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PROBING L-S CATEGORY WITH MAPS

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ABSTRACT. For any map $X \xrightarrow{f} Y$, we introduce two new homotopy invariants, $\operatorname{dcat} f$ and $\operatorname{rcat} f$. The classical category $\operatorname{cat} f$ is a lower bound for both, while $\operatorname{dcat} f \leq \operatorname{cat} X$ and $\operatorname{rcat} f \leq \operatorname{cat} Y$. When Y is an Eilenberg-Mac Lane space, f represents a cohomology class and $\operatorname{dcat} f$ often gives a good estimate for $\operatorname{cat} X$. We prove that if $\Omega \in H^n(M;\mathbb{Z})$ is the fundamental class of a compact, simply connected n-manifold, then $\operatorname{dcat} \Omega = \operatorname{cat} M$. Similarly, when X is sphere, then f is a homotopy class and while $\operatorname{cat} f = 1$, $\operatorname{rcat} f$ can be a good approximation to $\operatorname{cat} Y$. We show that if $\alpha \in \pi_2(\mathbb{C}P^n)$ is nonzero, then $\operatorname{rcat} \alpha = n$. Rational analogues are introduced and we prove that for $u \in H^*(X;\mathbb{Q})$, $\operatorname{dcat}_0 u = 1 \Leftrightarrow u^2 = 0$ and u is spherical.

1. INTRODUCTION

Let Lusternik-Schnirelmann category, $\operatorname{cat} X$, of space X is a subtle invariant which is usually difficult to compute. In particular, good lower bounds seem to require much more data than is provided by standard homotopy invariants such as the cohomology ring or homotopy groups. For example, Ginsburg and Toomer [Gi, To] defined lower bounds using the Milnor-Moore spectral sequence, while Felix, Halperin, and Lemairè [Fe-Ha, Ha-Le] employed the full force of the Sullivan model for their approximations.

Much older lower bounds are provided by the category, $\operatorname{cat} f$, of a map $X \xrightarrow{f} Y$, which was introduced originally by Fox [Fo]. Taking his point of view, we introduce two new L-S type invariants for a map $X \xrightarrow{f} Y$, which we call the domain-category, $\operatorname{dcat} f$ and the range category, $\operatorname{rcat} f$. These satisfy

$$\operatorname{cat} f \leq \operatorname{dcat} f \leq \operatorname{cat} X \quad \text{and} \quad \operatorname{cat} f \leq \operatorname{rcat} f \leq \operatorname{cat} Y.$$

When specialized to maps representing cohomology (when Y is an Eilenberg-Mac Lane space) or homotopy (when X is a sphere), in many cases these invariants provide excellent approximations to $\operatorname{cat} X$ or $\operatorname{cat} Y$. In particular, we prove

Theorem 2.8. *If $\Omega \in H^n(M;\mathbb{Z})$ is the fundamental class of a simply connected n-manifold, then* $\operatorname{dcat} \Omega = \operatorname{cat} M$.

This is to be contrasted with the classical category $\operatorname{cat} \Omega$, which is 1 in this case [Be-Ga].

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For homotopy classes $\alpha \in \pi_n(X)$, we always have $\operatorname{cat}\alpha = 1$. However, we show in Example 2.6 that if $\alpha \in \pi_2(\mathbb{C}P^n)$; where $\mathbb{C}P^n$ is the complex projective space, then $\operatorname{rcat}\alpha = n$.

Berstein and Ganea proved that for rational cohomology classes $u \in H^*(X, \mathbb{Q})$, $\operatorname{cat} u = \min\{m | u^{m+1} = 0\}$. By introducing rational analogues of dcat and rcat, we show (Theorem 3.5) that $\operatorname{dcat} u$ is at least as large as Toomer's invariant for u, which is often larger than $\operatorname{cat} u$, particularly when $\dim u$ is odd. We also prove

Theorem 3.8. *Suppose* $u \in H^n(X; \mathbb{Q})$ *is nonzero. Then* $\operatorname{dcat}_0 u = 1 \Leftrightarrow u$ *is spherical and* $u^2 = 0$.

In §2, we define $\operatorname{dcat} f$ and $\operatorname{rcat} f$ and show that they may be characterized in terms of the classical category of maps which commute, up to homotopy, with f. This allows us to give examples where $\operatorname{dcat} f > \operatorname{cat} f$, $\operatorname{rcat} f > \operatorname{cat} f$, and $\operatorname{dcat} f \neq \operatorname{rcat} f$. Section 3 studies the rational versions. Using work of Felix and Halperin, we give an algebraic description of $\operatorname{dcat} f_{\mathbb{Q}}$ and $\operatorname{rcat} f_{\mathbb{Q}}$ for rational maps $f_{\mathbb{Q}}: X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$. We can then prove some interesting results about the domain and range category of rational cohomology classes.

