TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 339, Number 1, September 1993

2-WEIGHTS FOR UNITARY GROUPS

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ABSTRACT. This paper gives a description of the local structures of 2-radical subgroups in a finite unitary group and proves Alperin's weight conjecture for finite unitary groups when the characteristic of modular representation is even.

INTRODUCTION

Let G be a finite group and r a prime. Denote $O_r(G)$ the largest normal r-subgroup of G. Following [3], we shall call an r-subgroup R of G a *radical* subgroup if $R = O_r(N(R))$, and a pair (R, φ) of an r-subgroup R and an irreducible character φ of $N(R)$ a *weight* of G if φ is trivial on R and in an r-block of defect 0 of $N(R)/R$, where $N(R)$ is the normalizer of R in G. Moreover, a weight (R, φ) is a B-weight for an r-block B of G if φ is contained in an r-block b of $N(R)$ such that $B = b^G$, that is, B corresponds to b by the Brauer homomorphism. Alperin in [2] conjectured that the number of weights of G should equal the number of modular irreducible representations. Moreover, this equality should hold block by block. Here a weight (R, φ) is identified with its conjugates in G. This conjecture has been proved by Alperin and Fong [3] for symmetric groups and for finite general linear groups when the characteristic r of modular representation is odd, and by the author [4] for finite general linear groups when r is even. In [5] the conjecture is proved for finite unitary groups when r is odd and in this paper the conjecture is proved for finite unitary groups when r is even. The defining characteristic of group may be assumed to be odd since the result is known when it is even.

If (R, φ) is a weight of G, then R is necessarily a radical subgroup of G. Thus the first step to describe a weight in [3, 5, and 4] is to determine the structures of radical subgroups in the given group. If q is a power of an odd prime, then these structures in a general linear group $\mathrm{GL}(n, q)$ are divided into two different parts in [4] according as 4 divides $q-1$ or $q+1$. Following [11], in the former case, we shall say that 2 is *linear* and in the latter case, 2 is *unitary*. It turns out that the structures of radical subgroups of a unitary group $\mathrm{U}(n, q)$ can be obtained by switching the two cases in the general linear group $\mathrm{GL}(n, q)$. Namely, the structures of radical subgroups in $\mathrm{U}(n, q)$ when 2 is linear are the same as those in $\mathrm{GL}(n, q)$ when 2 is unitary; those in $\mathrm{U}(n, q)$ when 2 is unitary is the same as those in $\mathrm{GL}(n, q)$ when 2 is linear. These are proved in §§1 and 2. In §3 we count the number of weights in a block and the

Received by the editors February 26, 1991 and, in revised form, June 4, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20C20; Secondary 20G40.

0002-9947/93 \$1.00 + \$.25 per page

conjecture is proved in (3E). Although the outlines of our proofs are similar to those in the case of the general linear group, the proofs in §§1 and 2 are both longer and more technical. The proofs in §3 can be obtained by modifying those in [4, §3] since both general linear groups and unitary groups have the similar local structures of radical subgroups.

I wish to thank Professor Paul Fong, my Ph.D. advisor, for many corrections and suggestions. I also wish to thank Professor Bhama Srinivasan for a lot of useful help.

1. The 2-groups of symplectic type

Throughout the paper we shall follow the notations of [4, 5, 6, and 11]. In particular, $2_\eta^{2\gamma+1}$ denotes the extraspecial group of order $2^{2\gamma+1}$ with type η, where $\eta = +$ or $-$. If $E \simeq 2_\eta^{2\gamma+1}$ with center $Z(E) = \langle z \rangle$, then it is generated by $x_1, x_2, \dots, x_{2\gamma-1}, x_{2\gamma}$ such that $[x_{2i-1}, x_{2i}] = x_{2i-1}^{-1}x_{2i}^{-1}x_{2i-1}x_{2i} = z$, $[x_{2i}, x_{2i+1}] = 1$ for $i = 1, \dots, \gamma$, $[x_i, x_j] = 1$ for $|i-j| \geq 2$, $|x_i| = 2$ for $i \geq 3$, and $|x_1| = |x_2| = 2$ or $|x_1| = |x_2| = 4$ according as $\eta = +$ or $-$, in the latter case $x_1^2 = x_2^2 = z$. Let S_β, D_β, and Q_β be respectively semidihedral, dihedral, and generalized quaternion groups of order 2^β. A 2-group R is called of *symplectic type* if R is a central product EP of an extraspecial group E and either a cyclic 2-group P or $P = S_\beta$, D_β, Q_β with $\beta \geq 4$. Here the center of E is identified with $\Omega_1(Z(P))$. Now we consider the embedding of R into a unitary group.

Again we denote $\operatorname{Aut} G$ the automorphism group of a finite group G, $\operatorname{Inn} G$ the group of inner automorphisms, and $\operatorname{Aut}^0 G$ the subgroup of $\operatorname{Aut} G$ acting trivially on $Z(G)$.

Suppose $R = EZ$ has symplectic type with Z cyclic. If $R > E$, then R can be rewritten as the central product of Z and an extraspecial group E with plus type, so that $\Omega_2(R)$ is a central product of a cyclic group of order 4 and E. If $R = E$, then $\Omega_2(R) = R$. In both cases, $\operatorname{Aut}^0 R = \operatorname{Aut}^0 \Omega_2(R)$. By [18, Theorem 1] and [16, §4; 15, pp. 406–407],

$$\operatorname{Aut}^0 \Omega_2(R)/\operatorname{Inn}\Omega_2(R) \simeq \begin{cases} \operatorname{Sp}(2\gamma, 2) & \text{if } R > E, \\ \operatorname{O}^\eta(2\gamma, 2) & \text{if } R = E. \end{cases}$$

Let $\mathbb{F}_q$ be the field of q elements with odd characteristic, and 2^{a+1} the exact power of 2 dividing $q^2 - 1$, so that $a \geq 2$. We shall say that 2 is *linear* or *unitary* according as 2^a divides $q-1$ or $q+1$.

Let $\Delta(T) = T^m + a_{m-1}T^{m-1} + \cdots + a_1T + a_0$ be a monic irreducible polynomial in $\mathbb{F}_{q^2}[T]$. Denote d_Δ the degree of polynomial Δ and define

$$\tilde{\Delta}(T) = (a_0^{-1})^q T^m (T^{-m} + a_{m-1}^q T^{-m+1} + \cdots + a_1^q T^{-1} + a_0^q).$$

In particular, ω is a root of $\Delta(T)$ if and only if ω^{-q} is a root of $\tilde{\Delta}(T)$. Thus $\Delta = \tilde{\Delta}$ if and only if Δ has odd degree d_Δ and the roots of Δ have order dividing $q^{d_\Delta} + 1$ (see [11, p. 111]). Let

$$\mathscr{F}_1 = \{\Delta : \Delta \in \mathbb{F}_{q^2}[T],\ \Delta \text{ is monic irreducible},\ \Delta \neq T,\ \Delta = \tilde{\Delta}\},$$
$$\mathscr{F}_2 = \{\Delta\tilde{\Delta} : \Delta \in \mathbb{F}_{q^2}[T],\ \Delta \text{ is monic irreducible},\ \Delta \neq T,\ \Delta \neq \tilde{\Delta}\},$$

and $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$. Thus any elementary divisor, in the sense of [11], of a unitary matrix lies in $\mathscr{F}$. We also define a sign ε_Γ for Γ in $\mathscr{F}$ by

$$\varepsilon_\Gamma = \begin{cases} -1 & \text{if } \Gamma \in \mathscr{F}_1, \\ 1 & \text{of } \Gamma \in \mathscr{F}_2. \end{cases}$$

Let V be a unitary space over $\mathbb{F}_{q^2}$ with a form $f(u, v)$, and $G = \mathrm{U}(V)$. An element of G is said to be *primary* if it has a unique elementary divisor.

(1A) *Let g be a primary 2-element of G with a unique elementary divisor $\Gamma \in \mathscr{F}$ of multiplicity m. Then either $g \in Z(G)$ or $C_G(g) \simeq \mathrm{GL}(m, q^{d_\Gamma})$. In particular, if 2 is linear and $|g| \geq 4$, then $C_G(g) \simeq \mathrm{GL}(m, q^{d_\Gamma})$.*

Proof. If $\Gamma \in \mathscr{F}_1$, then d_Γ is odd, $C_G(g) \simeq \mathrm{U}(m, q^{d_\Gamma})$, and $g \in Z(C_G(g))$. But $Z(G) \leq Z(C_G(g))$ and $|O_2(Z(G))| = |O_2(Z(C_G(g)))|$ by d_Γ odd, so $O_2(Z(G)) = O_2(Z(C_G(g)))$ and $g \in Z(G)$. If $\Gamma \in \mathscr{F}_2$, then

$$C_G(g) \simeq \mathrm{GL}(m, q^{d_\Gamma}).$$

Suppose 2 is linear and $|g| \geq 4$. Then $O_2(Z(G))$ has order 2, so that $g \notin Z(G)$ and then $\Gamma \in \mathscr{F}_2$. This completes the proof.

Let R be a 2-subgroup of $G = \mathrm{U}(V)$. Then R acts on the underlying space V of G. We shall say that an R-submodule W of V is *nondegenerate* or *totally isotropic* if W is respectively a nondegenerate or totally isotropic subspace of V.

(1B) *Let R be a 2-subgroup of G. Then V has an R-module decomposition*

$$V = V_1 \perp V_2 \perp \cdots \perp V_s \perp (U_1 \oplus U_1') \perp \cdots \perp (U_t \oplus U_t'), \tag{1.1}$$

where the V_i are nondegenerate simple R-submodules, the U_j and U_j' are totally isotropic simple R-submodules such that $U_j \oplus U_j'$ is nondegenerate and has no proper nondegenerate R-submodule. Moreover, if $Z(R)$ is cyclic and is not a subgroup of $Z(G)$, then $s = 0$.

Proof. Let W be a simple R-submodule of V of minimal dimension. Since the radical $\{v \in W : f(v, W) = 0\}$ of W is an R-submodule of W, it follows that W is either nondegenerate or totally isotropic. If W is nondegenerate, then $V = W \perp W^\perp$, where $W^\perp = \{v \in V : f(v, W) = 0\}$. The decomposition (1.1) then holds by induction, since $W^\perp$ is a nondegenerate R-submodule. If W is totally isotropic, then $W^\perp$ is an R-submodule of V and $V = W^\perp \oplus W'$ for some R-submodule W' of the same dimension as W, since V is a semisimple R-module. Moreover, $W \oplus W'$ is nondegenerate. Thus W' is either a nondegenerate or a totally isotropic simple R-module. If W' is nondegenerate, we can replace W by W' and appeal to the earlier case. Suppose W' is totally isotropic and $W \oplus W'$ has a proper nondegenerate R-submodule Y. Then Y is simple, so that we can replace W by Y and appeal to the earlier case again. Thus we may suppose $W \oplus W'$ has no proper nondegenerate R-submodule, so that $W \oplus W'$ is of the required form $U_j \oplus U_j'$, and we can apply induction to its orthogonal complement.

Suppose $Z(R)$ is cyclic and $Z(R) \nleq Z(G)$. If V has a nondegenerate simple R-submodule V_1, then the representation $\mathbf{F}$ of R in $\mathrm{U}(V_1)$ is irreducible, so that the generator g of $\mathbf{F}(Z(R))$ is primary with a unique elementary divisor

$\Gamma \in \mathcal{F}_2$ of multiplicity m by (1A). Thus $C_{\mathrm{U}(V_1)}(g) \simeq \mathrm{GL}(m, q^{d_\Gamma})$ and $\mathbf{F}(R) \leq C_{\mathrm{U}(V_1)}(g)$. So V_1 has a hyperbolic decomposition $V_1 = W_1 \oplus W_1'$ such that W_1 and W_1' are R-submodules of V_1. This is impossible. Thus the second half of (1B) follows.

We consider the groups $\mathrm{GL}(n, \varepsilon q)$, where $\varepsilon = \pm 1$. Here we are following the useful convention used by [6] in denoting $\mathrm{U}(n, q)$ as $\mathrm{GL}(n, -q)$. In the rest of this paper such terms as orthogonal, orthonormal, nondegenerate, totally isotropic, and isometric will have meaning only in contexts involving $\mathrm{GL}(n, -q)$ and unitary spaces, but no meaning in contexts involving $\mathrm{GL}(n, q)$ and linear spaces. The following four propositions are known results for general linear groups (cf. [4, 12, 13, and 14]) and we shall give a proof for both general linear and unitary groups.

(1C) *Let E be a quaternion group and $G = \mathrm{GL}(2, \varepsilon q)$. Then G contains a unique conjugacy class of subgroups isomorphic to E. In addition, let E be embedded as a subgroup of G, $N = N_G(E)$, and $C = C_G(E)$. If 4 divides $q + \varepsilon$, then*

$$C = Z(G), \qquad N/EZ(G) \simeq \mathrm{O}^-(2, 2).$$

Proof. Let $E = \langle x_1, x_2 \rangle$ and V the underlying space of G. If 4 divides $q - \varepsilon$, then $\mathbb{F}_{q^2}$ has an element w of order 4, so that with respect to an orthonormal basis of V,

$$X = \begin{pmatrix} w & \\ & -w \end{pmatrix}, \qquad Y = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \tag{1.2}$$

generates a quaternion subgroup of $\mathrm{GL}(2, \varepsilon q)$. Thus the mapping $x_1 \mapsto X$ and $x_2 \mapsto Y$ gives a faithful and irreducible representation of E in G. Suppose E is embedded as a subgroup of G. Since x_1 has order 4, at least one of the elements w and $w^3 = -w$ is its eigenvalue. Without loss of generality, we may suppose w is its eigenvalue. Let $V_j = \{v \in V : x_1 v = (-1)^{j+1} w v\}$ for $j = 1, 2$. Then V_j's are nondegenerate subspaces of V permuted by x_2 cyclically, since $x_1 x_2 = -x_2 x_1$. So both V_1 and V_2 have dimension 1. Suppose $\{v_1\}$ is an orthonormal basis of V_1, so that $\{v_1, x_2 v_1\}$ is an orthonormal basis of V and x_1, x_2 are given by (1.2) with respect to this basis. Thus G contains a unique conjugacy class of E.

Suppose 4 divides $q + \varepsilon$. Then by [7, pp. 142–143] a Sylow 2-subgroup P of $\mathrm{GL}(2, \varepsilon q)$ is semihedral and generated by two matrices X and Y satisfying the following conditions

$$\begin{gathered} |X| = 2^{a+1}, \qquad |Y| = 4, \qquad X^{2^a} = Y^2 = -I_2, \\ YXY^{-1} = X^{2^a - 1} = -X^{-1}. \end{gathered} \tag{1.3}$$

So $X^{2^{a-1}}$ has order 4 and $YX^{2^{a-1}}Y^{-1} = -X^{2^{a-1}}$. The mapping $x_1 \mapsto X^{2^{a-1}}$, $x_2 \mapsto Y$ gives a faithful and irreducible representation of E in G. Suppose E is embedded as a subgroup of G and suppose it is a subgroup of P. Since $P = \{X^i, X^i Y : 1 \leq i \leq 2^{a+1}\}$, its elements of order 4 are $\{\pm X^{2^{a-1}}, \pm X^{2i} Y\}$, where $1 \leq i \leq 2^{a-2}$. If $X^{2i}Y$ and $X^{2j}Y$ are generators of E, then $X^{2i}Y(X^{2j}Y)^{-1} = X^{2(i-j)}$ is an element of order 4 in E, so that $\langle X^{2(i-j)} \rangle = \langle X^{2^{a-1}} \rangle$ and $E = \langle X^{2^{a-1}}, X^{2i}Y \rangle$. It is clear that $X^{2^{a-1}}$ and $X^{2i}Y$ generate a quaternion subgroup of P, where $1 \leq i \leq 2^{a-1}$. The subgroup $\langle X^{2^{a-1}} \rangle$ of $E = \langle X^{2^{a-1}}, X^{2i}Y \rangle$ is called a *base* subgroup of E, in the sense of [10]. Since $X^{2^a} = -1$, P has 2^{a-2} quaternion subgroups and each contains $\langle X^{2^{a-1}} \rangle$ as a base subgroup. Since $X^{-1}X^{2i}YX = -X^{2i-2}Y$ for $1 \leq i \leq 2^a$, all the quaternion subgroups of P are conjugate in P. Each quaternion subgroup of G is contained in a Sylow 2-subgroup of G and all the Sylow 2-subgroups are conjugate in G, so that all the quaternion subgroups of G are conjugate in G.

