TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 339, Number 1, September 1993

THE LIMITING BEHAVIOR OF THE KOBAYASHI-ROYDEN PSEUDOMETRIC

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ABSTRACT. We study the limit of the sequence of Kobayashi metrics of Riemann surfaces (when these Riemann surfaces form an analytic fibration in such a way that the total space of fibration becomes a complex surface), as the fibers approach the center fiber which is not in general smooth. We prove that if the total space is a Stein surface and the smooth part of the center fiber contains a component biholomorphic to a quotient of the disk by a Fuchsian group of first kind, then the Kobayashi metrics of the near-by fibers converge to the Kobayashi metric of this component as fibers tend to the center fiber.

INTRODUCTION

Let $\Phi\colon M \to \Delta$ be a holomorphic mapping from a complex surface M on the disc $\Delta = \{z \in C \mid |z| < 1\}$. Suppose that for each $c \neq 0$ $\Gamma_c = \Phi^{-1}(c)$ is a smooth noncompact Riemann surface and Γ_0^* is a smooth part of $\Gamma_0 = \Phi^{-1}(0)$. We shall investigate relations between the Kobayashi-Royden pseudometric $k_{\Gamma_0^*}$ on Γ_0^* and the limit of the Kobayashi-Royden pseudometric on nearby fibers. More precisely, we shall study the problem when the equality

$$\lim_{c\to 0} k_{\Gamma_c} = k_{\Gamma_0^*} \tag{1}$$

holds. In general, it is not so. In [PS, §2.2] there is an example of such mapping $\Phi\colon M \to \Delta$, where M is a holomorphically convex region in C^2, every Γ_c is a disc, but $\lim_{c\to 0} k_{\Gamma_c} \neq k_{\Gamma_0}$. Zaidenberg found certain sufficient conditions, which imply (1) [Z]. But his result does not give the answer to the question whether (1) holds, when Φ is a polynomial of two complex variables and $M = \Phi^{-1}(\Delta)$. He supposed that the answer was positive. Let G be a Fuchsian group of the first kind. The Main Theorem of this paper says that, if M is a Stein surface and Γ_0^* contains a component R, which is biholomorphically equivalent to Δ/G, then $\lim_{c\to 0} k_{\Gamma_c} = k_R$. In particular, the Zaidenberg's conjecture is true. The last fact was announced in [Ka], where it was used to classify isotrivial polynomials on C^2.

The paper is organized as follows. We present some terminology and formulate our main results in the first section. The second section contains a technical lemma about Fuchsian groups and its corollaries needed for the proof of the Main Theorem. This lemma asserts that two noncommutative nonelliptic elements of a Fuchsian group cannot move any point $z \in \Delta$ by a distance less than

Received by the editors December 13, 1990 and, in revised form, June 21, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32H15.

a certain $\varepsilon > 0$ at the same time. Next we handle the case of hyperbolic fibers $\{\Gamma_{b_j}\}$ with $b_j \to 0$. We consider universal holomorphic covering $f_j: \Delta \to \Gamma_{b_j}$ and find out when $\{f_j\}$ converge to an unramified mapping $f: D \to \Gamma_0^*$ on a certain maximal region $D \subset \Delta$. We also prove that $f(D)$ is a component of Γ_0^* and, if $D = \Delta$, then the Kobayashi-Royden pseudometric on $f(D)$ coincides with the limit of the Kobayashi-Royden pseudometric of nearby fibers. The result of the forth section says that D is simply connected in the case when M is a Stein surface. The last section contains the proof of the Main Theorem.

It is a great pleasure to thank M. G. Zaidenberg for his question and stimulating discussions.

1. Formulation of the main theorem

First we fix terminology, notations and definitions that we shall use throughout the paper. Every manifold we are going to consider will be complex. If Y is a manifold, then TY is its holomorphic tangent bundle and T_yY is a tangent space at a point $y \in Y$. Put $\Delta_r = \{z \in C \mid |z| < r\}$, $\Delta = \Delta_1$, and $\Delta^* = \Delta - 0$. By a curve η in Y we mean a continuous mapping $\eta : [0, 1] \to Y$. A loop γ in Y is a curve with $\gamma(0) = \gamma(1)$. In other words, γ is a continuous mapping from $\partial\Delta$ to Y (as is frequently done, we use the symbol ∂ to denote boundaries). If $x \in \gamma(\partial\Delta)$, then we write $x \in \gamma$. Recall that a differential pseudometric on a complex manifold Y is a nonnegative homogeneous function on the tangent bundle TY, i.e., it is a function $p : TY \to \mathbf{R}$ such that $p(y, v) \geq 0$, $p(y, \lambda v) = |\lambda| p(y, v)$ for all $y \in Y$, $v \in T_yY$, and $\lambda \in C$. When p is continuous, we call the pseudometric continuous. If Y is connected and for each piecewise smooth curve η in Y there exists the integral $P(\eta) = \int_0^1 p(\eta(t), d\eta(t))\, dt$, one can define the integral pseudometric $P(x, y) = \inf_\eta\{P(\eta) | \eta(0) = x,\ \eta(1) = y\}$. Of course, the integral pseudometric exists, when a proper differential pseudometric is continuous. The Kobayashi-Royden differential pseudometric is given by the formula

$$k_Y(y, v) = \inf_r\{1/r \mid \phi \in \operatorname{Hol}(\Delta_r, Y), \phi(0) = y,\ d\phi(0) = v\}.$$

By Royden's theorem [R] it generates the integral pseudometric K_Y which coincides with the Kobayashi pseudometric on Y [Ko].

