0^2}^{-1} - (\mathcal{S}_{\lambda_0^2}^+)^{-1}$$

where $\mathcal{S}_{q - \lambda_0^2}^+$ is as in (2.1) with $q = 0$. Next we sketch how to prove that the map

$$(2.4) \quad \mathcal{S}_{q - \lambda_0^2} \rightarrow \mathcal{A}_{\lambda_0}(q)$$

is injective.

This is an old result of Berezanskii [B] who showed how to go from the far field $(\mathcal{A}_{\lambda_0}(q))$ to the near field $(\mathcal{S}_{q-\lambda_0^2})$. The main element is the asymptotic expansion of the outgoing Green's kernel, namely

$$(2.5) \quad G_q(x, y, \lambda_0) = \frac{e^{i\lambda_0|x|}}{|x|^{(n-1)/2}} \psi_+(\lambda_0, y, \theta) + O(|x|^{-(n-1)/2-1})$$

with $\theta = -x/|x|$ and ψ_+ the outgoing eigenfunction. Now if $\mathcal{A}_{\lambda_0}(q_1) = \mathcal{A}_{\lambda_0}(q_2)$, by (2.5) and (1.1) we get

$$(2.6) \quad G_{q_1}(x, y, \lambda_0) - G_{q_2}(x, y, \lambda_0) = O(|x|^{-(n-1)/2-1}|y|^{-(n-1)/2-1}).$$

Now

$$\varphi(x, y) = G_{q_1}(x, y, \lambda_0) - G_{q_2}(x, y, \lambda_0)$$

solves

$$(-\Delta_x - \lambda_0^2)\varphi = 0 \quad \text{for } |x| \geq R, |y| \geq R.$$

Therefore by Rellich's lemma we obtain that

$$G_{q_1}(x, y, \lambda_0) = G_{q_2}(x, y, \lambda_0) \quad \text{for } |x|, |y| \geq R$$

proving the injectivity of the map (2.4).

In this way, we have reduced the proof of Theorem 1 to the proof of Theorem 2.

Theorem 2 follows readily from the following

Proposition 2.1. Assume $\Omega, \lambda_0, \Omega_0^j, q_j, j = 1, 2$, as in Theorem 2 with $\Lambda_{q_1-\lambda_0^2} = \Lambda_{q_2-\lambda_0^2}$. Then $q_1 - q_2 \in C^0(\Omega)$.

Before giving a proof of Proposition 2.1, we need to recall the construction of special solutions of $(-\Delta + q)u = 0$, where $q \in L^\infty$ with compact support.

Proposition 2.2 [S-U, II]. Let $\zeta \in \mathbb{C}^2$ with $\zeta \cdot \zeta = 0$, $q \in L^\infty(\mathbb{R}^2)$ with compact support, and $-1 < \delta < 0$. Then there exists a constant $C_1 = C_1(\delta)$ such that for $|\zeta| > C_1 \|(1 + |x|^2)q\|_{L^\infty(\mathbb{R}^2)}$, there exists a unique solution of $(-\Delta + q)u_q = 0$ in \mathbb{R}^2 of the form

$$(2.7) \quad u_q(x, \zeta) = e^{x \cdot \zeta} (1 + \psi_q(x, \zeta))$$

with $\psi_q \in L_\delta^2(\mathbb{R}^2)$. Furthermore there exists a constant $C_2 = C_2(\delta, \|q\|_{L^\infty(\mathbb{R}^2)})$ such that

$$(2.8) \quad \|\psi_q\|_{L_\delta^2(\mathbb{R}^2)} \leq C_2/|\zeta|.$$

Here $L_\delta^2(\mathbb{R}^2)$ is the weighted L^2 space:

$$(2.9) \quad L_\delta^2(\mathbb{R}^2) = \left\{ f; \int_{\mathbb{R}^2} (1 + |x|^2)^\delta |f(x)|^2 dx < \infty \right\}.$$

If we choose

$$(2.10) \quad \zeta = \frac{1}{2}(ik + Jk), \quad k = (k_1, k_2) \in \mathbb{R}^2, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i = \sqrt{-1},$$

then a straightforward computation shows that

$$\bar{\partial}(\partial + (k_2 + ik_1))\psi_q - q\psi_q = q.$$

The above proposition follows directly from the lemma below.