By [4, (1A)] $\mathbb{F}_q$ is a splitting field of E, so that $C = Z(G)$. Since

$$\mathrm{Aut}^0 E/\mathrm{Inn}\, E \simeq \mathrm{O}^-(2, 2)$$

and each element of N induces an element of $\mathrm{Aut}^0 E$, it follows that $N/EC \simeq \mathrm{O}^-(2, 2)$ if and only if $|N/EC| = 6$. Denote C_β the cyclic group of order 2^β. Since 4 divides $q + \varepsilon$, the centralizer $C_G(C_2)$ of a subgroup C_2 of G is isomorphic to a Coxeter torus $\mathrm{GL}(1, q^2)$ of G, so that C_2 is a subgroup of the Sylow 2-subgroup C_{a+1} of $C_G(C_2)$ and $C_G(C_2) = C_G(C_{a+1})$. Thus if any two subgroups C_{a+1} and C'_{a+1} both contain a subgroup C_2, then $C_{a+1} = C'_{a+1}$. Fix a subgroup C_2. Let $H = N_G(C_2)$, C_{a+1} the Sylow 2-subgroup of $C_G(C_2)$, and P a Sylow 2-subgroup of G containing C_{a+1}. Since $C_G(C_2)$ is a Coxeter torus and H is its normalizer in G, all the normalizers of cyclic subgroups of order 4 in G are conjugate in G. We may suppose $C_{a+1} = \langle X \rangle$, $C_2 = \langle X^{2^{a-1}} \rangle$, and $P = \langle X, Y \rangle$, where X and Y are given by (1.2). Thus $P \leq H$, $N_G(P) = PZ(G) \leq H$, and $H = PC_G(C_2)$ since $|H/C_G(C_2)| = 2$ (cf. [11, p. 129]). So $N_G(H) = H$, $|H| = 2(q^2 - 1)$, and H has $\frac{1}{2}q(q - \varepsilon)$ conjugates in G. Moreover, a Sylow 2-subgroup of H is a Sylow 2-subgroup of G and so all quaternion subgroups of H are conjugate in H. Let Q be any quaternion subgroup of P. Then $Q = \langle X^{2^{a-1}}, X^{2k}Y \rangle$ for some $1 \leq k \leq 2^{a-2}$, so $g = X^{2^{a-2}}$ fixes $X^{2^{a-1}}$ and $gX^{2k}Yg^{-1} = \pm X^{2^{a-1}}X^{2k}Y$, so that $g \in N_H(Q)$. Each element of H maps $X^{2^{a-1}}$ either to itself or to $-X^{2^{a-1}}$ and the order 3 element of $\mathrm{Aut}^0 Q$ maps $X^{2^{a-1}}$ either to $\pm X^{2k}Y$ or to $\pm Y$. It follows that $N_H(Q) = \langle g, QZ(G) \rangle$, so that $|N_H(Q)| = 8(q - \varepsilon)$ and H has $\frac{1}{4}(q + \varepsilon)$ quaternion subgroups. Moreover, each quaternion subgroup Q' of H contains C_2 as a base subgroup, since Q' is contained in a Sylow 2-subgroup of H and each Sylow 2-subgroup of H contains C_{a+1}. Since $N_G(C_{a+1}) = H$ and G has exactly one conjugacy class of cyclic subgroups of order 2^{a+1}, G contains also $\frac{1}{2}q(q - \varepsilon)$ conjugates of C_{a+1}. For each conjugate H^x of H, denote C^x_{a+1} the unique subgroup of H^x of order 2^{a+1}, and denote C^x_2 the unique subgroup of order 4 of C^x_{a+1}. Then C^x_2 are all conjugates of C_2 in G, where x run over representatives of coset G/H. Each C^x_2 serves as a base subgroup of $\frac{1}{4}(q + \varepsilon)$ quaternion subgroups of H^x. All quaternion subgroups of G can be obtained in this way and each of them contains 3 subgroups of form C^x_2 as base subgroup. It follows that G has $\frac{1}{4}(q + \varepsilon)\frac{1}{2}q(q - \varepsilon)\frac{1}{3} = \frac{1}{24}q(q^2 - 1)$ quaternion subgroups, so that $|N_G(E)| = 24(q - \varepsilon)$ and $|N_G(E)/EC| = 6$. This completes the proof.

(1D) *Let E be an extraspecial 2-group of order $2^{2\gamma+1}$. Then $G = \mathrm{GL}(2^\gamma, \varepsilon q)$ contains a unique conjugacy class of subgroups isomorphic to E.*

Proof. Let $E_i = \langle x_{2i-1}, x_{2i} \rangle$, and V_i a linear space of dimension 2 over $\mathbb{F}_q$ if $\varepsilon = 1$, or a unitary space of dimension 2 over $\mathbb{F}_{q^2}$ if $\varepsilon = -1$, for $1 \leq i \leq \gamma$. Then E_i acts faithfully, irreducibly, and isometrically on V_i. Namely if E_i

is a dihedral group and $\{v_1^i, v_2^i\}$ is an orthonormal basis of V_i, then we may define

$$x_{2i-1}: v_j^i \mapsto (-1)^{j+1} v_j^i, \qquad x_{2i}: v_j^i \mapsto v_{j+1}^i,$$

where the subscripts on the basis vectors are naturally read modulo 2. In particular, $z = [x_{2i-1}, x_{2i}]: v_j^i \mapsto -v_j^i$.

Suppose E_1 is a quaternion group and V_1 is the underlying space of $\mathrm{GL}(2, \varepsilon q)$. Let X and Y be matrices of $\mathrm{GL}(2, \varepsilon q)$ defined by (1.2) or (1.3) according as 4 divides $q - \varepsilon$ or $q + \varepsilon$ with respect to an orthonormal basis $\{v_1^1, v_2^1\}$ of V_1. In the former case, a faithful and irreducible representation of E_1 on V_1 is given by the mapping $x_1 \mapsto X$ and $x_2 \mapsto Y$; in the latter case, it is given by $x_1 \mapsto X^{2^{a-1}}$ and $x_2 \mapsto Y$.

E then acts faithfully and irreducibly on $V = V_1 \otimes V_2 \otimes \cdots \otimes V_\gamma$, since E is a central product of E_i's and the element z in E_i is represented on V_i by the scalar matrix $-I$. We simplify notation and write

$$v_{j_1}^1 \otimes v_{j_2}^2 \otimes \cdots \otimes v_{j_\gamma}^\gamma = [j_1, j_2, \ldots, j_\gamma]$$

where $1 \leq j_i \leq 2$. So the 2^γ elements $[j_1, j_2, \ldots, j_\gamma]$ form an orthonormal basis for V, and

$$\begin{aligned} x_{2i-1}&: [j_1, j_2, \ldots, j_\gamma] \mapsto (-1)^{j_i+1}[j_1, j_2, \ldots, j_\gamma], \\ x_{2i}&: [j_1, j_2, \ldots, j_\gamma] \mapsto [j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1} \ldots, j_\gamma], \end{aligned} \tag{1.4}$$

except when E has minus type, in which case the actions of x_i for $i \geq 3$ are given by (1.4) and

$$\begin{aligned} x_1&: [j_1, j_2, \ldots, j_\gamma] \mapsto \begin{cases} (-1)^{j_1+1} w[j_1, j_2, \ldots, j_\gamma] & \text{if } 4|q-\varepsilon, \\ (X^{2^{a-1}} v_{j_1}^1) \otimes [j_2, j_3, \ldots, j_\gamma] & \text{if } 4|q+\varepsilon, \end{cases} \\ x_2&: [j_1, j_2, \ldots, j_\gamma] \mapsto \begin{cases} (-1)^{j_1}[j_1+1, j_2, \ldots, j_\gamma] & \text{if } 4|q-\varepsilon, \\ (Y v_{j_1}^1) \otimes [j_2, j_3, \ldots, j_\gamma] & \text{if } 4|q+\varepsilon. \end{cases} \end{aligned} \tag{1.5}$$

Since basic vectors are mapped onto orthonormal vectors by generating elements of E, E acts on V by isometries. Thus $\mathrm{GL}(2^\gamma, \varepsilon q)$ contains a copy of E.

To prove the uniqueness, we suppose E is embedded as a subgroup in G. It suffices to show that an orthonormal basis of the underlying space V of G exists such that the actions of x_i are given by (1.4) or (1.5). If $\gamma = 1$ and E has minus type, then the uniqueness follows by (1C). Suppose either $\gamma \geq 2$ or $\gamma = 1$ and E has plus type. Then the subspaces $W_j = \{v \in V: x_{2\gamma-1}v = (-1)^{j+1}v\}$ for $j = 1, 2$ are nondegenerate and permuted by $x_{2\gamma}$ cyclically since $x_{2\gamma-1}x_{2\gamma} = -x_{2\gamma}x_{2\gamma-1}$. In particular, W_1 and W_2 has the same dimension $2^{\gamma-1}$. If $\gamma = 1$, then E has plus type and W_1 has an orthonormal basis $\{v_1\}$. Thus $\{v_1, x_2v_1\}$ is an orthonormal basis of V and the actions of x_1 and x_2 are given by (1.4) with respect to this basis. Suppose $\gamma \geq 2$. Then the subgroup $\langle x_1, x_2, \ldots, x_{2\gamma-3}, x_{2\gamma-2}\rangle$ of E is an extraspecial group of order $2^{\gamma-1}$ with the same type as E and its acts faithfully and irreducibly on W_1. We may suppose by induction that $x_1, x_2, \ldots, x_{2\gamma-3}, x_{2\gamma-2}$ act on W_1 by (1.4) or (1.5) relative to the orthonormal basis $\{[j_1, j_2, \ldots, j_{\gamma-1}]: j_i = 1, 2\}$ of W_1. Then

$$\{[j_1, j_2, \ldots, j_\gamma] = x_{2\gamma}^{j_\gamma+1}[j_1, j_2, \ldots, j_{\gamma-1}]: j_i = 1, 2\}$$

is an orthonormal basis of V and $x_1, x_2, \ldots, x_{2\gamma}$ act on V by (1.4) or (1.5). Thus the uniqueness holds.

Remark. (1) Suppose E is embedded as a subgroup of $\mathrm{GL}(2^\gamma, \varepsilon q)$ and $\langle x_{2k-1}, x_{2k}\rangle \leq E$ is a dihedral group for some k. In the notation of (1D), we claim that V has an orthonormal basis $\{[j_1, j_2, \ldots, j_\gamma]'\}$, where $1 \leq j_i \leq 2$ such that the actions of x_{2i-1} and x_{2i} for $i \neq k$ are given by (1.4) or (1.5) with $[j_1, j_2, \ldots, j_\gamma]$ replaced by $[j_1, j_2, \ldots, j_\gamma]'$ and

$$x_{2k-1}\colon [j_1, \ldots, j_k, \ldots, j_\gamma]' \mapsto [j_1, \ldots, j_k+1, \ldots, j_\gamma]',$$
$$x_{2k}\colon [j_1, \ldots, j_k, \ldots, j_\gamma]' \mapsto (-1)^{j_k+1}[j_1, \ldots, j_k, \ldots, j_\gamma]'.$$

The proof of the remark is similar to that of the uniqueness above with $x_{2\gamma-1}$ replaced by x_{2k}, $x_{2\gamma}$ by x_{2k-1}, j_γ by j_k and some obvious modifications.

(2) Suppose E has plus type and $\mathbf{X}$ is a faithful representation of E in $\mathrm{U}(V)$ with exactly one Wedderburn component. Then $\mathbf{X}$ has degree $m2^\gamma$ for some $m \geq 1$ and there exists an orthonormal basis $\{[j_1, j_2, \ldots, j_\gamma]_k\}$ of V, where $1 \leq j_i \leq 2$ and $1 \leq k \leq m$ such that for each $1 \leq k \leq m$, the actions of x_{2i-1} and x_{2i} are given by (1.4) with $[j_1, j_2, \ldots, j_\gamma]$ replaced by $[j_1, j_2, \ldots, j_\gamma]_k$. It follows that in the decomposition (1.1) of V as an E-module, $V = M_1 \perp M_2 \perp \cdots \perp M_m$, where the M_k's are nondegenerate simple E-modules linearly generated by $\{[j_1, j_2, \ldots, j_\gamma]_k\colon 1 \leq j_i \leq 2\}$, so that E acts faithfully on each M_k. Moreover, if $\mathbf{X}'$ is another such representation of E in $\mathrm{U}(V)$, then $\mathbf{X}(E)$ and $\mathbf{X}'(E)$ are conjugate in $\mathrm{U}(V)$ by the uniqueness of (1D). The proof of this remark is similar to that of the uniqueness above. Since the unique faithful and irreducible representation of E has degree 2^γ, it follows that $\mathbf{X}$ has degree $m2^\gamma$ for some $m \geq 1$. For $j = 1, 2$, let $V_j' = \{v \in V\colon x_1 v = (-1)^{j+1} v\}$. Then the V_j''s are nondegenerate permuted by x_2 cyclically, so that $\dim V_1' = \dim V_2' = m2^{\gamma-1}$. If $\gamma = 1$, take an orthonormal basis $\{[1]_k\}$ of V_1', where $1 \leq k \leq m$ and let $[j_1]_k = x_2^{j_1+1}[1]_k$ for $1 \leq j_1 \leq 2$. Then $\{[j_1]_k\}$, where $1 \leq j_1 \leq 2$ and $1 \leq k \leq m$ is a required basis of V. Suppose $\gamma \geq 2$, so that $K = \langle x_3, x_4, \ldots, x_{2\gamma}\rangle$ is an extraspecial group with plus type and K acts on V_1' faithfully. The representation of K on V_1' has exactly one Wedderburn component. So by induction there exists an orthonormal basis $\{[j_2, j_3 \ldots, j_\gamma]_k\}$ of V_1' such that the actions of x_{2i-1} and x_{2i}, for $i \geq 2$ are given by (1.4) with $[j_2, j_3 \ldots, j_\gamma]$ replaced by $[j_2, j_3, \ldots, j_\gamma]_k$. Let $[j_1, j_2 \ldots, j_\gamma]_k = x_2^{j_1+1}[j_2, j_3, \ldots, j_\gamma]_k$. Then $\{[j_1, j_2 \ldots, j_\gamma]_k\}$, where $1 \leq j_i \leq 2$ and $1 \leq k \leq m$ is a required basis of V. This proves the remark.

(1E) *Suppose* 4 *divides* $q+\varepsilon$. *Let* $G = \mathrm{GL}(2^\gamma, \varepsilon q)$, *and* $E \simeq 2_\eta^{2\gamma+1}$ *embedded as a subgroup of* G. *Set* $C = C_G(E)$ *and* $N = N_G(E)$. *Then* $C_N(E) = C = Z(G)$ *and* $N/Z(N)E \simeq O^\eta(2\gamma, 2)$. *Moreover, each linear character of* $Z(N)$ *acting trivially on* $O_2(Z(N))$ *can be extended as a character of* N *acting trivially on* E.

Proof. Since $\mathbb{F}_q$ is a splitting field of E (see [4, (1A)]), it follows $C = C_N(E) = Z(G)$. The elements of N induce automorphisms in $\mathrm{Aut}^0 E$. We shall exhibit elements in N which together with E generate $\mathrm{Aut}^0 E$. Since $\mathrm{Aut}^0 E/\mathrm{Inn}\, E \simeq \mathrm{O}^\eta(2\gamma, 2)$, we need only exhibit elements in N which induce generators of $\mathrm{O}^\eta(2\gamma, 2)$ on $E/Z(E)$. The group $\mathrm{O}^\eta(2\gamma, 2)$ is generated by orthogonal transvections on $E/Z(E)$ except when $\eta = +$ and $\gamma = 2$, in which case, the subgroup generated by orthogonal transvections has index 2 in $O^+(4, 2)$ (see [9]). Thus first we show that N contains all orthogonal transvections on $E/Z(E)$. But every orthogonal transvection is uniquely determined by a nonsingular vector in $E/Z(E)$, so we need to investigate such a vector in $E/Z(E)$. By [18 or 15] the quadratic form $q(\overline{x})$ on $E/Z(E)$ is given as follows: if $x \in E$ and $x^2 = z^k$ for some $k \in \mathbb{Z}/2\mathbb{Z}$, then $q(\overline{x}) = k$, where $\overline{x} = xZ(E) \in E/Z(E)$. Thus $\overline{x}$ is nonsingular in $E/Z(E)$ if and only if x has order 4 and then the transvection T corresponding to $\overline{x}$ is given by $T\colon \overline{u} \mapsto \overline{u} + (\overline{u}, \overline{x})\overline{x}$ for all $\overline{u} \in E/Z(E)$, where $(\overline{u}, \overline{x}) = q(\overline{u} + \overline{x}) + q(\overline{u}) + q(\overline{x})$ is the bilinear form defined by the quadratic form. So it suffices to show that for each element $x \in E$ of order 4, there exists an element $g \in N$ such that $ghg^{-1} = \pm hx^k$ for $h \in E$, where $k = 1 + i + j \in \mathbb{Z}/2\mathbb{Z}$ with $h^2 = z^i$ and $(hx)^2 = z^j$. Such an element g will be called the *transvection* for x. It is clear that if x and u are elements of order 4 in E and they are conjugate under N, then the transvection for x exists in N if and only if the transvection for u exists in N. Thus we consider the N-conjugacy classes of elements of order 4 in E. We may suppose the action of E on the underlying space V is given by (1.4) or (1.5).

First suppose E has plus type.

(1) Let g be the element in G such that

$$g\colon [j_1, j_2, \dots, j_i, \dots, j_\gamma] \mapsto [j_i, j_2, \dots, j_1, \dots, j_\gamma].$$

Then $g^{-1}x_1g = x_{2i-1}$, $g^{-1}x_{2i-1}g = x_1$, $g^{-1}x_2g = x_{2i}$, $g^{-1}x_{2i}g = x_2$, and $g^{-1}x_kg = x_k$ for all other indices. It follows that N contains a subgroup inducing the symmetric group $\mathrm{S}(\gamma)$ on the set $\{E_1, E_2, \dots, E_\gamma\}$.