Throughout the paper $\Phi: M \to \Delta$ is a holomorphic mapping from a smooth complex surface M on Δ such that for $c \neq 0$ $\Gamma_c = \Phi^{-1}(c)$ is a smooth Riemann surface. We shall say that $\Phi: M \to \Delta$ is a family of Riemann surfaces. The fiber $\Gamma_0 = \Phi^{-1}(0)$ can contain singular points. Denote the smooth part of Γ_0 by Γ_0^*. Let $\beta = \{b_j\} \subset \Delta^*$ be a sequence that tends to zero, let R be a component of Γ_0^*. We say that $\lim_{j\to\infty} k_{\Gamma_{b_j}} = k_R$ (or $\overline{\lim}_{j\to\infty} k_{\Gamma_{b_j}} \leq k_R$), if for each sequence $\{w_j \in T\Gamma_{b_j}\}$ that converges to $w \in TR$ in the topology of TM the equality $\lim_{j\to\infty} k_{\Gamma_{b_j}}(w_j) = k_R(w)$ (or inequality $\overline{\lim}_{j\to\infty} k_{\Gamma_{b_j}}(w_j) \leq k_R(w)$) holds. If $\lim_{j\to\infty} k_{\Gamma_{b_j}} = k_R$ for each sequence β as above, then we say $\lim_{c\to 0} k_{\Gamma_c} = k_R$. In the same meaning $\overline{\lim}_{c\to\infty} k_{\Gamma_c} \leq k_R$. The following two results belong to Zaidenberg [Z].

Proposition 1.1. *For each component R of Γ_0^* the inequality $\overline{\lim}_{c\to\infty} k_{\Gamma_c} \leq k_R$ holds.*

Theorem 1.2. *Let $\overline{M}$ be a smooth compact surface and $\overline{\Gamma} \subset \overline{M}$ be an analytic curve in $\overline{M}$. Suppose that $M \subset \overline{M} - \overline{\Gamma}$, $\overline{\Gamma}_0 = \overline{\bigcap_{r>0} \Phi^{-1}(\Delta_r)}$, and $\Gamma_0 = \overline{\Gamma}_0 - \overline{\Gamma}$. If every component of Γ_0^* is hyperbolic, then $\lim_{c\to 0} k_{\Gamma_c} = k_R$.*

Zaidenberg conjectured that, if Φ is a polynomial on C^2 and $M = \Phi^{-1}(\Delta)$, then the assumption that all the components of Γ_0^* are hyperbolic can be omitted. We shall show that this hypothesis is correct. Recall that G is a Fuchsian group of the first kind, if the closure of the orbit $\{g(0)|g \in G\}$ in C contains $\partial\Delta$ [B]. In particular, in the polynomial case every hyperbolic component R of Γ_0^* has a representation $R \cong \Delta/G$, where G is a Fuchsian group of the first kind.

Main Theorem. *Let Φ: $M \to \Delta$ be a family of Riemann surfaces. Suppose that M is a Stein manifold and Γ_0^* contains a component R that is biholomorphically equivalent to Δ/G, where G is a Fuchsian group of the first kind. Then $\lim_{c\to 0} k_{\Gamma_c} = k_R$.*

Note that, if R is nonhyperbolic, such a fact follows from Proposition 1.1. Hence we have

Corollary. *Let Φ: $C^2 \to C$ be a polynomial. Then $\lim_{c\to 0} k_{\Gamma_c} = k_{\Gamma_0^*}$.*

We shall restrict ourselves to the case of connected fibers for $c \neq 0$ (in general case the proof is the same, but instead of Γ_c we have to use their components).

2. One property of Fuchsian groups

We shall denote the Kobayashi metric on Δ by K_Δ.

Lemma 2.1. *For every $r > 0$ there exists $\varepsilon > 0$ such that for every Fuchsian group G, noncommutative elements a', $b' \in G$, and a point $z \in \Delta$ satisfying $0 < K_\Delta(z, a'(z)) < r$, either $K_\Delta(z, b'(z)) > \varepsilon$ or z is a fixed point of the mapping $b' : \Delta \to \Delta$.*

Proof. Assume, to reach a contradiction, that for a certain $r > 0$ and each $\varepsilon > 0$ there exists a Fuchsian group G_ε, noncommutative elements a'_ε, $b'_\varepsilon \in G_\varepsilon$, and a point $z_\varepsilon \in \Delta$ such that $0 < K_\Delta(z_\varepsilon, a'_\varepsilon(z_\varepsilon)) < r$ and $0 < K_\Delta(z_\varepsilon, b'_\varepsilon(z)) < \varepsilon$. We shall show that for a sufficiently small ε the group G_ε cannot be discontinuous. Without loss of generality, we set $z_\varepsilon = 0$. Let id be the identity element of G_ε. Since G_ε is a discontinuous group, one can find elements a_ε and b_ε satisfying

$$K_\Delta(0, b_\varepsilon(0)) = \min\{K_\Delta(0, g(0))|g \in G_\varepsilon,\ g(0) \neq 0\}, \tag{2.1}$$

$$K_\Delta(0, a_\varepsilon(0)) = \min\{K_\Delta(0, g(0))|g \in G_\varepsilon, g(0) \neq 0,\ [b_\varepsilon, g] \neq \mathrm{id}\}. \tag{2.2}$$

The mapping a_ε and b_ε can be represented in the form

$$a_\varepsilon(z) = e^{i\theta_\varepsilon}(z + \alpha_\varepsilon)/(1 + \overline{\alpha}_\varepsilon z), \qquad |\theta_\varepsilon| \in [0, \pi],$$
$$b_\varepsilon(z) = e^{i\tau_\varepsilon}(z + \beta_\varepsilon)/(1 + \overline{\beta}_\varepsilon z), \qquad |\tau_\varepsilon| \in [0, \pi].$$

We shall omit the index ε from now on, if it does not cause misunderstanding. Let us consider b as a function of two variables z and β. Expand b in power series of z, β, and $\overline{\beta}$. Then $b(z) = e^{i\tau}z + e^{i\tau}\beta$ up to the nonlinear terms. Hence for every natural m one can find a neighborhood of the origin in $C^2 = \{(z, \beta)\}$ so that for all $n = 1, 2, \ldots, m$,

$$b^n(z) = e^{in\tau} z + \sum_{l=1}^{n} e^{il\tau}\beta + O(|z|^2 + |\beta|^2)$$

in this neighborhood. Thus

$$b^n(0) = \beta \sum_{l=1}^{n} e^{ilr} + O(|\beta|^2) = \beta e^{in\tau}(e^{in\tau} - 1)/(e^{i\tau} - 1) + O(|\beta|^2).$$