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2. Domain and range category

We restrict ourselves to well-pointed, simply connected spaces with the homotopy type of a CW complex of finite type. For this class of spaces, if $X \xrightarrow{f} Y$, is a continuous map, we can define $\operatorname{cat} f$ as follows, after Whitehead, Berstein, Ganea, and Gilbert [G]: Convert the inclusion of the fat wedge $T^{m+1}Y \to Y^{m+1}$ into a fibration and pull this back over the diagonal $\Delta_Y: Y \to Y^{m+1}$ to obtain a (Ganea) fibration $p_m: E_mY \to Y$. Then $\operatorname{cat} f$ is the least m such that we can lift f to a map $\beta: X \to E_mY$ so that the diagram below homotopy commutes.

$$\begin{tikzcd} & E_mY \arrow[d, "p_m"] \\ X \arrow[ur, dashed, "\varphi"] \arrow[r, "f"'] & Y \end{tikzcd}$$

The definitions of domain and range category are variations on this description of $\operatorname{cat} f$. Suppose $X \xrightarrow{f} Y$, and $p_m: E_mX \to X$ and $q_m: E_mY \to Y$ are mth-Ganea fibrations for X and Y respectively.

Definition 2.1. The *domain-category of* f, denoted $\operatorname{dcat} f$, is the least m such that, in the diagram

$$\begin{tikzcd} E_mX \arrow[d, "p_m"] & \\ X \arrow[u, dashed, bend left=60, "\varphi"] \arrow[r, "f"] & Y \end{tikzcd}$$

the map φ exists so that $f \circ p_m \circ \varphi \simeq f$.

Definition 2.2. The *range-category of* f, denoted $\operatorname{rcat} f$, is the least m such that, in the diagram

$$\begin{tikzcd} & E_mY \arrow[d, "q_m"] \\ X \arrow[r, "f"] & Y \arrow[u, dashed, bend right, "\psi"'] \end{tikzcd}$$

the map ψ exists so that $f \simeq q_m \circ \psi \circ f$.

We summarize some straightforward properties of the domain and range categories in the following theorem. All are immediate from the definitions.

Theorem 2.3. *For maps* $X \xrightarrow{f} Y$ *and* $Y \xrightarrow{g} Z$.

(i) $\operatorname{dcat} f$ *and* $\operatorname{rcat} f$ *depend only on the homotopy class of* f.
(ii) $\operatorname{cat} f \leq \operatorname{dcat} f \operatorname{cat} X$ *and* $\operatorname{cat} f \leq \operatorname{rcat} f \leq \operatorname{cat} Y$.
In particular, $\operatorname{cat} \mathrm{id}_X = \operatorname{dcat} \mathrm{id}_X = \operatorname{rcat} \mathrm{id}_X$.
(iii) $\operatorname{dcat}(g \circ f) \leq \operatorname{dcat} f$ *and* $\operatorname{rcat}(g \circ f) \leq \operatorname{rcat} g$.

A useful characterization of dcat and rcat in terms of the classical category of a map is described in

Theorem 2.4. *Suppose* $X \xrightarrow{f} Y$. *Then*

(i) $\operatorname{dcat} f = \min\{\operatorname{cat} s | X \xrightarrow{s} X$ *and* $f \circ s \simeq f\}$.
(ii) $\operatorname{rcat} f = \min\{\operatorname{cat} s | Y \xrightarrow{s} Y$ *and* $s \circ f \simeq f\}$.

Proof. (i) Suppose $X \xrightarrow{s} X$ with $f \circ s \simeq f$ and $\operatorname{cat} s = m$. Now $\operatorname{cat} s = m$ implies that there is $\beta\colon X \to E_mX$ with $p_m \circ \beta \simeq s$. Then $f \circ p_m \circ \beta \simeq f \circ s \simeq f$ and so $\operatorname{dcat} f \leq m$. If, on the other hand, $\operatorname{dcat} f = m$, there is $\varphi\colon X \to E_mX$ with $f \circ p_m \circ \varphi \simeq f$. Let $s = p_m \circ \varphi$. Then $\operatorname{cat} s \leq m$ and $f \circ s \simeq f$. Similar arguments prove (ii).

Example 2.5. Consider the map h defined as the composition $S^2 \times S^2 \xrightarrow{f} S^4 \xrightarrow{g} S^4 \xrightarrow{g} \mathbb{H}P^3$. Here, f extends $S^2 \vee S^2 \to *$ and satisfies $f^*(\alpha_4) = \alpha_2 \times \alpha_2'$ (α_2, α_2' denoting generators in $H^2(S^2 \vee S^2; \mathbb{Z})$) and g is the inclusion $S^4 \to S^4 \cup e^8 \cup e^{12} = \mathbb{H}P^3$ into the quaternionic projective space. We show that $\operatorname{cat} h = 1$, $\operatorname{dcat} h = 2$, and $\operatorname{rcat} h = 3$.