Lemma 2.1 [S-U, II]. *Let $g \in L^\infty(\mathbb{R}^2)$, with compact support and $f \in L^2_{\delta+1}(\mathbb{R}^2)$, $-1 < \delta < 0$, and $l \in \mathbb{C}$. Then there exists a constant $C_3 = C_3(\delta)$ such that for $|l| > C_3\|(1+|x|^2)g\|_{L^\infty(\mathbb{R}^2)}$ there exists a unique function $\omega \in L^2_\delta(\mathbb{R}^2)$ satisfying*

$$(2.11) \quad \bar{\partial}(\partial + l)\omega - g\omega = f \quad \text{in } \mathbb{R}^2.$$

Moreover

$$\|\omega\|_{L^2_\delta(\mathbb{R}^2)} \leq (C_4/|l|)\|f\|_{L^2_{\delta+1}(\mathbb{R}^2)},$$

where $C_4 = C_4(\delta, \|g\|_{L^\infty(\mathbb{R}^2)})$ is a constant.

In [S-U, II] it was also proven that if we assume in addition that $q \in C^1(\mathbb{R}^2)$ then ψ_q has an asymptotic expansion for $|\zeta|$ large of the form

$$(2.12) \quad \psi_q(x, \zeta) = \frac{a(x)}{k_2 + ik_1} + \frac{b(x, k)}{(k_2 + ik_1)^2}, \quad x \text{ in } \mathbb{R}^2,$$

with $a, b \in L^2_\delta(\mathbb{R}^2)$, $-1 < \delta < 0$. Moreover,

$$\|a\|_{L^2_\delta(\mathbb{R}^2)} + \|b\|_{L^2_\delta(\mathbb{R}^2)} < C_5$$

where $C_5 = C_5(\delta, \|q\|_{C^1(\mathbb{R}^2)})$ is a constant.

Here a satisfies

$$(2.13) \quad \bar{\partial}a = q$$

and b satisfies

$$(2.14) \quad \bar{\partial}(\partial + k_2 + ik_1)b - qb = k(qa - 4\partial q).$$

In our case the difficulty in proving an expansion of the form (2.12) is that we cannot directly use the proof in [S-U, II] to conclude $b \in L^2_\delta(\mathbb{R}^2)$ since the right-hand side of (2.14) is not even a function. A very similar problem was considered in [Su, II]. An expansion of the form (2.12) was proven there if ∂q has delta type singularities on the boundary of simply connected smooth subdomains of Ω . In our case, we need to prove such an expansion for potentials having jump type singularities across general smooth subdomain of Ω , not just simply connected ones. However, this difficulty can be dealt with, as stated in the next proposition. The proof of this result will be given in the next section.

Proposition 2.3. *Let $q \in L^\infty(\mathbb{R}^2)$ with compact support and $\Omega_0 \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Let $q \in C^1(\bar{\Omega}_0)$, and $q \in C^1(\mathbb{R}^2 \setminus \Omega_0)$. Let $\zeta \in \mathbb{C}^2$ with $\zeta \cdot \zeta = 0$, $-1 < \delta < 0$. Then for $|\zeta| > C_1\|(1+|x|^2)q\|_{L^\infty(\mathbb{R}^2)}$, the unique solution of $(-\Delta + q)u = 0$ as in (2.7) satisfies*

$$(2.15) \quad \psi_q(x, \zeta) = \frac{a(x)}{k_2 + ik_1} + \frac{b(x, k)}{(k_2 + ik_1)^2}, \quad x \text{ in } \mathbb{R}^2,$$

with

$$\|a\|_{L^2_\delta(\mathbb{R}^2)} + \|b(x, k)\|_{L^2_\delta(\mathbb{R}^2)} < C_6$$

and $C_6 = C_6(\delta, \|q\|_{C^1(\bar{\Omega}_0)}, \|q\|_{C^1(\mathbb{R}^2 \setminus \Omega_0)})$ is a constant.

