PARABOLIC SYSTEMS: THE $GF(3)$ CASE

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Abstract. Parabolic systems defined over $GF(q)$ have been classified by Timmesfeld for $q \geq 4$ and by Stroth for $q = 2$ (see references). We deal with the case $q = 3$.

Parabolic systems have been classified by Niles, Timmesfeld, Stroth, and Heiss, if the field of definition is $GF(2)$ or has at least four elements. [Ni, Tim1, Tim2, Tim5, Tim7, St1, St2, St3, He]. We treat the $GF(3)$ case, where only partial results by Thiel exist so far [Th]. Our result says that strong parabolic systems in characteristic 3 have spherical diagram, and therefore essentially generate only finite groups of Lie type with the same diagram. This is the content of Theorem A. If we drop the assumption that the parabolic systems have to be strong, some infinite families of systems occur, whose diagrams are

or complete bipartite graphs with only double or triple bonds, and the systems are classified. This is Theorem B. The results of this paper are used in the determination of locally finite classical Tits chamber systems with a transitive group of automorphisms having finite chamber stabilizers. This classification, in turn, could be used in the proof of the theorem of Kantor, Liebler, and Tits that determines all classical affine buildings of rank at least 3 having a discrete chamber-transitive group of automorphisms.

The organization of the paper is as follows. The proof of Theorem A is given in §3, while the proof of Theorem B is contained in §4. Definitions, notation and some preliminaries are given in §1, while in §2 the relevant $FF$-modules for some Lie-type groups defined over $GF(3)$ are determined.

1. Definitions, notation, preliminaries

We are mainly concerned with characteristic 3, hence our notation and the definitions reflect this fact. Let $G$ be a finite group, we set $\bar{G} := O^3'(G)$, and $\bar{G} = G/O_3(G)$. If $S$ is a subgroup of $G$, by $S_G$ we denote the largest normal subgroup of $G$ contained in $S$. If $\{X_i, i \in I\}$ is a system of subgroups of $G$, we set $X_{ij}$ for the group generated by $X_i$ and $X_j$. If $X$ is a finite simple group of Lie type, $PSL_2(3)$ or $^2G_2(3)$, or a direct product of such groups (these

Received by the editors August 3, 1989 and, in revised form, April 16, 1991.

1991 Mathematics Subject Classification. Primary 20D25, 51D20.

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0002-9947/93 $1.00 + .25$ per page

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groups will be denoted either by their symbol or their name as a matrix group, if they are classical), then any finite group \( G \) satisfying \( G = \widetilde{G} \) and \( G/Z(G) \cong X \) is said to be a group of Lie-type \( X \). If \( X \) has Lie rank \( n \), then \( G \) is said to be a rank \( n \) Lie-type group.

A group of order \( 2^4 \cdot 3 \) with \( G = \overline{G} \) and \( O_2(G) \) elementary abelian is named \( D \) (there is only one isomorphism type, and \( D \) is a product of two subgroups isomorphic to \( \text{PSL}_2(3) \)). A finite group \( G \) with \( G = \tilde{G} \) whose Sylow 3-subgroups have three elements and \( G/Z(G) \) is isomorphic to \( D \) is called of type \( D \).

(1.1) **Definition.** Let \( G \) be a group generated by finite subgroups \( X_1, \ldots, X_n \) satisfying the following conditions:

(i) \( \cap X_i \) contains a 3-group \( S \) such that \( S \in \text{Syl}_3(X_{ij}) \) for all \( i, j \leq n \).

(ii) \( X_i \) is a rank 1 Lie-type group in characteristic 3 for \( i \leq n \).

(iii) \( X_{ij} \) is a rank 2 Lie-type group in characteristic 3 for \( i \neq j \) or is of type \( D \).

Then \( X = \{X_1, \ldots, X_n\} \) is called a parabolic system of rank \( n \) in characteristic 3 in \( G \). If type \( D \) never occurs in (iii), the parabolic system is said to be strong.

To a (strong) parabolic system \( X \) of rank \( n \) in characteristic 3 there belongs a diagram that serves as a “type” of the system. Vertices (nodes) of the diagram are the indices \( i \in I \), and no bond (resp. a bond of strength 1, 2, or 3—i.e., a single, double or triple bond) is drawn between the vertices \( i \) and \( j \), if the type of \( X_{ij} \) is a direct product of two groups that are rank 1 groups or \( D \) (resp. is \( A_2(q) \), resp. is \( B_2(q) \), \( 2A_3(q) \) or \( 2A_4(q) \), resp. is \( G_2(q) \) or \( 3D_4(q) \)) for some power \( q \) of the prime 3. The diagram contains exactly the same information as a Coxeter matrix \( M = (m(i, j))_{i,j} \), where the entries \( m(i, j) \) for \( i \neq j \) are equal to 2 (resp. 3, resp. 4, resp. 6) and we will use both ways to describe the diagrams of parabolic systems. Forgetting about the strength of the bonds in the diagram, we get the graph of the diagram and may talk about connected components of the diagram. In the whole paper, we always assume that together with a (strong) parabolic system \( X \) we are given the 3-group \( S \) occurring in the definition, and the diagram \( \Delta \).

The following theorems are listed for easy reference. They were proved by Timmesfeld in arbitrary characteristic; we need only the characteristic 3, so we state them in a somewhat restricted form.

(1.2) **Theorem.** Let \( X = \{X_1, \ldots, X_n\}, n \geq 3, \) be a parabolic system in characteristic 3 in the group \( G \) having a connected spherical diagram \( \Delta \). Assume \( S_G = 1 \). Then \( G_0 = \langle \tilde{X}_1, \ldots, \tilde{X}_n \rangle \) is a normal subgroup of \( G \) and the following holds:

(a) \( G_0 \) is a finite group of Lie type in characteristic 3 with diagram \( \Delta \).

(b) \( S \) is a Sylow 3-subgroup of \( G_0 \).

(c) the groups \( X_i \) are “essentially” the rank 1 parabolic subgroups of \( G_0 \) containing the Borel subgroup \( B \) of \( G_0 \) normalizing \( S \), i.e., the groups \( X_i \) are of the form \( B\tilde{X}_i \).

**Proof.** \([\text{Tim5}, (3.2)]\).

As an immediate consequence we get that a parabolic system in characteristic
3 is automatically strong, if all subdiagrams of type \( o \circ (A_1 \times A_1) \) of \( \Delta \) are contained in connected spherical subdiagrams of \( \Delta \).

The nonconnected diagrams are treated in the following theorem. See also the beginning of §4.

**Theorem.** Let \( \{X_i, i \in I\} \) be a strong parabolic system in characteristic 3 in the group \( G \) having diagram \( \Delta \). Let \( \Delta_j, j \in J \), be the connected components of \( \Delta \), and let \( Y_j = \langle O^3(X_i), i \in \Delta_j \rangle \). Assume \( S_G = 1 \). Then the subgroups \( Y_j \) are normal in \( G \) and commute pairwise.

**Proof.** [Tim2, (4.4)].

In §3, we will need to have a list of all connected nonspherical diagrams all of whose proper subdiagrams are spherical.

**Let \( \Delta \) be a connected nonspherical Coxeter diagram of rank at least 4, whose proper subdiagrams are spherical. Assume \( \Delta \) contains only single or double bonds \( (m(i, j) \leq 4 \text{ for all } i, j) \). Then \( \Delta \) is one of the following:

(a) the extended Dynkin diagram of type \( \tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4 \) \( (r \geq 3) \),

(b) 

\[
\begin{array}{c}
\begin{array}{c}
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\circ \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
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\circ \\
\circ \\
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\circ
\end{array}
\end{array}
\]

**Proof.** Clear.

**Corollary.** Let \( X = \{X_1, \ldots, X_n\} \) be a (strong) parabolic system in characteristic 3 in the group \( G \) with diagram \( \Delta \). Suppose \( X_i = \tilde{X}_i \) holds for all \( i \) and \( S_G = 1 \). Let, for \( i = 1, \ldots, n \), \( t_i \) be involutions in \( X_i \) normalizing \( S \) that commute pairwise.

(a) Assume that \( m(n, n - 1) = m(n - 1, n - 2) = 3 \) and \( m(n, i) = m(n - 2, i) = 2 \) for \( i \leq n - 3 \). Then \( t_n t_{n-2} \in Z(G) \).

(b) Assume \( \tilde{X}_n \) is isomorphic to \( SL_2(3) \) or \( SL_2(9) \), and \( t_n \) centralizes \( \tilde{X}_i \) for all \( i \) with \( m(i, n) \neq 2 \). Then \( t_n \in Z(G) \).

**Proof.** Clearly, the elements \( t_i \) normalize the subgroups \( \tilde{X}_j \) for all \( j \). Consider the case (a). Being involutions, the elements \( t_n \) and \( t_{n-2} \) centralize \( \tilde{X}_n \) (resp. \( \tilde{X}_{n-2} \)) and are contained in the commutator subgroup of \( \tilde{X}_n, \tilde{X}_{n-2} \) respectively. Hence they also centralize all \( X_j \) where \( m(n, j) = m(n-2, j) = 2 \). Inspection of the group \( \langle X_n, X_{n-1}, X_{n-2} \rangle \) shows that \( t := t_n t_{n-2} \) also centralizes \( \tilde{X}_{n-1} \). The result (a) now follows from (1.5).
Consider case (b). The same argument as above shows \([t_n, X_1] = 1\). Again the result follows from (1.5).

The next facts are clear but will be needed in §3.

(1.7) Let \(G\) be a perfect central extension of \(\text{PSp}_6(3)\) or of \(\Omega_7(3)\). Let \(B\) be a Borel subgroup of \(G\), and let \(X_1, X_2, X_3\) be the three rank 1 parabolic subgroups of \(G\) containing \(B\) corresponding to the diagram

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array}
\]

Then the following holds:

(i) \(X_{23} \cong \text{PSp}_4(3)\) if and only if \(G \cong \Omega_7(3)\).

Let now \(X_{23}\) be isomorphic to \(\text{Sp}_4(3)\), and let \(t\) be an involution in \(X_{23}\) centralizing \(X_{23}\). Then

(ii) If \(t \in \widetilde{X}_2\), then \(G \cong \text{Spin}_7(3)\) and \(t \in Z(G)\).

(iii) If \(t \in \widetilde{X}_1\), then \(G \cong \text{Sp}_6(3)\) or \(\text{PSp}_6(3)\).

(iv) \(G \cong \text{PSp}_6(3)\) if and only if \(X_{13} \cong \text{SL}_2(3) * \text{SL}_2(3)\).

Proof. Easy exercise.

(1.8) Let \(G\) be \(\text{Sp}_{2n}(3)\), \(B\) some Borel subgroup of \(G\) and \(X_1, \ldots, X_n\) the rank 1 parabolic subgroups of \(G\) containing \(B\) corresponding to the diagram

\[
\begin{array}{ccc}
1 & 2 & \ldots & n-1 & n \\
\end{array}
\]

Let \(H\) be some Cartan subgroup of \(G\) contained in \(B\) and \(t_i\) the involution in \(H \cap \widetilde{X}_i\). Then \(t := t_1t_2\ldots\) generates the center of \(G\).