Now $H^*(\mathbb{H}P^3; \mathbb{Z}) \cong \mathbb{Z}[u]/(u^4)$ with degree $u = 4$. Indeed, if $\tilde{u}\colon \mathbb{H}P^3 \to K(\mathbb{Z}, 4)$ represents u (the latter being an Eilenberg-Mac Lane space), then $S^4 \xrightarrow{g} \mathbb{H}P^3 \xrightarrow{\tilde{u}} K(\mathbb{Z}, 4)$ also represents α_4, the generator of $H^4(S^4; \mathbb{Z})$, since we may choose u so that $g^*(u) = \alpha_4$. Moreover, estimates from Theorem 1.11 of [Be-Ga] establish $\operatorname{cat}(S^2 \times S^2) = 2$, $\operatorname{cat} S^4 = 1$, and $\operatorname{cat} \mathbb{H}P^3 = 3$. Clearly,

$$\begin{aligned} \operatorname{cat} h &= \operatorname{cat}(g \circ f) \leq \min\{\operatorname{cat} f, \operatorname{cat} g\} \\ &\leq \min\{\operatorname{cat}(S^2 \times S^2), \operatorname{cat} S^4, \operatorname{cat} \mathbb{H}P^3\} = 1. \end{aligned}$$

If $\operatorname{cat} h = 0$ then h is homotopic to a constant and so $h^*u = 0$. But $h^*u = f^*g^*u = f^*\alpha_4 = \alpha_2 \times \alpha_2' \neq 0$, and so $\operatorname{cat} h = 1$.

Let $S^2 \times S^2 \xrightarrow{s_1} S^2 \times S^2$ and $\mathbb{H}P^3 \xrightarrow{s_2} HP^3$ be such that $h \circ s_1 \simeq h$, $s_2 \circ h \simeq h$, $\operatorname{cat} s_1 = \operatorname{dcat} h$, and $\operatorname{cat} s_2 = \operatorname{rcat} h$. Then $\operatorname{cat} s_1 \leq 2$ and $\operatorname{cat} s_2 \leq 3$.

Suppose that $\operatorname{dcat} h = \operatorname{cat} s_1 = 1$. Then

$$s_1^*(\alpha_2 \times \alpha_2') = s_1^*[(\alpha_2 \times 1) \cup (1 \times \alpha_2')] = s_1^*(\alpha_2 \times 1) \cup s_1^*(1 \times \alpha_2') = 0,$$

since the cup-length in $s_1^* \widetilde{H}(S^2 \times S^2; \mathbb{Z})$ is at most $\operatorname{cat} s_1 = 1$. But $s_1^*(\alpha_2 \times \alpha_2') = s_1^* h^* u = h^* u = \alpha_2 \times \alpha_2'$. Hence $\operatorname{cat} s_1 \geq 2$ and so $\operatorname{dcat} h = \operatorname{cat} s_1 = 2$.

Similarly, suppose $\operatorname{rcat} h = \operatorname{cat} s_2 \leq 2$. Then, as $s_2^* u = \lambda u$ for some $\lambda \in \mathbb{Z}$, $s_2^*(u^3) = \lambda^3 u^3 = (s_2^* u)^3 = 0$ by a cup-length bound. Thus $\lambda = 0$ and so $s_2^* u = 0$. But this contradicts $0 \neq \alpha_2 \times \alpha_2' = h^* u = h^* s_2^* u$ and so we must have $\operatorname{rcat} h = 3$.

Example 2.6. Consider the complex projective space $\mathbb{C}P^n$. Computations as in Example 2.5 show that if $\alpha \in \pi_2(\mathbb{C}P^n)$ is the class of a generator, then $\operatorname{rcat} \alpha = n$. This is in contrast to the fact that $\operatorname{cat} \alpha = 1$. Similarly, if u is a generator of $H^2(\mathbb{C}P^n, \mathbb{Z})$, straightforward calculations show that $\operatorname{dcat}(u^k) = n$ for any k, $1 \leq k \leq n$. However, $\operatorname{cat}(u^k) < n$ for any $k > 1$, and in particular, $\operatorname{cat}(u^n) = 1$.

Remarks 2.7. These examples show that standard bounds of the form $\operatorname{cat} f \leq (\dim X)/p$ for $X \xrightarrow{f} Y$ (where Y is $p-1$ connected, $p \geq 2$) are not valid for dcat or rcat. In Example 2.6, $\mathbb{H}P^3$ is 3 connected but $\operatorname{dcat} h$ and $\operatorname{rcat} h$ both exceed $(\dim S^2 \times S^2)/p = 4/4 = 1$.

The following results show that when applied to cohomology and homotopy classes of a space, dcat and rcat can provide excellent lower bounds for its L-S category.

Theorem 2.8. *Suppose M is a compact, simply connected n-dimensional manifold and let $\Omega \in H^n(M, \mathbb{Z})$ be the fundamental class. Then* $\operatorname{dcat} \Omega = \operatorname{cat} M$.