(2) Let $\{[j_1, j_2, j_3, \dots, \gamma_\gamma]'\}$ be the basis of V given by the Remark (1) above with $k = 1$, and g the element in G such that

$$g\colon [j_1, j_2, j_3, \dots, j_\gamma]' \mapsto [j_1, j_2, j_3, \dots, j_\gamma].$$

Then $g^{-1}x_1g = x_2$, $g^{-1}x_2g = x_1$, and $g^{-1}x_kg = x_k$ for $k \geq 3$. Since $x_2 = x_1x_1x_2$ and $x_1 = -x_2x_1x_2$, the element g is the transvection for x_1x_2 in N.

(3) Let g be the element in G such that

$$g\colon [j_1, j_2, \dots, j_\gamma] \mapsto [j_1 + j_2 + 1, j_2, \dots, j_\gamma].$$

Then $g^{-1}x_1g = x_1x_3$, $g^{-1}x_4g = x_2x_4$, and $g^{-1}x_kg$ for all other indices. Since $\langle x_1, x_3, \dots, x_{2\gamma-1}\rangle$ and $\langle x_2, x_4, \dots, x_{2\gamma}\rangle$ give a hyperbolic decomposition of $E/Z(E)$, g induces

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & & & \\ & I & & \\ & & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \\ & & & I \end{pmatrix}$$

relative to this decomposition of $E/Z(E)$. By (1) we may replace E_1 and E_2 by E_i and E_j for $1 \leq i < j \leq \gamma$. Thus there is a subgroup of N inducing

$$\left\langle \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}(\gamma, 2) \right\rangle \tag{1.6}$$

on $E/Z(E)$.

In order that N contain all orthogonal transvections on $E/Z(E)$, it then suffices to show that every element x in E of order 4 is conjugate with x_1x_2 under N. Moreover, we shall show that every noncentral element y of order 2 in E is conjugate with x_1 in N. Suppose $\gamma = 1$. Then $x = \pm x_1x_2$ and $y = \pm x_1$ or $\pm x_2$. If $x = -x_1x_2$, then $x_2xx_2 = x_1x_2$, so that x is conjugate with x_1x_2 in N. If $y = \pm x_2$, then $g^{-1}yg = \pm x_1$ for some $g \in N$ given by (2). We may suppose $y = \pm x_1$. If $y = -x_1$, then $x_2yx_2 = x_1$. Thus y is conjugate with x_1 in this case. Suppose $\gamma \geq 2$. Let $D = \langle x_1, x_2, x_5, \dots, x_{2\gamma} \rangle$, then $D \simeq 2_+^{2\gamma-1}$. It follows by (1.4) that $L = C_G(\langle x_3, x_4 \rangle) \simeq \mathrm{GL}(2^{\gamma-1}, \varepsilon q)$ and $D \leq L$. Thus by induction every element of order 4 in D is conjugate with x_1x_2 under $N_L(D)$ and every noncentral element of order 2 in D is conjugate with x_1 under $N_L(D)$. It is clear that $N_L(D) \leq N$ and centralizes $\langle x_3, x_4 \rangle$. If x and y are elements of D, then x is conjugate with x_1x_2 and y is conjugate with x_1 under N. If $x \notin D$, then $x = x_3x'$, x_4x', or x_3x_4x' for some element x' of D. Suppose $x = x_3x'$ or $x = x_4x'$. In the latter case, take the element $g \in N$ which is a product of some elements given by (1) and (2) such that $g^{-1}x_4g = x_3$, $g^{-1}x_3g = x_4$, and $g^{-1}x_ig = x_i$ for other indices. Thus $g^{-1}xg = x_3x'$, so that we may suppose $x = x_3x'$. Thus x' has order 4, so that $h^{-1}x'h = x_1x_2$ for some $h \in N_L(D)$ and $h^{-1}xh = x_3x_1x_2$. Thus $g^{-1}h^{-1}xhg = x_1x_2$ for the element $g \in N$ given by (3). Suppose $x = x_3x_4x'$ for some element $x' \in D$ of order 2 or 1. If $x' \in Z(D)$, then $x = \pm x_3x_4$ and $g^{-1}xg \in D$ for some element $g \in N$ given by (1), so that x is conjugate with x_1x_2 in N. If x' is a noncentral element, then $h^{-1}x'h = x_1$ for some $h \in N_L(D)$, so that $h^{-1}xh = x_3x_4x_1$ and $g^{-1}h^{-1}xhg = x_1x_2x_4$ for the element g by (3). By the argument above x is conjugate with x_1x_2 under N. Similarly if $y \notin D$, then $y = x_3y'$, x_4y', or x_3x_4y' for some $y' \in D$. Suppose $y = x_3y'$ or x_4y'. In the latter case, $g^{-1}yg = x_3y'$ for an element $g \in N$ given by (1) and (2), so that we may suppose $y = x_3y'$. Thus y' has order 2 or 1. In the case $y' \in Z(D)$, $g^{-1}yg \in D$ for some element g given by (1). Thus y is conjugate with x_1 in N. Suppose y' is a noncentral element of D. Then $h^{-1}y'h = x_1$ for some $h \in N_L(D)$, so that $h^{-1}yh = x_3x_1$ and $g^{-1}h^{-1}yhg = x_1$, where g is the element given by (3). Suppose $y = x_3x_4y'$, so that y' has order 4 and we may suppose $y' = x_1x_2$ by replacing y' by $h^{-1}y'h$ for some $h \in N_L(D)$. Thus $y = x_3x_4x_1x_2$ and then $g^{-1}xg = x_4x_2$, where g is the element given by (3). By the argument above, y is conjugate with x_1 in N. It follows that N contains a subgroup H inducing a subgroup of $\mathrm{O}^{\eta}(2\gamma, 2)$ which is generated by all orthogonal transvections on $E/Z(E)$.

Suppose E has minus type and by (1C) we may suppose $\gamma \geq 2$. Then $x_3, x_4, \dots, x_{2\gamma}$ generate an extraspecial group K of order $2^{2\gamma-1}$ with plus type, so that N contains the elements given in (1), (2), and (3) with x_1 replaced by x_3, x_2 by x_4, and some obvious modifications. For example, the action of the element g given by (1) is defined by

$$g\colon [j_1, j_2, \dots, j_i, \dots, j_\gamma] \mapsto [j_1, j_i, \dots, j_2, \dots, j_\gamma],$$

where $\{[j_1, j_2, \dots, j_i, \dots, j_\gamma]\}$ is the basis of V given in (1D). It follows that all the elements given by (1), (2), and (3) act trivially on x_1 and x_2. In particular, the element g given by (2) is the transvection for x_3x_4 in N. If y is a noncentral element of order 2 in E, then y lies in K. It follows by (1.5) that $C_G(\langle x_1, x_2 \rangle) \simeq \mathrm{GL}(2^{\gamma-1}, \varepsilon q)$ and $K \leq C_G(\langle x_1, x_2 \rangle)$. Apply a similar proof to K and $C_G(\langle x_1, x_2\rangle)$, so that y is conjugate to x_3 under $N_{C_G(\langle x_1, x_2\rangle)}(K) \leq N$. Thus every noncentral element of order 2 in E is conjugate to x_3 in N. Let $D = \langle x_1, x_2, x_5, x_6, \dots, x_{2\gamma}\rangle$, so that $D \simeq 2^{2\gamma-1}$. By (1.4) $L = C_G(\langle x_3, x_4\rangle) \simeq \mathrm{GL}(2^{\gamma-1}, \varepsilon q)$ and $D \leq L$. If $\gamma = 2$, then each element of order 4 in D are conjugate with x_1 in $N_L(D)$ by (1C). Suppose $\gamma \geq 3$. Then each noncentral element of order 2 in D is conjugate with x_5 under $N_L(D)$ by applying the proof above to D and L. By induction we may suppose each element of order 4 in D are conjugate with x_1 under $N_L(D)$. It is clear that $N_L(D) \leq N$ and $N_L(D)$ centralizes $\langle x_3, x_4\rangle$. Let X be the matrix given by (1.3) and g an element in G such that

$$(1.7)\qquad g\colon [j_1, j_2, \dots, j_\gamma] \mapsto (-1)^{j_2+1}((X^{2^{a-1}})^{j_2} v^1_{j_1}) \otimes [j_2, j_3, \dots, j_\gamma].$$

Then $g^{-1}x_1x_4g = x_3x_4$, $g^{-1}x_2g = -x_3x_2$, and $g^{-1}x_ig = x_i$ for $i \neq 2, 4$, so that $g \in N$. Suppose x is an element of E of order 4. If $x \in D$, then x is conjugate with x_1 in $N_L(D)$. If $x \notin D$, then $x = x_3x'$, x_4x', or x_3x_4x' for some $x' \in D$. Suppose $x = x_3x'$ or x_4x'. Then in the former case $g^{-1}xg = x_4x'$ for some element $g \in N$ given by (1) and (2) with x_1 replaced by x_3, x_2 by x_4, and some obvious modifications. Thus we may suppose $x = x_4x'$, so that x' has order 4 and $h^{-1}x'h = x_1$ for some $h \in N_L(D)$. So $g^{-1}h^{-1}xhg = x_3x_4$, where g is the element given by (1.7). Finally suppose $x = x_3x_4x'$ for some $x' \in D$. A similar proof to above shows that we may suppose x' is a noncentral element of order 2, so that $h^{-1}x'h = x_5$ for some $h \in N_L(D)$ and hence $g^{-1}h^{-1}xhg = x_3x_4$ for some element g given by (3). Thus each element of E of order 4 is conjugate with x_3x_4 in N. Since the transvection for x_3x_4 is given by (2), N contains a subgroup H inducing a group on $E/Z(E)$ generated by all orthogonal transvections.

It follows that $N = HZ(G)$ and $N/EZ(G) \simeq \mathrm{O}^\eta(2\gamma, 2)$, except $\eta = +$ and $\gamma = 2$, in which case $H/EZ(E)$ is a subgroup of $\mathrm{O}^+(4, 2)$ of index 2. Let g be an element of G such that

$$(1.8)\qquad g\colon [j_1, j_2] \mapsto \begin{cases} -[2, 2] & \text{if } j_1 = j_2 = 2, \\ [j_1, j_2] & \text{otherwise.} \end{cases}$$

Then $g^{-1}x_2g = x_2x_3$, $g^{-1}x_4g = x_1x_4$, and $g^{-1}x_ig = x_i$ for $i = 1, 3$. Thus $g \in N$ and the subgroup generated by elements given by (1.8) and (3) induces a Borel subgroup of $\mathrm{O}^+(4, 2)$, the subgroup generated by the elements (1) and (2) induces a Weyl group on $E/Z(E)$. Let H' be the subgroup generated by elements (1), (2), (3), (1.8), E, and $Z(G)$. Then $Z(H') = Z(G)$ and $H'/EZ(G) \simeq \mathrm{O}^+(4, 2)$, so that $N = H'$.

To prove the last assertion, suppose ξ is a linear character of $Z(N) = Z(G)$ acting trivially on $O_2(Z(N))$. Let S be the subgroup of G whose elements has determinant 1. Then $S = \mathrm{SL}(2^\gamma, q)$ or $\mathrm{SU}(2^\gamma, q)$ according as $\varepsilon = 1$ or -1. For any element $g \in Z(N) \cap S$, $g = uI$ for some $u \in \mathbb{F}_{q^2}$, so that $\deg g = u^{2^\gamma} = 1$ and $g \in O_2(Z(N))$. Thus $Z(N)S$ is a central product of $Z(N)$ and S over $O_2(Z(N)) \cap S$. Let $\tilde{\xi}$ be the tensor product of ξ and the trivial character of S. Then $\tilde{\xi}$ is an irreducible character of $Z(N)S$ acting trivially on $E \cap S$ and G stabilizes $\tilde{\xi}$. Since $G/Z(N)S$ is a cyclic group, $\tilde{\xi}$ can be extended as a character of G which is trivial on E by Clifford theory, so that the restriction of the latter to N is a required extension of ξ. This completes the proof.

(1F) *Suppose* 4 *divides* $q-\varepsilon$. *Let* $G = \mathrm{GL}(2^\gamma, \varepsilon q)$ *and* $R = EZ$ *a subgroup of* G *of symplectic type, where* $Z = O_2(Z(G))$ *and* E *is an extraspecial subgroup of order* $2^{2\gamma+1}$. *Set* $C = C_G(R)$ *and* $N = N_G(R)$. *Then* $C_N(R) = C = Z(G)$ *and* $N/Z(N)R \simeq \mathrm{Sp}(2\gamma, 2)$. *Moreover, each linear character of* $Z(N)$ *acting trivially on* $O_2(Z(N))$ *can be extended as a character of* N *acting trivially on* R.

Proof. The statement $C = C_G(R) = Z(G)$ is a consequence of the fact that R is an absolutely irreducible subgroup of $\mathrm{GL}(2^\gamma, \varepsilon q)$. The proof of the last assertion is the same as that of (1E). Each element of N induces an automorphism in $\mathrm{Aut}^0\,\Omega_2(R) = \mathrm{Aut}^0 R$. Since $R > E$, we may suppose E has plus type and acts on the underlying space V as in (1.4). Set $W = \Omega_2(R)$. Then $W = \langle\rho\rangle E$, where $\rho = wI$ and $w \in \mathbb{F}_{q^2}$ has order 4. By [18] or [15] the alternating from $(\overline{u}, \overline{x})$ on $W/Z(W)$ is induced by commutation: If $[u, x] = z^k$, then $(\overline{u}, \overline{x}) = k$, where u, x are elements of W, $\overline{u} = uZ(W)$, $\overline{x} = xZ(W)$, and $k \in \mathbb{Z}/2\mathbb{Z}$. The group $\mathrm{Sp}(2\gamma, 2)$ is generated by all symplectic transvections (see [9]) and each nonzero vector of $W/Z(W)$ uniquely determines a symplectic transvection which is defined the same as the orthogonal transvection above. It is clear that the elements defined by (1), (2) and (3) in the proof of (1E) are elements of N. It follows by the same proof to that of (1E) that every element of order 4 in E is conjugate with x_1x_2 and every noncentral element of order 2 in E is conjugate with x_1 under N. We claim that ρx_1 is conjugate with x_1x_2 in N. Indeed, let g be the element in G such that

$$g\colon [j_1, j_2, \dots, j_\gamma] \mapsto (-1)^{j_1+1} w^{j_1}[j_1, j_2, \dots, j_\gamma]. \tag{1.9}$$

Then $g^{-1}\rho x_2 g = x_1x_2$, and $g^{-1}x_k g = x_k$ for all other indices. Thus the claim holds. It follows that N induces a transitive action on the nonzero vectors in $W/Z(W)$. The element g given by (1.9) induces a symplectic transvection on $W/Z(W)$ corresponding to $\overline{x}_2$, so that N induces a subgroup of $\mathrm{Sp}(2\gamma, 2)$ containing all symplectic transvections. Thus N induces $\mathrm{Sp}(2\gamma, 2)$ on $W/Z(W)$ and then $N/RC \simeq \mathrm{Sp}(2\gamma, 2)$. This completes the proof.

The following proposition is proved in [4] for general linear groups and we shall give a proof for unitary groups.

(1G) *Let* $P = S_\beta, D_\beta$, *or* Q_β *with* $\beta \geq 4$, *and let* $\mathbf{W}$ *be a faithful and irreducible representation of* P *in* $G = \mathrm{U}(n, q)$ *such that* $O_2(C(\mathbf{W}(P))) \leq \mathbf{W}(P)$. *Then* 2 *is linear,* $n = 2$, *and* $\beta \leq a+2$. *Moreover, if* $P = S_\beta$, *then* $\beta = a+2$ *and* $\mathbf{W}(P)$ *is a Sylow* 2*-subgroup of* G; *if* $P = D_\beta$ *or* Q_β, *then there exists an element* $x \in G$ *such that* $|x| = 2^\beta$, x *normalizes* $\mathbf{W}(P)$ *and* $x \in C_G([\mathbf{W}(P), \mathbf{W}(P)])$.

Proof. Let $N = N_G(\mathbf{W}(P))$ and $C = C_G(\mathbf{W}(P))$. Since $O_2(Z(G)) \leq O_2(C) \leq \mathbf{W}(P)$, it follows that $O_2(Z(G)) \leq Z(\mathbf{W}(P))$. But $Z(\mathbf{W}(P))$ has order 2. Thus $O_2(Z(G)) \leq Z(\mathbf{W}(P)) = \{\pm I_n\}$, so that 2 is linear. Suppose σ and τ are generators of $\mathbf{W}(P)$, where $|\sigma| = 2^{\beta-1} \geq 8$ and $|\tau| = 2$ or 4 according as $P \neq Q_\beta$ or $P = Q_\beta$ and in the latter case $\tau^2 = -I_n$. Let $K = \langle\sigma\rangle$. By (1B) the underlying space V of G has a K-module decomposition (1.1) such that

$s=0$. But if M is a simple K-submodule of V, then $V=M+\tau M$ as V is a simple P-module. Thus V has a decomposition $V=U\oplus U'$, where U and U' are totally isotropic simple K-modules. It follows that σ is primary with a unique elementary divisor $\Gamma\in\mathscr{F}_2$ and $C_G(\mathbf{W}(K))\simeq \mathrm{GL}(1, q^{d_\Gamma})$ is Coxeter torus of G. If $\beta-1\leq a+1$, then $|\sigma|=2^{\beta-1}$ and it divides q^2-1, so that $d_\Gamma=2$ since U is a simple K-module. Thus $\mathbf{W}(P)$ is a subgroup of a Sylow 2-subgroup of $G=\mathrm{U}(2, q)$. By [7, p. 143] a Sylow 2-subgroup of G is semidihedral of order 2^{a+2} and by [14, 5.4.3] a semidihedral group has no proper semidihedral subgroups. Thus if $P=S_\beta$, then $\mathbf{W}(P)$ is a Sylow 2-subgroup of G; if $P=D_\beta$ or Q_β, then $\mathbf{W}(P)$ is a subgroup of a Sylow 2-subgroup D of G. Let L be a subgroup of D containing $\mathbf{W}(P)$ such that $(L:\mathbf{W}(P))=2$, so that $L\leq N$. The same proof as that of [4, (1D)] can be applied here to show that there exists an element $x\in L$ such that $|x|=2^\beta$ and $x\in C_G([\mathbf{W}(P), \mathbf{W}(P)])$.