It is easy to check that for each $\tau_0 \neq 2\pi k$ there is a neighborhood U of τ_0 and integer $n \geq 2$ so that for every $\tau \in U$,

$$|(e^{in\tau} - 1)/(e^{i\tau} - 1)| < 1. \tag{2.3}$$

Now one can see that $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed, the preceding assumption implies

$$0 < |\alpha| < \tilde{r}, \qquad 0 < |\beta| < \tilde{\varepsilon},$$

where $\tilde{\varepsilon} = (e^\varepsilon + 1)$ and $\tilde{r} = (e^r - 1)/(e^r + 1)$. Thus $\lim_{\varepsilon \to 0} |\beta_\varepsilon| = 0$. Assume $\overline{\lim}_{\varepsilon \to 0} |\tau_\varepsilon| > 1/m$. Then by (2.3) we can find $n \leq m$ with $|b^n(0)| < |b(0)|$. This contradicts (2.1). Thus $b_\varepsilon(z) \to z$ uniformly on compact subsets of Δ as $\varepsilon \to 0$. Let $\overline{\lim}_{\varepsilon \to 0} |\alpha_\varepsilon| = \alpha^0$. Since $|\alpha_\varepsilon| < \tilde{r}$, $a_\varepsilon \circ b_\varepsilon \circ a_\varepsilon^{-1}(z) \to z$ as $\varepsilon \to 0$. In particular, for any sufficiently small ε we have $|a_\varepsilon b_\varepsilon a_\varepsilon^{-1}(0)| < \alpha^0/2$. This implies either $\alpha^0 = 0$ or b and aba^{-1} are commutative. We shall prove that the last case does not hold. One can represent a and b as mappings of the upper half-plane. Then, if a is a hyperbolic element, we may put $a(z) = \lambda z$ with $\lambda > 0$ and if a is a parabolic element, we may put $a(z) = z + 1$ [A]. In both cases for any $b(z) = (pz + q)/(tz + s)$ with $p, q, t, s \in \mathbf{R}$ the direct computation shows that $[aba^{-1}, b] = \mathrm{id}$, iff $[a, b] = \mathrm{id}$. When a is an elliptic element, one may consider a as a mapping $a\colon \Delta \to \Delta$ given by the formula $a(z) = \lambda z$ with $\lambda^n = 1$ for a certain natural n. Again it is easy to show that $[aba^{-1}, b] = \mathrm{id}$, iff $[a, b] = \mathrm{id}$ for any Möbius transformation $b\colon \Delta \to \Delta$. But this a contradicts (2.2). Therefore $\lim_{\varepsilon \to 0} \alpha_\varepsilon = 0$. Same arguments as above show that $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$. Hence for any sufficiently small ε we have $|e^{i\theta_\varepsilon} - 1| + |e^{i\tau_\varepsilon} - 1| < 1/2$ and for an arbitrarily small α the following inequality holds

$$|b^{-1}a^{-1}ba(0)| \approx |e^{i\tau} - 1|\,|\alpha| + |e^{i\theta} - 1|\,|\beta| < |\alpha|/2 < |a(0)|$$

but $b^{-1}a^{-1}ba$ and b are not commutative, since $[a^{-1}, ba, b] \neq \mathrm{id}$. This is a contradiction. □

Corollary 2.2. *For every $r > 0$ there exists $\varepsilon > 0$ such that for every hyperbolic Riemann surface R, for every point $x \in R$, and for every couple of loops γ and μ that generate noncommutative elements of the fundamental group $\pi_1(R, x)$, the inequalities $K_R(\gamma) < \varepsilon$ and $K_R(\mu) < r$ do not hold simultaneously.*

The next three lemmas enable us to restate this corollary in a form which will be convenient for our following needs.

Lemma 2.3. *Let γ be a noncontractible loop on a Riemann surface R. Suppose that the corresponding element of the fundamental group $\pi_1(R)$ has a representation $[\gamma] = [\mu]^n$, where $[\mu] \in \pi_1(R)$ and the natural number $n \geq 2$. Then γ has points of self-intersection.*

Proof. Let H be the upper half-plane, and let $f: H \to R$ be a universal holomorphic covering. Then we can define the Möbius transformation $b: H \to H$ corresponding to $[\mu]$. If b is a hyperbolic transformation, one can choose f so that $b(z) = \lambda z$ with $\lambda > 0$ [A]. Let z_0 be a point in the inverse image of a point $x_0 \in \gamma$. Obviously, each curve in H that connects the points z_0 and $\lambda^n z_0$ contains points z' and z'' such that $z' = \lambda z''$. But this means that γ has the point of self-intersection $f(z')$. If b is parabolic, we may suppose that $b(z) = z + 1$. Again each curve that connects the points z_0 and $z_0 + n$ contains points z' and $z'' = z' + 1$. This implies the desired conclusion. □

Lemma 2.4. *Let γ and μ be disjoint noncontractible loops in a Riemann surface R. Suppose that neither γ nor μ has points of self-intersection. Then γ and μ are homotopically equivalent, iff there is a region $U \subset R$ such that $\partial U = \gamma \cup \mu$ and U is topologically an annulus.*