(1.9) Let \(G\) be a perfect central extension of \(\Omega_{2n+1}(3)\) or of \(\Omega_{2n+2}(3)\), let \(B\) be some Borel subgroup of \(G\) and \(X_1, \ldots, X_n\) the rank 1 parabolic subgroups of \(G\) containing \(B\) corresponding to the diagram

\[
\begin{array}{ccc}
1 & 2 & \ldots & n-1 & n \\
\end{array}
\]

Assume \(t\) is an involution in \(\widetilde{X}_n\) centralizing \(\widetilde{X}_n\). Then \(t \in Z(G)\).

2. SOME \(FF\)-MODULES

In this section, we want to collect material that will be helpful to treat some cases in §3. There, the situation is similar to the \(GF(2)\)-case [St2, Tim7] where Niles' construction of a Tits system does not work. One considers the amalgam of two properly chosen "maximal parabolics" \(G_1\) and \(G_2\) of the parabolic system instead, and tries to get contradictions by comparing the action of both parabolics on their composition factors in the common 3-group \(S\). In this situation, (definitions will be given in §3), one can sometimes assume that one of these composition factors is a so-called \(FF\)-module for \(G_1\) resp. \(G_2\). Therefore, it is helpful to have a list of all \(FF\)-modules for certain Lie-type groups to work with. But whereas in the characteristic 2 such an enemies' list is available [Co], we have to determine some \(FF\)-modules for Lie-type groups defined over \(GF(3)\) ourselves.
Let us recall the definition, $p$ is an arbitrary prime here. Let $G$ be a finite group that acts faithfully on the elementary abelian $p$-group $V$. Assume there is a nontrivial $p$-subgroup $A$ of $G$ having the property

$$|V| \leq |A||C_V(A)|.$$  

Assume $A$ is elementary abelian; then $A$ is called an offending subgroup of $G$ on $V$, and $V$ is called a failure-of-factorization module (FF-module) in characteristic $p$ for $G$.

In the determination of irreducible FF-modules $V$ for a specific group $G$, one is almost done as soon as the $GF(p)$-dimension of $V$ is under control. Therefore one wants to get hold of a nice offending subgroup $A$ such that $G$ is generated by few conjugates of $A$.

(2.1) **Lemma.** Let $V$ be an FF-module in characteristic $p$ for the finite group $G$. Let $U$ be a $p$-subgroup of $G$ containing an offending subgroup. Assume $N_G(U)$ acts irreducibly on $U/\phi(U)$ and $\phi(U)$. Then $U$ or $\phi(U)$ satisfy condition (FF) on $V$.

**Proof.** Set $C = \{X, X \leq U\}$ and apply [CD].

(2.2) **Lemma.** Let $G$ be a finite simple group of Lie type in characteristic $p$, let $S \in \text{Syl}_p(G)$ and $B = N_G(S)$ some borel subgroup of $G$, $H$ some complement to $S$ in $B$. Let $P = U \cdot L$ be a maximal parabolic subgroup of $G$ containing $B$. Assume

(i) $G$ is of type $B_n$, $C_n$, $^2A_n$ or $^2D_{n+1}$ ($n \geq 2$), or
(ii) $G$ is of type $D_n$ and $L$ of type $D_{n-1}$ ($n \geq 3$).

Then there is an element $g \in G$ such that $G = \langle U, U^g \rangle$. Furthermore in case (i), $g$ can be chosen to centralize every involution in $H$.

**Proof.** Let $(G, B, N, R)$ be the Tits system with the given $B$, and with $N$ normalizing $H$, and let $g \in N$ be an element mapping onto the longest element $w_0$ in the Weyl group $W = N/H$ (with respect to $R$). Then in case (i), $w_0$ acts as $-1$ on the root system ($W$ is of type $C$), and hence $L$, which we may assume to be generated by $H$ and some root subgroups only permuted by $g$, is normalized by $g$. Also in case (ii), we may assume $L$ is normalized by $g$.

But certainly $P$ is not normalized by $g$, hence $G = \langle U, U^g, L \rangle$. Now, the subgroup $\langle U, U^g \rangle$ of $G$ is normalized by $G$, and the first result follows. Assume hypothesis (i), and let $P_i = U_i \cdot L_i$ be any parabolic subgroup of $G$ containing $B$, with Levi decomposition adjusted to $H$. Then again $g$ normalizes $L_1$, hence $L_1 \cap H$, and hence the section assertion follows from [Ni, (4.1)].

(2.3) **Lemma.** Let $V$ be an irreducible $GF(3)$-module for the finite Lie-type group $G$ in characteristic 3 of type $B_n$, $C_n$, $n \geq 2$, or $^2D_n$, $n \geq 3$. Let $P = U \cdot L$ be a maximal parabolic subgroup of $G$ containing the Borel subgroup $B$, and let $h \in B$ be an involution.

(a) $P$ centralizes $C_V(U)$, if and only if $P$ centralizes $V/[V, U]$.

(b) Assume $h$ centralizes $C_V(U)$. Then $[V, h] \leq [V, U]$.

**Proof.** Take $g \in G$ as in (2.2). Then as in the proof of (2.2), we can see that $h$ is centralized and $L$ is normalized by $g$. Then $h$ centralizes $C_V(U^g) = C_V(U)^g$, which by [Tim4] complements $[V, U]$. The results (a) and (b) follow.
(2.4) **Lemma.** Let $G$ be a finite group of Lie type in characteristic 3, $V$ some irreducible $FF$-module in characteristic 3 for $G$. Then there is an element in $G$ with minimal polynomial $(X - 1)^2$ on $V$.

**Proof.** [Tim3, (2.3)].

Elements with minimal polynomial $(X - 1)^2$ on some module $V$ are said to be quadratic on $V$. A group $A$ is said to be quadratic on $V$, if $[V, A, A] = 0$. If a group $G$ acts faithfully on a module $V$ such that it contains a quadratic element on $V$, then $V$ is called a quadratic module for $G$. Quadratic irreducible $K[G]$-modules for finite Lie type-groups in odd characteristic $p$, $K$ some algebraically closed field in characteristic $p$, have been determined by Premet and Suprunenko [PS]. We recall the part of [PS] that is needed in §3. (Actually, these quadratic modules have already been determined by Thompson in unpublished parts of his quadratic pairs paper, but we prefer to refer to the easily accessible [PS].)

(2.5) **Theorem.** Let $G$ be a finite group of Lie type in characteristic 3 with connected diagram; let $V$ be an irreducible $GF(3)$-module for $G$, and assume there is some quadratic element in $G$. Then, if $G$ is a Chevalley group, $V$ is a “fundamental module” for $G$, in particular there is a maximal parabolic subgroup $P = U \cdot L$ of $G$ such that $C_V(U)$ is centralized by $P'$. More precisely:

(i) If $G$ is of type $A_n(3)$, $V$ is an exterior power of the natural module.
(ii) If $G$ is of type $B_n(3)$, $V$ is the natural or spin module.
(iii) If $G$ is of type $C_n(3)$, $V$ is an exterior power of the natural module.
(iv) If $G$ is of type $D_n(3)$, $V$ is the natural or a half spin module.
(v) If $G$ is of type $2D_n(3)$, $V$ is the natural $\Omega^-_{2n}(3)$-module or the $GF(9)$-spin module got from the embedding of $G$ into $\Omega^+_{2n}(9)$.

**Proof.** For (i) to (iv), see [PS, Theorem 1]. Since $GF(3)$ is a splitting field for $G$ in any case, all modules already exist over $GF(3)$. For (v), see [PS, Theorems 1 and 2]; the (half) spin module for $\Omega^+_{2n}(9)$ cannot be written over $GF(3)$ when it is restricted to $\Omega^-_{2n}(3)$, whereas the natural module can be written over $GF(3)$, if it is restricted to $G$.

Let us determine some irreducible $FF$-modules in characteristic 3 for some Lie-type groups defined over $GF(3)$. We assume always that we are given some Sylow 3-subgroup $S$ of our Lie-types groups in characteristic 3, the Borel group $B = N_G(S)$ and the (parabolic) system of rank 1 subgroups $X_i$ of $G$ containing $B$ corresponding to the given diagram. Maximal parabolics $G_i$ (also corresponding to the diagram) will be given in a Levi decomposition $G_i = U_i \cdot L_i$.

We start with rank 3.

(2.6) **Lemma.** Let $G$ be of type $A_3(3)$ with diagram

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 1 -- 2 -- 3
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Let $V$ be an irreducible $GF(3)$-module for $G$.

(i) If $V$ is quadratic for $G$, then $V$ is a natural, dual, or orthogonal module for $G$. 
If $V$ is $\mathbb{F}F$ with $A \leq U_2$ offending, then $V$ is a natural or dual module, in particular $\tilde{G} \cong \text{SL}_4(3)$.

(iii) Let $V$ be an orthogonal module for $G$ and $A$ some offending subgroup. Then $A$ is conjugate to $U_1$ or $U_3$ and $[V, A] = C_V(A)$, $|V| = |A||C_V(A)|$.

Proof. (i) follows from (2.5). In (ii), $V$ is quadratic by (2.4), hence let us assume $V$ is an orthogonal module. By (2.1), $U_2$ itself is offending on $V$, which is certainly not the case. This contradiction proves (ii). Let us still assume $V$ is an orthogonal module for $G$, and let $A \leq S$ being offending. Claim $|A| = [V/C_V(A)] = [V, A] = 3^3$, and $A$ is quadratic. Then the statement follows easily. Let $B$ be an offending subgroup of $S$ with $|B| |V/C_V(B)|$ maximal. Then by [Tim3, (2.3)], we may assume $B$ is quadratic, and $|B| |V/C_V(B)| = |V|$, $C_V(B) = [V, B]$ of order $3^3$ follows, since $[V, B]$ is a singular subspace of $V$. Since $|B| |V/C_V(B)| \leq |V|$ for every quadratic subgroup on $V$, we may assume the given $A$ contains $U_1$ or $U_3$. But since $J(S) = U_2$ is not offending on $V$, we get the claim.

(2.7) Lemma. Let $G$ be of type $B_3(3)$ with diagram

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1 -- 2 = 3
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Let $V$ be an irreducible $GF(3)$-module for $G$.

(i) If $V$ is quadratic for $G$, then $V$ is a 7-dimensional natural module or an 8-dimensional spin module for $G$.

(ii) If $V$ is an $FF$-module for $G$ with $A \leq U_1$ offending, then $V$ is the spin module; in particular $Z(G) \cap X \neq 1$; if $A \leq U_3$ is offending on $V$, then $V$ is the natural module for $G$.

(iii) If $V$ is the spin module for $G$ and $A \leq U_1$ is offending and quadratic on $V$, then $C_V(A) = [V, A] = [V, U_1]$.

