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SOME COMPLETE Σ_2^1 SETS IN HARMONIC ANALYSIS

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Abstract. We prove that several specific pointsets are complete Σ_2^1 (complete PCA). For example, the class of N_0-sets, which is a hereditary class of thin sets that occurs in harmonic analysis, is a pointset in the space of compact subsets of the unit circle; we prove that this pointset is complete Σ_2^1. We also consider some other aspects of descriptive set theory, such as the nonexistence of Borel (and consistently with ZFC, the nonexistence of universally measurable) uniformizing functions for several specific relations. For example, there is no Borel way (and consistently, no measurable way) to choose for each N_0-set, a trigonometric series witnessing that it is an N_0-set.

1. Introduction

This paper is about some connections between descriptive set theory and three topics in analysis: Pointwise convergence of sequences of functions, increasing unions of members of a given collection of compact sets, and thin sets in harmonic analysis. The main purpose of the paper is to show that several specific pointsets are complete Σ_2^1. We also discuss some other aspects of descriptive set theory, such as the nonexistence of Borel-measurable uniformizing functions for several specific relations.

In this paper, all spaces are Polish. A pointset is Σ_1^1 if it is the projection of a Borel set; it is Π_n^1 if it is the complement of a Σ_n^1 set; it is Σ_{n+1}^1 if it is the projection of a Π_n^1 set. Another name for Σ_1^1, Π_1^1, Σ_2^1, and Π_2^1 sets is A (analytic), CA (coanalytic), PCA and CPCA sets, respectively. These collections of pointsets can also be viewed in terms of definability. A set $P \subset X$ is Σ_2^1 iff it has a definition of the form

$$x \in P \leftrightarrow \exists y \, \forall z((x, y, z) \in B),$$

where B is a Borel set in a product space $X \times Y \times Z$; P is Π_2^1 iff it has a definition of the form

$$x \in P \leftrightarrow \forall y \, \exists z((x, y, z) \in B),$$

for B Borel. This subject is presented in Kuratowski [18] and Moschovakis [20]. We tend to follow Moschovakis [20] in notation, terminology, etc.

A Σ_2^1 set S in a space X is called *complete* Σ_2^1 if for any Σ_2^1 subset Q of the Cantor set, 2^ω, there is a continuous $h: 2^\omega \to X$ such that $Q = h^{-1}[S]$.

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Such an h is said to *reduce* Q to S. A complete $\mathbf{\Sigma}_2^1$ set is true $\mathbf{\Sigma}_2^1$, that is, it is not $\mathbf{\Pi}_2^1$. In §§2–4 we show that several natural examples of $\mathbf{\Sigma}_2^1$ sets are in fact complete $\mathbf{\Sigma}_2^1$.

In §§5, 6 we consider some other descriptive set theoretic facts about these examples. In §5 we show that there can be no simply definable uniformizing function (that is, selection or choice function) for certain natural relations associated with the above examples, and that it is consistent with ZFC that there is no measurable selection. It is known that ZFC, the usual set of axioms for set theory, is not sufficient to answer many questions about $\mathbf{\Sigma}_2^1$ sets—either answer to the question is consistent. In §6 we point out that these consistency results are applicable to the specific examples of true $\mathbf{\Sigma}_2^1$ sets discussed in this paper, e.g., to some classes of thin sets in harmonic analysis.

There are two technical lemmas which are used. We prove these lemmas in §§7, 8.

2. Pointwise convergent subsequences

Consider the Polish space $(C[0, 1])^\omega$, that is, the topological product of countably many copies of $C[0, 1]$. In this space, consider the following two pointsets.

$$S_1 = \{(f_n)\colon \text{ Some subsequence of } (f_n) \text{ converges pointwise}\},$$
$$S_2 = \{(f_n)\colon \text{ Some subsequence of } (f_n) \text{ converges pointwise to a continuous limit}\}.$$

Both S_1 and S_2 are $\mathbf{\Sigma}_2^1$ sets. (The classification $\mathbf{\Sigma}_2^1$ refers to the topology of the Polish space, that is, the topology of uniform convergence, not to the topology of pointwise convergence.)

Theorem 2.1. *S_1 and S_2 are both complete $\mathbf{\Sigma}_2^1$.*

For S_2, this theorem was proved in Becker [3]. For S_1, the theorem was announced in Becker [3, 4], but no proof was given. We now give another proof for S_2 and the first proof for S_1.