Proof. Let $x_1 \in \gamma$ and $y_1 \in \mu$. Choose a curve $\nu_1: [0, 1] \to R$ so that $\nu_1(0) = x_1$, $u_1(1) = y_1$, ν_1 has no points of self-intersection and ν_1 intersects $\gamma \cup \mu$ at the points x_1 and y_1 only. Choose an analogous curve ν_2 so that ν_2 connects points $x_2 \in \gamma$ and $y_2 \in \mu$, and ν_2 is sufficiently close to, but disjoint from ν_1. Then $\gamma - (x_1 \cup x_2)$ consists of two components γ_1 and γ_2, and γ_1 is small enough. In the same way $\mu - (y_1 \cup y_2) = \mu_1 \cup \mu_2$, and μ_1 is small. Then there exists an open disc $D \subset R$ with $\partial D = \nu_1 \cup \nu_2 \cup \gamma_1 \cup \mu_1$. One can construct the loop $\eta = \nu_1 \cup \mu_2 \cup \nu_2 \cup \gamma_2$. Since γ and μ are homotopically equivalent, η must be contractible. By our construction η has no points of self-intersection. This implies the existence of the disc $U \subset R$ with $\partial U = \eta$. If $U \supset D$, then $U - \overline{D}$ contains the two components U_1 and U_2. Each of them is a disc, $\partial U_1 = \gamma$ and $\partial U_2 = \mu$. This contradicts the assumption that γ and μ are noncontractible. Hence $U \cap D = \varnothing$. Obviously, $\overline{D} \cup \overline{U}$ is topologically a closed annulus and $\partial(\overline{U} \cup \overline{D}) = \gamma \cup \mu$. This completes the proof of the lemma.

Lemma 2.5. *Let γ and μ be noncontractible loops on a Riemann surface R, and neither γ nor μ has points of self-intersection. Suppose that $R - (\gamma \cup \mu)$ does not contain components that are topologically an annulus. Then for each $\varepsilon > 0$ there exists $r > 0$ such that, if $K_R(\gamma) < \varepsilon$ and $K_R(\mu) < \varepsilon$, then the distance between γ and μ in the Kobayashi metric is greater than r.*

Proof. Let $\nu: [0, 1] \to R$ be a curve that connects γ and μ so that $K_R(\nu)$ coincides with the distance between γ and μ. Let $\nu(0) = x_0 \in \gamma$. By Corollary 2.2 it is enough to verify that γ and $\gamma' = \nu^{-1} \circ \mu \circ \nu$ generate noncommutative elements $[\gamma]$ and $[\gamma']$ in $\pi_1(R, x_0)$. Since the group $\pi_1(R, x_0)$ is free, $[\gamma]$ and $[\gamma']$ are commutative, iff they belong to a cyclic subgroup. This implies that $[\gamma] = [\nu]^n$ and $[\gamma'] = [\nu]^l$ for a certain $[\nu] \in \pi_1(R, x_0)$. By Lemma 2.3 $k = l = 1$. Hence $[\gamma] = [\gamma']$. Therefore γ and μ must be homotopically equivalent. But this contradicts Lemma 2.4. □

3. Limiting behavior of hyperbolic metric

From now on by R we denote a connected hyperbolic component of Γ_0^*.

Lemma 3.1. *Let α be a sequence of points in Δ^* that tends to zero. Suppose that for each $c \in \alpha$ the fiber Γ_c is a hyperbolic Riemann surface. Then for a certain infinite subsequence $\beta = \{b_j\} \subset \alpha$ there exists a differential pseudometric α_β*

on R such that $\alpha_\beta = \lim_{j\to\infty} k_{\Gamma_{b_j}}$. Moreover, α_β is a continuous pseudometric and the equality $\alpha_\beta(v) = 0$ for a vector $v \in TR$ implies $\alpha_\beta \equiv 0$.

Proof. Let $\phi: \Delta \to R$ be a holomorphic embedding and $\phi(\Delta) = U$. It is easy to construct holomorphic embeddings $\phi_j: \Delta \to U_j \subset \Gamma_{b_j}$ so that $\phi_j(z) \to \phi(z)$ as $j \to \infty$ (e.g., see [Z]). Let ν_z denote the point $(z, d/dz) \in T\Delta$. We set $s_z^j = \phi_{j*}(\nu_z)$ and $s_z = \phi_*(\nu_z)$ (where ϕ_{j*} and ϕ_* are the induced mappings of the tangent bundles). Then $s_z^j \to s_z$ in topology of TM. Let $f_j: \Delta \to \Gamma_{b_j}$ be a universal holomorphic covering. Choose a connected component V_j of $f_j^{-1}(U_j)$ and a holomorphic mapping $g_j: \Delta \to \Delta$ so that the restriction of $g_j \circ \phi_j^{-1} \circ f_j$ to V_j is the identity mapping. One may suppose that $0 \in V_j$ and $g_j(0) = 0$. Let $\tilde{s}_z^j \in TV_j$ belong to the inverse image of the vector s_z^j under the mapping f_{j*}. Then $g_{j*}(\nu_z) = \tilde{s}_z^j$. On the other hand $g_{j*}(\nu_z) = g_j'(z)\nu_{g_j(z)}$. Hence, taking into consideration the equalities $k_\Delta(\tilde{s}_z^j) = k_{\Gamma_{b_j}}(s_z^j)$ and $k_\Delta(\nu_z) = 1/(1-|z|^2)$, we have $k_{\Gamma_{b_j}}(s_z^j) = |g_j'(z)|/(1-|g_j(z)|^2)$. Passing to a subsequence, if necessary, we suppose that $g_j(z) \to g(z)$ uniformly on compact subsets of Δ. By Hurwitz's theorem either $g'(z) \neq 0$ for every $z \in \Delta$ or $g'(z) \equiv 0$ (in the last case $g(z) \equiv 0$, since $g(0) = 0$). Therefore $\lim_{j\to\infty} k_{\Gamma_{b_j}}(s_z^j) = |g'(z)|/(1-|g(z)|^2)$. Let $s_z^j = (x_j(z), t_j(z))$, where $x_j(z) \in U_j$ and $t_j(z) \in T_{x_j(z)}U_j$ (the notation, $s_z = (x(z), t(z))$ has the same meaning). A sequence $\{v_j | v_j = (x(z_j), \lambda_j t_j(z_j)); \lambda \in C\}$ converges to $v = (x(z), \lambda t(z))$ in the topology of TM, iff $z_j \to z$ and $\lambda_j \to \lambda$. Hence $\lim_{j\to\infty} k_{\Gamma_{b_j}}(v_j) = |\lambda g'(z)|/(1-|g(z)|^2)$ and a proper limiting pseudometric exists on U. Let $\{U^j\}$ be a cover on R and each U^j be an open disc. We can repeat the above construction of the limiting pseudometric for each U^j instead of U. Application of the diagonal process completes the proof of the lemma. □