Proof. Let $M \xrightarrow{s} M$ be such that $\operatorname{cat} s = \operatorname{dcat} \Omega$ and $s^* \Omega = \Omega$. Since M satisfies Poincaré duality, let $D\colon H^p(M, \mathbb{Z}) \to H_{n-p}(M, \mathbb{Z})$ denote the isomorphism induced by capping with the fundamental homology class $\theta = D(1)$. Since $H_n(M, \mathbb{Z}) = \mathbb{Z}\langle\theta\rangle$ and $s^*\Omega = \Omega$, if s_* denotes the induced map on homology, we have $s_* D s^* = D$. As D is an isomorphism, s_* is surjective. But $H_*(M, \mathbb{Z})$ is a finitely generated abelian group so s_* is an isomorphism. By the Whitehead theorem, s is a homotopy equivalence. But if s' is a homotopy inverse, $\operatorname{cat} M = \operatorname{cat} \mathrm{id} = \operatorname{cat}(s' \circ s) \leq \operatorname{cat} s$ so that $\operatorname{cat} s = \operatorname{cat} M$. This proves $\operatorname{dcat} \Omega = \operatorname{cat} M$.

When the structure of a space X depends on more than a single class, we still have

Theorem 2.9. *Let k denote $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ with p prime. Let $\{u_i\}$ be a collection of homogeneous generators of $H^*(X, k)$ as a k-algebra. If $\tilde{u}_i\colon X \to K(k, |u_i|)$ represents u_i and $u\colon X \to \prod_i K(k, |u_i|)$ is $u = (\tilde{u}_1, \tilde{u}_2, \ldots)$, then* $\operatorname{dcat} u \geq \operatorname{cat} X_p$, *where X_p is the localization of X at p, or the rationalization if $k = \mathbb{Q}$.*

Proof. Suppose $X \xrightarrow{s} X$ is such that $\operatorname{dcat} u = \operatorname{cat} s$ and $u \circ s \simeq u$. Then $s^* \tilde{u}_i = \tilde{u}_i$ and so s^* is an isomorphism. Then the localized map $s_p\colon X_p \to X_p$ is a homotopy equivalence, so $\operatorname{cat} s_p = \operatorname{cat} X_p$. But $s_p = s \circ l_p$ where l_p is the localization map $X \to X_p$, and so $\operatorname{cat} s_p \leq \operatorname{cat} s$. Hence $\operatorname{dcat} u \geq \operatorname{cat} X_p$.

An analogous result holds for rcat and homotopy.

Theorem 2.10. *Let $\alpha_i \in \pi_{n_i}(X)$ generate $\pi_*(X)$ and let $A\colon \bigvee_i S^{n_i} \to X$ be the map which is α_i on each summand. Then* $\operatorname{rcat} A = \operatorname{cat} X$.

Proof. Any map $X \xrightarrow{s} X$ which commutes with A induces an isomorphism on homotopy groups and so is a homotopy equivalence. This implies that $\operatorname{dcat} A = \operatorname{cat} X$.

3. Rational category for maps

In [Fe-Ha], Felix and Halperin used Sullivan's minimal models to provide a useful algebraic characterization of $\operatorname{cat} X$ when X is a rational space. We will define the rational domain and range categories of a map using their methods.

For the homotopy theory of commutative graded differential algebras (CGDAs), we refer the reader to an excellent summary in [Fe-Ha] and for complete details to [Ha1] or [B-G]. For our purposes, we recall the following.

Sullivan defined a functor A which associates to each space X the CGDA over $\mathbb{Q}$, $(A(X), d)$, which consists of the compatible rational differential forms on the singular simplices of X. This CGDA computes the rational cohomology of X: $H(A(X), d) \cong H(X; \mathbb{Q})$. Any continuous map $X \xrightarrow{f} Y$ gives a CGDA morphism $(A(X), d) \xleftarrow{A(f)} (A(Y), d)$. If $(A, d) \xrightarrow{\varphi} (B, d)$ is a morphism of CGDAs with $H^0(A) = H^0(B) = \mathbb{Q}$, then *a Sullivan minimal model of* φ is a diagram

$$\begin{array}{ccc} (A, d) & \xrightarrow{\varphi} & (B, d) \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \psi \simeq} & \\ (A \otimes \Lambda W, d) & & \end{array}$$

where the $\simeq$ indicates that ψ is an isomorphism on cohomology and $(A, d) \xrightarrow{i} (A \otimes \Lambda W, d)$ is a KS extension as defined in [Ha1]. In particular, $i(a) = a \otimes 1$ and the graded vector space W has a well ordered, homogeneous basis $\{w_\alpha | \alpha \in I\}$ such that $dw_\alpha \in A \otimes \Lambda W_{<\alpha}$ and $\alpha < \beta \Rightarrow \deg w_\alpha \le \deg w_\beta$. Here, ΛW is the free CGA on W and $W_{<\alpha}$ denotes $\operatorname{span}\{w_\beta | \beta < \alpha\}$. The diagram is determined up to isomorphism by φ and we say that i *represents* φ.