Given $q \in L^\infty(\Omega)$ and the corresponding special solution ψ_q as in (2.4) (we extend $q = 0$ outside Ω). We consider the function T_q considered in the

$\bar{\partial}$ approach to the inverse scattering problem by Beals and Coifman [B-C] and Ablowitz and Nachman [N-A], which is defined by

$$(2.16) \quad T_q(k) = \int_{\Omega} e^{ix \cdot k} q(x) (1 + \psi_q(x, \xi_k)) dx$$

where $k = (k_1, k_2) \in \mathbb{R}^2$, $|k|$ large and

$$(2.17) \quad \xi_k = \frac{1}{2}(Jk + ik), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

An important fact about T_q is that knowledge of Λ_q determines T_q uniquely as a function of k for $|k|$ large (see [N-S-U]). The proof of this fact follows readily by integrating by parts and a result in [S-U, II]. Namely, $\psi_q(x, \xi_k)$ satisfies

$$(2.18) \quad \Delta \psi_q + 2\xi_k \cdot \nabla \psi_q = q \psi_q + q.$$

Multiplying (2.18) by $e^{ix \cdot k}$, integrating both sides of (2.15) over Ω , and using Green's theorem we get

$$(2.19) \quad \begin{aligned} T_q(k) &= \int_{\Omega} e^{ix \cdot k} q(x) (1 + \psi_q(x, \xi_k)) dx \\ &= \int_{\partial\Omega} \left[e^{ix \cdot k} \frac{\partial \psi_q}{\partial \nu} - \psi_q \frac{\partial}{\partial \nu} (e^{ix \cdot k}) + 2e^{ix \cdot k} (\xi_k \cdot \nu) \psi_q \right] dS. \end{aligned}$$

Thus, $T_q(k)$ depends only on the boundary values of ψ_q and of $\partial \psi_q / \partial \nu$. Applying Proposition 2.1 of [S-U, II] we have that these boundary values are determined uniquely by Λ_q . Therefore T_q is determined uniquely by Λ_q .

In the proof below we shall show that T_q contains all the information about the location of the singularity of q as well as the jump of q across the curve of discontinuity. A very interesting problem is to reconstruct the location of singularities of the potential and the jump across the surface of discontinuity from T_q .

Proof of Theorem 2 using Proposition 2.3. Let $R = C_1 \|(1 + |x|^2)q\|_{L^\infty}$ where C_1 is as in Proposition 2.2. We assume $R > 1$. We shall show that

$$(2.20) \quad |k|^s (T_q(k) - \hat{q}(k)) \in L^2(|k| > R), \quad 0 \leq s < 1.$$

By (2.19),

$$T_q(k) - \hat{q}(k) = \int_{\Omega} e^{ix \cdot k} q \psi_q(x, \xi_k) dx.$$

By Proposition 2.3 we have that

$$\begin{aligned} |k|^s \left| \int_{\Omega} e^{ix \cdot k} q \psi_q(x, \xi_k) dx \right| &\leq \frac{1}{|k|^{1-s}} \left| \int_{\Omega} e^{ix \cdot k} q(x) a(x) dx \right| \\ &\quad + \frac{1}{|k|^{2-s}} \int_{\Omega} |q(x) b(x, k)| dx. \end{aligned}$$

Thus we get

$$\begin{aligned} & \left(\int_{|k|>R} |k|^{2s} |T_q(k) - \hat{q}(k)|^2 dk \right)^{1/2} \leq \left\| \int_{\Omega} e^{ix \cdot k} q(x) a(x) dx \right\|_{L^2(\mathbb{R}^2)} \\ & + \left(\int_{|k|>R} \left(\frac{1}{|k|^{2-s}} \right)^2 \left(\int_{\Omega} |q(x) b(x, k)| dx \right)^2 dk \right)^{1/2} \\ & = I_1 + I_2. \end{aligned}$$