Suppose $\beta-1>a+1$. Since U is a simple K-module, the commuting algebra of K on U is isomorphic to $\mathbb{F}_{q^{2^{\beta-a-1}}}$, so that $C_G(\mathbf{W}(\sigma))\simeq \mathrm{GL}(1, q^{2^{\beta-a-1}})$ and $d_\Gamma=2^{\beta-a-1}$. If $T=C_G(W(K))$, then $\mathbf{W}(K)=O_2(T)$. Let $\alpha=\beta-a-1$, so that $\alpha\geq 2$ and $d_\Gamma=2^\alpha$. Since T is a Coxeter torus of G, it follows that $N_G(T)=\langle\zeta, T\rangle$, where ζ acts on T by $t\mapsto t^{-q}$ (cf. [11, p. 129]). Thus $N_G(T)/T$ is cyclic of order 2^α.

Suppose $R=S_\beta$, so that $\tau\sigma\tau^{-1}=-\sigma^{-1}$ and $\tau^2=1$. Thus τ induces an element of order 2 in $N_G(T)/T$. Since $N_G(T)/T\simeq\langle\zeta\rangle$ is cyclic, it follows that $\tau=\zeta^{2^{\alpha-1}}t'$ for some $t'\in T$, so that τ and $\zeta^{2^{\alpha-1}}$ induce the same action on T. Since $\alpha-1\geq 1$, τ acts on T by $t\mapsto t^{(-1)^{2^{\alpha-1}}q^{2^{\alpha-1}}}=t^{q^{2^{\alpha-1}}}$, so that $\sigma^{q^{2^{\alpha-1}}}=-\sigma^{-1}$ and $\sigma^{q^{2^{\alpha-1}}+1}=-1$ since $\tau\sigma\tau^{-1}=-\sigma^{-1}$. Since 2 is linear, it follows that 2 is the exact power of 2 dividing $q^{2^{\alpha-1}}+1$, so that σ has order 4. This is a contradiction.

If $R=D_\beta$ or Q_β, then $\tau\sigma\tau^{-1}=\sigma^{-1}$ and $\tau^2\in T$, so that τ induces an element of order 2 in $N_G(T)/T$. Thus $\tau=\zeta^{2^{\alpha-1}}t'$ for some $t'\in T$, so that τ and $\zeta^{2^{\alpha-1}}$ induce the same action on T. Thus τ acts on T by $t\mapsto t^{q^{2^{\alpha-1}}}$ and $\sigma^{q^{2^{\alpha-1}}}=\sigma^{-1}$, so that $\sigma^{q^{2^{\alpha-1}}+1}=1$ and $|\sigma|=2$. This is impossible and (1G) follows.

Now we consider the embedding of other groups of symplectic type in a unitary group. In the following two propositions, suppose $R=EP$ is a central product of E and P over $Z(E)=Z(P)$, where $P=S_\beta, D_\beta$ or Q_β with $\beta\geq 4$, and $E\simeq 2_\eta^{2\gamma+1}$. The first proposition can be proved by replacing GL by U and some obvious modifications in the proof of [4, (1E)].

Suppose 2 is linear. Let $\mathbf{Y}$ be a faithful and irreducible representations of E in $\mathrm{U}(2^\gamma, q)$, N_1 the underlying space of $\mathrm{U}(2^\gamma, q)$, $N=N_1\perp N_1\perp\cdots\perp N_1$ (m copies), and $\mathbf{X}=m\mathbf{Y}$ the faithful representation of E in $\mathrm{U}(N)$. Let $\mathbf{W}$ be a faithful and irreducible representation of P in $\mathrm{U}(2, q)$, M the underlying space of $\mathbf{W}$, and $V=N\otimes M$. Then R acts faithfully on V and we denote by $\mathbf{F}$ the representation of R in $\mathrm{U}(V)$. The central product $\mathrm{U}(N)\mathrm{U}(M)$ of $\mathrm{U}(N)$ and $\mathrm{U}(M)$ over $Z(\mathrm{U}(N))=Z(\mathrm{U}(M))$ also acts faithfully on V. For simplicity of notation, we denote again by $\mathbf{F}$ the representation of $\mathrm{U}(N)\mathrm{U}(M)$ in $\mathrm{U}(V)$.

(1H) *With the notation above, let* $N(\mathbf{X}(E))$, $N(\mathbf{W}(P))$, *and* $N(\mathbf{F}(R))$ *be the normalizers of* $\mathbf{X}(E)$, $\mathbf{W}(P)$, *and* $\mathbf{F}(R)$ *in* $\mathrm{U}(N)$, $\mathrm{U}(M)$, *and* $\mathrm{U}(V)$ *respectively. In addition, let* $\mathbf{F}(R)^0 = C_{\mathbf{F}(R)}([\mathbf{F}(R), \mathbf{F}(R)])$, *and*

$$N^0(\mathbf{W}(P)) = \{x \in N(\mathbf{W}(P)) : [x, [\mathbf{W}(P), \mathbf{W}(P)]] = 1\},$$
$$N^0(\mathbf{F}(R)) = \{x \in N(\mathbf{F}(R)) : [x, [\mathbf{F}(R), \mathbf{F}(R)]] = 1\}.$$

Then $\mathbf{F}(N(\mathbf{X}(E))) \le N^0(\mathbf{F}(R))$ *and* $\mathbf{F}(N^0(\mathbf{W}(P))) \le Z(C_{N^0(\mathbf{F}(R))}(\mathbf{F}(R)^0))$. *In particular, if* $P = D_\beta$ *or* Q_β, *then* $\mathbf{F}(R)$ *is not radical in* $\mathrm{U}(V)$.

(1I) *Suppose* 2 *is linear. Let* $G = \mathrm{U}(2^{\gamma+1}, q)$, $P = S_{a+2}$, *and* $R = EP$.

(a) *There exists a faithful and absolutely irreducible representation* $\mathbf{T}$ *of* R *in* G. *Moreover,* R *is independent of the type of* E *and* G *contains a unique conjugacy class of subgroups isomorphic to* R.

(b) *Identity* R *with* $\mathbf{T}(R)$ *and let*

$$R^0 = C_R([R, R]), \qquad N = N_G(R), \qquad N^0 = \{g \in N : [g, [R, R]] = 1\}.$$

Then R^0 *is a central product of a cyclic group of order* $2^{\alpha+1}$ *and an extraspecial group of order* $2^{2\gamma+1}$, $R \cap N^0 = R^0$, $Z(N^0) = Z(G)Z(R^0)$, *and* $N^0/Z(N^0) \simeq \mathrm{Aut}^0 R^0$. *In particular,* $N^0/R^0Z(N^0) \simeq \mathrm{Sp}(2\gamma, 2)$. *Moreover, each linear character of* $Z(N^0)$ *acting trivially on* $O_2(Z(N^0))$ *can be extended to a character of* N^0 *acting trivially on* R^0.

Proof. (a) With the assumption of (1H), suppose $P = S_{a+2}$ and $\mathbf{X} = \mathbf{Y}$ is irreducible. Denote by $\mathbf{T}$ the representation $\mathbf{F}$ in (1H). Then $\mathbf{T}$ is a faithful and absolutely irreducible representation of R in G. The same proof as that of [4, (1F), (a)] shows that R is independent of the type of E, so that we may suppose E has plus type. Suppose $\mathbf{T}'$ is another faithful and irreducible representation of R in G. Then both $\mathbf{T}|_E$ and $\mathbf{T}'|_E$ have exactly one Wedderburn component. By the Remark (2) after (1D), $\mathbf{T}(E)$ and $\mathbf{T}'(E)$ are conjugate in G and we may suppose $\mathbf{T}(E) = \mathbf{T}'(E)$. Thus $\mathbf{T}(P)$ and $\mathbf{T}'(P)$ are Sylow 2-subgroups of $C_G(\mathbf{T}(E))$, so that they are conjugate in $C_G(\mathbf{T}(E))$ and then $\mathbf{T}(R)$ and $\mathbf{T}'(R)$ are conjugate in G.

(b) The rest of the proof is similar to that of [4, (1F), (b)].

(1J) *Let either* $R = E$ *or* $R = EP$ *and* $G = \mathrm{U}(n, q) = \mathrm{U}(V)$, *where* $E \simeq 2_\eta^{2\gamma+1}$ *and* $P \simeq S_\beta$, D_β, *or* Q_β *with* $\beta \ge 4$, *and let* $\mathbf{J}$ *be a faithful representation of* R *in* G *and* $C = C_G(\mathbf{J}(R))$.

Suppose in the decomposition (1.1) *of* V *as an* R*-module all the nondegenerate components are isomorphic and* $\mathbf{J}(R)$ *is radical in* G. *Then* 2 *is linear and all the nondegenerate components are simple. Moreover, if* $R = EP$, *then* $P = S_{\alpha+2}$ *and* $\mathbf{J}(R)$ *is uniquely determined up to conjugacy in* G.

More general, if $R = E$ *and* $\mathbf{J}$ *has exactly one Wedderburn component, then all the nondegenerate components of* V *in* (1.1) *are simple, so that* $\mathbf{J}(R)$ *is uniquely determined up to conjugacy in* G.

Proof. Let $E = \langle x_1, x_2, \dots, x_{2\gamma}\rangle$ and $P = \langle \sigma, \tau\rangle$, so that $|\sigma| = 2^{\beta-1}$, $|\tau| = 2$ or 4 according as $P \ne Q_\beta$ or $P = Q_\beta$, and $\tau\sigma\tau = \pm\sigma^{-1}$. Since $\mathbf{J}(R)$ is radical, it follows $O_2(Z(G)) \le O_2(N_G(\mathbf{J}(R))) = \mathbf{J}(R)$ and $O_2(Z(G)) \le Z(\mathbf{J}(R))$. Thus 2 is linear. Suppose in the decomposition (1.1) of V, $V = mV_1$, where the nondegenerate R-submodule V_1 is either simple or $V_1 = U_1 \oplus U_1'$ for totally isotropic simple R-modules U_1 and U_1'. Moreover, V_1 has no proper nondegenerate R-submodule. Let $\mathbf{Y}$ be the representation of R on V_1, $G_1 = \mathrm{U}(V_1)$, and $C_1 = C_{G_1}(\mathbf{Y}(R))$. In addition, let E_0 be a dihedral group of order 8, $D = RE_0$ the central product of R and E_0 over $Z(R) = Z(E_0)$, and $R_1 = \langle x_3, x_4, \dots, x_{2\gamma}\rangle$, so that $R_1 \simeq 2_+^{2\gamma-1}$. Suppose $V_1 = U_1 \oplus U_1'$. We shall show that V_1 has a proper nondegenerate R-submodule and induce a contradiction.

First consider $R = E$. Thus R has a unique faithful and irreducible representation of degree 2^γ over any finite field of odd characteristic, so that U_1 and U_1' are isomorphic R-modules and $\mathbf{Y}$ has exactly one Wedderburn component. If $R \simeq 2_+^{2\gamma+1}$, then V_1 has a proper nondegenerate R-submodule by the Remark (2) of (1D). This is impossible and (1J) holds in this case. Suppose $R \simeq 2_-^{2\gamma+1}$. Since U_1 has dimension 2^γ over $\mathbb{F}_{q^2}$, it follows $C_{G_1}(g) \simeq \mathrm{GL}(2^\gamma, q^2)$, where $g = \mathbf{Y}(x_1)$. Thus $\mathbf{Y}(x_2)$ induces a field automorphism of order 2 on $C_{G_1}(g)$ and $\mathbf{Y}$ induces a faithful representation of R_1 in $\mathrm{GL}(2^\gamma, q^2)$ which has one Wedderburn component. By [4, (1A)] $\mathbb{F}_{q^2}$ is a splitting field of R_1, so that $C_{C_{G_1}(g)}(\mathbf{Y}(R)_1)) \simeq \mathrm{GL}(2, q^2)$ and $\mathbf{Y}(x_2)$ induces a field automorphism of order 2 on it. Thus the fixed-point set of the automorphism on $C_{C_{G_1}(g)}(\mathbf{Y}(R_1))$ is isomorphic to $\mathrm{U}(2, q)$, so that $C_1 \simeq \mathrm{U}(2, q)$. By (1D) E_0 has a faithful and irreducible representation in $C_1 \le G_1$, so that E_0 has a faithful representation in G_1. Denote again by $\mathbf{Y}$ the representation of E_0 in G_1. Then $K = \mathbf{Y}(R)\mathbf{Y}(E_0)$ is a central product of $\mathbf{Y}(R)$ and $\mathbf{Y}(E_0)$ over $Z(\mathbf{Y}(R)) = Z(\mathbf{Y}(E_0))$, so that $K \simeq 2^{2\gamma+3}$. Since K is a subgroup of G_1 and V_1 has dimension $2^{\gamma+1}$, the natural representation of K in G_1 induces a faithful and irreducible representation of D in G_1. Denote again by $\mathbf{Y}$ the representation. In addition, let M_1 and M_2 be nondegenerate subspaces of V_1 of dimension 2^γ such that $V_1 = M_1 \perp M_2$. By (1D) there exists a faithful and irreducible representation $\mathbf{X}$ of R in $\mathrm{U}(2^\gamma, q)$. Identify $\mathrm{U}(2^\gamma, q)$ with $\mathrm{U}(M_1)$ and $\mathrm{U}(M_2)$. Then R acts on M_1 and M_2 through $\mathbf{X}$, and on $M_1 \perp M_2$ through $\mathbf{Y}' = 2\mathbf{X}$. Thus $\mathbf{Y}'$ is a faithful representation of R on V_1 and the M_i are nondegenerate simple $\mathbf{Y}'(R)$-modules. If $\mathbf{Y}(R)$ and $\mathbf{Y}'(R)$ are conjugate in G_1, then we may suppose $\mathbf{Y}(R) = \mathbf{Y}'(R)$ and then the M_i are nondegenerate $\mathbf{Y}(R)$-submodule. In order that V_1 be simple, it then suffices to show that $\mathbf{Y}(R)$ and $\mathbf{Y}'(R)$ are conjugate in G_1. Let $C_1' = C_{G_1}(\mathbf{Y}'(R))$. Then $C_1' \simeq \mathrm{U}(2, q)$ and E_0 has a faithful and irreducible representation in $C_1' \le G_1$, so that E_0 has a faithful representation, denoted again by $\mathbf{Y}'$, in G_1. Thus $K' = \mathbf{Y}'(R)\mathbf{Y}'(E_0)$ is a central product of $\mathbf{Y}'(R)$ and $\mathbf{Y}'(E_0)$ over $Z(\mathbf{Y}'(R)) = \mathbf{Y}'(E_0)$, and the natural representation of K' in G_1 also induces a faithful and irreducible representation of D, denoted again by $\mathbf{Y}'$, in G_1. Thus both $\mathbf{Y}$ and $\mathbf{Y}'$ are faithful and irreducible representations of $D \simeq 2^{2\gamma+3}$ in G_1. Since $D = RE_0$ is a central product of R and E_0, both $\mathbf{Y}|_{E_0}$ and $\mathbf{Y}'|_{E_0}$ have exactly one Wedderburn component. By the Remark (2) of (1D) $\mathbf{Y}(E_0)$ and $\mathbf{Y}'(E_0)$ are conjugate in G_1, so that we may suppose $\mathbf{Y}(E_0) = \mathbf{Y}'(E_0)$ and then both $\mathbf{Y}(R)$ and $\mathbf{Y}'(R)$ are subgroups of $C_{G_1}(\mathbf{Y}(E_0)) \simeq \mathrm{U}(2^\gamma, q)$. By (1D) $\mathbf{Y}(R)$ and $\mathbf{Y}'(R)$ are conjugate in $C_{G_1}(\mathbf{Y}(E_0))$, so that they are conjugate in G_1. It follows that V_1 has a proper nondegenerate R-submodule. This is impossible. Note in the proof above we only suppose $V_1 = U_1 \oplus U_1'$ has no proper nondegenerate R-submodule and R acts on V_1 faithfully.

Suppose $\mathbf{J}$ has one Wedderburn component and in the decomposition (1.1)

V has a nondegenerate R-submodule of the form $V' = U \oplus U'$, where U and U' are totally isotropic simple R-submodules and V' has no proper nondegenerate R-submodule. Then R acts faithfully on V'. Repeating the proof above with V_1 replacing by V', we can get that V' has a proper nondegenerate R-submodule. This is impossible. Thus all the nondegenerate components of V in (1.1) are simple, so that by (1D) we can suppose all the irreducible representations of R on the components have the same images and then $\mathbf{J}(R)$ is uniquely determined up to conjugacy in G.