If X is a space, the Sullivan minimal model of $(\mathbb{Q}, 0) \to (A(X), d)$ (obtained from a basepoint) is of the form $(\mathbb{Q}, 0) \to (\Lambda W, d) \xrightarrow{\simeq} (A(X), d)$, and $(\Lambda W, d)$ is called the *Sullivan minimal model of* X. If $X \xrightarrow{f} Y$ is a map, then a standard lifting lemma [Ha1, Theorem 5.19] applied to $A(f)$ gives a unique homotopy class of morphisms $(\Lambda W_Y, d) \xrightarrow{\varphi} (\Lambda W_X, d)$ between minimal models, any of which is called a *Sullivan representative of* f.

Sullivan proved that $(\Lambda W_X, d)$ carries the rational homotopy type of X (i.e., the homotopy type of $X_{\mathbb{Q}}$). In particular, besides computing the rational cohomology of X, we have, as graded vector spaces, $W_X^k \cong \operatorname{Hom}_{\mathbb{Z}}(\pi_k(X), \mathbb{Q})$. Moreover, every morphism between Sullivan minimal models of spaces induces a unique homotopy class of maps between their localizations at $\mathbb{Q}$, the Sullivan representative of any such may being homotopic to the original morphism.

Now let m be a positive integer, $(\Lambda W, d)$ be a Sullivan model of a space X and $E_m X \to X$ denote the mth Ganea fibration. Felix and Halperin showed

[Fe-Ha, Proposition 2.7 and Theorem 3.1] that one has a commutative diagram

(3.1)
$$\begin{tikzcd}
(\Lambda W/\Lambda^{>m}W, d) \arrow[r, "\pi"] & (\Lambda W, d) \arrow[d, "j"] \arrow[r, "\pi"] & (\Lambda W/\Lambda^{>m}W, d) \\
(\Lambda W \otimes \Lambda V, d) \arrow[u, "\simeq"] \arrow[r, "\xi"] & (\Lambda W \otimes \Lambda U, d) \arrow[ul, "i"] \arrow[ur, "i"] \arrow[r, "\eta"] & (\Lambda W \otimes \Lambda V, d) \arrow[u, "\simeq"']
\end{tikzcd}$$

in which i represents the projection π and j is a Sullivan representative of $E_m X \to X$. As in [Fe-Ha], define $\operatorname{cat}_0 X$ to be the least integer m such that there is a morphism of CGDAs $\Lambda W \otimes \Lambda V \xrightarrow{r} \Lambda W$ satisfying $r \circ i \simeq \mathrm{id}_{\Lambda W}$. ($R$ is called a *retract.*) The above establishes one of their main results, namely, that $\operatorname{cat}_0 X = \operatorname{cat} X_{\mathbf{Q}}$.

In this spirit, let $X \xrightarrow{f} Y$ be a map of spaces and suppose

$$(\Lambda T, d) \xrightarrow{\varphi} (\Lambda S, d)$$

is a Sullivan representative of f. Consider the diagram

(3.2)
$$\begin{tikzcd}
(\Lambda T/\Lambda^{>m}T, d) & (\Lambda T, d) \arrow[l, "\pi"'] \arrow[d, "i"] \arrow[r, "\varphi"] & (\Lambda S, d) \arrow[d, "j"] \arrow[r, "\pi"] & (\Lambda S/\Lambda^{>m}S, d) \\
 & (\Lambda T \otimes \Lambda U, d) \arrow[ul, "\simeq"] & (\Lambda S \otimes \Lambda Z, d) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

where i and j are Sullivan representatives of Ganea projections.

Definition 3.3. (i) $\operatorname{cat}_0 f$ is the least m such that a CGDA morphism

$$(\Lambda T \otimes \Lambda U, d) \xrightarrow{r} (\Lambda S, d)$$

exists satisfying $r \circ i \simeq \varphi$.

(ii) $\operatorname{dcat}_0 f$ is the least m such that a CGDA morphism

$$(\Lambda S \otimes \Lambda Z, d) \xrightarrow{r} (\Lambda S, d)$$

exists satisfying $r \circ j \circ \varphi \simeq \varphi$.

(iii) $\operatorname{rcat}_0 f$ is the least m such that a CGDA map

$$(\Lambda T \otimes \Lambda U, d) \xrightarrow{r} (\Lambda T, d)$$

exists satisfying $\varphi \circ r \circ i \simeq \varphi$.

Remark. If $(\Lambda T, d) \xrightarrow{\varphi} (\Lambda S, d)$ is a CGDA morphism then $\operatorname{cat}_0 \varphi$, $\operatorname{dcat}_0 \varphi$, and $\operatorname{rcat}_0 \varphi$ can also be defined by the above.

Now, if $X_{\mathbf{Q}}: f_{\mathbf{Q}} \to X_{\mathbf{Q}}$, is a localization of $X \xrightarrow{f} Y$, we have, (referring to (3.2) when necessary)

Proposition 3.4.