By using Plancherel's Theorem we obtain

$$I_1 = \left\| \int_{\mathbb{R}^2} e^{ix \cdot k} q(x) a(x) dx \right\|_{L^2(\mathbb{R}^2)} = (2\pi)^2 \|q(x) a(x)\|_{L^2(\mathbb{R}^2)}$$

and by the estimate (2.12) on $a(x)$ we get

$$I_1 \leq \|q\|_{L^\infty(\Omega)} \|a(x)\|_{L^2_s(\mathbb{R}^2)} < +\infty.$$

Using the estimate on $b(x, k)$ in Proposition 2.3 we obtain

$$I_2 \leq \|q\|_{L^\infty(\Omega)} \left(\int_{|k|>R} \left(\frac{1}{|k|^{2-s}} \right)^2 \|b(\cdot, k)\|_{L^2(\Omega)}^2 dk \right)^{1/2} \leq C \|q\|_{L^\infty(\Omega)} < \infty$$

for some constant $C < \infty$. Thus, we have proven (2.20). Now let q_j , $j = 1, 2$, be given satisfying the hypotheses of Theorem 2. The assumption $\Lambda_{q_1} = \Lambda_{q_2}$ implies

$$T_{q_1}(k) = T_{q_2}(k) \quad \text{for } |k| \text{ large.}$$

Thus

$$\hat{q}_1(k) - \hat{q}_2(k) = \int_{\Omega} e^{ix \cdot k} q_2 \psi_{q_2} dx - \int_{\Omega} e^{ix \cdot k} q_1 \psi_{q_1} dx$$

and by (2.20)

$$|k|^s (\hat{q}_1(k) - \hat{q}_2(k)) \in L^2(|k| > R)$$

for $0 \leq s < 1$. This implies that

$$\hat{q}_1(k) - \hat{q}_2(k) \in L^2_s(\mathbb{R}^2)$$

and, consequently,

$$(2.21) \quad q_1(x) - q_2(x) \in H^s(\Omega) \quad \text{for all } s, 0 \leq s < 1.$$

To prove that q_1 and q_2 have the same location of singularity as well as jump across the discontinuities, it is enough to show that

$$(2.22) \quad q_1 - q_2 \in C^0(\Omega).$$

This follows from (2.18) and the following trace lemma.

Lemma 2.2. *Let h be a piecewise C^1 function defined on Ω . Suppose $h \in H^s(\Omega)$, $s > 1/2$. Then $h \in C^0(\Omega)$.*

Proof. Suppose h is not in $C^0(\Omega)$. Since h is piecewise C^1 there exists a disc $B \subset \Omega$ and a simple C^1 curve $L \subset B$ such that when we restrict h on B , h is C^1 on $B \setminus L$ and h has jump type discontinuities on L . We may assume, without loss of generality that $B = \{|x| \leq \varepsilon\}$ and L is a portion of the x_1 -axis in B . Furthermore, by multiplying h by a suitable compactly

supported, positive function, one may assume further that h itself is supported in B . Now consider the function:

$$F: x_2 \rightarrow h(\cdot, x_2) \in L^2(\mathbb{R}_{x_1}), \quad x_2 \neq 0,$$

where $h(\cdot, x_2)$ is defined as a function of x_1 . By assumption, $h \in H^s(\mathbb{R}^2)$, $s > 1/2$. Thus by the standard trace theorem, F must be continuous as a map from \mathbb{R}_{x_2} , to $L^2(\mathbb{R}_{x_1})$. But this is not the case since

$$\lim_{x_2 \rightarrow 0^+} h(\cdot, x_2) \neq \lim_{x_2 \rightarrow 0^-} h(\cdot, x_2). \quad \square$$

3. PROOF OF PROPOSITION 2.3

This section is devoted to the proof of Proposition 2.3. We shall show that the special solutions ψ_q have expansions of the form (2.12) for $|k|$ large, where q has a jump type discontinuity across the boundary of a subdomain $\Omega_0 \subset \Omega$. The existence of such an expansion is proven in [Su, II] when Ω_0 is a simply connected domain. In the case that Ω_0 is not simply connected some difficulties arise and the method in [Su, II] cannot be applied directly.