Definition. Let $\beta = \{b_j\} \subset \Delta^*$ be a sequence that converges to zero, and let every fiber Γ_{b_j} hyperbolic. We shall say that β is an *admissible sequence* if there is a continuous differential pseudometric a_β on R such that $a_\beta = \lim_{j\to\infty} k_{\Gamma_{b_j}}$ and the quality $a_\beta(v) = 0$ for a vector $v \in TR$ implies $a_\beta \equiv 0$. We will denote the corresponding integral pseudometric by A_β, and throughout the rest of the paper we will fix these notations β, a_β, and A_β for the above objects.

Lemma 3.2. *Suppose that the a_β is a metric. Let $F = \{f_j\}$, where $f_j: \Delta \to \Gamma_{b_j}$ is a holomorphic universal covering with $f_j(0) \to x_0 \in R$ as $j \to \infty$. Then there is a nonempty open subset $D \subset \Delta$ that contains 0 and a subsequence $F_1 \subset F$ that converges to a mapping $f: D \to R$. Moreover*

(i) *$f: D \to R$ is an unramified covering.*

(ii) *F transforms the metric $k_\Delta|_D$ into the metric a_β.*

Proof. First assume that f exists and prove (ii). Let $z \in D$, $x = f(z)$, and $x_j = f_j(z)$. Then $x_j \to x$ as $j \to \infty$. Choose a sequence $\{v_j | v_j \in T_{x_j}\Gamma_{b_j}\}$ that converges to a nonzero vector $v \in T_xR$ in the topology of TM. Let $\tilde{v}_j \in T_z\Delta$ belong to the inverse image of v_j under the mapping f_{j*}. Since β is admissible, $k_\Delta(\tilde{v}_j) = k_{\Gamma_{b_j}}(v_j) \to a_\beta(v)$. Thus for every j, $k_\Delta(\tilde{v}_j)$ is less than a certain common constant. Hence we may suppose that there is the limiting vector $\tilde{v}$ for the sequence $\{\tilde{v}_j\}$. Clearly, $k_\Delta(\tilde{v}) = a_\beta(v)$ and $f_*(\tilde{v}) = v$.

This implies (ii). Property (ii) means that, if f exists, then it must be locally homeomorphic. Let P be a sufficiently small neighborhood of x_0 such that P is biholomorphically equivalent to a ball, and all $U_j = P \cap \Gamma_{b_j}$ and $U = \Gamma_0 \cap P$ are discs. For every manifold N we will denote by $B(y, r, N) \subset N$ the ball of radius r in the metric K_N with the center at y. Let $B(x, r, A_\beta, R) \subset R$ be the analogous ball in the metric A_β with the center at x. Since a_β is a metric, there exists $r > 0$ such that $\overline{B(x_0, r, A_\beta, R)} \subset U$. Hence $\overline{B(f_j(0), r, \Gamma_{b_j})} \subset U_j$, when j is sufficiently large. The restriction of f_j to $\widetilde{H}_0 = B(0, r, \Delta)$ is a homeomorphism between $\widetilde{H}_0$ and $B(f_j(0), r, \Gamma_{b_j})$. The family $\{f_j|_{\widetilde{H}_0} : \widetilde{H}_0 \to P\}$ is normal. Pick out a converging subsequence $F_1 \subset F$ in this family. Let $f: \widetilde{H}_0 \to B(x, r, A_\beta, R)$ be the limiting mapping. We have proved that $D \supset \widetilde{H}_0$, i.e., D is not empty. Since f is locally homeomorphic and each $f_j|_{\widetilde{H}_0}$ is a homeomorphism, one can easily check that the limiting mapping $f|_{\widetilde{H}_0}$ is also homeomorphism. Set $H = B(x_0, r/3, A_\beta, R)$. Suppose there is a point $z \in D - \widetilde{H}_0$ with $y = f(z) \in H$. Let $H_1 = B(y, 2r/3, A_\beta, R)$ and $\widetilde{H}_1 = B(z, 2r/3, \Delta)$. Clearly $H_1 \subset U$. Repeating the above arguments we can choose a subsequence $F_2 \subset F_1$ so that the restriction F_2 to $\widetilde{H}_1$ converges to a homeomorphism $g: \widetilde{H}_1 \to H_1$. For every subsequence $F_3 \subset F_1 - F_2$ that converges to a mapping $h: \widetilde{H}_1 \to H_1$ we have $g|_{\widetilde{H}_1 \cap D} = h|_{\widetilde{H}_1 \cap D} = f|_{\widetilde{H}_1 \cap D}$. By the uniqueness theorem $h = g$. Thus one can take F_1 itself as F_2 and $D \supset \widetilde{H}_1$. Since $H_1 \supset H$, $\widetilde{H}_1$ contains a disc $\widetilde{H}$ such that $f|_{\widetilde{H}}: \widetilde{H} \to H$ is a homeomorphism and $\widetilde{H}_0 \cap \widetilde{H} = \varnothing$ (indeed, $z \notin \widetilde{H}_0$ and the restriction of f to $\widetilde{H}_0$ is also a homeomorphism). We can consider H as a neighborhood of x_0. Of course, analogous arguments enable us to find such a neighborhood for every point $x \in F(D)$. Hence $f: D \to F(D)$ is an unramified covering. In particular, $f(D)$ is an open set.