(iv) If $V$ is the natural module for $G$ and $A$ is quadratic and offending on $V$, then $|A| = |V/C_V(A)| = [V, A] = 3^3$ and $A$ is conjugate to $Z(U_3)$; $A$ is not contained in $U_2$.

Proof. By (2.4) and (2.5), (i) holds. Hence for (ii), only the natural and spin modules have to be investigated. Using (2.1), we can assume $A = U_1$. But then (ii) follows easily. For (iii), we are done, if $A$ contains elements of rank 4 on $V$, hence assume all nontrivial elements in $A$ have rank 2 on $V$, thus $A$ is a singular subspace of the natural module $U_1$ for $L_1$. Then $|A| \leq 3^2$, and since $G$ does not possess transvections on $V$, we have $|A| = 3^2$, and $C_V(A) = C_V(a)$ for all nontrivial $a \in A$. This is clearly impossible.

Finally, assume $V$ is the natural (orthogonal) module for $G = \Omega_7(3)$, and $A$ is quadratic and offending on $V$. Since $A$ is quadratic, $[V, A]$ is a singular subspace of $V$, and $C_V(A) = [V, A]$. Now clearly $[V, A]$ is of order $3^3$, and also $|A| = 3^3 = |V/C_V(A)|$. Since $A$ centralizes $[V, A]$, $A$ is conjugate to $Z(U_3)$. Assume $A \leq U_2$. Then $[V, A]$ is contained in the 5-space $[V, U_2]$ and contains the singular 2-space $C_V(U_2)$, which is the radical of the space $[V, U_2]$. Now we may assume $A$ equals $Z(U_3)$, since $G_2$ is transitive on the singular 3-spaces containing $C_V(U_2)$. But $Z(U_3)$ is not contained in $U_2$, a contradiction. Hence (iv) is proved.
(2.8) **Lemma.** Let $G$ be of type $C_3(3)$ with diagram

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1 -- 2 -- 3
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Let $V$ be an irreducible $GF(3)$-module for $G$.

(i) If $V$ is quadratic for $G$, then $V$ is the 6-dimensional natural module, the 13-dimensional nontrivial composition factor of the exterior square of the natural module, or the 14-dimensional nontrivial composition factor of the third exterior power of the natural module for $G$. In particular, if $G \cong PSp_6(3)$, then $\dim(V) = 13$.

(ii) If $V$ is an FF-module for $G$ with $A \leq U_1$ or $A \leq U_3$ offending, then $V$ is the natural module, in particular $G \cong Sp_6(3)$.

**Proof.** (i) follows from (2.5). Assume $A \leq U_1$ is offending on $V$. By (2.1) we may assume $A = Z(U_1)$, whence $G$ contains transvections on $V$, and $V$ is certainly the natural module, or $U_1$ satisfies (FF) on $V$, whence $\dim(V) \leq 10$ by (2.2), and again $V$ is the natural module. If $A \leq U_3$, we may assume $A = U_3$ by (2.1), and (2.2) implies $\dim(V) \leq 12$. Again the result follows from (i).

We have to treat also some higher rank cases.

(2.9) **Lemma.** Let $G$ be of type $D_n(3)$ with diagram

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\[ n-1 \]
\[ 1 \ldots \ldots \rightarrow \bigcirc \] (\( n \geq 3 \))
\[ \bigcirc \rightarrow \ldots \rightarrow \bigcirc \]
\[ n \]

Let $V$ be an irreducible FF-module for $G$ in characteristic 3 with $A \leq U_1$ offending. Then $V$ is a spin module for $G$ (for $n = 4$, we may also view the natural module as a spin module). In particular, unless $n = 4$, we have that $Z(G) \cap (X_{n-2}X_n)$ is not reduced to 1.

**Proof.** The case $n = 3$ is just (2.6)(ii), while the case $n = 4$ follows from (2.5). Hence we may assume $n \geq 5$, and assume $V$ is the natural (orthogonal) module for $G$. But by (2.1) $U_1$ is offending on $V$, which is impossible. Hence $V$ is a (half) spin module for $G$, and now the action of $G$ on $V$ forces $\overline{X_{n-2}X_n}$ to act as $\text{Sl}_4(3)$, whence the claim by (1.6).

(2.10) **Lemma.** Let $G$ be of type $B_n(3)$ with diagram

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\[ \bigcirc \rightarrow \ldots \rightarrow \bigcirc \rightarrow \bigcirc \] (\( n \geq 4 \))
\[ \bigcirc \rightarrow \ldots \rightarrow \bigcirc \]
\[ \bigcirc \rightarrow \ldots \rightarrow \bigcirc \]
\[ n-1 \]

Let $V$ be an irreducible $GF(3)$-module for $G$.

(i) If $V$ is quadratic, $V$ is the natural or spin module for $G$.

(ii) If $V$ is an FF-module for $G$ with $A \leq U_1$ offending, then $V$ is the spin module, and in particular $Z(G) \cap \overline{X_n}$ is not reduced to 1.

**Proof.** (i) follows from (2.5), whereas by (2.1) $V$ must be the spin module in (ii). The last assertion follows from (2.7)(ii) by the action of $G$ on its spin module, which behaves somehow “inductive” with respect to maximal parabolic subgroups of type $A$.

(2.11) **Lemma.** Let $G$ be of type $C_n(3)$ with diagram

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\[ \bigcirc \rightarrow \ldots \rightarrow \bigcirc \rightarrow \bigcirc \] (\( n \geq 4 \))
\[ \bigcirc \rightarrow \ldots \rightarrow \bigcirc \]
\[ \bigcirc \rightarrow \ldots \rightarrow \bigcirc \]
\[ n-1 \]
``
Let $V$ be an irreducible FF-module in characteristic 3 for $G$. Assume $A$ is some offending subgroup with $A \leq U_1$ or $A \leq U_n$. Then $V$ is the natural module for $G$, in particular $Z(G)$ is nontrivial.

Proof. Assume first $A \leq U_1$. Then by (2.1) either $Z(U_1)$ induces transvections on $V$, whence obviously the result follows, or $U_1$ satisfies $(FF)$ on $V$. Then by (2.2), $\dim(V) \leq 4n-2$. By (2.4) and (2.5), we know that $V$ is a fundamental module for $G$. Let $i$ be such that $C_V(U_1)$ is 1-dimensional. Then $i = 1$ implies $V$ is the natural module for $G$, hence assume $i \geq 2$. Certainly $i \neq 2$, since for $i = 2$ we know $\dim(V) = n(2n-1) - 1$ contradicting $n \geq 4$. But for $i \geq 2$, $C_V(U_1)$ is neither a trivial nor a natural module for $L_1$, and so we get an easy contradiction to $\dim(V) \leq 4n-2$.

Hence we may assume $A$ is contained in $U_n$, and so also $U_n$ is offending on $V$ by (2.1). Assume $V$ is not the natural module for $G$, and choose $n$ minimal with respect to this. Then by (2.8) we may assume inductively that $C_V(U_1)$ is a natural module for $L_1$, using [Tim3, (2.2)]. Since $V$ is a fundamental module, $C_V(U_2)$ must be 1-dimensional, and $\dim(V) = n(2n-1) - 1$. But from (2.2) we know $\dim(V) \leq n(n+1)$, a contradiction to $n \geq 4$.

(2.12) Lemma. Let $G$ be of type $^2D_n(3)$ with diagram

\[
\begin{array}{c}
\circ & \cdots & \circ \\
\circ & & \circ \\
\end{array}
\]

Let $V$ be an irreducible GF(3)-module for $G$.

(i) If $V$ is quadratic, then $V$ is a natural $\Omega^{2n}(3)$-module for $G$ or a (half) spin module (over GF(9)) for $\Omega^{2n}_{+}(9)$ restricted to $G$.

(ii) If $V$ is an FF-module for $G$ with $A \leq U_1$ offending, then $V$ is a spin module, in particular $Z(G) \cap X_{n-1}$ is not reduced to 1.

Proof. The first assertion follows from (2.5), and clearly $U_1$ is not offending on the natural module, hence $V$ is the spin module in (ii) by (2.1). The last assertion again follows from the “inductive” action, hence needs only to be verified for $n = 3$, where it is clear.

In a certain situation in §3, one does not get along with the knowledge of FF-modules, but has to build up a bit more of the 3-group $S$. The argument needed is due to Timmesfeld. We state what together with (2.2) is sufficient for that situation (in 3.7).

(2.13) Lemma. Let $G$ be a finite group with $G/O_3(G)$ isomorphic to PSL$_2(3)$, SL$_2(3)$, PSL$_2(9)$ or SL$_2(9)$. Let $t \in G$ be an element satisfying $\langle tO_3(G) \rangle = Z(G/O_3(G))$. Let $V$ be a GF(3)-module for $G$ with proper GF(3)-subspace $W$ such that the following holds:

(i) $W$ is invariant under some Sylow 3-subgroup $S$ of $G$.


(iii) $V = \langle W^G \rangle$.

Then the following holds:

(1) There is a $G$-composition factor in $V$ which is a nontrivial PSL$_2(3)$ (resp. PSL$_2(9)$) module for $G/O_3(G)$.

(2) There is no quadratic element in $G/O_3(G)$.

Proof. Clearly PSL$_2(3^3)$ has no quadratic module in characteristic 3. Hence it is enough to show (1). We may therefore assume $t$ is an involution, $G = O_3(G)C_G(t)$, and $G = S \cdot O^3(C_G(t))$. 

\[\]
By way of contradiction, we assume that every $G$-composition factor on $V$ is either faithful for $(t)$ or trivial for $G$. But this means that $O^3(C_G(t))$ acts trivially on $C_V(t)$. Consider $U = (C_W(t)^G)$. By (ii) and (iii), $U = V$, whereas the above shows $U \leq W$. This contradiction finishes the proof.

3. Strong parabolic systems in characteristic 3

This section is devoted to the proof of Theorem A, as announced in the introduction.

**Theorem A.** Let $X = \{X_1, X_2, \ldots, X_n\}$ be a strong parabolic system in characteristic 3 in the group $G$ with connected diagram $\Delta$ of rank at least 3. Then $\Delta$ is spherical and for $G_0 = \langle X_1, X_2, \ldots, X_n \rangle$ we have the following: $G_0$ is a normal subgroup of $G$, and $G_0/S_G$ is a Lie-type group in characteristic 3 with same diagram $\Delta$.

**Proof.** First of all, if $\Delta$ is spherical, the rest of the statement is clear by Theorem (1.2). Hence we only have to show that $\Delta$ is spherical.

Assume the contrary, then we may assume the rank $n$ of $X$ is minimal with respect to being a counterexample, therefore the connected components $\Delta_j$ of all proper subdiagrams are spherical, and so the groups $\langle X_i, i \in \Delta_j \rangle$ are (mod the largest normal subgroup in $S$) Lie-type groups in characteristic 3 with diagram $\Delta_j$. In our contradiction proof, we surely may assume $X_i = \widetilde{X}_i$ for all $i \in I = \{1, 2, \ldots, n\}$, hence $G = G_0$, since $G_0$ is a normal subgroup of $G$ by the argument in [Ni, (4.4)], and $S_G = Z(G) = 1$.