First we claim that, under the hypotheses of Proposition 2.3, the function q can always be written as a sum of a finite number of other functions that carry only jump type discontinuity across some simply-connected domains. We state a simple lemma describing precisely such a decomposition of q .

Lemma 3.1. *Let q be a function in $L^\infty(\Omega)$ such that $q \in C^1(\Omega - \Omega_0)$ and $q \in C^1(\overline{\Omega_0})$, where $\overline{\Omega_0} \subset \Omega$ is a subdomain with smooth boundary. Then there exists a simply-connected domain D_j , and functions q_0, q_j , $1 \leq j \leq m$, such that*

- (1) $q = q_0 + \sum_{j=1}^m q_j$.
- (2) $q_0 \in C^1(\Omega)$.
- (3) $q_j|_{D_j} \in C^1(\overline{D_j})$, $q_j|_{\Omega \setminus D_j} \equiv 0$, $1 \leq j \leq m$.

Proof. The proof of this lemma is elementary but cumbersome. We only give a detailed proof for a special case. The complete proof can be done using induction.

Let D_1, D_2 , and D_3 be simply-connected domains with smooth boundary such that $\overline{D_1} \cap \overline{D_2} = \emptyset$, $\overline{D_j} \subseteq \Omega$, $j = 1, 2$, and $\overline{D_3} \subset D_2$. We consider the case that $\Omega_0 = D_1 \cup (D_2 \setminus D_3)$.

Let $q_0 \in C^1(\Omega)$ be a C^1 extension of $q|_{\Omega \setminus \Omega_0}$ to Ω , i.e., $q_0(x) = q(x)$ for $x \in \Omega \setminus \Omega_0$. Then let

$$q_1(x) = \begin{cases} q(x) - q_0(x), & x \in D_1, \\ 0, & x \in \Omega \setminus D_1. \end{cases}$$

Next, let $q_2^* \in C^1(D_2)$ be a C^1 extension of $q(x) - q_0(x)|_{D_2 \setminus D_3}$ to D_2 and set

$$q_2(x) = \begin{cases} q_2^*, & x \in D_2, \\ 0, & x \in \Omega \setminus D_2. \end{cases}$$

Finally set

$$q_3(x) = \begin{cases} q(x) - q_2(x), & x \in D_3, \\ 0, & x \in \Omega \setminus D_3. \end{cases}$$

Clearly, q_0, q_1, q_2 , and q_3 satisfy the condition of Lemma 3.1.

Hereafter, we assume $q = q_0 + \sum_{j=1}^m q_j$ where $q_0, q_j, 1 \leq j \leq m$, are given by the above lemma. For ψ_q to have an expansion of the form (2.2), a and b must satisfy

$$(3.1) \quad \bar{\partial}a = q,$$

$$(3.2) \quad \bar{\partial}(\partial + (k_2 + ik_1))b - qb = k(qa - 4\partial q).$$

As we mentioned earlier, the difficulty in obtaining the expansion (2.2) lies in the equation (3.2). More specifically, the discontinuity of q introduces a delta type singularity in ∂q , while the rest of the right-hand side of (3.2) gives no problem. We divide b into two parts

$$(3.3) \quad b = b_1 + b_2$$

where

$$(3.4) \quad \bar{\partial}(\partial + (k_2 + ik_1))b_1 - qb_1 = kqa - 4k\partial q_0,$$

$$(3.5) \quad \bar{\partial}(\partial + (k_2 + ik_1))b_2 - qb_2 = -4k \sum_{j=1}^m \partial q_j.$$

By Lemma 3.1, we note that $\partial q_0 \in L^\infty$ with compact support. (We define $(\partial q_0)(x) = 0$ when $x \in \mathbb{R}^2 \setminus \Omega$.) Then by Lemma 2.1, there exists a solution b_1 , satisfying (3.4) and

$$(3.6) \quad \|b_1(\cdot, k)\|_{L^2_\delta(\mathbb{R}^2)} \leq C$$

where C is independent of k .