Finally suppose $R = EP$. If $g = \mathbf{Y}(\sigma)$, then $C_{G_1}(g) \simeq \mathrm{GL}(2^{\gamma+1}, q^{2\delta})$ for some integer $\delta \geq 1$, so that $\mathbf{Y}$ induces a faithful representation of E in $C_{G_1}(g)$ with one Wedderburn component. Thus $C_{G_1}(\langle \mathbf{Y}(g), \mathbf{Y}(E)\rangle) \simeq \mathrm{GL}(2, q^{2\delta})$ and $\mathbf{Y}(\tau)$ induces a field automorphism of order 2 on it. The fixed-point set of the automorphism on $C_{G_1}(\langle \mathbf{Y}(g), \mathbf{Y}(E)\rangle)$ is isomorphic to $\mathrm{U}(2, q^{\delta})$ and $C_1 \simeq \mathrm{U}(2, q^{\delta})$. By (1D) E_0 has a faithful and absolutely irreducible representation in $C_1 \leq G_1$, so that E_0 is embedded as a subgroup in G_1. Denote again by $\mathbf{Y}$ the representation of E_0 in G_1. Thus $K = \mathbf{Y}(R)\mathbf{Y}(E_0)$ is a central product of $\mathbf{Y}(R)$ and $\mathbf{Y}(E_0)$ over $Z(\mathbf{Y}(R)) = Z(\mathbf{Y}(E_0))$ and the natural representation in K in G_1 induces a faithful and irreducible representation of D in G_1. Denote again by $\mathbf{Y}$ the representation. Since D is a central product of EE_0 and P, $\mathbf{Y}|_{EE_0}$ has one Wedderburn component. By the proof above we may suppose all the components in the decomposition (1.1) of V_1 as an (EE_0)-module are isomorphic nondegenerate simple R-submodules, so that by (1D) we can identify these components by a conjugate in G_1. Thus $C_{G_1}(\mathbf{Y}(EE_0)) \simeq \mathrm{U}(s, q)$ and $\mathbf{Y}$ induces a faithful and irreducible representation $\mathbf{W}$ of P in $C_{G_1}(\mathbf{Y}(EE_0))$, where s is an integer such that $s2^{\gamma+1} = \dim V_1$. Since $C_{G_1}(\mathbf{Y}(D)) = C_{C_1}(\mathbf{Y}(E_0)) \simeq \mathrm{U}(1, q^{\delta})$ and $O_2(\mathrm{U}(1, q^{\delta}))$ has order 2, it follows that $C_{C_{G_1}(\mathbf{Y}(EE_0))}(\mathbf{W}(P)) \simeq \mathrm{U}(1, q^{\delta})$ and then $O_2(C_{C_{G_1}(\mathbf{Y}(EE_0))}(\mathbf{W}(P))) \leq \mathbf{W}(P)$. By (1G) $s = 2$ and $\beta \leq a + 2$, so that $\dim V_1 = 2^{\gamma+2}$. Moreover, if $P = S_\beta$, then $\beta = a + 2$. By the proof above $C_{G_1}(\mathbf{Y}(P)) \simeq \mathrm{U}(2^{\gamma+1}, q)$ and $\mathbf{Y}$ induces a faithful representation $\mathbf{X}'$ of E in $C_{G_1}(\mathbf{Y}(P))$ which has one Wedderburn component. Apply the proof above to $\mathbf{X}'$ and $C_{G_1}(\mathbf{Y}(P))$. Then all the nondegenerate components in the decomposition (1.1) of the underlying space N of $\mathbf{X}'$ as an E-module are simple, so that $N = N_1 \perp N_2$, where N_1 and N_2 are simple $\mathbf{X}'(E)$-submodule. Since $\mathbf{Y}$ is the tensor product of $\mathbf{X}'$ and $\mathbf{W}$, $V_1 = N \otimes M$ and $N_1 \otimes M$ is a proper nondegenerate R-submodule of V_1, where M is the underling space of $\mathbf{W}$. This is a contradiction. Thus V_1 is simple and $\mathbf{Y}$ is irreducible of degree 2^γ. Similar proof to above shows that $C_{G_1}(\mathbf{Y}(E)) \simeq \mathrm{U}(2, q)$ and $\mathbf{Y}$ induces an irreducible representation $\mathbf{W}$ of P in $C_{G_1}(\mathbf{Y}(E))$. Moreover, $C_G(\mathbf{J}(P)) \simeq \mathrm{U}(m2^\gamma, q)$, $\mathbf{J}$ induces a faithful representation $\mathbf{X}$ of E in $C_G(\mathbf{J}(P))$ with one Wedderburn component, and all the nondegenerate components in the decomposition (1.1) of the underlying space of $C_G(\mathbf{J}(P))$ as an E-module are simple. Now $\mathbf{J}$ is the tensor product of $\mathbf{X}$ and $\mathbf{W}$. By (1H) and $\mathbf{J}(R)$ radical, $P = S_\beta$ and then $\beta = a + 2$. Thus (1J) follows by (1I), (a).

Let Z_α be a cyclic group of order $2^{a+\alpha} \geq 8$ if $\alpha \geq 1$, of order $2^a \geq 4$ if 2 is unitary and $\alpha = 0$ but of order 2 if 2 is linear and $\alpha = 0$. Let $E_\gamma Z_\alpha$ be a central product of an extraspecial group $E_\gamma \simeq 2_\eta^{2\gamma+1}$ and Z_α over $Z(E_\gamma) = \Omega_1(Z_\alpha)$.

Define

$$\varepsilon_\alpha = \begin{cases} -1 & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha \geq 1. \end{cases}$$

Then $E_\gamma Z_\alpha$ can be embedded as a subgroup of $\mathrm{GL}(2^\gamma, \varepsilon_\alpha q^{2^\alpha})$ such that Z_α is identified with $O_2(Z(\mathrm{GL}(2^\gamma, \varepsilon_\alpha q^{2^\alpha})))$. Moreover, if $\alpha = 0$, then $\mathrm{GL}(2^\gamma, \varepsilon_\alpha q^{2^\alpha}) = \mathrm{U}(2^\gamma, q)$; if $\alpha \geq 1$ and g is a primary element of order $2^{a+\alpha}$ in $\mathrm{U}(2^{\alpha+\gamma}, q)$, then $C_{\mathrm{U}(2^{\alpha+\gamma}, q)}(g) \simeq \mathrm{GL}(2^\gamma, q^{2^\alpha})$ and we can identify these two groups. Thus $\mathrm{GL}(2^\gamma, \varepsilon_\alpha q^{2^\alpha})$ is embedded as a subgroup of $\mathrm{U}(2^{\alpha+\gamma}, q)$ such that a generator of Z_α is primary as an element of $\mathrm{U}(2^{\alpha+\gamma}, q)$. Denote H_γ the normalizer of $E_\gamma Z_\alpha$ in $\mathrm{GL}(2^\gamma, \varepsilon_\alpha q^{2^\alpha})$, so that by (1E) and (1F) $H_\gamma/Z(H_\gamma) \simeq \mathrm{Aut}^0 \Omega_2(E_\gamma Z_\alpha)$, and $E_\gamma Z_\alpha$, H_γ are absolutely irreducible over $\mathbb{F}_{q^2}$ or $\mathbb{F}_{q^{2^\alpha}}$ according as $\alpha = 0$ or $\alpha \geq 1$. Moreover, each linear character of $Z(H_\gamma)$ acting trivially on $O_2(Z(H_\gamma))$ can be extended as a character of H_γ acting trivially on $E_\gamma Z_\alpha$. The images $R_{\alpha,\gamma}$ of $E_\gamma Z_\alpha$ and $H_{\alpha,\gamma}$ of H_γ under the composition

$$K \hookrightarrow \mathrm{GL}(2^\gamma, \varepsilon_\alpha q^{2^\alpha}) \hookrightarrow \mathrm{U}(2^{\alpha+\gamma}, q),$$

where $K = E_\gamma Z_\alpha$ or H_γ, is then determined up to conjugacy in $\mathrm{U}(2^{\alpha+\gamma}, q)$.

We identify E_γ and Z_α with their images in $\mathrm{U}(2^{\alpha+\gamma}, q)$. So $R_{\alpha,\gamma} = E_\gamma Z_\alpha$. If $\alpha \geq 1$ and $Z_\alpha = \langle y \rangle$, then we claim there exists $\sigma \in \mathrm{U}(2^{\alpha+\gamma}, q)$ such that σ normalizes $R_{\alpha,\gamma}$ and $\sigma y \sigma^{-1} = y^{-q}$. Indeed there exists $\tau \in \mathrm{U}(2^{\alpha+\gamma}, q)$ such that $\tau y \tau^{-1} = y^{-q}$ and τ induces a field automorphism of $C_{\mathrm{U}(2^{\alpha+\gamma}, q)}(Z_\alpha) \simeq \mathrm{GL}(2^\gamma, q^{2^\alpha})$. The embedding of $R_{\alpha,\gamma}$ in $C_{\mathrm{U}(2^{\alpha+\gamma}, q)}(Z_\alpha)$ can be viewed as an embedding of $R_{\alpha,\gamma}$ in $\mathrm{GL}(2^\gamma, q^{2^\alpha})$ in which y is represented by a scalar multiple of the identity matrix. So $\tau E_\gamma \tau^{-1}$, E_γ are extraspecial subgroups of $\mathrm{GL}(2^\gamma, q^{2^\alpha})$ with same type, and $h\tau E_\gamma \tau^{-1} h^{-1} = E_\gamma$ for some $h \in \mathrm{GL}(2^\gamma, q^{2^\alpha})$ by (1D). Thus $\sigma = h\tau$ normalizes $R_{\alpha,\gamma}$ and $\sigma y \sigma^{-1} = y^{-q}$. Thus the claim holds. A similar proof shows that $R_{\alpha,\gamma}$ is uniquely determined up to conjugacy in $\mathrm{U}(2^{\alpha+\gamma}, q)$, since all cyclic subgroups of order $2^{a+\alpha}$ generated by a primary element are conjugate in $\mathrm{U}(2^{\alpha+\gamma}, q)$.

(1K) *Let $R = E_\gamma Z$ be embedded as subgroup of $G = \mathrm{U}(n, q)$, where Z is cyclic and $|Z| \geq 4$. Suppose the underlying space V of G has one component of nondegenerate R-module in the decomposition* (1.1), *i.e. V is either a simple R-module or decomposes as $V = U \oplus U'$, where U and U' are totally isotropic simple R-modules and V has no proper nondegenerate R-submodule. If $Z = O_2(Z(C_G(Z)))$, then $|Z| = 2^{a+\alpha}$, $n = 2^{\alpha+\gamma}$, and R is of the form $R_{\alpha,\gamma}$ as a subgroup of G.*

Proof. If V is a simple R-module, then by (1B) $Z \leq Z(G)$, so that 2 is unitary since $|Z| \geq 4$. Thus V is a simple E_γ-module and then $n = 2^\gamma$. Since $Z = O_2(Z(C_G(Z)))$ and $Z(C_G(Z)) = Z(G)$, it follows $Z = O_2(Z(G))$ and $|Z| = 2^a$. Thus R is of the form $R_{0,\gamma}$. Suppose $V = U \oplus U'$ and V has no proper nondegenerate R-submodule. If $Z \leq Z(G)$, then each E_γ-submodule of V is an R-submodule. Thus U and U' are simple E_γ-modules acted faithfully by E_γ. Since E_γ has a unique such module over $\mathbb{F}_{q^2}$, the representation of E_γ in G has one Wedderburn component, so that by (1J) V has a proper nondegenerate R-submodule. This is a contradiction and so $Z \nleq Z(G)$. Let $\mathbf{Y}$ be the representation of R in $\mathrm{GL}(U)$. Then $\mathbf{Y}$ is irreducible and a generator of $\mathbf{Y}(Z)$ is primary as an element of $\mathrm{GL}(U)$, so that $C_{\mathrm{GL}(U)}(\mathbf{Y}(Z)) \simeq \mathrm{GL}(n, q^{2\delta})$ for some integers n and δ. Thus $\mathbf{Y}(E_\gamma)$ is an irreducible subgroup of $C_{\mathrm{GL}(U)}(\mathbf{Y}(Z))$, so that $n = 2^\gamma$ and then 2^γ is the multiplicity of the unique elementary divisor Γ of a generator of Z. By (1A) $C_G(Z) \simeq \mathrm{GL}(2^\gamma, q^{d_\Gamma})$ and so $d_\Gamma = 2\delta$. Since $Z = O_2(Z(C_G(Z)))$, it follows that $|Z| = 2^{a+\alpha}$ for some $\alpha \geq 1$, so that $\delta = 2^{\alpha-1}$ and R has the form $R_{\alpha,\gamma}$ as a subgroup of G. This completes the proof.

For each $m \geq 1$, the images $R_{m,\alpha,\gamma}$ and $H_{m,\alpha,\gamma}$ of $R_{\alpha,\gamma}$ and $H_{\alpha,\gamma}$ under the m-fold diagonal mapping in $\mathrm{U}(m2^{\alpha+\gamma}, q)$ given by

$$(1.10) \qquad g \mapsto \begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix}, \qquad g \in R_{\alpha,\gamma}, \text{ or } H_{\alpha,\gamma},$$

is also respectively determined up to conjugacy. Denote again E_γ and Z_α the images of E_γ and Z_α under the diagonal mapping (1.10). Thus $Z_\alpha = Z(R_{m,\alpha,\gamma})$ and E_γ is a subgroup of $C_{\mathrm{U}(m2^{\alpha+\gamma},q)}(Z_\alpha) \simeq \mathrm{GL}(m2^\gamma, \varepsilon_\alpha q^{2^\alpha})$. It follows by (1J) and [4, (1A)] that E_γ is uniquely determined up to conjugacy in $C_{\mathrm{U}(m2^{\alpha+\gamma},q)}(Z_\alpha)$, so that $R_{m,\alpha,\gamma}$ is uniquely determined up to conjugacy in $\mathrm{U}(m2^{\alpha+\gamma}, q)$, since all the cyclic subgroups of order $2^{a+\alpha}$ generated by a primary element are conjugate in $\mathrm{U}(m2^{\alpha+\gamma}, q)$. It is clear that

$$H_{m,\alpha,\gamma}/Z(H_{m,\alpha,\gamma})R_{m,\alpha\gamma} \simeq \begin{cases} \mathrm{Sp}(2\gamma, 2) & \text{if either 2 is unitary or } \alpha \geq 1; \\ \mathrm{O}^\eta(2\gamma, 2) & \text{if 2 is linear and } \alpha = 0, \end{cases}$$

where η is the type of E_γ.

(1L) *Let* $G = \mathrm{U}(m2^{\alpha+\gamma}, q)$, $R = R_{m,\alpha,\gamma}$, $H = H_{m,\alpha,\gamma}$, $Z = Z_\alpha = Z(R_{m,\alpha,\gamma})$, *and let*

$$C = C_G(R), \qquad N = N_G(R), \qquad N^0 = \{g \in N : [g, Z] = 1\}.$$

Then the following hold:

(1) $C \simeq \mathrm{GL}(m, \varepsilon_\alpha q^{2^\alpha}) \otimes I_\gamma$, $[H, C] = 1$, $H \cap C = Z(H) \leq Z(C)$, $N^0 = HC$, *where* I_γ *is the identity matrix of size* 2^γ *and* $\mathrm{GL}(m, \varepsilon_\alpha q^{2^\alpha}) \otimes I_\gamma = \{g \otimes I_\gamma : g \in \mathrm{GL}(m, \varepsilon_\alpha q^{2^\alpha})\}$. *Moreover, each linear character of* $Z(H)$ *acting trivially on* $O_2(Z(H))$ *can be extended as a character of* H *which acts trivially on* R.

(2) N/N^0 *is cyclic of order* 2^α.

Proof. (1) If $\alpha = 0$, then $Z \leq Z(G)$, so that the underlying space $V_{\alpha,\gamma}$ of $R_{\alpha,\gamma}$ is a simple $R_{\alpha,\gamma}$ module. The commuting algebras of $R_{\alpha,\gamma}$ and $H_{\alpha,\gamma}$ are isomorphic to $\mathbb{F}_{q^2}$, so that $C \simeq \mathrm{U}(m, q)$ and $C_G(H) \simeq \mathrm{U}(m, q)$. Thus $[H, C] = 1$ and $C \simeq \mathrm{U}(m, q) \otimes I_\gamma$. Suppose $\alpha \geq 1$. Then the underlying space $V_{\alpha,\gamma}$ of $R_{\alpha,\gamma}$ decomposes as $V_{\alpha,\gamma} = U \oplus U'$ for some totally isotropic simple R-modules U and U'. The commuting algebra of $R_{\alpha,\gamma}$ and $H_{\alpha,\gamma}$ on U are isomorphic to $\mathbb{F}_{q^{2^\alpha}}$. It follows that $C \simeq C_G(H) \simeq \mathrm{GL}(m, q^{2^\alpha})$ and $C = C_G(H)$, since $C_G(H) \leq C_G(R)$. Thus $[H, C] = 1$ and $Z(H) \leq H \cap C$ since $R \leq H$. But $H \cap C \leq Z(H)$ and so $Z(H) = H \cap C$. By (1E) and (1F) $H/Z(H) \simeq \mathrm{Aut}^0 R$. The elements of N^0 induce the automorphisms of R trivial on Z. Thus for each element g of N^0, there is $h \in H$ such that

$gh \in C$, so that $N^0 = HC$. The last assertion follows by $H \simeq H_{\alpha,\gamma}$ and $R \simeq R_{\alpha,\gamma}$.