By (1.4), the diagram $\Delta$ is either one of the extended diagrams $\widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r, \widetilde{D}_r$ ($r = n - 1$), $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, one of the exceptional diagrams, or is of rank 3. (If $\Delta$ contains a triple bond, the rank $n$ clearly has to be 3.)

We now try and construct a Tits system inside our group $G_0$ following the method introduced by Niles in [Ni, §4]. In those cases, where the construction is possible, we end up with a Tits system $(G_0, B, N, R)$ of type $\Delta$ that has the property that $B$ is finite while $W = N/B \cap N$ is infinite, since the type $\Delta$ is nonspherical. This together is impossible by [Tim1, (2.7)]. In the construction, we keep as close to Niles' notation as possible. We already have $X_i = \widetilde{X}_i$, hence also $X_{ij} = \widetilde{X}_{ij}$ for all $i \neq j$. Let $B_i$ denote the normalizer of $S$ in $X_i$ for $i = 1, 2, \ldots, n$. Then $\langle B_1, B_2 \rangle$ covers the Borel subgroup normalizing $S$ in the Lie-type groups $\widetilde{X}_{ij}$ for $i \neq j$ by [Ni, (4.1)] and the group $B := \langle B_i, i = 1, 2, \ldots, n \rangle$ has the following properties:

- $B$ normalizes $X_i$ and $X_{ij}$ for all $i, j$.
- $B/S$ is a finite abelian $3'$-group.

The just-defined group $B$ will be the $B$ of the Tits system to be constructed. Let us now change the strong parabolic system $X$ slightly to avoid notational length. We replace the rank 1 parabolics $X_i$ by $X_i \cdot B$ but call these again $X_i$. Of course, we get another strong parabolic system with the same diagram $\Delta$ and the rank 1 and rank 2 parabolics of the system differ from the old ones.
only by some abelian 3'-part at the top. This part can, by the way, only induce diagonal automorphisms on the \( \bar{X} \), since it induces diagonal automorphisms on the \( \bar{X} \).

Now pick a complement \( H \) to \( S \) in \( B \), and define \( N_i \) as the normalizer of \( H \) in \( X_i \) for \( i = 1, 2, \ldots, n \). Then Niles’ arguments of [Ni, §4] apply directly to our situation and give:

For \( N := \langle N_i, i = 1, 2, \ldots, n \rangle \) we have \( G_0 = \langle B, N \rangle \).

\( B \cap N \) is normal in \( N \) and \( N_i(B \cap N)/B \cap N \) is of order 2 for all \( i \).

Let \( r_i \) denote the nontrivial coset of \( B \cap N \) in \( N_i(B \cap N) \), and \( R := \{ r_i, i = 1, 2, \ldots, n \} \); then \( (G_0, B, N, R) \) is a Tits system (of type \( \Delta \), of course) provided the following conditions are satisfied in our groups \( X_i \) and \( X_{ij} \):

\[ (**): \text{The centralizer of } H \text{ in } S/S_{X_i} \text{ is trivial.} \]

\[ (*): \text{If } X \text{ is an } H\text{-invariant normal subgroup of } S \text{ with } X \cdot S_{X_i} = X \cdot S_{X_{ij}} = S \text{ then also } X \cdot S_{X_{ij}} = S. \]

Therefore in our contradiction proof we may assume that at least one \( X_i \) does not satisfy \((**)\), or at least one \( X_{ij} \) fails to satisfy condition \((*)\). In particular, from Niles’ Theorem B (in [Ni]) we know that at least one \( \bar{X}_i \) must be of type \( A_1(3) \). But we need a bit more detailed information (in our situation!).

\( i) \ X_i \) does not satisfy \((**)\) if and only if \( X_i/O_3(X_i) \) is a central extension of \( \text{PSL}_2(3) \).

Proof. The if part is trivial, so assume \( X_i \) does not satisfy \((**)\) for some \( i \). Since \( \Delta \) is connected, there is \( j \) such that \( m(i, j) \) is not 2, hence \( \bar{X}_i \) cannot be of type \( 2G_2(3) \). Therefore [Ni, (3.2)] tells that \( \bar{X}_i \) is of type \( A_1(3) \). But if \( X_i/O_3(X_i) \) has some homomorphic image isomorphic \( \text{PGL}_2(3) \), then certainly \((**)\) holds, hence the claim.

\( ii) \ X_{ij} \) does not satisfy \((*)\) if and only if \( \bar{X}_{ij} \) is of type \( G_2(3) \) or of type \( A_1(3) \times A_1(3) \) and \( X_{ij} \) has no homomorphic image \( \text{PSL}_2(3) \times \text{PGL}_2(3) \) or \( \text{PGL}_2(3) \times \text{PGL}_2(3) \).

Proof. Again the if part is easy. Hence assume \( X_{ij} \) does not satisfy \((*)\) for some \( i \neq j \). Summing up the propositions in [Ni, §3], we get that \( \bar{X}_{ij} \) is either of type \( G_2(3) \) or of type \( A_1(3) \times L \), where \( L \) is a rank 1 Lie-type group in characteristic 3. Assume \( \bar{X}_{ij} \) is of the second type and \( L \) is not \( A_1(3) \). But then in \( X_{ij} \) the two unipotent radicals \( X_{X_i} \) and \( S_{X_{ij}} \mod O_3(X_{ij}) \) are just centralizer and commutator of \( H \cap \bar{X}_{ij} \) with \( S \), and \((*)\) holds. Hence we assume \( L = A_1(3) \) and \((ii)\) follows easily.

As an immediate consequence, we note the following

\[ (3.1) \text{If } \Delta \text{ is} \]

\[ \begin{array}{ccc}
1 & - & 2 \\
& & 3 \\
\end{array} \]

then \( \bar{X}_{12} \) is not of type \( 3D_4(3) \).

Proof. Assume the contrary. Then, as can easily be seen in \( X_{12} \) and \( X_{23} \), \((**)\) is satisfied in \( X_i \), \( i = 1, 2, 3 \). And also \((*)\) holds in \( X_{ij} \) for all \( i \), \( j \) by \((ii)\). Hence Niles’ construction works, a contradiction to the above remarks.
If $X_{ij}$ is of type $A_1(3) \times A_1(3)$, we still have the chance to prove condition (i) for ($\ast\ast$) by embedding the subdiagram $\circ \circ$ in a suitable rank 3 subdiagram of $\Delta$, as follows.

(iii) Assume $\widetilde{X}_{ij}$ is of type $A_1(3) \times A_1(3)$, then ($\ast$) holds in $X_{ij}$ provided there is a vertex $k$ in $\Delta$ such that for the subdiagram $\Delta_{ijk}$ on $\{i, j, k\}$ one of the following holds:

(a) $\Delta_{ijk}$ is

\[
\begin{array}{ccc}
\circ & \circ & \circ \\
i & j & k
\end{array}
\]

(b) $\Delta_{ijk}$ is

\[
\begin{array}{ccc}
\circ & \circ & \circ \\
i & j & k
\end{array}
\]

and ($\ast\ast$) holds for $X_j \cap \widetilde{X}_{jk}$.

\textbf{Proof.} In case (a) the group $X_i(X_j \cap \widetilde{X}_{jk})$ has certainly a homomorphic image $PSL_2(3) \times PGL_2(3)$, whence ($\ast$) holds in $X_{ij}$. In case (b), the same holds under the additional hypothesis.

We are now able to rule out quite a lot of the diagrams left.

(3.2) (a) $\Delta$ is not $\widetilde{A}_r$ ($r \neq 3$), $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$ or

(b) If $\Delta$ is of type $\widetilde{C}_r$, $r \geq 3$, at least one of the groups $\overline{G}_1$, $\overline{G}_n$ where $G_n = \langle X_1, \ldots, X_{n-1} \rangle$, $G_1 = \langle X_2, \ldots, X_n \rangle$ is of type $C_{n-1}(3)$.

\textbf{Proof.} Assume ($\ast\ast$) is not satisfied in $X_i$ for some $i$. Then choose $j$ such that $m(i, j) > 2$. This is possible, since $\Delta$ is connected. Now $\widetilde{X}_{ij}$ must be of type $C_2(3)$, since in all other possible rank 3 groups in characteristic 3 the rank 1 parabolics of type $A_1(3)$ have homomorphic images $PGL_2(3)$. (a) follows. But it is also easily seen, that in the diagrams in (a) every subdiagram $\circ \circ$ can be embedded into a subdiagram $\circ \circ \circ \circ$, whence for all $X_{ij}$ ($\ast$) holds in view of (iii). Hence (a). Assume $\Delta$ is of type $\widetilde{C}_r$, $r \geq 3$, and both maximal parabolics $G_1$, $G_n$ of type $C_{n-1}$ are not of type $C_{n-1}(3)$. Then as above, ($\ast\ast$) holds for all $X_i$, and by (iii)(b) also ($\ast$) for all $X_{ij}$. Hence (b) follows.

Using work by Timmesfeld [Tim6, Tim7], we can also rule out the rank 3 case.

(3.3) $\Delta$ is not of rank 3.

\textbf{Proof.} Assume the contrary. Then it follows from [Tim6, (2.3)] and Theorem 2 that $\Delta$ has to be a string. In [Tim7, Theorem 1], Timmesfeld also shows that in the cases, where $\Delta$ is a string (say $m(1, 3) = 2$), $O_3(X_{12})$ (resp. $O_3(X_{23})$) are centralized by $\widetilde{X}_{12}$ (resp. $\widetilde{X}_{23}$). In fact, he gives a list of all parabolic systems in rank 3, that have a connected diagram. By inspection, a contradiction follows. (Recall that the parabolic system $X$ Ris assumed to be strong!)
PARABOLIC SYSTEMS: THE GF(3) CASE

It should be remarked, that the situation in (3.3) is highly restricted, so in fact one uses only a very small part of [Tim7].

The discussion above shows that in $\Delta$ there are only bonds of strength 1 or 2, leaving us with the following possibilities: $\Delta$ is of type $B_r$, $C_r$ or $D_r$ ($r \geq 3$) or $\Delta$ is one of

![Diagram](image)

We have to use a different method now, to get a contradiction, since in the remaining cases (i), (ii), and (iii) do not apply. We choose in our diagram $\Delta$ over $I = \{1, \ldots, n\}$ two maximal subdiagrams, $\Delta_n = \{1, \ldots, n-1\}$ and $\Delta_1 = \{2, \ldots, n\}$, say. Then, the groups $G_n = \langle X_1, \ldots, X_{n-1} \rangle$ and $G_1 = \langle X_2, \ldots, X_n \rangle$ intersect in $G_{1,n} = \langle X_2, \ldots, X_{n-1} \rangle$, which maps onto a maximal parabolic subgroup of both Lie-type groups $G_1$ and $G_n$ (by (1.2), since $\Delta_1$ and $\Delta_n$ are spherical; the intersection contains a maximal parabolic subgroup of each group and if this containment was proper, two rank 1 parabolics of $X$ would have to coincide in $G$, which is certainly not the case).