To solve (3.5), we divide b_2 into two parts again:

$$(3.7) \quad b_2 = b_3 + b_4$$

where

$$(3.8) \quad \bar{\partial}(\partial + (k_2 + ik_1))b_3 = 4k \sum_{j=1}^m \partial q_j,$$

$$(3.9) \quad \bar{\partial}(\partial + (k_2 + ik_1))b_4 - qb_4 = qb_3.$$

The crux of the matter is to solve (3.8). Suppose for the moment that one gets a solution b_3 in $L^2_\delta(\mathbb{R}^2)$ for (3.8). Then by Lemma 2.4 again, there will be a solution b_4 for (3.9) and the whole problem is solved.

Let l be a simple closed and smooth curve in \mathbb{R}^2 , and let $h \in L^\infty(l)$. We define the distribution $\delta_{h,l} \in D'(\mathbb{R}^2)$ by

$$(3.10) \quad \delta_{h,l}(\varphi) = \int_l h(\tau)\varphi(\tau) d\tau, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2).$$

Let f be a function on \mathbb{R}^2 such that f is C^1 everywhere except in l , where f has a jump type discontinuity across l . We denote by $[f]_l$ its jump, that is

$$(3.11) \quad [f]_l(\sigma) = \lim_{\substack{x \rightarrow \sigma \\ \text{from inside } l}} f(x) - \lim_{\substack{x \rightarrow \sigma \\ \text{from outside } l}} f(x).$$

Using these notations we can write

$$\partial q_j = \delta_{(\nu(x_1) - i\nu(x_2))[q_j]/2, \partial D_j} + p_j, \quad 1 \leq j \leq m,$$

where $\nu(x_1)$, $\nu(x_2)$ denote the x_1 and x_2 components of ν and

$$(3.12) \quad p_j = \begin{cases} \partial q_j, & x \in D_j \text{ or } \mathbb{R}^2 \setminus \overline{D_j}, \\ 0, & x \in \partial D_j, \end{cases} \quad 1 \leq j \leq m,$$

and $z = x_1 + ix_2$. Then we split b_3 into

$$(3.13) \quad b_3 = \tilde{b}_3 + \sum_{j=1}^m b_3^j,$$

where

$$(3.14) \quad \bar{\partial}(\partial + (k_2 + ik_1))\tilde{b}_3 = 4k \sum_{j=1}^m p_j,$$

$$(3.15) \quad \bar{\partial}(\partial + (k_2 + ik_1))b_3^j = 4k\delta_{(\nu(x_1) - i\nu(x_2))[q_j]/2, \partial D_j}, \quad 1 \leq j \leq m.$$

From (3.12) we conclude $p_j \in L^\infty(\Omega)$, thus we can get a solution for (3.14) using Lemma 2.4. The existence of a solution for (3.15) follows directly from the proposition below.

Proposition 3.1. *Let l be a simple closed and smooth curve in \mathbb{R}^2 and let $h \in H^1(l)$. There exists a unique function ω in $L_\delta^2(\mathbb{R}^2)$, $-1 < \delta < 0$, satisfying*

$$(3.16) \quad \bar{\partial}(\partial + (k_1 + ik_2))\omega = \delta_{h,l}$$

provided that $k = k_1 + ik_2 \neq 0$. Moreover

$$(3.17) \quad \|\omega\|_{L_\delta^2(\mathbb{R}^2)} \leq (C/|k|)\|h\|_{L^2(l)}^{1/2}\|h\|_{H^1(l)}^{1/2}.$$

Proof. The proof of this proposition is similar to the proof of Lemma 5.1 in [Su, II]. We only give an outline of the proof. Let us recall two lemmas. Let L be either ∂ or $\bar{\partial}$.

Lemma 3.2 [S-U, II]. *Let $f \in L_{\delta+1}^2(\mathbb{R}^2)$, $-1 < \delta < 0$. Then there exists a unique solution $u \in L_\delta^2(\mathbb{R}^2)$ satisfying*

$$(3.18) \quad Lu = f.$$

Moreover

$$(3.19) \quad \|u\|_{L_\delta^2(\mathbb{R}^2)} + \|\nabla u\|_{L_{\delta+1}^2(\mathbb{R}^2)} \leq C\|f\|_{L_{\delta+1}^2(\mathbb{R}^2)},$$

where $C = C(\delta)$ is a constant.