(2) If $\alpha = 0$, then $Z = O_2(Z(G))$ and so $N = N^0$. Suppose $\alpha \geq 1$. Since $Z = Z(R)$, the elements of N induce automorphisms of Z. Let y be a generator of $Z(R_{\alpha,\gamma}) \leq \mathrm{U}(2^{\alpha+\gamma}, q)$. Then there exists $\sigma \in \mathrm{U}(2^{\alpha+\gamma}, q)$ such that σ normalizes $R_{\alpha,\gamma}$ and $\sigma y \sigma^{-1} = y^{-q}$. Let ρ and w be the images of σ and y under the m-fold diagonal mapping (1.10). Then $\rho \in N$, $Z = \langle w \rangle$, and $\rho w \rho^{-1} = w^{-q}$. For each $g \in N$, $gwg^{-1} = w^i$ for some $i \geq 1$. Thus w and w^i are conjugate in G, so that $i = (-q)^l$ for some $l \geq 0$ as w is primary. Thus replacing g by $\rho^{-l}g$, we may suppose g fixes w and then $g \in N^0$. It follows that $N = \langle \rho, N^0 \rangle$. This completes the proof.

Remark. Suppose 2 is linear and $\alpha = 0$. Then $R \simeq 2_\eta^{2\gamma+1}$, $N = N^0 = HC$, $H \trianglelefteq N$, and $H/Z(H)R \simeq \mathrm{O}^{\pm}(2\gamma, 2)$. If $(R_{m,\alpha,\gamma}, \varphi)$ is a weight, then each irreducible constituent φ_0 of the restriction of φ to H has defect 0 as a character of H/R. An irreducible constituent of the restriction of φ_0 to $Z(H)$ is a linear character ξ of $Z(H)$ acting trivially on $R \cap Z(H) = Z(R) = O_2(Z(H))$, so that it has an extension $\tilde{\xi}$ to H which is trivial on R. Thus $\varphi_0 \tilde{\xi}^{-1}$ is an irreducible character of defect 0 of $H/Z(H)R$. For $\gamma \geq 2$ denote $\Omega^\eta(2\gamma, 2)$ the subgroup of index 2 in $\mathrm{O}^\eta(2\gamma, 2)$ such that $\Omega^+(2\gamma, 2) \simeq D_\gamma(2)$ and $\Omega^-(2\gamma, 2) \simeq {}^2D_\gamma(2)$. Then $\Omega^\eta(2\gamma, 2)$ has exactly one irreducible character of defect 0, i.e. the Steinberg character. Thus $\mathrm{O}^\eta(2\gamma, 2)$ has no irreducible character of defect 0, so that no such weight of $\mathrm{U}(m2^\gamma, q)$ exists. If $\gamma = 1$ and E_γ has plus type, then $H/R \simeq \mathbb{Z}/2\mathbb{Z}$ and so no such weight exists either. If $\gamma = 1$ and E_γ has minus type, then $H/R \simeq \mathrm{O}^-(2, 2) = \mathrm{GL}(2, 2)$ and the Steinberg character St is the only irreducible character of defect 0 and so $\varphi_0 \tilde{\xi}^{-1} = \mathrm{St}$.

Suppose 2 is linear and 2^a is the exact power of 2 dividing $q - 1$. Let $E_\gamma P$ be the central product of an extraspecial group $E_\gamma \simeq 2_\eta^{2\gamma+1}$ and a semidihedral group $P = S_{a+2}$ of order 2^{a+2} over $Z(E_\gamma) = Z(P)$. Then there exists a faithful and absolutely irreducible representation $\mathbf{T}$ of $E_\gamma P$ in $\mathrm{U}(2^{\gamma+1}, q)$ by (1I). The image $S_{1,\gamma}$ of $E_\gamma P$ in $\mathrm{U}(2^{\gamma+1}, q)$ is uniquely determined up to conjugacy, and independent of the type η. Thus we may suppose E_γ has plus type. Denote again P and E_γ the images $\mathbf{T}(P)$ and $\mathbf{T}(E)$ in $\mathrm{U}(2^{\gamma+1}, q)$. Let $S_{1,\gamma}^0 = C_{S_{1,\gamma}}([S_{1,\gamma}, S_{1,\gamma}])$ and $L_{1,\gamma}$ the subgroup of $N_{\mathrm{U}(2^{\gamma+1},q)}(S_{1,\gamma})$ which acts trivially on $[S_{1,\gamma}, S_{1,\gamma}]$. By (1I), (b)

$$[L_{1,\gamma}, Z(S_{1,\gamma}^0)] = 1, \quad Z(L_{1,\gamma}) = Z(\mathrm{U}(2^{\gamma+1}, q))Z(S_{1,\gamma}^0),$$
$$C_{\mathrm{U}(2^{\gamma+1},q)}(L_{1,\gamma}S_{1,\gamma}) = C_{\mathrm{U}(2^{\gamma+1},q)}(S_{1,\gamma}) = Z(\mathrm{U}(2^{\gamma+1}, q)),$$

and

$$L_{1,\gamma}/Z(L_{1,\gamma}) \simeq \mathrm{Aut}^0 S_{1,\gamma}^0.$$

Moreover, each linear character of $L_{1,\gamma}$ acting trivially on $O_2(Z(L_{1,\gamma}))$ can be extended as a character of $L_{1,\gamma}$ acting trivially on $S_{1,\gamma}^0$.

For each $m \geq 1$, the images $S_{m,1,\gamma}$ and $L_{m,1,\gamma}$ of $S_{1,\gamma}$ and $L_{1,\gamma}$ under the m-fold diagonal mapping in $\mathrm{U}(m2^{\gamma+1}, q)$ given by

$$(1.11)\qquad g \mapsto \begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix}, \qquad g \in S_{1,\gamma}, \text{ or } L_{1,\gamma},$$

is also determined up to conjugacy and $S_{m,1,\gamma}$ is uniquely determined up to conjugacy in $\mathrm{U}(m2^{\gamma+1}, q)$ by (1J). Let $S^0_{m,1,\gamma} = C_{S_{m,1,\gamma}}([S_{m,1,\gamma}, S_{m,1,\gamma}])$. Then $L_{m,1,\gamma}$ normalizes $S_{m,1,\gamma}$, $[L_{m,1,\gamma}, Z(S^0_{m,1,\gamma})] = 1$, and $Z(L_{m,1,\gamma}) = Z(\mathrm{GL}(m2^{\gamma+1}, q))Z(S^0_{m,1,\gamma})$. Moreover, $S^0_{m,1,\gamma} \trianglelefteq L_{m,1,\gamma}$ and

$$L_{m,1,\gamma}/Z(L_{m,1,\gamma}) \simeq \mathrm{Aut}^0 S^0_{m,1,\gamma}.$$

In particular,

$$L_{m,1,\gamma}/S^0_{m,1,\gamma}Z(L_{m,1,\gamma}) \simeq \mathrm{Sp}(2\gamma, 2).$$

Denote again by P and E_γ the images of P and E_γ under the m-fold diagonal mapping (1.11). Let $P = \langle \tau, \sigma \rangle$, so that $|\sigma| = 2^{a+1}$, $|\tau| = 2$, and $\tau\sigma\tau^{-1} = -\sigma^{-1}$. Thus $[S_{m,1,\gamma}, S_{m,1,\gamma}] = \langle \sigma^2 \rangle$, $S^0_{m,1,\gamma} = \langle \sigma \rangle E_\gamma$, $Z(S^0_{m,1,\gamma}) = \langle \sigma \rangle$, and $S_{m,1,\gamma} = \langle \tau, S^0_{m,1,\gamma} \rangle$.

(1M) *Let* $G = \mathrm{U}(m2^{\gamma+1}, q)$, $S = S_{m,1,\gamma}$, $L = L_{m,1,\gamma}$, *and* $S^0 = S^0_{m,1,\gamma}$, *and let*

$$C = C_G(S), \qquad N = N_G(S), \qquad N^0 = \{g \in N : [g, Z(S^0)] = 1\}.$$

Then the following hold:

(1) $C \simeq \mathrm{U}(m, q) \otimes I_{\gamma+1}$, $Z(C) = Z(G)$, $[L, C] = 1$, $L \cap CS^0 = S^0Z(L)$, $L \cap S = S^0$, $N^0 = CL$, *and* $Z(N^0) = Z(L) = Z(G)Z(S^0)$, *where* $I_{\gamma+1}$ *is the identity matrix of size* $2^{\gamma+1}$ *and* $\mathrm{U}(m, q) \otimes I_{\gamma+1}$ *is defined similarly to* (1L). *Moreover, each linear character of* $Z(L)$ *acting trivially on* $O_2(Z(L))$ *has an extension to* L *trivial on* S^0.

(2) $N^0 = \{g \in N : [g, \sigma^2] = 1\}$, $N^0 \trianglelefteq N$, *and* $N = \langle \tau, N^0 \rangle$.

Proof. (1) Since $\mathbf{T}$ is absolutely irreducible, $C \simeq \mathrm{U}(m, q) \otimes I_{\gamma+1}$ and $Z(C) = Z(G)$. It is clear that $N^0 \cap S = L \cap S = S^0$ and $LC \le N^0$. Since $C_G(LS) \simeq \mathrm{U}(m, q)$ and $C_G(LS) \le C_G(S)$, it follows $C_G(LS) = C$ and so $[L, C] = 1$. The rest of proof is the same as that of [4, (1I), (1)].

(2) Let $N^1 = \{g \in N : [g, \sigma^2] = 1\}$. Since σ^2 has order $2^a \ge 4$, $C(\sigma^2) \simeq \mathrm{GL}(2^\gamma, q^2)$ and $C(\sigma) \simeq \mathrm{GL}(2^\gamma, q^2)$. So $C(\sigma^2) = C(\sigma)$ since $C(\sigma) \le C(\sigma^2)$. It follows that $N^1 \le C(\sigma)$ and then $N^1 = N^0$ as $N^0 \le N^1$. Since $[S, S] = \langle \sigma^2 \rangle$ and N normalizes $[S, S]$, it follows that $N^0 \trianglelefteq N$. Now 2^a is the exact power of 2 dividing $q-1$ and σ^2 has order 2^a, so that $(\sigma^2)^{-q} = (\sigma^2)^{-1}$. Since $\tau\sigma^2\tau^{-1} = \sigma^{-2}$, it follows $\tau\sigma^2\tau^{-1} = (\sigma^2)^{-q}$. For any $h \in N$, $h\langle \sigma^2 \rangle h^{-1} = \langle \sigma^2 \rangle$ and so $h\sigma^2h^{-1} = (\sigma^2)^i$ for some $i \ge 1$. Thus $h\sigma^2h^{-1} = (\sigma^2)^{(-q)^l}$ for some $l \ge 0$ since σ^2 is primary. Replacing h by $\tau^{-l}h \in N$, we may suppose $h\sigma^2h^{-1} = \sigma^2$, and then $h \in N^0$. Thus $N = \langle \tau, N^0 \rangle$ and this completes the proof.

2. The radical 2-subgroups

In this section, we shall describe the structures of radical subgroups of unitary groups.

For each $\alpha \geq 0$, $\gamma \geq 0$, $m \geq 1$, and $1 \leq i \leq 2$, define

$$R^i_{m,\alpha,\gamma} = \begin{cases} S_{m,1,\gamma-1} & \text{if 2 is linear, } \alpha = 0,\ \gamma \geq 1, \text{ and } i = 2, \\ R_{m,\alpha,\gamma} & \text{otherwise,} \end{cases}$$

where $R_{m,\alpha,\gamma}$ and $S_{m,1,\gamma-1}$ are subgroups of $\mathrm{U}(m2^{\alpha+\gamma}, q)$ defined in (1L) and (1M). Thus if 2 is linear, $\alpha = 0$, and $\gamma \geq 1$, then $R^1_{m,\alpha,\gamma} = R_{m,0,\gamma}$ and $R^2_{m,\alpha,\gamma} = S_{m,1,\gamma-1}$. The centralizer $C^i_{m,\alpha,\gamma}$ and normalizer $N^i_{m,\alpha,\gamma}$ of $R^i_{m,\alpha,\gamma}$ in $\mathrm{U}(m2^{\alpha+\gamma}, q)$ are given by (1L) and (1M).

For each integer $c \geq 0$, let A_c denote the elementary abelian 2-subgroup of order 2^c represented by its regular permutation representation. For any sequence $\mathbf{c} = (c_1, c_2, \dots, c_t)$ of nonnegative integers, let $A_{\mathbf{c}} = A_{c_1} \wr A_{c_2} \wr \cdots \wr A_{c_t}$, and let

$$R^i_{m,\alpha,\gamma,\mathbf{c}} = R^i_{m,\alpha,\gamma} \wr A_{\mathbf{c}}, \qquad i = 1, 2,$$

be the wreath product in $\mathrm{U}(d, q)$, where $d = m2^{\alpha+\gamma+c_1+c_2+\cdots+c_t}$. Then $R^i_{m,\alpha,\gamma,\mathbf{c}}$ is determined up to conjugacy in $\mathrm{U}(d, q)$. It is clear that $[V, R^i_{m,\alpha,\gamma}] = V$, and $A_{\mathbf{c}}$ acts transitively on the set of underlying spaces of the factors of the base subgroup of $R^i_{m,\alpha,\gamma,\mathbf{c}}$. Here V is the underlying space of $R^i_{m,\alpha,\gamma}$ and $[V, R^i_{m,\alpha,\gamma}]$ is the set of vectors of V moved by $R^i_{m,\alpha,\gamma}$. By [3, (1.4)] with obvious modifications,

$$C_{\mathrm{U}(d,q)}(R^i_{m,\alpha,\gamma,\mathbf{c}}) = C^i_{m,\alpha,\gamma} \otimes I_{\mathbf{c}},$$

where $I_{\mathbf{c}}$ is the identity matrix of size $n = 2^{c_1+c_2+\cdots+c_t}$ and $C^i_{m,\alpha,\gamma} \otimes I_{\mathbf{c}} = \{g \otimes I_{\mathbf{c}} : g \in C^i_{m,\alpha,\gamma}\}$. Moreover, the following hold:

$$\begin{aligned} &N_{\mathrm{U}(d,q)}(R^i_{m,\alpha,\gamma,\mathbf{c}}) = (N^i_{m,\alpha,\gamma}/R^i_{m,\alpha,\gamma}) \otimes N_{\mathrm{S}(n)}(A_{c_1} \wr \cdots \wr A_{c_t}), \\ (2.1)\qquad &N_{\mathrm{U}(d,q)}(R^i_{m,\alpha,\gamma,\mathbf{c}})/R^i_{m,\alpha,\gamma,\mathbf{c}} \\ &\quad = (N^i_{m,\alpha,\gamma}/R^i_{m,\alpha,\gamma}) \times \mathrm{GL}(c_1, 2) \times \cdots \times \mathrm{GL}(c_t, 2), \end{aligned}$$

except when 2 is linear, $\alpha = \gamma = 0$, and $c_1 = 1$, in which case $R^1_{m,0,0,\mathbf{c}} = R^1_{m,0,1} \wr A_{\mathbf{c}'}$, and

$$\begin{aligned} &N_{\mathrm{U}(d,q)}(R^2_{m,0,0,\mathbf{c}}) = (N^1_{m,0,1}/R^1_{m,0,1}) \otimes N_{\mathrm{S}(n-2)}(A_{c_2} \wr \cdots \wr A_{c_t}), \\ (2.2)\qquad &N_{\mathrm{U}(d,q)}(R^1_{m,0,0,\mathbf{c}})/R^1_{m,0,1} \\ &\quad = (N^1_{m,0,1}/R^1_{m,0,1,}) \times \mathrm{GL}(c_2, 2) \times \cdots \times \mathrm{GL}(c_t, 2), \end{aligned}$$

where $R^1_{m,0,1} = \langle -I_m \rangle \wr A_{c_1}$ is dihedral of order 8, and $\mathbf{c}' = (c_2, \dots, c_t)$. In the latter case $R^1_{m,0,0,\mathbf{c}}$ is not radical by (1L). Here $(N^i_{m,\alpha,\gamma}/R^i_{m,\alpha,\gamma}) \otimes N_{\mathrm{S}(n)}(A_{\mathbf{c}})$ is defined as [3, (1.5)]. Before proving these equations, we first state a lemma which can be proved by replacing GL by U in the proof of [4, (2A)].

(2A) *Let $X \leq \mathrm{U}(m, q)$, $Y = A_{\mathbf{c}} \leq \mathrm{S}(n)$, where $\mathbf{c} = (c_1, c_2, \dots, c_t)$ and $n = 2^{c_1+c_2+\cdots+c_t}$, and let $R = X \wr Y \leq \mathrm{U}(mn, q)$, $D = X_1 \times X_2 \times \cdots \times X_n$ the base subgroup of R, and $V_1, V_2, \dots, V_n$ the underlying spaces of $X_1, X_2, \dots, X_n$.*

(a) *If either X is nonabelian or there exists $w \in Z(X)$ such that $|w| \geq 3$, then every normal abelian subgroup of R is contained in D.*

(b) *Suppose $X = \langle -I_m \rangle$ and $Y = A_{c_1}$. If $c_1 \geq 2$, then $C_R([R, R]) = D$ and R is generated by normal abelian subgroups of R. If $c_1 = 1$, then $R =$*

$R^1_{m,0,1} \leq \mathrm{U}(2m, q)$ *and* R *is dihedral of order* 8. *In particular,* R *is nonradical in* $\mathrm{U}(2m, q)$.