(i) $\operatorname{cat}_0 f = \operatorname{cat} f_{\mathbf{Q}} \le \operatorname{cat} f$.
(ii) $\operatorname{dcat}_0 f = \operatorname{dcat} f_{\mathbf{Q}} \le \operatorname{dcat} f$.
(iii) $\operatorname{rcat}_0 f = \operatorname{rcat} f_{\mathbf{Q}} \le \operatorname{rcat} f$.
(iv) $\operatorname{cat}_0 f \le \operatorname{dcat}_0 f$ *and* $\operatorname{cat}_0 f \le \operatorname{rcat}_0 f$.
(v) $\operatorname{dcat}_0 f = \min\{\operatorname{cat}_0 s \mid (\Lambda T, d) \xrightarrow{s} (\Lambda T, d)$ *satisfies* $s \circ \varphi \simeq \varphi\}$.
(vi) $\operatorname{rcat}_0 f = \min\{\operatorname{cat}_0 s \mid (\Lambda S, d) \xrightarrow{s} (\Lambda S, d)$ *satisfies* $\varphi \circ s \simeq \varphi\}$.

Proof. (i) If we localize the diagram given by $\operatorname{cat} f \le m$ at $\mathbb{Q}$, and note that $(E_m Y)_{\mathbb{Q}} \simeq E_m Y_{\mathbb{Q}}$, we obtain one establishing $\operatorname{cat} f_{\mathbb{Q}} \le m$. Then, any Sullivan representative of $X_{\mathbb{Q}} \xrightarrow{\beta} E_m Y_{\mathbb{Q}}$ (from $\operatorname{cat} f_{\mathbb{Q}} \le m$), precomposed with the map ξ of the Y-version of diagram (3.1), gives a morphism showing that $\operatorname{cat}_0 f \le m$. This establishes $\operatorname{cat}_0 f \le \operatorname{cat} f_{\mathbb{Q}} \le \operatorname{cat} f$. Conversely, a morphism guaranteeing that $\operatorname{cat}_0 f \le m$, precomposed with the map η of diagram (3.1) (for Y), gives a lift of $f_{\mathbb{Q}}$ to $E_m Y_{\mathbb{Q}}$ by Sullivan and so we have $\operatorname{cat} f_{\mathbb{Q}} \le m$. This shows that $\operatorname{cat}_0 f = \operatorname{cat} f_{\mathbb{Q}}$.

The proofs of (ii) and (iii) are similar, and parts (iv), (v), and (vi) are clear from the definitions.

Remarks. If Y is rational (so that $Y \simeq Y_{\mathbb{Q}}$), then for any map $X \xrightarrow{f} Y$, we see that the $\operatorname{cat} f = \operatorname{cat}_0 f$. In particular, if $u \in [X, K(\mathbb{Q}, n)]$ represents a rational cohomology class, we have $\operatorname{cat} u = \operatorname{cat}_0 u$. Using this fact, we easily recover the characterization $\operatorname{cat} u = \min\{n | u^{n+1} = 0\}$ of Berstein and Ganea for rational cohomology classes.

We now investigate some properties of $\operatorname{dcat}_0 u$ when u is a cohomology class of $H^*(X;\mathbb{Q})$. Suppose (E_i^p, d_i) is the Milnor-Moore spectral sequence for X over $\mathbb{Q}$. Then, after Toomer [To], define

$$e_0 u \equiv \max\{p | u \text{ can be represented in } E_\infty^p\}.$$

Toomer's invariant for X, $e_0 X$, is then just the maximum value of $e_0 u$ for u in $H^*(X;\mathbb{Q})$. If $(\Lambda W, d)$ is a Sullivan minimal model for X, we can compute $e_0 u$ easily. Indeed, by [Fe-Ha, Proposition 9.1], $e_0 u \equiv \max\{p | u$ has a representative in $\Lambda^{\ge p} W\}$.

Toomer established the inequality $e_0 X \le \operatorname{cat}_0 X$. On the other hand, it is not always true that $e_0 u \le \operatorname{cat}_0 u = \operatorname{cat} u$. Consider the generator $u_5 \in H^5(S^2 \times S^3;\mathbb{Q})$. Since $u_5 = (u_2 \times 1) \cup (1 \times u_3)$ where u_2 and u_3 are generators of $H^2(S^2, \mathbb{Q})$ and $H^3(S^3, \mathbb{Q})$, we see that $e_0 u_5 \ge 2$. Moreover, since $\operatorname{cat}(S^2 \times S^2) = 2$, this means that $e_0 u_5 = 2$. But, as $u_5^2 = 0$, we have $\operatorname{cat} u = 1$. However, the rational domain category of a cohomology class does satisfy such an inequality.

Theorem 3.5. *If* $u \in H^n(X;\mathbb{Q})$, *then* $e_0 u \le \operatorname{dcat}_0 u$.