Lemma 3.3 [Su, II]. *Let $l = \{x \in \mathbb{R}^2, |x| = 1\}$, $h \in H^1(l)$, $-1 < \delta < 0$. Then there exists a unique $v \in L_\delta^2(\mathbb{R}^2)$ satisfies*

$$(3.20) \quad Lv = \delta_{f,l}.$$

Moreover,

$$(3.21) \quad \|v\|_{L_\delta^2(\mathbb{R}^2)} \leq C\|f\|_{L^2(l)},$$

$$(3.22) \quad \|\nabla v\|_{L^2(|x|<1)} + \|\nabla v\|_{L^2_{\delta+1}(|x|>1)} \leq C \|f\|_{L^2(l)}^{1/2} \|f\|_{H^1(l)}^{1/2}.$$

We construct

$$(3.23) \quad \omega = \frac{1}{k}(A + e^{-2i \operatorname{Im}(kz)} B).$$

A computation shows that ω solves (3.16) if

$$(3.24) \quad \bar{\partial} A = \delta_{h,l},$$

$$(3.25) \quad \partial B = -e^{2i \operatorname{Im}(kz)} \partial A.$$

By a conformal change of variables, we may assume that l is the unit circle of \mathbb{R}^2 . (Note that $\bar{\partial}$ and ∂ is invariant under such a change of variables.) Using Lemma 3.3 we get a unique solution A satisfying

$$(3.26) \quad \|A\|_{L^2_{\delta}(\mathbb{R}^2)} + \|\nabla A\|_{L^2_{\delta+1}(\mathbb{R}^2 \setminus l)} \leq C \|h\|_{L^2(l)}^{1/2} \|h\|_{H^1(l)}^{1/2}.$$

To prove (3.20), we note that (assuming l is the unit circle) $\partial A = \delta_{(\bar{z}/z)h,l} + A_1$, where

$$A_1 = \begin{cases} \partial A, & z \in \mathbb{R}^2 \setminus l, \\ 0, & z \in l, \end{cases}$$

is a function in $L^2_{\delta+1}(\mathbb{R}^2)$. We write

$$(3.27) \quad B = B_1 + B_2,$$

where

$$(3.28) \quad \partial B_1 = -e^{2i \operatorname{Im}(kz)} \delta_{(\bar{z}/z)h,l} = \delta_{[-e^{2i \operatorname{Im}(kz)}(\bar{z}/z)h],l},$$

$$(3.29) \quad \partial B_2 = A_1.$$

By using Lemma 3.3, we get a solution $B_1 \in L^2_{\delta}(\mathbb{R}^2)$ and

$$(3.30) \quad \|B_1\|_{L^2_{\delta}(\mathbb{R}^2)} \leq C \|h\|_{L^2(l)}^{1/2} \|h\|_{H^1(l)}^{1/2}.$$

Since $A_1 \in L^2_{\delta+1}(\mathbb{R}^2)$ is a regular function, one can solve (3.28) by using Lemma 3.3 to get a solution B_2 satisfying

$$(3.31) \quad \|B_2\|_{L^2_{\delta}(\mathbb{R}^2)} \leq C \|\nabla A\|_{L^2_{\delta+1}(\mathbb{R}^2 \setminus l)}.$$

Combining the estimates (3.26), (3.30), and (3.31) one gets the estimate (3.17).

The existence of the expansion (3.2) is now proven. Uniqueness follows from Lemma 2.1. \square

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Note added in proof. We have recently generalized Theorems 1 and 2 to a more general class of singular potentials, see [Su-U, III].

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DEPARTMENT OF MATHEMATICS AND STATISTICS, WICHITA STATE UNIVERSITY, WICHITA, KANSAS 67208

E-mail address: sun@twsumv.bitnet

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195

E-mail address: gunther@math.washington.edu