To check the equality $R = f(D)$ it is enough to prove that the set $f(D)$ is closed in R. Let x belong to the closure of $f(D)$ in R. Let P' be a sufficiently small neighborhood of x. Suppose that P' is biholomorphically equivalent to a ball, and $U' = P' \cap \Gamma_0$ and $\{U_j' = P' \cap \Gamma_{b_j}\}$ are discs. Choose $r > 0$ with $\overline{B(x, r, A_\beta, R)} \subset U'$. Then we can find a point $y \in f(D) \cap B(x, r/3, A_\beta, R)$. As we have seen, in this case $f(D) \supset B(y, 2r/3, A_\beta, R)$. Hence, $x \in f(D)$, which is the desired conclusion. □

Corollary 3.3. *If the assumptions of Lemma* 3.2 *hold and* $D = \Delta$, *then* $k_R = \lim_{j\to\infty} k_{\Gamma_{b_j}}$.

Proof. We shall use the notation of the proof of Lemma 3.2. If $D = \Delta$, then $f: D \to R$ is a universal holomorphic covering and

$$\lim_{j\to\infty} k_{\Gamma_{b_j}}(v_j) = \lim_{j\to\infty} k_\Delta(\tilde{v}_j) = k_\Delta(\tilde{v}) = k_D(\tilde{v}) = k_R(v). \quad \square$$

4. Stein case

From now on M is a Stein surface, and we will use the same notations R, $\beta = \{b_j\}$, a_β, $f_j: \Delta \to \Gamma_{b_j}$, $F = \{f_j\}$ and $f: D \to R$ as in the preceding section. Let a Riemann surface A be topologically an annulus. Denote the minimum of lengths of noncontractible loops in A by $l(A)$.

Proposition 4.1. *To each number* $t > 0$ *corresponds a positive number* $r < 1$ *so that the assumptions*:

(i) L *is a compact in* Δ;

(ii) $0 \in L$;

(iii) $\Delta - L$ *is topologically an annulus*;

(iv) $l(\Delta - L) < t$

imply that $L \subset \Delta_r$.

Proof. Assume that the contrary. Then for a certain t and every $r < 1$ there is compact L_r that contains a point z_r with $|z_r| > r$ and satisfies (i)–(iv). Clearly $l(L - \Delta)$ is greater than $2K_\Delta(0, z_r)$. But $K_\Delta(0, z_r) \to \infty$ $K_\Delta(0, z_r) \to \infty$ as $r \to 1$, and we have a contradiction with (iv). □

According to [S] the Stein subvariety R has a tubular neighborhood $V \subset M$ that is biholomorphically equivalent to a neighborhood of the zero section in the normal bundle to R in M. Thus we have a holomorphic retraction $\tau: V \to R$. Let Q be a region in R with the compact closure. Then for a sufficiently small ε and every $c \in \Delta_\varepsilon$ the restriction τ to $\tau^{-1}(Q) \cap \Gamma_c$ is a holomorphic unramified covering, whose multiplicity over Q is equal to the multiplicity to zero of the function Φ on R.

Lemma 4.2. *Let* γ *be a loop in* R *without points of self-intersection. Let* $\{\bar\phi_j | \bar\phi_j : \bar\Delta \to \Gamma_{b_j}\}$ *be continuous embeddings that are holomorphic on* Δ. *Suppose that* $\gamma_j = \bar\phi_j(\partial\Delta)$ *belong to* $\tau^{-1}(\gamma)$. *Then* γ *is contractible.*

Proof. We shall consider the Stein manifold M as a closed analytic submanifold in C^n (e.g., see [GR]). Then each $\bar\phi_j$ has the following coordinate representation $\phi_j(z) = (\bar\phi_{j1}, \dots, \bar\phi_{jn}))$. Denote the restriction $\bar\phi_j$ to Δ by ϕ_j, and let $\phi' = (\phi'_{j1}, \dots, \phi'_{jn})$ be the derivation of ϕ_j. As usual we shall use the symbol $\|\phi'_j(z)\|$ to denote the Euclidean length of the vector $\phi'_j(z)$. Suppose that the functions $\|\phi'_j\|$ converges to zero uniformly on compact subsets of Δ. Then there exists a sequence of points $\{z_j\} \subset \Delta$ with $|z_j| \to 1$ that satisfies $\|\phi'_j(z_j)\| \geq t/(1 - |z_j|^2)$ for a certain positive t. Indeed, otherwise it is easy to show that the maximal Euclidean distance between the points of γ_j, tends to zero as $j \to \infty$. But γ_j is close to $\tau(\gamma_j)$. This implies that γ must be a constant mapping, and we have a contradiction. Put $\bar\psi_j = \phi_j \circ \mu_j$, where $\mu_j(z) = (z + z_j)/(1 + \bar z_j z)$. Let $\psi_j = \bar\psi_j|_\Delta$. The loop γ belongs to a ball B in C^n. Hence for an arbitrary large j we have $\bar\psi_j(\partial\Delta) \subset B$. By the Maximum Principle $\bar\psi_j(\Delta) \subset B$. Therefore the family $\{\psi_j\}$ is normal. Passing to a subsequence, if necessary, we can suppose that $\{\psi_j\}$ converge to a mapping $\psi: \Delta \to \overline{R}$. Obviously, $\|\psi'_j(0)\| \geq t$, and, therefore, ψ is not constant. According to [Z, Lemma 2.2] $\psi(\Delta) \subset R$. Using a Möbius transformation again, if necessary, one may suppose that $\psi(0) \notin \gamma$. Choose an arbitrary small neighborhood N of γ in R so that N is topologically an annulus and $\psi(0) \notin N$. Then $N - \gamma$ consists of two components N_1 and N_2, which are also annuli. Let μ_k be the component of the boundary of N_k other than γ. Obviously, $\psi_j(\Delta)$ must contain a component of either $\tau^{-1}(N_1) \cap \Gamma_{b_j}$ or $\tau^{-1}(N_2) \cap T_{b_j}$. Denote this component by L_j. Passing to a subsequence, we may suppose that $\tau(L_j) = N_1$ and $\tau|_{L_j}$ is a s-sheeted unramified covering, where s does not exceed the multiplicity of zero of the function Φ on R. Hence the Riemann surfaces $\{L_j\}$ are pairwise biholomorphically equivalent, and $l(L_j) = l(\psi_j^{-1}(L_j))$ does not depend on j. Since $0 \notin \psi_j^{-1}(L_j)$, we see by Proposition 4.1 that there is a positive $r < 1$ such that $\Delta - \psi_j^{-1}(L_j) \subset \Delta_r$. Hence $\mu_1 \subset \psi(\Delta_r)$. This implies that μ_1 is contractible, and, therefore, γ is also contractible. □