Proof. Suppose $\operatorname{dcat}_0 u \le m$. Then we have a diagram

$$\begin{tikzcd}
(\Lambda W, d) \arrow[d, "i"] & & (\Lambda \nu, 0) \arrow[ll, "\varphi"'] \\
(\Lambda W \otimes \Lambda V, d) \arrow[u, bend left, "r"] \arrow[rr, "\theta", "\simeq"'] & & (\Lambda W / \Lambda^{>m} W, d),
\end{tikzcd}$$

with $r \circ i \circ \varphi \simeq \varphi$, and $\varphi(\nu)$ is some cocycle in ΛW with $[\varphi(\nu)] = u$. If $e_0 u > m$, we may choose $\varphi(\nu) \in \Lambda^{\ge m+1} W$. Now, $u = \varphi^*[\nu] = r^* i^* u$. But $\theta i \varphi(\nu) = 0$, so $\theta^* i^* u = 0$; however, θ^* is an isomorphism so we conclude $i^* u = 0$. This contradicts $u = r^* i^* u$ unless u is zero, in which case $e_0 u = 0$. In either case, $e_0 u \le m$.

Remark. It is possible to have $e_0 u < \operatorname{dcat}_0 u$. Let $L = (\mathbb{C}P^2 \vee S^2) \cup_\omega e^7$ (where $\omega = [\alpha, \beta]$ for $\alpha \in \pi_5(\mathbb{C}P^2)$, $\beta \in \pi_2(S^2)$ be Lemaire and Sigrist's space [Le]. A straightforward computation, using for example the model for L in [Fe-Ha], shows that if $\Omega \in H^7(L)$ is a generator, then $e_0\Omega = 2$ but $\operatorname{dcat}_0 \Omega = \operatorname{dcat} \Omega = 3$.

A space X is *rationally elliptic* if both $\operatorname{cat}_0 X$ and $\dim(\pi_*(X) \otimes \mathbb{Q})$ are finite. For these spaces, Halperin [Ha2] has shown that $H^*(X; \mathbb{Q})$ is a Poincaré duality algebra. We have the

Theorem 3.6. *Suppose $H^*(X; \mathbb{Q})$ satisfies Poincaré duality, and let $\Omega \in H^n(X; \mathbb{Q})$ be the fundamental class. Then* $\operatorname{dcat}_0 \Omega = \operatorname{cat}_0 X$.

Proof. Any map $X \xrightarrow{s} X$ which satisfies $s^*\Omega = \Omega$ is and isomorphism on rational cohomology, and hence $\operatorname{cat}_0 s = \operatorname{cat}_0 X$. This shows that $\operatorname{dcat}_0 \Omega = \operatorname{cat}_0 X$.

Remark. Whether or not we can always attain $\operatorname{cat}_0 X$ with a single cohomology class in the case when $\dim(\pi_*(X) \otimes \mathbb{Q}) = \infty$ is open. However, in all cases checked by the author, it has been possible. (For example, Lemaire's space L is not rationally elliptic but $\operatorname{dcat}_0 \Omega = \operatorname{cat}_0 L = \operatorname{cat} L = 3$.)

If $u^2 = 0$ for a nonzero rational cohomology class, then $\operatorname{cat}_0 u = 1$. In the following, we show that this remains true for dcat_0 if and only if u is a *spherical* class. By this we mean that there is $S^n \xrightarrow{\gamma} X$ so that $S^n \xrightarrow{\gamma} X \xrightarrow{u} K(\mathbb{Q}, n)$ represents a generator in $H^n(S^n; \mathbb{Q})$. Equivalently, $u \in H^*(X; \mathbb{Q})$ is spherical if $e_0 u = 1$. First, we prove

Lemma 3.7. *Suppose $(\Lambda V, d) \xrightarrow{i} (\Lambda V \otimes \Lambda W, d)$ is a KS extension such that there is $k \geq 1$ with* (a) *$e_0(\Lambda V, d) \leq k$ and* (b) *$d\colon W \to \Lambda^{\geq k+1}(V \oplus W)$. Then* $\operatorname{cat}_0 i = \operatorname{dcat}_0 i \leq \operatorname{cat}_0(\Lambda V, d) \leq \operatorname{cat}_0(\Lambda V \otimes \Lambda W)$.

Proof. We construct $(\Lambda V \otimes \Lambda W, d) \xrightarrow{\varphi} (\Lambda V, d)$ extending the identity on $(\Lambda V, d)$ as follows. Let $\{w_\alpha | \alpha \in I\}$ be a KS basis for W. Then if 0 is the first element of I, we have $dw_0 \in \Lambda^{\geq k+1} V$. But since $e_0(\Lambda V, d) \leq k$, dw_0 is already a boundary in ΛV, say $dw_0 = db$ for $b \in \Lambda V$. Define $\varphi w_0 = b$. Then φ commutes with the differential. Now assume φ has been defined on $\Lambda V \otimes \Lambda W_{<\alpha}$. To define $\varphi(w_\alpha)$ note that $dw_\alpha \in \Lambda^{\geq k+1}(V \oplus W)$ and so $\varphi(dw_\alpha) \in \Lambda^{\geq k+1} V$, which allows us to define $\varphi(w_\alpha)$ so as to satisfy $d\varphi(w_\alpha) = \varphi(dw_\alpha)$. Now define $\Lambda V \otimes \Lambda W \xrightarrow{s} \Lambda V \otimes \Lambda W$ by $s = i \circ \varphi$. Then $s \circ i = i \circ \varphi \circ i = i$ since φ extends $\mathrm{id}_{\Lambda V}$. This shows that $\operatorname{dcat}_0 i \leq \operatorname{cat}_0 s \leq \min\{\operatorname{cat}_0 i, \operatorname{cat}_0 \varphi\} \leq \operatorname{cat}_0(\Lambda V, d)$. Moreover, since $\mathrm{id}_{\Lambda V} = \varphi \circ i$, $\operatorname{cat}_0(\mathrm{id}_{\Lambda V}) = \operatorname{cat}_0(\Lambda V, d) \leq \operatorname{cat}_0 \varphi \leq \operatorname{cat}_0(\Lambda V \otimes \Lambda W)$.