Now we prove (2.1) and (2.2). Let $R^i_{m,\alpha,\gamma,\mathbf{c}} = X \wr Y$ where $X = R^i_{m,\alpha,\gamma}$ and $Y = A_{\mathbf{c}}$. First we consider (2.1), so that either $X \neq \langle -I_m \rangle$ or $X = \langle -I_m \rangle$, but $c_1 \geq 2$. Let K be the subgroup of $X \wr Y$ generated by all normal abelian subgroups of $X \wr Y$ and $A(X \wr Y) = Z(C_K([K, K]))$. Then a similar proof to that of [4, (2.1)] shows that $A(X \wr Y) = Z(X^0)^n$ and $A(R^i_{m,\alpha,\gamma})$ is elementary abelian if and only if 2 is linear and $i = 1$, where $X^0 = C_X([X, X])$. Thus $N_{\mathrm{U}(d,q)}(X \wr Y)$ normalizes $Z(X^0)^n$. Let $\mathscr{E} = \{[V, x]: x \in Z(X^0)^n, x \neq 1\}$ be partially ordered by inclusion, where $[V, x] = (x-1)V$. Then the minimal elements in this ordering are the underlying spaces of the factors of the base subgroup $D = (X)^n$. So $N_{\mathrm{U}(d,q)}(X \wr Y)$ induces a permutation group on these spaces, and the equations (2.1) follow by [3, (1.5), (2,1)] with obvious modifications.

Finally suppose $X = \langle -I_m \rangle$ and $c_1 = 1$. Let $\mathbf{c}' = (c_2, \ldots, c_t)$, $X' = X \wr A_{c_1}$, and $Y' = A_{\mathbf{c}'}$. Then $X \wr Y = X' \wr Y'$ and $X' = R^1_{m,0,1} \leq \mathrm{GL}(2m, q)$. Thus (2.2) follows by (2.1).

We shall call $R^i_{m,\alpha,\gamma,\mathbf{c}}$ a *basic* subgroup of $\mathrm{U}(d, q)$ except when 2 is linear, $\alpha = \gamma = 0$, and $c_1 = 1$. In addition, we shall call $\deg R^i_{m,\alpha,\gamma,\mathbf{c}} = d$ the *degree* of $R^i_{m,\alpha,\gamma,\mathbf{c}}$ and $l(R^i_{m,\alpha,\gamma,\mathbf{c}}) = t$, the *length* of $R^i_{m,\alpha,\gamma,\mathbf{c}}$.

(2B) *Let* R *be a radical* 2*-subgroup of* $G = \mathrm{U}(V)$ *and* $N = N_G(R)$. *Then there exists a corresponding decomposition*

$$V = V_1 \perp \cdots \perp V_s \perp V_{s+1} \perp \cdots \perp V_t,$$
$$R = R_1 \times \cdots \times R_s \times R_{s+1} \times \cdots \times R_t$$

such that $R_i = \{\pm 1_{V_i}\}$ *for* $1 \leq i \leq s$, *and* R_i *are basic subgroup of* $\mathrm{U}(V_i)$ *for* $i \geq s+1$. *Moreover, if* 2 *is unitary, then* $s = 0$.

Proof. Since R is radical in G, it follows that $O_2(Z(G)) \leq O_2(N) = R$, so that $[V, R] = V$ and $s = 0$ if 2 is unitary. By (1B) we may write

$$V = m_1V_1 \perp m_2V_2 \perp \cdots \perp m_uV_u \perp n_1(U_1 \oplus U_1') \perp \cdots \perp n_v(U_v \oplus U_v'),$$

where the V_i represent representatives of isomorphic classes of nondegenerate simple R-modules, U_j and U_j' represent representatives of isomorphic classes of totally isotropic simple R-submodules occurring in V, and m_i, n_j are the multiplicities of V_i, $U_j \oplus U_j'$ in V. Moreover by (1B) we may suppose $U_j \oplus U_j'$ has no proper nondegenerate R-submodule. For simplicity of notation we rewrite this as

$$V = m_1V_1 \perp m_2V_2 \perp \cdots \perp m_uV_u \perp m_{u+1}V_{u+1} \perp \cdots \perp m_{u+v}V_{u+v},$$

where $m_i = n_i$, $V_i = U_i \oplus U_i'$ for $i > u$. Let $\mathbf{T}$ be the natural representation of R on V, and let $\mathbf{F}_i$ be the representation of R on V_i. Thus

$$\mathbf{T} = \begin{pmatrix} m_1\mathbf{F}_1 & & & \\ & m_2\mathbf{F}_2 & & \\ & & \ddots & \\ & & & m_{u+v}\mathbf{F}_{u+v} \end{pmatrix}.$$

Let R_i be the group of linear operators which agree with an element of R on m_iV_i and are the identity on m_jV_j for $j \neq i$. Then N induces a permutation group on the set of pairs (m_iV_i, R_i), so that

$$R \leq N \cap (R_1 \times R_2 \times \cdots \times R_{u+v}) \trianglelefteq N.$$

Since R is radical, $R = R_1 \times R_2 \times \cdots \times R_{u+v}$. Let $N_i = N_{\mathrm{U}(m_iV_i)}(R_i)$, so that $R_i \leq O_2(N_i)$ and

$$R \leq N \cap (O_2(N_1) \times O_2(N_2) \times \cdots \times O_2(N_{u+v})) \trianglelefteq N.$$

Again, since R is radical, it follows that $R_i = O_2(N_i)$ and each R_i is radical in $\mathrm{U}(m_iV_i)$ for all i. By induction we may suppose $u+v = 1$ and $V = m_1V_1$.

Suppose R has a characteristic noncyclic abelian subgroup A. As an A-module, V_1 decomposes by (1B) as

$$V_1 = u_1X_1 \perp u_2X_2 \perp \cdots \perp u_mX_m,$$

where X_i is either a nondegenerate simple A-submodule or a sum $Y_i \oplus Y_i'$ of totally isotropic simple A-submodules Y_i, Y_i', and u_i is the multiplicity of X_i in V_1. So R induces a permutation group on the set

$$\Omega = \{u_1X_1, u_2X_2, \ldots, u_mX_m\}.$$

The sum of each R-orbit is a nondegenerate R-submodule of V_1. If V_1 is a simple R-module, then Ω has exactly one R-orbit. If $V_1 = U_1 \oplus U_1'$, then V_1 has no proper nondegenerate R-submodule, so that Ω also has exactly one R-orbit. Thus R acts transitively on Ω and the X_i's are either all nondegenerate simple R-modules or all sums $Y_i \oplus Y_i'$ of totally isotropic simple R-modules Y_i, Y_i'. In particular, $u_i = u_j$ for all $1 \leq i, j \leq m$. Thus V has a corresponding decomposition and may be rewritten as

$$V = nX_1 \perp nX_2 \perp \cdots \perp nX_m, \tag{2.3}$$

where the X_i are mutually nonisomorphic A-submodules. Here the multiplicities are all equal since $V = m_1V_1$. Let $\mathbf{E}_i$ be the representation of A on nX_i. Thus

$$\mathbf{T}|_A = \begin{pmatrix} \mathbf{E}_1 & & & \\ & \mathbf{E}_2 & & \\ & & \ddots & \\ & & & \mathbf{E}_m \end{pmatrix}.$$

If X_i is a nondegenerate simple A-submodule, then $m \geq 2$ since A is noncyclic. The same conclusion holds if X_i is a sum of totally isotropic A-submodules Y_i and Y_i', since the representation of A on Y_i' is the contragredient of the representation of A on Y_i composed with a field automorphism. Let

$$N_1 = \{g \in N \colon gnX_1 = nX_1\}, \qquad R_1 = \{g \in R \colon gnX_1 = nX_1\}.$$

Then $A \leq R_1 \trianglelefteq N_1$ and $\mathbf{E}_1$ extends to a representation, denoted again by $\mathbf{E}_1$, of N_1 on nX_1. Since A is characteristic in R, N induces a permutation group L on $\Omega' = \{nX_1, nX_2, \ldots, nX_m\}$. The subgroup K of L corresponding to the subgroup R of N is transitive on Ω'. Moreover

$$R \leq \mathbf{E}_1(R_1) \wr K, \qquad N \leq \mathbf{E}_1(N_1) \wr L.$$

An argument similar to that of [3, (4A)] shows that N normalizes $\mathbf{E}_1(R_1)\wr K$. We sketch the proof as follows: Every element g of N has the form

$$g = \begin{pmatrix} \mathbf{E}_1(g_1) & & & \\ & \mathbf{E}_1(g_2) & & \\ & & \ddots & \\ & & & \mathbf{E}_1(g_m) \end{pmatrix} \pi(g),$$

where the $g_i \in N_1$ and $\pi(g) \in L$. Since $R_1 \trianglelefteq N_1$, g normalizes the base subgroup $\mathbf{E}_1(R_1)^m$ of $\mathbf{E}_1(R_1)\wr K$. So N normalizes $\mathbf{E}_1(R_1)\wr K$, since $\mathbf{E}_1(R_1)\wr K$ is generated by its base subgroup and R. Thus $R \leq (\mathbf{E}_1(R_1) \wr K) \cap N \trianglelefteq N$. Since R is radical, it follows that $R = \mathbf{E}_1(R_1)\wr K$. Now N permutes the spaces $nX_1, nX_2, \dots, nX_m$. By [3, (1.5)] with obvious modifications

$$N = (N_{\mathrm{U}(nX_1)}(\mathbf{E}_1(R_1))/\mathbf{E}_1(R_1)) \otimes N_{\mathbf{S}(m)}(K),$$
$$N/R = (N_{\mathrm{U}(nX_1)}(\mathbf{E}_1(R_1))/\mathbf{E}_1(R_1)) \times N_{\mathbf{S}(m)}(K)/K.$$

Thus $\mathbf{E}_1(R_1)$ and K are radical subgroups of $\mathrm{U}(nX_1)$ and $\mathbf{S}(m)$ respectively. Since K is transitive on Ω', it follows that $K = A_{\mathbf{c}}$ for some $\mathbf{c}$ by [3, (2A)]. By induction, there exist decompositions

$$nX_1 = M_1 \perp M_2 \perp \cdots \perp M_w, \qquad \mathbf{E}_1(R_1) = S_1 \times S_2 \times \cdots \times S_w,$$

where each S_i is a basic subgroup of $\mathrm{U}(M_i)$ for $1 \leq i \leq w$. Since $R = \mathbf{E}_1(R_1)\wr A_{\mathbf{c}}$, $Z(R) \simeq Z(\mathbf{E}_1(R_1))$ is cyclic. So $w = 1$ and $R = S_1 \wr A_{\mathbf{c}}$. Moreover, since $A \leq R_1$, $|R_1| \neq 2$, so that $|S_1| \neq 2$ and then R is a basic subgroup of $\mathrm{U}(V)$.

Thus we may suppose that every characteristic abelian subgroup of R is cyclic. By a result of P. Hall, [14, 5.4.9], R is the central product EP of E and P over $\Omega_1(Z(P)) = Z(E)$, where E is an extraspecial 2-group of order $2^{2\gamma+1}$, and P is one of the following groups: a cyclic group, a semidihedral group S_β, a dihedral group D_β, or a generalized quaternion group Q_β. Moreover, S_β, D_β, and Q_β have order $2^\beta \geq 16$. By (1J) either P is cyclic or $P = S_{a+2}$ and the latter case occurs only if 2 is linear, so that $R = S_{m_1, 1, \gamma}$. If $R = E$, then by (1J) again 2 is linear and $R = R^1_{m_1, 0, \gamma}$.

Suppose P is cyclic generated by g and $|P| \geq 4$, so that $P = Z(R)$. Thus N normalizes $C_G(\mathbf{T}(P))$ and $Z(C_G(\mathbf{T}(P))) \trianglelefteq N$. Since $P \leq O_2(Z(C_G(\mathbf{T}(P))))$ and R is radical, it follows that $O_2(Z(C_G(\mathbf{T}(P)))) \leq O_2(N) = R$, so that

$$O_2(Z(C_G(\mathbf{T}(P)))) = Z(R) = P,$$

since $R \leq C_G(\mathbf{T}(P))$. Let $\mathbf{X}$ be the representation of R on V_1, where $V = m_1 V_1$. Then $\mathbf{T} = m_1\mathbf{X}$. As an element of $\mathrm{U}(V_1)$, $\mathbf{X}(g)$ is primary with a unique elementary divisor $\Gamma \in \mathscr{F}$ of multiplicity u. So $C_{\mathrm{U}(V_1)}(\mathbf{X}(g)) \simeq \mathrm{GL}(u, \varepsilon_\Gamma q^{d_\Gamma})$ and $C_G(\mathbf{T}(g)) \simeq \mathrm{GL}(m_1 u, \varepsilon_\Gamma q^{d_\Gamma})$. Thus

$$Z(C_G(\mathbf{T}(g))) \simeq Z(C_{\mathrm{U}(V_1)}(\mathbf{X}(g))) \simeq \mathrm{GL}(1, \varepsilon_\Gamma q^{d_\Gamma}),$$

so that $|\mathbf{X}(P)| = |O_2(Z(C_{\mathrm{U}(V_1)}(\mathbf{X}(g))))|$ and then $\mathbf{X}(P) = O_2(Z(C_{\mathrm{U}(V_1)}(\mathbf{X}(g))))$, since $\mathbf{X}(P) \leq Z(C_{\mathrm{U}(V_1)}(\mathbf{X}(g)))$. By (1K) $\mathbf{X}(R) = R_{\alpha,\gamma}$ in $\mathrm{U}(V_1)$, so that $R = R^1_{m_1, \alpha, \gamma}$ in G. This proves (2B).

(2C) *Let* (R, φ) *be a weight of* $G = \mathrm{U}(V)$ *and*

$$V = V_1 \perp \cdots \perp V_s \perp V_{s+1} \perp \cdots \perp V_t,$$
$$R = R_1 \times \cdots \times R_s \times R_{s+1} \times \cdots \times R_t$$

be the corresponding decomposition of (2B). *Let*

$$V(k, m, \alpha, \gamma, \mathbf{c}) = \sum_i V_i, \; R(k, m, \alpha, \gamma, \mathbf{c}) = \prod_i R_i, \; G(k, m, \alpha, \gamma, \mathbf{c}) = \mathrm{U}(V(k, m, \alpha, \gamma, \mathbf{c})),$$

where i *runs over the indices such that* $R_i = R^k_{m,\alpha,\gamma,\mathbf{c}}$. *Then*

$$N(R) = \prod_{k,m,\alpha,\gamma,\mathbf{c}} N_{G(k,m,\alpha,\gamma,\mathbf{c})}(R(k, m, \alpha, \gamma, \mathbf{c})),$$
$$N(R)/R = \prod_{k,m,\alpha,\gamma,\mathbf{c}} N_{G(k,m,\alpha,\gamma,\mathbf{c})}(R(k, m, \alpha, \gamma, \mathbf{c}))/R(k, m, \alpha, \gamma, \mathbf{c}).$$

Moreover

$$N_{G(k,m,\alpha,\gamma,\mathbf{c})}(R(k, m, \alpha, \gamma, \mathbf{c})) = N^k_{m,\alpha,\gamma,\mathbf{c}} \wr \mathbf{S}(u),$$
$$N_{G(k,m,\alpha,\gamma,\mathbf{c})}(R(k, m, \alpha, \gamma, \mathbf{c}))/R(k, m, \alpha, \gamma, \mathbf{c}) = (N^k_{m,\alpha,\gamma,\mathbf{c}}/R^k_{m,\alpha,\gamma,\mathbf{c}}) \wr \mathbf{S}(u),$$

where if $V_{m,\alpha,\gamma,\mathbf{c}}$ *is the underlying space of* $R^k_{m,\alpha,\gamma,\mathbf{c}}$ *then* $N^k_{m,\alpha,\gamma,\mathbf{c}}$ *is the normalizer of* $R^k_{m,\alpha,\gamma,\mathbf{c}}$ *in* $\mathrm{U}(V_{m,\alpha,\gamma,\mathbf{c}})$, *and* u *is the number of basic components* $R^k_{m,\alpha,\gamma,\mathbf{c}}$ *in* $R(k, m, \alpha, \gamma, \mathbf{c})$.