Now we consider the coset graph $\Gamma(1, n) = \Gamma(G; G_1, G_n)$. The arguments to follow will give a contradiction independent of the particular group $G$ only using the way $G_1$ and $G_n$ are amalgamated, i.e., the way their intersection $G_{1,n}$ is embedded in these groups. Hence we may without loss assume in our contradiction proof that $G$ is the amalgamated sum of $G_1$ and $G_n$, amalgamated along $G_{1,n}$. (Now we left the strong parabolic system $X$, since we replaced $X_{1,n}$ by the (infinite) group $X_1 \ast_B X_2$, but still have $S_G = Z(G) = 1$ and $G = \langle S_G \ast B \rangle$)

Recall the structure of the groups $G_i$: we have $G_i = G_i \ast B$ and $G_i$ is finite Lie-type group in characteristic 3 of type $\Delta_i$; set $M_i := O_3(G_i)$ which is $S_{G_i}$. The advantage of assuming $G$ is $G_1 \ast_{G_{1,n}} G_n$ lies in the fact that $G_i$, $i = 1, n$, are now self-normalizing in $G$, and the graph $\Gamma(1, n)$, on which $G$ acts faithfully by right multiplication, is a tree. The vertices of $\Gamma = \Gamma(1, n)$ are cosets $G_{ix}$, $i \in \{1, n\}$ and $x \in G$ (two different vertices being adjacent, if their intersection is not empty), and the stabilizer in $G$ of the vertex $G_{ix}$ is just $G_i^x$.

Vertices of $\Gamma$ will be denoted by Greek letters, and if $\alpha$ is $G_{ix}$, then its stabilizer $G_\alpha$ in $G$ will be $G_i^x$. Moreover, if $Z_i$ is a normal subgroup of $G_i$, then the normal subgroup $Z_\alpha := Z_i^x$ of $G_\alpha$ is well defined. In particular, we have $M_\alpha = O_3(G_\alpha)$, and certainly $M_\alpha$ fixes all vertices adjacent to $\alpha$. On the locally finite connected graph $\Gamma$, we have a natural distance function $d(\alpha, \beta)$ defined as the minimal length of a path from $\alpha$ to $\beta$ in $\Gamma$, which is even unique since $\Gamma$ is a tree, and will be denoted by $(d, \alpha + 1, \ldots, \beta - 1, \beta)$.

Assume now that $M_1 \neq 1 \neq M_n$, and $Z_i$ is a $G_i$-invariant nontrivial elementary abelian subgroup of $Z(M_i)$, $i = 1, n$. Then $M_\alpha \neq 1$ for all $\alpha$, and we have defined $Z_\alpha$ for all vertices $\alpha$ as above. Clearly $Z_\alpha$ is not contained in all $M_\beta$, $\beta$ in $\Gamma$, since $G$ acts faithfully on $\Gamma$, hence for $\alpha$ we have $d_\alpha := \min\{d(\alpha, \beta) : Z_\alpha \leq M_\beta\}$ and even $d := \min\{d_\alpha : \alpha \in \Gamma\}$. A pair $(\alpha, \delta)$ of vertices is called critical, if $d(\alpha, \delta) = d$ and $Z_\delta \not\leq M_\delta$. Since $G$ is transitive on the two types of vertices (cosets of $G_1$ and $G_n$), we obviously
have four (different) possibilities for critical pairs, corresponding to the orbits of $\alpha$ and $\delta$. If $\alpha$ is in the orbit of the vertex $G_i$ and $\delta$ in the orbit of $G_j$, $i, j \in \{1, n\}$, then we say the critical pair $(\alpha, \delta)$ is of type $(i, j)$.

This notation is the setting for the so-called amalgam method and is used in the many papers written on amalgams recently; one of the fundamental properties of critical pairs $(\alpha, \delta)$ is the following:

$(\ast)$ Assume $\overline{G_i}$ acts nontrivially on $Z_i$ for $G_i$ in the orbit of $\alpha$ and in the orbit of $\delta$, then $Z_\alpha$ (or $Z_\delta$) is an FF-module in characteristic 3 for $\overline{G_\alpha}$ (resp. $\overline{G_\delta}$), and $Z_\delta$ (resp. $Z_\alpha$) induces a quadratic offending subgroup on $Z_\alpha$ (resp. $Z_\delta$).

Hence for the choice of $Z_i$ one is led to pick a $G_i$-invariant subgroup $Z_i$ of $\Omega_i(Z(M_i))$, $i \in \{1, n\}$, with nontrivial $\overline{G_i}$-action, if possible. If $\Omega_i(Z(M_i))$ is a trivial $G_i$-module, hence lies in the center of $S$ (and $\overline{G_i}$), we denote this situation by $G_i \leq N(Z)$. If not, we put $G_i \nleq N(Z)$. Of course, the case $G_1 \leq N(Z)$ and $G_n \nleq N(Z)$ lead to $M_1 \cap M_n = 1$, since otherwise $\Omega_i(Z(M_i)) \cap \Omega_n(Z(M_n))$ would be a nontrivial normal subgroup of $G$ contained in $S$. Then $M_1M_n$ is a direct product and $M_1M_n/M_1$ is contained in the unipotent radical of $G_{1,n}/M_1$. This bounds the order of $M_n$, often implies that $M_i$ is trivial under the action of $\overline{G_i}$, $i = 1, n$, then $M_1M_n/M_1$ is trivial for $\overline{G_{1,n}}$ and therefore one usually gets a contradiction.

We will treat the remaining cases now one after the other, by choosing the two nodes (denoted $1, n$ above) in the diagram $\Delta$ properly, forming the corresponding coset graph $\Gamma$ of the amalgamated sum $S$ and investigating the action of $G$ on $\Gamma$. Since in some situation we have to collect information from more than one amalgam and do not want to change labelling in the diagram $\Gamma$, we will be free to pick two nodes $i, j$ and form the tree $\Gamma(i, j)$ in exactly the same manner as $\Gamma(1, n)$. In every case, we have to give the labelling of $\Delta$, the choice of $\Gamma(i, j)$ and to define $Z_i$ and $Z_j$, if $M_i$ and $M_j$ are not trivial.

(3.4) $\Delta$ is not $A_3 = D_3$.

Proof. Since at least one $\overline{X_i}$ is of type $A_1(3)$, we know that all $\overline{G_i}$ are of type $A_3(3)$. Assume $M_1 = 1$. Then all $M_i$ are equal to $1$, and $|S| = 3^6$. But then $J(S)$ is normalized by $X_{13}, X_{24}$ hence by $G$, a contradiction. Hence $M_i \neq 1$ for $i = 1, 2, 3, 4$, and we may choose labelling of the diagram such that $G_1 \nleq N(Z)$ and $G_3 \nleq N(Z)$. Without loss, $\Omega_1(Z(M_1))$ is an FF-module in characteristic 3 for $\overline{G_1}$, with offending subgroup contained in $O_{3}(X_{24})/M_1$. Now there is also a $\overline{G_1}$-composition factor of $\Omega_1(Z(M_1))$ with the same properties, and by (2.6)(ii) $\overline{X_{24}}/O_{3}(X_{24}) \cong \text{SL}_{2}(3) \times \text{SL}_{2}(3)$. But now some involution $t$ in $H$ centralizes $\overline{X_i}$ for $i = 1, 2, 3, 4$ by (1.6) and (1.5) gives a contradiction.

(3.5) $\Delta$ is not
\textbf{Proof.} As in the proof of (3.4) we may assume \( M_i \neq 1 \) and \( G_i \) is of type \( D_{n-1}(3) \) for \( i = 1, 2, n-1, n \) (recall \( n \geq 5 \)). Therefore we may also assume that \( G_1 \notin N(Z) \) and \( G_2 \notin N(Z) \). Hence we may assume by (2.9), that in \( \Omega_1(Z(M_1)) \) there is a \( G_1 \)-spin module involved. For \( n \geq 6 \), (2.9) implies that some involution in \( H \cap \cong_{n-2} \) centralizes \( \cong_i, \ i \geq 2 \). But now again (1.6) and (1.5) yield a contradiction. For \( n = 5 \), the same contradiction follows with some involution in \( H \cap \cong L \), where \( L \) is \( X_{24}, X_{25}, \) or \( X_{45} \).

(3.6) \( \Delta \) is not

\[
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (n) at (3,0) {n};
  \node (n-1) at (2,0) {n-1};
  \node (D_{n-1}) at (3.5,0) {\(D_{n-1}, n \geq 5\)};
  \node (\ldots) at (1.5,0) {\ldots};
  \draw (1) -- (2);
  \draw (2) -- (n-1);
  \draw (n-1) -- (n);
\end{tikzpicture}
\]

\textbf{Proof.} Assume the contrary. Then by the structure of \( \cong_{n-1, n} \) the groups \( \cong_1 \) and \( \cong_2 \) must have the same type (compare (1.7)): \( C_{n-1}(3), B_{n-1}(3), \) or \( 2D_{n}(3) \). By (1.6) we may assume \( \cong_{12} \) is isomorphic to \( SL_2(3) \ast SL_2(3) \). Assume \( M_1 \cap M_2 = 1 \). If now \( M_1 \neq 1 \), then \( G_1 \) is of type \( C_{n-1}(3) \) and \( |M_1| = |M_2| = 3 \). Comparing the orders of \( D_{n-1}(3) \) and \( C_{n-1}(3) \), one sees that \( M_n \) cannot be 1, but must be contained in \( O_3(G_1, n) \), a contradiction to the action of \( G_{1,n} \) on \( O_3(G_1, n) \). If \( M_1 = M_2 = 1 \), this contradiction is got rightaway.

Hence we may assume \( M_1, M_2, \) and \( M_n \) to be nontrivial, and \( G_i \leq N(Z) \) for at most one \( i \in \{1, 2, n\} \). Assume \( G_i \leq N(Z) \). Then we consider the graph \( \Gamma(2, n) \), taking \( Z_i := \Omega_i(Z(M_i)) \), \( i = 2, \ldots, n \). Since \( H \cap \cong_n \) centralizes \( \Omega_1(Z(S)) \) by the structure of \( \cong_{12} \), by (2.5) \( Z_n \) cannot be a quadratic module for \( G_n \), and any critical pair \((\alpha, \delta)\) must be of type (2.2). Considering \( G_1 \leq N(Z) \) and using (2.5), some composition factor of \( Z_2 \) must be a natural module for \( \cong_2 = Sp_{2n-2}(3) \) or \( \Omega_{2n-1}(3) \) or \( \Omega_{2n}(3) \). But again the action of \( X_{12} \) on \( \Omega_1(Z(S)) \) gives a contradiction. Hence we may assume \( G_1, G_2 \notin N(Z) \). Now without loss some composition factor of \( \Omega_1(Z(M_1)) \), is an irreducible \( FF \)-module in characteristic 3 for \( \cong_i \) with offending subgroup contained in \( O_3(G_{12}/M_1) \). If \( G_1 \) has type \( C_{n-1}(3) \), we get a contradiction using (2.11), (1.8), and (1.5). If \( \cong_1 \) has type \( 2D_{n}(3) \), the contradiction follows with (2.12), (1.9), and (1.5). If \( \cong_1 \) is of type \( B_{n-1}(3) \), the contradiction follows from (2.10), (1.9), and (1.5).