Lemma 4.3. *The pseudometric a_β generated by an admissible sequence β is a metric on R in the case when R is different from Δ, Δ^*, or an annulus.*

Proof. Let $\gamma_1, \dots, \gamma_k$ be disjoint noncontractible loops in R without points of self-intersection such that they are not pairwise homotopically equivalent, for each i the set $R - \gamma_i$ is not connected, and every γ_i is a component of the boundary of a compact $L \subset R$. Let L_j be a component of $\tau^{-1}(L) \cap \Gamma_{b_j}$. One may suppose that $\tau|_{L_j}: L_j \to L$ is a s-sheeted unramified covering for all j. Let $\{\gamma_{ij}^l | l = 1, \dots, l_{ij} \leq s\}$ be the components of $\tau^{-1}(\gamma_i) \cap L_j$. If R has a positive genus, we can suppose that L contains a loop μ without points of self-intersection so that $L - \mu$ is connected and $\mu \cap \bigcup_{j=1}^k \gamma_j = \varnothing$. In this case we denote one of the components of $\gamma^{-1}(\mu) \cap \Gamma_{b_j}$ by μ_j. Assume, to reach a contradiction, that $a_\beta \equiv 0$. Then $K_{\Gamma_{b_j}}(\gamma_{ij}^l)$, $K_{\Gamma_{b_j}}(\mu_j) \to 0$ as $j \to \infty$ and the distance between each pair of these loops in the Kobayashi metric on Γ_{b_j} also tends to zero. By Lemma 2.5 all of these loops must be homotopically equivalent. Since $\tau|_{L_j}: L_j \to L$ is an unramified covering, $L_j - \mu_j$ is connected. Hence by Lemma 2.4 μ_j cannot be homotopically equivalent to any component of the boundary of L_j, or in other words, to any γ_{ij}^l. Therefore it remains to consider the case when R is biholomorphically equivalent to a region in C. Then under the assumptions of the lemma one may suppose that $k \geq 3$. Thus we have, at least, three loops γ_1, γ_2, and γ_3. By Lemma 2.5 there is a region $U_j \subset \Gamma_{b_j}$ such that $\partial U_j = \gamma_{1j}^1 \cup \gamma_{2j}^1$ and U_j is topologically an annulus. Note that U_j does not belong to L_j (otherwise, using Lemmas 2.3 and 2.4 it is easy to show that γ_1 and γ_2 are homotopically equivalent). Moreover, since the component of $\Gamma_{b_j} - L_j$ whose boundary contains γ_{3j}^1 is different from a disc according to Lemma 4.2, U_j does not contain L_j. Hence U_j is a component of $\Gamma_{b_j} - L_j$. Taking γ_{3j}^1 instead of γ_{2j}^1 we can construct a component V_j of $\Gamma_{b_j} - L_j$ so that $\partial V_j = \gamma_{1j}^1 \cup \gamma_{3j}^1$ and V_j is topologically an annulus. Since $\partial V_j \cap \partial U_j = \gamma_{1j}^1$, $V_j = U_j$. Then $\partial U_j = \partial V_j$, and this leads to a contradiction. Therefore a_β is not trivial. By Lemma 3.1 $a_\beta(v) \neq 0$ for each $v \in TR$. This completes the proof of the lemma. □

Lemma 4.4. *Let M be a Stein surface and let D be the same as in Lemma* 3.2. *Then D is simply connected.*

Proof. Assume that D is not simply connected. Then there is a couple of discs d and d' such that $\bar{d} \subset \Delta$, $d' \subset d$, d does not belong to D, and $\bar{d} - d' \subset D$. We again consider M as a submanifold in C^n. The set $f(\bar{d} - d')$ belongs to a certain ball in C^n. Same arguments as in Lemma 4.2 show that the family $\{f_j|_d\}$ is normal. Let $\tilde{f}: d \to \bar{R}$ be a limiting mapping. This mapping is unique, since it coincides with f on $d - d'$. In particular, it is nonconstant. The set $f(d)$ does not contain singular points of Γ_0, because otherwise $f_j(d)$ must intersect Γ_0 for an arbitrary large j [Z]. Hence $\tilde{f}(d) \subset R$, i.e., $d \subset D$. But this contradicts our assumption. □

Corollary 4.5. *Lemma* 4.2 *holds without the condition that* γ *has no point of self-intersection.*

5. Proof of the main theorem

We keep the same notation R, β, a_β, $F = \{f_j\}$, $f\colon D \to R$ as in the preceding section. By Lemmas 3.2, 4.3, and 4.4 we suppose that the family F converges to the mapping $f\colon D \to R$ on a nonempty simply connected region $D \subset \Delta$ with $0 \in D$, and f is an unramified covering, which transforms the metric $k_\Delta|_D$ into the metric a_β. Let G_j be the Fuchsian group such that $f_j(z) = f_j(z')$ iff $z' = g(z)$ for a certain $g \in G_j$. We say that a Möbius transformation h is *limiting* for $\{G_j\}$, if there is a sequence $\{g_j | g_j \in G_j\}$ that converges to h uniformly on compact subsets of Δ. Let G be the group of holomorphic one-to-one mappings D to D such that $f(z) = f(z')$, if $z' = g(z)$ for a certain $g \in G$.