Proof. Let $N = N(R)$ and $\mathscr{D} = \{[V, x]: x \in Z(R), x \neq 1\}$, which is partially ordered by inclusion. Then N induces a permutation group on $\mathscr{D}$. The minimal elements in this ordering are the spaces V_i, so N permutes the pairs $\{(V_i, R_i)\}$. Let K_i be the subgroup of R_i generated by all normal abelian subgroups of R_i, $A(R_i) = C_{K_i}([K_i, K_i])$, and $\mathscr{E}_i = \{[V, g]: g \in A(R_i), g \neq 1\}$ partially ordered by inclusion. If $R_i = R^k_{m,\alpha,\gamma,\mathbf{c}}$, then the minimal elements of $\mathscr{E}_i$ have dimension $m2^{\alpha+\gamma}$ as shown in the proof of (2.1). We claim that $(m, 2) = 1$. Indeed let $C = C_G(R)$ and θ an irreducible constituent of the restriction $\varphi|_{CR}$ of φ to CR. Then $R \leq \ker\theta$ and θ has defect 0 as a character of CR/R. Let $C_i = C_{\mathrm{U}(V_i)}(R_i)$. Then $C = C_1 \times C_2 \times \cdots \times C_t$ and $\theta = \theta_1 \times \theta_2 \times \cdots \times \theta_t$, where θ_i is an irreducible character of C_iR_i/R_i of defect 0. As a character of C_i, θ_i falls into a block b_i of C_i with defect group $Z(R_i)$ such that θ_i is the canonical character of b_i. Now $C_i \simeq \mathrm{GL}(m, \varepsilon_\alpha q^{2^\alpha})$. By a theorem of Broué, [6, (4.18)], there is a semisimple $2'$-element $s \in C_i$ such that $Z(R_i)$ is a Sylow 2-subgroup of $C_{C_i}(s)$. This forces $C_{C_i}(s) \simeq \mathrm{GL}(1, \varepsilon_\alpha q^{m2^\alpha})$. If $\alpha \geq 1$, then $Z(R_i)$ has order $2^{a+\alpha}$ and $C_{C_i}(s) \simeq \mathrm{GL}(1, q^{m2^\alpha})$, so that m is odd. If $\alpha = 0$, then $C_i \simeq \mathrm{U}(m, q)$, $C_{C_i}(s) \simeq \mathrm{U}(1, q^m)$, and $|Z(R_i)| = 2^a$ or 2 according as 2 is unitary or linear. Thus s is primary in C_i with a unique elementary divisor Γ of multiplicity 1, so that $\Gamma \in \mathscr{F}_1$, and then $m = d_\Gamma$ is odd. Now the rest of the proof is the same as that of [4, (2C)].

Remark. By (2C) if (R, φ) is a weight, then there exists an irreducible character φ_0, covered by φ, of $N_{G(k,m,\alpha,\gamma,\mathbf{c})}(R(k, m, \alpha, \gamma, \mathbf{c}))$, so that φ_0 is trivial on $R(k, m, \alpha, \gamma, \mathbf{c})$ and φ_0 has defect 0 as a character of

$$N_{G(k,m,\alpha,\gamma,\mathbf{c})}(R(k, m, \alpha, \gamma, \mathbf{c}))/R(k, m, \alpha, \gamma, \mathbf{c}) = (N^k_{m,\alpha,\gamma,\mathbf{c}}/R^k_{m,\alpha,\gamma,\mathbf{c}}) \wr \mathbf{S}(u).$$

As shown in the proof of [3, (2C)], there exists an irreducible character ψ of $N^k_{m,\alpha,\gamma,\mathbf{c}}$ covered by φ_0. Thus ψ has defect 0 as a character of

$$N^k_{m,\alpha,\gamma,\mathbf{c}}/R^k_{m,\alpha,\gamma,\mathbf{c}}.$$

By (2.1), there exists an irreducible character ψ_0 of $N^k_{m,\alpha,\gamma}$ covered by ψ. So ψ_0 has defect 0 as a character of $N^k_{m,\alpha,\gamma}/R^k_{m,\alpha,\gamma}$. Suppose 2 is linear $\alpha = 0$ and $k = 1$. Then by the remark after (1L), this only occurs when $R^1_{m,\alpha,\gamma} = R_{m,0,1}$ is a quaternion group.

Given $m \geq 1$, $\alpha \geq 0$, $\gamma \geq 0$, and a sequence $\mathbf{c} = (c_1, c_2, \dots, c_t)$ of nonnegative integers c_i. Let $\mathbf{c}' = (c_2, \dots, c_t)$. Define $D_{m,\alpha,\gamma,\mathbf{c}}$ as follows: If 2 is unitary, then $D_{m,\alpha,\gamma,\mathbf{c}} = R_{m,\alpha,\gamma,\mathbf{c}}$. Suppose 2 is linear. Then

$$D_{m,\alpha,\gamma,\mathbf{c}} = \begin{cases} R_{m,\alpha,\gamma,\mathbf{c}} & \text{if } \alpha \geq 1, \\ S_{1,\gamma-1,\mathbf{c}} & \text{if } \alpha = 0, \text{ and } \gamma \geq 1, \\ R_{m,\alpha,\gamma,\mathbf{c}} & \text{if } \alpha = \gamma = 0 \text{ and } c_1 \neq 1, \\ R_{m,0,1,\mathbf{c}'} & \text{if } \alpha = \gamma = 0 \text{ and } c_1 = 1, \end{cases}$$

where $R_{m,0,1}$ is a quaternion group. In addition let $D_{m,\alpha,\gamma} = D_{m,\alpha,\gamma,0}$. By the remark above, the components R_i in the decomposition of (2C) can be supposed to have the form $D_{m,\alpha,\gamma,\mathbf{c}}$.

3. The 2-weights

Let H be a subgroup of a finite group G, $K \trianglelefteq H$, R a normal 2-subgroup of H with $R \leq K$, and θ an irreducible character of K trivial on R. Following [4], we denote the sets of irreducible characters of H which cover θ and which have defect 0 as characters of H/R by $\operatorname{Irr}^0(H, \theta)$. We also denote by $N(\theta)$ the stabilizer of θ in $N(R)$. By [3, p. 3] we can enumerate the weights for a block B of G as follows: Let R be a radical subgroup of G, b a block of $C(R)R$ with defect group R and $B = b^G$, and θ the canonical character of b. Then each $\psi \in \operatorname{Irr}^0(N(\theta), \theta)$ gives rise to a B-weight $(R, I(\psi))$ of G, where $I(\psi) = \operatorname{Ind}^N_{N(\theta)}(\psi)$ is the induction mapping. All B-weights of G are obtained by letting R run over representatives for the G-conjugacy classes of radical subgroups, and for each such R letting b run over representatives for the $N(R)$-conjugacy classes of blocks of $C(R)R$ such that b has defect group R and $b^G = B$.

A Brauer pair (R, b) of a finite group G consists of a 2-subgroup R of G and a block b of $C(R)$. If G is a unitary group over $\mathbb{F}_{q^2}$, then the Brauer pairs (R, b) of G have been labeled by ordered triples $(R, s, -)$ in [6, (3.2)], where s is a semisimple $2'$-element of the dual group G^* of G, and—is an empty set. Moreover, by [6, (3.4)] each block B of G is labeled by a pair $(s, -)$. Since $G^* \simeq G$, we may identify G^* with G.

Let $\mathscr{F}'$ be the set of polynomials Γ in $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$ whose roots have odd orders. Given Γ in $\mathscr{F}$, let α_Γ be the exponent such that $2^{\alpha_\Gamma} = (d_\Gamma)_2$ and m_Γ the integer such that $m_\Gamma 2^{\alpha_\Gamma} = d_\Gamma$. By [6, (3.8)] each Γ in $\mathscr{F}'$ determines a block B_Γ of $G_\Gamma = \mathrm{U}(d_\Gamma, q)$ with label $(\Gamma, -)$, where Γ represents a semisimple element of G_Γ with an elementary divisor Γ of multiplicity 1 and no other elementary divisors. By [6, (4.18)] a defect group R_Γ of B_Γ exists as a subgroup of $C_{G_\Gamma}(\Gamma) \simeq \mathrm{GL}(1, \varepsilon_\Gamma q^{d_\Gamma})$. If $\Gamma \in \mathscr{F}_1$, then $\varepsilon_\Gamma = -1$, d_Γ is odd, $\alpha_\Gamma = 0$, and $m_\Gamma = d_\Gamma$, so that $C_{G_\Gamma}(\Gamma) \simeq \mathrm{U}(1, q^{d_\Gamma})$ and $R_\Gamma = O_2(Z(G_\Gamma))$. Thus R_Γ has the form $D_{m_\Gamma, \alpha_\Gamma, 0}$. If $\Gamma \in \mathscr{F}_2$, then $\varepsilon_\Gamma = 1$, d_Γ is even, and $\alpha_\Gamma \geq 1$, so that $C_{G_\Gamma}(\Gamma) \simeq \mathrm{GL}(1, q^{d_\Gamma})$ and R_Γ has the form $D_{m_\Gamma, \alpha_\Gamma, 0}$. Let $C_\Gamma = C_{G_\Gamma}(R_\Gamma)$, and $N_\Gamma = N_{G_\Gamma}(R_\Gamma)$. Then $C_\Gamma \simeq \mathrm{GL}(m_\Gamma, \varepsilon_\Gamma q^{2^{\alpha_\Gamma}})$ and N_Γ/C_Γ is cyclic of order 2^{α_Γ} by (1L). Let b_Γ be a block of C_Γ with defect group R_Γ and $b_\Gamma^{G_\Gamma} = B_\Gamma$, let θ_Γ be the canonical character of b_Γ, and $N(\theta_\Gamma)$ the stabilizer of θ_Γ in N_Γ. Then θ_Γ acts trivially on R_Γ and has defect 0 as a character of C_Γ/R_Γ. The pair $(R_\Gamma, \theta_\Gamma)$ is determined up to conjugacy in N_Γ by Brauer's First Main Theorem. Since R_Γ is a defect group of B_Γ, $(N(\theta_\Gamma): C_\Gamma)$ is odd, so that $N(\theta_\Gamma) = C_\Gamma$. Conversely let B be a block of $\mathrm{U}(2^\alpha m, q)$ with defect group $R = D_{m,\alpha,0}$, $(s, -)$ the label of B, and b a block of $C_G(R)$ such that $b^{\mathrm{U}(2^\alpha m, q)} = B$. By [6, (4.18)] we may suppose R is a Sylow 2-subgroup of $C_{\mathrm{U}(2^\alpha m, q)}(s)$. This forces $C_{\mathrm{U}(2^\alpha m, q)}(s) \simeq \mathrm{GL}(1, \varepsilon_\alpha q^l)$ for some $l \geq 1$, so that s has a unique elementary divisor $\Gamma \in \mathscr{F}'$ with multiplicity 1. If $\alpha \geq 1$, then $|R_{\alpha,\gamma}| = 2^{a+\alpha}$ and $C_{\mathrm{U}(2^\alpha m, q)}(s) \simeq \mathrm{GL}(1, q^{d_\Gamma})$, so that $l = d_\Gamma = 2^\alpha m$ and $|\mathrm{GL}(1, q^{m2^\alpha})|_2 = 2^{a+\alpha}$. Thus m is odd, $m = m_\Gamma$, and $\alpha = \alpha_\Gamma$. If $\alpha = 0$, then $R_{\alpha,\gamma} = O_2(Z(\mathrm{U}(m, q)))$ and $C_{\mathrm{U}(m,q)}(s) \simeq \mathrm{U}(1, q^{d_\Gamma})$, so that $d_\Gamma = m$. Since $\Gamma \in \mathscr{F}_1$, d_Γ is odd and so $m = m_\Gamma$, $\alpha = \alpha_\Gamma$. Thus Γ and B correspond in the preceding manner. In particular, $\mathrm{U}(2^\alpha m, q) = G_\Gamma$, $R_{\alpha,\gamma}$ has the form R_Γ as a subgroup of G_Γ, $B = B_\Gamma$, and s, Γ are conjugate in G_Γ.

(3A) *Given $\Gamma \in \mathscr{F}'$. Let $G = \mathrm{U}(2^\gamma d_\Gamma, q)$ and $R = D_{m_\Gamma, \alpha_\Gamma, \gamma} \leq G$ or $G = \mathrm{U}(2 d_\Gamma, q)$ and $R = R_{m_\Gamma, 0, 1} \leq G$, where $R_{m_\Gamma, 0, 1}$ is a quaternion group. Let $C = C_G(R)$ and $N = N_G(R)$. Then R is a basic subgroup of G and $C = C_\Gamma \otimes I$, where I is the identity matrix of order 2^γ or 2 according as $R = D_{m_\Gamma, \alpha_\Gamma, \gamma}$ or $R = R_{m_\Gamma, 0, 1}$. The irreducible character of C defined by $\theta(c \otimes I) = \theta_\Gamma(c)$ for $c \in C_\Gamma$ is then a character of defect 0 of $C/Z(R)$ and $|\mathrm{Irr}^0(N(\theta), \theta)| = 1$.*

The proof of (3A) is the same as that of [4, (3A)].

Let $\Gamma \in \mathscr{F}'$, and let $G = \mathrm{U}(2^d d_\Gamma, q)$ and $R = D_{m_\Gamma, \alpha_\Gamma, \gamma, \mathbf{c}}$ be a basic subgroup of G, where $\mathbf{c} = (c_1, c_2, \ldots, c_t)$, and $d = \gamma + c_1 + c_2 + \cdots + c_t$. In addition, let $\mathbf{c}' = (c_2, \ldots, c_t)$. Then $C = C_G(R) = C_\Gamma \otimes I_\gamma \otimes I_{\mathbf{c}}$, except when 2 is linear, $\alpha = \gamma = 0$, and $c_1 = 1$, in which case, $C = C_G(R) = C_\Gamma \otimes I_2 \otimes I_{\mathbf{c}'}$, where I_γ, $I_{\mathbf{c}}$, I_2, and $I_{\mathbf{c}'}$ are identity matrices of orders 2^γ, $2^{c_1+c_2+\cdots+c_t}$, 2, and $2^{c_2+\cdots+c_t}$ respectively. The irreducible character of C defined by

$$(3.1)\quad \begin{cases} \theta(c \otimes I_2 \otimes I_{\mathbf{c}'}) = \theta_\Gamma(c) & \text{if 2 is linear}, \ \alpha = \gamma = 0, \quad \text{and} \quad c_1 = 1, \\ \theta(c \otimes I_\gamma \otimes I_{\mathbf{c}}) = \theta_\Gamma(c) & \text{otherwise}, \end{cases}$$

for $c \in C_\Gamma$ is then a character of defect 0 of CR/R. We shall say that the pair (R, θ) is of *type* Γ. If b is the block of C containing θ, then (R, b) has a label $(R, 2^d\Gamma, -)$, so that the block $B = b^G$ of G has the label $(2^d\Gamma, -)$. Regard b as a block of CR. Then b has a defect group R. Moreover, using the following lemma (3B) and the same proof as [4, (3C), (2)], we can show that CR has exactly one $N(R)$-conjugacy class of blocks b such that $B = b^G$.

(3B) *Let* $G = \mathrm{U}(n, q)$, R *a basic subgroup of* G, (R, φ) *a weight of* G, *and* θ *an irreducible character of* $C_G(R)$ *covered by* φ. *Then* (R, θ) *has type* Γ *for some* $\Gamma \in \mathscr{F}'$.

The proof of (3B) can be obtained by using the remark before (3A) and replacing GL by U in the proof of [4, (3B)] with some obvious modifications.

Therefore we can count the number of B-weights by letting $R = D_{m_\Gamma, \alpha_\Gamma, \gamma, \mathbf{c}}$ run over the basic subgroups of G with degree $2^d d_\Gamma$. Using (3A) and replacing GL by U in the proof of [4, (3C)] with some obvious modifications, we can get the following proposition.

(3C) *Let* B *be a block of* $G = \mathrm{U}(2^d d_\Gamma, q)$ *labeled by* $(2^d\Gamma, -)$. *Then there are exactly* 2^d B*-weights* (R, φ), *where* R *runs over the basic subgroups of* G *with degree* $2^d d_\Gamma$.

For each $\Gamma \in \mathscr{F}'$ and $d \geq 0$, let $\mathscr{C}_{\Gamma, d} = \{\varphi_{\Gamma, d, j} : 1 \leq j \leq 2^d\}$ be the set of characters associated with basic subgroups of $\mathrm{U}(2^d d_\Gamma, q)$ in (3C).

(3D) *Let* $\Gamma \in \mathscr{F}'$, $G = \mathrm{U}(w_\Gamma d_\Gamma, q)$, *for some integer* $w_\Gamma \geq 1$, *and* B *the block of* G *labeled by* $(w_\Gamma\Gamma, -)$. *Then the number of* B*-weights is the number* f_Γ *of assignments*

$$\coprod_{d \geq 0} \mathscr{C}_{\Gamma, d} \to \{2 - \mathit{cores}\}, \qquad \varphi_{\Gamma, d, j} \mapsto \kappa_{\Gamma, d, j},$$

such that

$$\sum_{d \geq 0} 2^d \sum_{j=1}^{2^d} |\kappa_{\Gamma, d, j}| = w_\Gamma.$$

The proof of (3D) is the same as that of [4, (3D)] with $\mathrm{GL}(V_i)$ replaced by $\mathrm{U}(V_i)$.

The main theorem of this paper is the following theorem which can be proved by replacing $\mathrm{GL}(V_i)$ by $\mathrm{U}(V_i)$, $\mathscr{F}$ by $\mathscr{F}'$, and $\mathrm{GL}(V_\Gamma)$ by $\mathrm{U}(V_\Gamma)$ in the proof of [4, (3E)].

(3E) *Let* B *be a block of* $G = \mathrm{U}(V)$ *with label* $(s, -)$, $\prod_\Gamma s_\Gamma$ *the primary decomposition of* s, $\sum_\Gamma V_\Gamma$ *the corresponding decomposition of* V, *and* w_Γ *the integer such that* $\dim V_\Gamma = d_\Gamma w_\Gamma$. *Then the following hold*:

(1) *The number of* B*-weights of* G *is* $\prod_\Gamma f_\Gamma$, *where* f_Γ *is given by* (3D). *In particular,* f_Γ *is the number of partitions of* w_Γ.

(2) *The number of* B*-weights of* G *is* $l(B)$.

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