(3.7) \( \Delta \) is not

\[
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (n-1) at (2,0) {n-1};
  \node (n) at (3,0) {n};
  \node (C_{n-1}) at (3.5,0) {\(C_{n-1}, n \geq 4\)};
  \node (\ldots) at (1.5,0) {\ldots};
  \draw (1) -- (2);
  \draw (2) -- (n-1);
  \draw (n-1) -- (n);
\end{tikzpicture}
\]
Proof. Note first, that $G_1$ and $G_2$ are of type $B_{n-1}(3)$, $C_{n-1}(3)$, or $2D_n(3)$ and $H$ is an abelian 2-group. By (3.2)(b), one of $G_1$ and $G_2$ is of type $C_{n-1}(3)$. Assume first $M_1 = 1$. Then $S$ is isomorphic to a Sylow 3-group of $\text{Sp}_{2n-2}(3)$, $\Omega_{2n-1}(3)$, or $\Omega_{2n}^{-}(3)$, and now clearly also $M_n = 1$ and $G_1$ and $G_2$ are both of type $C_{n-1}(3)$, since the type is determined by the structure of the Sylow 3-subgroup. But then $Z(S)$ is normalized by $G_{12}$ and $G_{n-1,n}$, hence by $G$, a contradiction. Hence $M_1$ and $M_n$ are both nontrivial.

Assume $G_1 \leq N(Z)$ and $G_n \leq N(Z)$. Then the introductory remarks show that $M_1 \cap M_n = 1$ and $G_{1,n}$ (viewed as a Levi complement of a maximal parabolic of the group of type $C_{n-1}(3)$) must act trivially on its unipotent radical, of course a contradiction.

Assume $G_1 \not\leq N(Z)$ and $G_n \leq N(Z)$. If now $\Omega_i(Z(S)) \not\leq M_n$, then $\widetilde{G}_n = M_n \cdot C_n$, where $C_n$ is the centralizer of $M_n$ in $\widetilde{G}_n$, and $\widetilde{G}_{1,n}$ has only central composition factors in $M_n$, hence the only noncentral $G_{1,n}$-composition factors of $O_3(G_{1,n})$ are contained in $O_3(G_{1,n}/M_n)$. But $G_{1,n}$ has noncentral composition factors in $\Omega_i(Z(M_1))$ and $O_3(G_{1,n}/M_1)$ and we get a contradiction for all possible types of $G_1$ and $G_{1,n}$. Hence $\Omega_i(Z(S)) \leq M_n$, and therefore $\Omega_i(Z(S)) = \Omega_i(Z(M_n))$. Now $G_1$ cannot have a trivial submodule in $\Omega_i(Z(M_1))$, and $G_{1,n}$ normalizes $\Omega_i(Z(M_1)) \cap Z(S)$.

Consider $\Gamma = \Gamma(1,2)$ with $Z_1$ some irreducible $G_1$-submodule of $\Omega_i(Z(M_1))$, and $Z_2$ the (unique) irreducible $G_{1,2}$-submodule of $Z_1$. Clearly, $Z_2$ is a nontrivial $G_2$-module. Let $(\alpha, \delta)$ be a critical pair in $\Gamma$. Clearly, $\alpha$ is of type 1; if now also $\delta$ is of type 1, then $Z_1$ is an FF-module in characteristic 3 for $\widetilde{G}_1$, and by the action of $G_{1,n}$ cannot be a natural module for $\widetilde{G}_1$ of type $C_{n-1}(3)$, $B_{n-1}(3)$, or $2D_n(3)$. Hence in this case by (2.10), (2.11), and (2.12), $\widetilde{G}_1$ must be of type $B_{n-1}(3)$ or $2D_n(3)$ and $Z_1$ a spin module. In this case, however, (1.6)(b) gives a contradiction. Therefore $(\alpha, \delta)$ is of type (1.2). The same argument as above shows $[Z_\alpha, Z_\delta] = 1$. Assume $d = 1$. Then we may take $(\alpha, \delta) = (1,2)$ and $Z_1/Z_1 \cap M_2$ is centralized by $\widetilde{G}_{1,2}$. But then by (2.3), $G_{1,2}$ fixes $Z_1 \cap Z(S)$, a contradiction. Hence $d \geq 3$, and in particular $V_2 := \langle Z_1^{G_1} \rangle = \langle Z_1^{G_2} \rangle$ is elementary abelian. Since $X_1$ does not fix $Z_1$, $X_1$ acts as $(P)\text{SL}_2(3)$, or $(P)\text{SL}_2(9)$ on $V_2$. Let $t = \Omega_i(H \cap X_1)$, then $t$ centralizes $Z_1 \cap Z(S)$, hence $Z_1/[Z_1, S]$ by (2.3), and we may apply (2.13) to $X_1$, $t$, $V_2$, and $Z_1$. It follows that elements in $X_1$ acting quadratically on $V_2$ are contained in $O_3(X_1)$. But this contradicts the quadratic action of $\langle Z_\alpha^{G_{1,1}} \rangle$ on $\langle Z_\delta^{G_{1,1}} \rangle$.

Hence we may assume $G_1 \not\leq N(Z)$ and $G_n \leq N(Z)$. Clearly, $Z_1 \cap Z_n \geq \Omega_i(Z(S))$. Consider $\Gamma = \Gamma(1, n)$ with $Z_1 = \Omega_i(Z(M_i))$, $i = 1, n$. Without loss, $Z_1$ is an FF-module for $\widetilde{G}_1$, and hence by (1.6), (1.9), (2.10), (2.11), and (2.12), all noncentral $\widetilde{G}_1$-composition factors in $Z_1$ are natural modules for $\widetilde{G}_1$, which is of type $C_{n-1}(3)$, $B_{n-1}(3)$ or $2D_n(3)$, and $G_{1,2}$ is the normalizer in $G_1$ of $\Omega_i(Z(S))$. Now $G_n$ does not have a trivial submodule on $Z_n$, and hence we may choose $Z_n$ an irreducible nontrivial $G_n$-submodule and $Z_{n-1} \leq Z_n$ an irreducible nontrivial $G_{n-1}$-submodule for $\Gamma(n, n-1)$. Now the same arguments as above show that either $Z_n$ is a quadratic module for $\widetilde{G}_n$, which contradicts the action of $G_{1,2}$, or we get a contradiction to (2.13).

(3.8) $\Delta$ is not
Proof. Assume the contrary. Since one of the $X_i$ is of type $A_4(3)$, by the structure of $X_{34}$ we get that (without loss) $\overline{G_1}$ is of type $B_3(3)$ and $\overline{G_2}$ is of type $C_4(3)$. Now clearly $M_1 \neq 1$ and $M_2 \neq 1$, (trivially also $M_4 \neq 1 \neq M_3$) and the structure of $X_{13}$ together with (1.6) and (1.7) tells $\overline{G_2} \cong PSp_6(3)$ and $\overline{G_3} \cong PSL_4(3)$. This implies easily $\overline{G_1} \cong \text{Spin}_7(3)$ and $\overline{G_3} \cong \text{SL}_4(3)$ by (1.7). And obviously $G_i \leq N(Z)$ can hold for at most one $i$, and if so, $\Omega_1(Z(M_i)) = \Omega_1(Z(S))$.

Assume first $G_2 \not\leq N(Z)$ and $G_4 \not\leq N(Z)$. Then the $G_i$-module $\Omega_i(Z(M_i))$ involves a natural $\text{Sp}_6(3)$ (resp. $\text{SL}_4(3)$)-module by (2.6) and (2.8), $i = 2$ (resp. 4), which is certainly a contradiction.

Assume next $G_4 \leq N(Z)$. Consider the graph $\Gamma = \Gamma(1,2)$ with irreducible nontrivial $G_i$-submodules $Z_i$ of $\Omega_i(Z(M_i))$, $i = 1, 2$. Let $(\alpha, \delta)$ be a critical pair, and let the order be chosen so that $Z_\delta$ is an $FF$-module for $G_\delta$ with $Z_\alpha M_\delta / M_\delta$ offending. Then by (2.8)(ii) the type of $(\alpha, \delta)$ is not $(2,2)$ or $(1,2)$; and it is not $(2,1)$ either: by (2.7) the group $Z_\delta M_\alpha / M_\alpha$ acting quadratically on $Z_\alpha$ would have order at least $3^4$, being contained in $O_3(G_{\alpha-1/\alpha+1/M_\alpha})$, which is certainly impossible. Hence its type is $(1,1)$, and again $Z_1$ is an 8-dimensional spin module for $G_1$. Let $\alpha - 1$ be any vertex of $\Gamma$ adjacent to $\alpha$ different from $\alpha + 1$. Then, since $(\alpha - 1, \delta - 1)$ is not critical, $Z_{\alpha - 1}$ is contained in $M_{\delta - 1}$, hence in $G_\delta$ and moreover $[Z_\delta, Z_\alpha] = [Z_\delta, O_3(G_\delta, \delta - 1)] = [Z_\delta, Z_\alpha Z_{\alpha - 1}]$ by (2.7)(iii). In particular, $[Z_\delta, Z_{\alpha - 1}] \leq Z_\alpha$. Since $\alpha - 1$ was chosen arbitrarily, we get $[V_\alpha, Z_\delta] \leq Z_\alpha$ for the $G_\alpha$-module $V_\alpha = \langle Z_{\alpha - 1}^\sigma \rangle$. Hence $G_\alpha$ acts trivially on $V_\alpha / Z_\alpha$, and hence also $\overline{G_1}$ on $Z_2 / Z_1 \cap Z_2$. Now by (2.3), $Z_2$ must be contained in $Z_1$, which is clearly impossible.

Assume finally $G_2 \leq N(Z)$. Then $\Omega_1(Z(S)) = \Omega_1(Z(M_2))$, and we pick irreducible nontrivial $G_i$-submodules $Z_i$ in $\Omega_i(Z(M_i))$ for $i = 1, 3, 4$. Consider the graph $\Gamma = \Gamma(1,4)$ first. Let $(\alpha, \delta)$ be a critical pair. We want to show that $Z_1$ is an $FF$-module for $G_1$, which must be a natural 7-dimensional module then by (2.7)(ii). Hence assume, there is no critical pair of type $(1,1)$ and $(\alpha, \delta)$ is of type $(1,4)$ or $(4,1)$. Then $Z_1$ is still quadratic and the result follows from (2.7) and the action of $G_{12}$ on $Z_1$. Hence we may assume $(\alpha, \delta)$ is of type $(4,4)$, $Z_4$ is an orthogonal module for $G_4$ by (2.6) and the action of $G_{24}$ on $Z_4$, and $Z_\alpha$ and $Z_\delta$ both offend on each other by (2.6)(iii).