Lemma 5.1. *The set* H *of limiting Möbius transformations is a subgroup of* G *of finite index.*

Proof. By construction, H is a group and for each pair z, $z' \in D$ the equality $h(z) = z'$ for an element $h \in H$ implies $f(z) = f(z')$. Hence $H \subset G$. As in the preceding section $\tau\colon V \to R$ is a holomorphic retraction of a Stein neighborhood V of R. Consider all the loops $\{\gamma\colon \partial\Delta \to R\}$ such that $\gamma(1) = f(0)$ and for an arbitrary large j there is a loop γ_j in Γ_{b_j} with $\gamma_j(1) = f_j(0)$ and $\gamma = \tau \circ \gamma_j$. These loops generate a subsgroup H_1 of finite index in $\pi_1(R, f(0))$. This index does not exceed the multiplicity of zero of the function Φ on R. Since $\pi_1 \cong G$, one can consider H_1 as a subgroup in G as well. Let γ be a loop in R with $[\gamma] \in H_1$ and $\{\gamma_j \in \Gamma_{b_j}\}$ be the corresponding loops, which converge to γ uniformly. Consider the mappings $\nu_j\colon \mathbf{R} \to \Delta$ and $\nu\colon \mathbf{R} \to D$ such that $f_j \circ \nu_j(t) = \gamma_j(e^{2\pi i t})$, $f \circ \nu(t) = \gamma(e^{2\pi i t})$, and $\nu(0) = \nu_j(0) = 0$. Since $\gamma_j \to \gamma$ and $f_j \to f$, one can see that $\nu_j \to \nu$ uniformly. By $\widetilde{\gamma}_j$ and $\widetilde{\gamma}$ we will denote the elements of the Fuchsian groups G_j and G that correspond $[\gamma_j]$ and $[\gamma]$ respectively. Clearly, $\widetilde{\gamma}_j^k(t) = \nu_j(t+k)$ and $\tilde{\gamma}^k(t) = \nu(t+k)$ for each integer k. This means $\tilde{\gamma}_j \to \tilde{\gamma}$ as $j \to \infty$. Hence $H_1 \subset H$ and H is a subgroup of G of finite index. □

Let $\widetilde{R} \to R$ be an unramified covering that corresponds to the subgroup $H \subset \pi_1(R)$. Then, since D is simply connected, the mapping $\tilde{f}\colon D \to D/H \cong \widetilde{R}$ is a universal holomorphic covering. Recall that by the hypotheses of Main Theorem G is isomorphic to a Fuchsian group of the first kind G', acting on Δ. More precisely, there is a biholomorphic mapping $\varphi\colon \Delta \to D$ such that φ generates isomorphism between G and G'. Therefore H is isomorphic to a subgroup H' of finite order in G'. Hence H' is a Fuchsian group of the first kind as well. According to [G, §3, Lemma 3] it is easy to check now that, since the closure of the orbits $\{h'(0) | h' \in H'\}$ coincides with $\partial\Delta$, the closure of orbits $\{h(0) | h \in H)$ must coincide with ∂D. Assume that z is a point of $\partial D \cap \Delta$. Choose an arbitrary small neighborhood U of z and element $\tilde{v}$, $\tilde{\eta} \in H$ so that $\tilde{\nu}(0)$ and $\tilde{\eta} \circ \tilde{\nu}(0) \in U \cap D$. Let $\tilde{\mu}$, $\tilde{\gamma} \in H$ be noncommutative elements. Then $\tilde{\eta}$, $\tilde{\nu}^{-1}$, $\tilde{\gamma} \circ \tilde{\nu}^{-1}$, $\tilde{\mu} \circ \tilde{\nu}^{-1}$ cannot belong to a cyclic subgroup of H. Hence one of the pairs $\tilde{\eta}$ and $\tilde{\nu}^{-1}$, $\tilde{\eta}$ and $\tilde{\gamma} \circ \tilde{\nu}^{-1}$ or $\tilde{\eta}$ and $\tilde{\mu} \circ \tilde{\nu}^{-1}$ are not commutative. Consider the corresponding noncommutative pair of elements in G_j for a sufficiently large

j. Put $z' = \tilde{\nu}(0)$. Application of Lemma 2.1 to the above pair and the point z' leads to a contradiction. Thus $D = \Delta$ and by Corollary 3.3 $k_R = \lim_{j\to\infty} k_{\Gamma_{b_j}}$. This implies immediately that for every sequence $\{b_j\} \subset \Delta^*$ with hyperbolic fibers $\{\Gamma_{b_j}\}$ and $b_j \to 0$ $k_R = \lim_{j\to\infty} k_{\Gamma_{b_j}}$. The last thing we need to confirm is that if there exists a sequence $\{b_j\} \to 0$ with nonhyperbolic fibers $\{\Gamma_{b_j}\}$ then R cannot be hyperbolic. Assume that such a sequence exists. Then Γ_{b_j} is biholomorphically equivalent to C or C^*. Hence R has no handle, for if it had, then all of the fibers Γ_{b_j} would have handles as well for sufficiently large j. Since a Fuchsian group of the first kind corresponds to the Riemann surface R, R is different from Δ, Δ^* or an annulus. Thus $\pi_1(R)$ has, at least, two generators $[\gamma_1]$ and $[\gamma_2]$. One may suppose that the loops γ_1 and γ_2 have no points of self-intersection. Note that the proof of Lemma 4.2 does not use the assumption that $\{\Gamma_{b_j}\}$ are hyperbolic, i.e., it remains true without this assumption. Thus, since Γ_{b_j} is biholomorphically equivalent to C or C^* either γ_1^k or γ_2^k must be approximated by contractible loops in $\{\Gamma_{b_j}\}$ for a certain integer k. This contradicts Lemma 4.2. Hence there is no sequence $\{b_j\} \to 0$ with nonhyperbolic fibers $\{\Gamma_{b_j}\}$. The main theorem is proved.

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