Now the promised

Theorem 3.8. *Suppose $u \in H^n(X; \mathbb{Q})$ is nonzero. Then* $\operatorname{dcat}_0 u = 1 \Leftrightarrow u$ *is spherical $u^2 = 0$.*

Proof. If $\operatorname{dcat}_0 u = 1$, then $\operatorname{cat}_0 u = 1$ and so $u^2 = 0$. Moreover, by Theorem 3.5, $e_0 u = 1$. Thus, u is spherical.

Now suppose u is spherical and $u^2 = 1$. Let $(\Lambda \nu, 0) \xrightarrow{\varphi} (\Lambda W, d)$ be a Sullivan representative for u with $\deg \nu = n$ and $[\varphi \nu] = u$. As u is spherical, $\varphi \nu = \omega + b$ with $0 \neq \omega \in W$ and $b \in \Lambda^{\geq 2} W$. Consider the case when n is even: since $u^2 = 0$, there is $c \in \Lambda W$ with $dc = (\omega + b)^2$; in particular $c = c_1 + c_2$ with $0 \neq c_1 \in W$ and $c_2 \in \Lambda^{\geq 2} W$ since $d\colon W \to \Lambda^{\geq 2} W$. Define a CGDA $(\Lambda(\nu, x), d)$ by $d\nu = 0$ and $dx = \nu^2$ (this is the Sullivan model of S^n if n is even) and extend φ to a map Φ: $(\Lambda(\nu, x), d) \to (\Lambda W, d)$ by setting $\Phi(x) = c$. Now let

$$\begin{tikzcd} \Lambda(\nu, x) \arrow[r, "\Phi'"] \arrow[d, "\Phi"'] & (\Lambda(\nu, x) \otimes \Lambda U, d) \arrow[dl, "\simeq\theta"] \\ (\Lambda W, d) & \end{tikzcd}$$

be a Sullivan minimal model for Φ. Since Φ is injective on homotopy, $d\colon U \to \Lambda^{\geq 2}(\langle \nu, x\rangle \oplus U)$. Apply Lemma 3.7 with $k = 1$ to Φ'. This shows that $\mathrm{dcat}_0 \Phi' = 1$. But since $(\Lambda(\nu, x) \otimes \Lambda U, d)$ is also minimal, the map θ above is actually an isomorphism, so $\mathrm{dcat}_0 \Phi = \mathrm{dcat}_0 \Phi' = 1$. But φ factors as $\Phi \circ j$ where $(\Lambda(\nu, 0)) \xrightarrow{j} (\Lambda(\nu, x), d)$. Thus $\mathrm{dcat}_0 \varphi \leq \mathrm{dcat}_0 \Phi = 1$. Thus $\mathrm{dcat}_0 u = 1$. The case n odd is handled similarly, but we do not need x, as $e_0(\Lambda(\nu, 0)) = 1$ in that case.

We may generalize this to

Theorem 3.9. *Suppose $f\colon X \to \bigvee_\alpha S^{n_\alpha}$ induces a surjective map on rational homotopy groups. Then $\mathrm{dcat}_0 f \leq 1$.*

Proof. Let $(\Lambda W, d) \xleftarrow{\varphi} (\Lambda Z, d)$ be a Sullivan representative of f. Now suppose

$$\begin{tikzcd} \Lambda(Z, d) \arrow[r, "i"] \arrow[d, "\varphi"'] & (\Lambda Z \otimes \Lambda Y, d) \arrow[dl, "\simeq"] \\ (\Lambda W, d) & \end{tikzcd}$$

is a Sullivan minimal model of φ. Since $\pi_* f \otimes \mathbb{Q}$ is onto, the derivative in $(\Lambda Z \otimes \Lambda Y, d)$ satisfies $d\colon Y \to \Lambda^{\geq 2}(Z \oplus Y)$. Moreover, $\mathrm{cat}_0(\bigvee_\alpha S^{n_\alpha}) = e_0(\bigvee_\alpha S^{n_\alpha}) = 1$, so we may apply Lemma 3.7 with $k = 1$ to obtain $\mathrm{dcat}_0 i \leq 1$. But i is also a Sullivan representative of f, because $(\Lambda Z \otimes \Lambda Y, d)$ is minimal, thus we may conclude that $\mathrm{dcat}_0 f \leq 1$.

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