Now the same proof as in the case $G_4 \leq N(Z)$ implies $G_4$ centralizes $\langle Z_1^\sigma \rangle / Z_4$. But this contradicts the action of $G_{14}$ on $Z_1 / Z_1 \cap Z_4$. Hence indeed we have $Z_1$ a natural module for $G_1$.

Consider now the graph $\Gamma = \Gamma(1,2)$ with the same $Z_1$, and $Z_2$ the centralizer of $S$ on $Z_1$. Then pick a critical pair $(\alpha, \delta)$, it must have type $(1,2)$ or $(1,1)$. The second case, however, contradicts (2.7), hence the type is $(1,2)$. Assume $d = 1$. Then $[Z_1, M_2], Z_1 = 1$ and all noncentral $G_2$-composition factors of $M_2$ are quadratic, hence 13-dimensional by (2.8). If, however, $d \geq 3$, then $V_2 = \langle Z_1^\sigma \rangle$ is abelian, and $V_{\alpha + 1}$ and $V_\delta$ act quadratically on each.
other. Again, (2.8) tells that all noncentral $G_2$-composition factors on $V_2$ are 13-dimensional. Let $V$ be a minimal nontrivial $G_{12}$-submodule of $V_2$. Then by what we just said and the action of $G_{12}$ on the 13-dimensional fundamental module the noncentral $G_{12}$-composition factor of $V$ is 4-dimensional. But inside $Z_1$, we see a 6-dimensional submodule with 5-dimensional noncentral composition factor. This contradiction finishes the proof.

(3.9) $\Delta$ is not

\[ \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \]

\[ \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \]

Proof. Assume the contrary; clearly (***) holds for all $X_i$, hence at least one rank 2 group of type $A_1(3) \times A_1(3)$ is involved in the parabolic system, and we know immediately the types of $G_i$, $i = 1, \ldots, 4$. Without loss $G_1$ is of type $B_3(3)$ and $G_2$ is of type $C_3(3)$ by the structure of $X_{34}$ and (1.7). Also by (1.7) and the structure of $X_{13}$, and $X_{24}$, we get $G_1 \cong G_3 \cong \text{Spin}_7(3)$ and $G_2 \cong G_4 \cong \text{PSp}_6(3)$. Clearly $M_i \neq 1$ for $i = 1, 2, 3, 4$ since Sylow 3-subgroups of $B_3(3)$ and $C_3(3)$ are not isomorphic. Also $\Omega_i(Z(S)) \leq M_i$ for all $i$, and finally $G_i \leq N(Z)$ can be true for at most one $i$.

Assume $G_2 \leq N(Z)$ and $G_4 \not\leq N(Z)$. Then consider the graph $\Gamma(2, 4)$ with $Z_i = \Omega_i(Z(M_i))$. We immediately get a (quadratic) $FF$-module for $G_2$ or $G_4$ and hence by (2.8), noncentral composition factors of, say, $Z_2$ are isomorphic to the 13-dimensional fundamental module $V$ for $G_2$, moreover an offending subgroup is contained in the unipotent radical $U$ of a line stabilizer $P$. By (2.1), and since no transvections are reduced on by $G_2$, also $U$ satisfies $(FF)$ on $V$. This is impossible, since $P$ fixes a point on $V$.

Hence assume $G_2 \leq N(Z)$. Then consider the graph $\Gamma(1, 4)$. Let $Z_i$ be irreducible $G_i$-submodules of $\Omega_i(Z(M_i))$, $i = 1, 4$, and let $(\alpha, \delta)$ be a critical pair. By (2.8), the type is not (4, 4). By (2.7), $Z_1$ is a spin module for $G_1$, and hence again by (2.7), and the structure of $O_3(G_{14}/M_4)$, we get that the type of $(\alpha, \delta)$ is (1, 1). Therefore again $[Z_\alpha Z_{\alpha - 1}, Z_\delta] \leq Z_\alpha$ follows, and hence $G_\alpha$ acts trivially on $(Z_{\alpha - 1})Z_\alpha/Z_\alpha$, a contradiction to the action of $G_{14}$ on $Z_{14}/Z_1 \cap Z_{14}$.

4. Parabolic systems that are not strong

In this section, we consider parabolic systems in characteristic 3 that do not have to be strong any more. That means, $X_{ij}$ of type $D$ is allowed. The following lemma will be used later in the proof of Theorem B, but also indicates why we may restrict our interest to the case of connected diagrams.

(4.1) Lemma. Let $\{X_1, X_2, X_3\}$ be a parabolic system in characteristic 3 in $G$ with diagram

\[ \begin{array}{c}
1 \\
2 \\
3
\end{array} \]

\[ \begin{array}{c}
1 \\
2 \\
3
\end{array} \]
i.e., \( m(1, 2) = m(1, 3) = 2 \) and \( m(2, 3) > 2 \). Then the parabolic system is strong.

**Proof.** Assume by way of contradiction that \( \bar{X}_{12} \) is of type \( D \). Of course, we may assume \( X_i = \bar{X}_i \) for \( i = 1, 2, 3 \) and \( S_G = Z(G) = 1 \). Then \( X_1 \) is not contained in \( X_{23} \) and clearly \( X_{23} \) has index 4 in \( G \). Let \( K \) be the largest normal subgroup of \( G \) contained in \( X_{23} \), then \( S \not\subseteq K \), and even \( \text{PSL}_2(3) \) is a subgroup of \( X_1K/K \). Since \( O_3(X_{23}) \) is not contained in \( K \), there is \( x \in O_3(X_{23}) \leq O_3(X_2) = O_3(X_1) \) not contained in \( K \). This contradicts the structure of \( X_1 \).

If \( X \) is a parabolic system in characteristic 3 in a group \( G \) having a diagram \( \Delta \) that is not connected, then we may apply a version of Theorem (1.3) to get a decomposition of \( O^3(G) \) corresponding to the decomposition of \( \Delta \).

Consider now a parabolic system \( X \) in characteristic 3 in the group \( G \) that is not strong, i.e. there are \( i, j \) in the diagram \( \Delta \) such that \( \bar{X}_{ij} \) is of type \( D \), and assume \( \Delta \) is connected. Then it is interesting, how the vertices \( i, j \) are “embedded” in \( \Delta \).

(4.2) **Lemma.** Let \( \{X_1, X_2, X_3\} \) be a parabolic system in characteristic 3 in \( G \), with connected diagram \( \Delta \) and \( \bar{X}_{13} \) of type \( D \). Assume \( S_G = Z(G) = 1 \). Then \( \bar{X}_{12} \cong \bar{X}_{23} \) and one of the following holds:

(a) \( m(1, 2) = 4 \) and \( \bar{X}_{12} \) is isomorphic to \( \text{PSp}_4(3) \), \( Z_3 \times \text{PSp}_4(3) \), or \( U_4(3) \).

(b) \( m(1, 2) = 6 \) and \( \bar{X}_{12} \) is isomorphic to \( G_2(3) \) or \( 3D_4(3) \).

**Proof.** As already used in the proof of (3.3), work by Timmesfeld [Tim7] shows that \( \bar{X}_{12} \) and \( \bar{X}_{23} \) act trivially on \( O_3(X_{12}) \) (resp. \( O_3(X_{23}) \)). Now inspection of the outcome of [Tim7, Theorem 1] gives the desired result.

It should be mentioned that unless \( \bar{X}_{12} \) is of type \( G_2(3) \) or \( \text{PSp}_4(3) \), the types of the \( X_i \) (i.e., the labelling) is uniquely determined in (4.2).

We come now to the proof of Theorem B.

**Theorem B.** Let \( X = \{X_1, \ldots, X_n\} \) be a parabolic system in characteristic 3 in \( G \), with connected diagram \( \Delta \), \( n \) at least 3. Then either the system is strong (and \( \Delta \) is spherical by Theorem A) or \( \Delta \) is one of the following:

(i) a complete bipartite graph with only triple or only double bonds

(ii)

\[
\begin{array}{c}
0 \\
1 \\
2 \\
\vdots
\end{array}
\]

\( r + s \\
\)  
\( r, s \geq 2 \),  

hence of type \( Y(r, s) \) in the notation of [St].

**Proof.** We may assume \( X \) is not strong. Then assume first that there is a triple bond contained in \( \Delta \), say \( m(i, j) = 6 \) for some \( i, j \in \Delta \). Let \( k \) be an arbitrary vertex in \( \Delta \) different from \( i \) and \( j \).

Claim the subdiagram on \( \{i, j, k\} \) is either

\[
\begin{array}{c}
i \\
\vdots
\end{array}
\]

or

\[
\begin{array}{c}
\vdots \\
\vdots
\end{array}\]
The claim follows from (4.2), if \( k \) is connected to \( i \) or \( j \) in \( \Delta \), since the subsystem \( \{X_i, X_j, X_k\} \) is not strong by Theorem A. Hence, by way of contradiction, we may assume \( k \) is at distance 2 from \( \{i, j\} \) in \( \Delta \). Thus, we have a vertex \( v \in \Delta \), connected to \( i \) or \( j \) and to \( k \), while \( k \) is connected to neither \( i \) nor \( j \). Clearly, by Theorem A, the system \( \{X_i, X_j, X_k, X_v\} \) is not strong, and also the system \( \{X_i, X_j, X_v\} \) is not strong. Without loss, the diagram on \( \{i, j, v\} \) is

\[
\begin{array}{cccc}
  & i & j & v \\
\end{array}
\]

and hence also the system \( \{X_i, X_j, X_k\} \) is not strong and its diagram is also

\[
\begin{array}{cccc}
  & j & v & k \\
\end{array}
\]

Hence the diagram on \( \{i, j, v, k\} \) is

\[
\begin{array}{cccc}
  & i & j & v & k \\
\end{array}
\]

and we get a contradiction to (4.1).

This contradiction proves the claim and it follows that all bonds in \( \Delta \) are triple bonds, every vertex being adjacent to either \( i \) or \( j \), and by (4.2) case (i) follows. So we may assume there are no triple bonds contained in \( \Delta \). If there are no single bonds either contained in \( \Delta \), then the same argument as above gives case (i) again.

So we may assume there are single bonds contained in \( \Delta \). Let \( i, j \) in \( \Delta \) such that the \( X_{ij} \) is of type \( D \). Then \( i \) and \( j \) are connected in \( \Delta \) by a path \( (i, k, \ldots, v, j) \), and by (4.1) and (4.2) we have the subdiagram

\[
\begin{array}{cccc}
  & i & k & j \\
\end{array}
\]

Consider the graph \( \tilde{\Delta} \) got from \( \Delta \) by first removing all single bonds, then all isolated vertices.

**Claim:** \( \Delta \) is a star with central vertex \( k \).

It is clear that \( \tilde{\Delta} \) is connected, since \( \Delta \) does not contain subdiagrams of type

\[
\begin{array}{cccc}
  & x & y \\
\end{array}
\]

or circuits. The corresponding subsystem would have to be strong, contradicting Theorem A.

For the same reason, for \( v, w \), in \( \Delta \) that are no adjacent in \( \tilde{\Delta} \), we also have \( m(v, w) = 2 \) (they are not adjacent in \( \Delta \), and hence the above argument shows that \( \tilde{\Delta} \) is a complete bipartite graph with only double bonds. Let \( t \) be a vertex in \( \Delta-\Delta \) that is adjacent to some \( h \) in \( \tilde{\Delta} \). Certainly, \( h \) is contained in a subdiagram of type

\[
\begin{array}{cccc}
  & h & x & y \\
\end{array}
\]

or

\[
\begin{array}{cccc}
  & x & y & h \\
\end{array}
\]

in \( \tilde{\Delta} \). In the first case, by (4.1), \( m(t, x) = 3 \) or \( m(t, y) = 3 \), both contradicting Theorem A. Hence \( \tilde{\Delta} \) contains a vertex \( h \), that is not at distance 2 from any
other vertex of $\bar{\Delta}$, hence $\bar{\Delta}$ must be a star, and clearly $k$ must be the central vertex, hence the claim follows.

Moreover, $k$ must be equal to $h$. This implies that any vertex in $\Delta-\bar{\Delta}$ that is adjacent to some vertex in $\Delta$, is adjacent to $k$ and to no other vertex. But since $\Delta$ does not contain triangles nor subdiagrams of type $B_3$ by Theorem A, we also get that $t$ was unique. But now $\Delta-\bar{\Delta}$ is a connected subdiagram of $\Delta$ containing only single bonds, and hence is spherical by (4.1) and Theorem A. If it is of Type $A_r$, we get conclusion (ii).

Hence assume it is of type $D_1$ or $E_1$, then however $\Delta$ contains a subdiagram of type $B_f$, a final contradiction.

Let still $X = \{X_1, \ldots, X_n\}$ be a parabolic system in characteristic 3 in $G$, that is not strong, but has a connected diagram. We want to say a bit more on these systems.

(4.3) Assume some $X_{ij}$ is isomorphic to $3D_4(3)$. Then $\Delta$ is a star. If $j$ is the central vertex of $\Delta$, $X_j$ is of type $L_2(27)$ and $S_G = S_{X_{jk}}$ for all $j \neq k$.

Proof. Any connected subdiagram on $\{i, j, k\}$, say, looks like

```
\begin{array}{c}
\bullet_i \quad \bullet_j \quad \bullet_k
\end{array}
```

by Theorem B, and is clearly not strong, hence $X_j$ is of type $L_2(27)$. Obviously, $\Delta$ is a star with central vertex $j$. By (4.2), $S_{X_{ij}} = S_{X_{jk}}$ for all $k \neq j$, hence this group is normal in $G$.

(4.4) Assume some $X_{ij}$ is of type $G_2(3)$. Then $S_G = S_{X_{ij}}$ for all $i \neq j$ adjacent in $\Delta$.

Proof. Easy application of (4.2).

(4.5) Assume $\Delta$ is a complete bipartite graph with only double bonds involved. Assume some $X_{ij}$ is of type $2A_3(3)$. Then $\Delta$ is a star, with central vertex $j$, say, and $X_j$ is of type $A_1(9)$, and $S_G = S_{X_{jk}}$ for all $k \neq j$.

Proof. Same as (4.3).

Let us fix some notation for the rest of the paper. Recall that the parabolic system $X$ is defined on the index set $I = \{1, 2, \ldots, n\}$. For any nonempty subset $J$ of $I$ we set $X_J = \{X_i, i \in J\}$, and $Q_J = S_{X_J}$. If $J$ consists of all vertices of $I$ but $i$, we set (as in §3) $G_i := X_J$ and $M_i = Q_J$.

(4.6) Let $\Delta$ be of type $Y(r, s)$, $r, s \geq 2$. Let $J = \{1, \ldots, r\}$ and $J_i = J \cup \{r+i\}$ for all $1 \leq i \leq s$. Then all groups $\overline{X_J}$ are of the same type $(B_{r+1}(3), C_{r+1}(3))$ or $2A_{2r+1}(3)$), and $S_G = S_{X_{J_i}}$ for $i = 1, 2, \ldots, s$.

Proof. We may assume $S_G = Z(G) = 1$. Since $\Delta$ is not spherical, the system is not strong, and hence obviously $\overline{X_{r+i,r+j}}$ is of type $D$ for all $1 \leq i \neq j \leq s$. The type of $\overline{X_J}$ is clearly determined by the system $\{X_r, X_{r+i}\}$ (for the case $B/C$ see (1.7)), hence by (4.2) independent of $i$. For the last claim, assume $S_G = 1$. We may assume $s = 2$. Consider first the case $r = 2$.

We have to show $M_3 = M_4 = 1$. Assume $M_3M_4 > M_4$. Then $M_3M_4$ is a normal 3-subgroup of $X_{12}$, hence is contained in $Q_{12}$ and equals $Q_{12}$, unless $G_3$ and $G_4$ are of type $B_3(3)$. But $M_3 \leq Q_4 = Q_3$, since the system is not
strong, and we get a contradiction, if \( M_3 M_4 = Q_{12} \), because in \( G_4 \) we see that \( Q_{12}/M_4 \) is not contained in \( Q_3/M_4 \).

Hence we may assume \( G_3 \) and \( G_4 \) are of type \( B_3(3) \) and \( M_3 M_4 \) is the unique \( X_{12}\)-invariant subgroup of \( Q_{12} \) with \( M_4 < M_3 M_4 < Q_{12} \).

Assume \( M_3 \leq M_1 \). Then \( M_3 M_4 \) is contained in \( M_1 M_4 = Q_{23} \) and \( Q_{12} \), a contradiction to the above.

Therefore, we may assume \( M_3 \nleq M_1 \), hence \( M_3 M_1 / M_1 \) has order 3 by (4.2). Since \( M_3 M_4 / M_3 \cong M_4 / M_3 \cap M_4 \) is a nontrivial \( X_{12}\)-module, we have nontrivial action of \( G_i \) on \( M_i \) for \( i = 3, 4 \), and of course also for \( i = 1 \). We again apply the amalgam method to get a contradiction.

Assume first \( G_1 \leq N(Z) \). Then by the discussion above \( \Omega_1(Z(S)) = \Omega_1(Z(M_i)) \) and we consider the graph \( \Gamma(3, 4) \) with irreducible nontrivial submodules \( Z_i \) of \( \Omega_1(Z(M_i)) \), \( i = 3, 4 \). Without loss, \( Z_3 \) is an \( FF\)-module for \( G_3 \), and hence a natural module by (2.7).

Consider \( \Gamma(3, 2) \) with the natural module \( Z_3 \) and \( Z_2 \) contained in \( Z_3 \). Then a critical pair \( (\alpha, \delta) \) is of type \( (3, 3) \) or \( (3, 2) \) by the choice of \( Z_2 \). But type \( (3, 2) \) is impossible, since \( O_3(G_3, 2) = M_2 \), while type \( (3, 3) \) contradicts (2.7)(iv). Hence we may finally assume \( G_1 \nleq N(Z) \) and also \( G_4 \nleq N(Z) \).

Consider \( \Gamma(1, 4) \) with \( Z_i = \Omega_1(Z(M_i)) \), \( i = 1, 4 \). Let \( (\alpha, \delta) \) be a critical pair. Then it is not of type \( (4, 4) \) by (2.7), (1.9), and (1.6). But if it is of type \( (1, 4) \) or \( (4, 1) \), then transvections are induced on the quadratic module \( Z_4 \) and we get a contradiction, by (2.7).

Hence it is of type \( (1, 1) \), and the usual argument yields that \( V_4/Z_4 \) is trivial for \( G_4 \), contradicting the action of \( X_{23} \) on \( Z_1/Z_1 \cap Z_4 \). This finishes the case \( r = 2 \), and we use induction on \( r \), still \( s = 2 \). But now clearly \( M_{r+2} \leq Q_2, ..., r, r+1 = M_i \) and also \( M_{r+1} \leq M_1 \), and we get \( M_{r+1} = M_{r+2} = 1 \) as in the case \( r = 2 \).

(4.7) Let \( \Delta \) be a complete bipartite graph with only double bonds involved. Assume \( \overline{X}_{ij} \) is of type \( C_2(3) \) for some \( i, j \). Then \( |S_{X_{ij}} : S_G| \leq 3 \).

Proof. We may assume \( S_G = Z(G) = 1 \). For \( n = 3 \), this is contained in (4.2), so consider the case \( n = 4 \) next. Assume first \( \Delta \) is

Then we may assume without loss \( |Q_{23} : M_1| = 3 \). Also \( |Q_{34} : M_1| = 3 \) and \( |S : M_1| = 3^5 \). Now, \( \overline{X}_3 \) is isomorphic to \( SL_2(3) \), and \( X_{12} \cong X_{23} \cong X_{34} \cong X_{41} \), hence \( M_4 = Q_{12} = Q_{23} \) and \( M_2 = Q_{34} = Q_{41} \). Therefore, \( M_1 \) is contained in \( M_2 \) and \( M_4 \), and also \( M_3 \leq M_2 \cap M_4 \). So, since \( |M_2 M_4 : M_1|, |M_2 M_4 : M_3| \) are at most \( 3^2 \), we have either \( M_1 = M_3 \), or \( M_1 M_3 = M_2 \cap M_4 \). In both cases, \( M_1 M_3 = S_G = 1 \). The claim follows. Assume now that \( \Delta \) is

and assume \( M_1 \leq M_2 \). Then clearly \( M_1 M_2 = Q_{34} \), and if \( M_2 \leq M_1 \), then \( |S : M_1| = 3^4 \), and also \( M_1 = Q_{24} \), whence \( M_3 \leq M_1 \). Now \( M_2 = M_3 \) or
$M_2M_3 = M_1$, so in any case $M_2M_3 = S_G = 1$ and again the claim follows. Hence assume finally in rank 4 that for all pairs $(i,j)$ with $i,j$ different elements of {1, 2, 3} we have $M_i \leq M_j$ and $M_j \leq M_i$. Then, however, $M_i \cap M_j \leq M_k$ for $\{i, j, k\} = \{1, 2, 3\}$, and therefore $M_i \cap M_j \cap M_k = M_i \cap M_j$ for all choices of $i, j$. Hence $M_i \cap M_2 \cap M_3$ is normal in $G$, and $M_i \cap M_j = S_G = 1$. But now the action of $G_4$ on $M_4$ gives a contradiction. Thus we are lead to $n$ at least 5.

For given $i, j$ we pick three more vertices 1, 2, 3 in $\Delta$, by the above $|Q_{ij} : M_k| \leq 3$ for $k = 1, 2, 3$. If now $M_1$ is contained properly in $M_2$, then also $M_3$ is contained properly in $M_2$, and we get the claim $M_2 = M_1M_3 = S_G = 1$. But if no $M_k$ is properly contained in $M_h$ for any different $k, h$ in \{1, 2, 3\}, then we get $Q_{ij} = M_1M_2 = M_1M_3 = M_2M_3 = S_G = 1$, again the result.

References

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