Lemma 2.2. *Let $Q \subset 2^\omega$ be any $\mathbf{\Sigma}_2^1$ set. There exists a sequence of continuous functions $g_n\colon 2^\omega \times 2^\omega \to 2$ such that for all $w \in 2^\omega$, the following are equivalent.*

(a) $w \in Q$.

(b) *There is a subsequence (g_{n_i}) of (g_n) such that for every $x \in 2^\omega$, $g_{n_i}(w, x) \to 0$.*

Lemma 2.2 will be proved in §7. We use Lemma 2.2 to prove Theorem 2.1, and indeed to prove all the results in this paper.

Proof of Theorem 2.1. Let $Q \subset 2^\omega$ be an arbitrary $\mathbf{\Sigma}_2^1$ set. We reduce Q to S_1 and S_2, simultaneously. In other words, for every $w \in 2^\omega$, we define a sequence of functions (f_n^w) from $[0, 1]$ into $\mathbf{R}$ such that:

(a) The function $h\colon 2^\omega \to (C[0, 1])^\omega$ given by $h(w) = (f_n^w)$ is continuous.

(b) If $w \in Q$ then some subsequence of (f_n^w) converges pointwise to a continuous limit (in fact, to 0).

(c) If $w \notin Q$ then no subsequence of (f_n^w) converges pointwise.

To do this, let (g_n) satisfy Lemma 2.2 for this set Q. Let $F_n^w\colon 2^\omega \to \mathbf{R}$ be the function $F_n^w(x) = n \cdot g_n(w, x)$. Identify 2^ω with the Cantor middle third set in $[0, 1]$, and let f_n^w be such that $f_n^w \restriction 2^\omega = F_n^w$ and f_n^w is linear on each interval of $[0, 1] \setminus 2^\omega$. It is easy to check that (a)–(c) hold. □

The functions f_n^w produced in the above proof are obviously not bounded. But by combining the proof of Theorem 2.1 with the method of Kaufman [12], we can get the f_n^w's to be uniformly bounded. Thus, viewing $C[0, 1]$ as a Banach space, we also obtain a proof of Theorem 2.3.

Let

$$S_3 = \{(f_n): \text{ Some subsequence of } (f_n) \text{ is weakly Cauchy}\},$$

$$S_4 = \{(f_n): \text{ Some subsequence of } (f_n) \text{ is weakly convergent}\}.$$

Theorem 2.3. *S_3 and S_4 are both complete Σ_2^1.*

In Banach spaces other than $C[0, 1]$, the situation may be very different. For example, if X is a separable-dual space or X is l_1, then the sets of weakly Cauchy and weakly convergent sequences are Borel sets in X^ω. (For l_1 this follows from Schur's Theorem; see Diestel [6].) Hence for such an X, the analogs of S_3 and S_4 are Σ_1^1 sets. We do not know of any characterization of which separable Banach spaces satisfy 2.3.

3. Increasing unions of compact sets

Let X be a compact Polish space and let $\mathscr{K}(X)$ denote the space of nonempty compact subsets of X with the Hausdorff metric δ:

$$\delta(K, K') = \sup\{d(x, K), d(y, K') : x \in K', \ y \in K\}.$$

$\mathscr{K}(X)$ is also compact and Polish. In §§3, 4 we will be considering the complexity of pointsets in spaces of the form $\mathscr{K}(X)$.

Let $C \subset \mathscr{K}(X)$ be a hereditary class of compact sets, and define C^σ and $C\uparrow$ as follows:

$$C^\sigma = \left\{ K \in \mathscr{K}(X) : \text{ There exists a sequence } (K_n) \text{ of members of } C \text{ such that } K \subset \bigcup_n K_n \right\},$$

$$C\uparrow = \left\{ K \in \mathscr{K}(X) : \text{ There exists a sequence } (K_n) \text{ of members of } C \text{ such that } K \subset \bigcup_n K_n \text{ and for all } n,\ K_n \subset K_{n+1} \right\}.$$

Of course if C is closed under finite unions, then $C\uparrow = C^\sigma$.

If C is Π_1^1 then C^σ is also Π_1^1. (*Proof.* $K \in C^\sigma$ iff for every compact $K' \subset K$ there is a basic open neighborhood N of X such that $N \cap K' \neq \varnothing$ and $\overline{N} \cap K' \in C$.) For some interesting C's, C^σ is known to be complete Π_1^1. For a thorough analysis of these Π_1^1 σ-ideals, see Kechris [13] and Kechris, Louveau-Woodin [15].

If C is Σ_2^1 then $C\uparrow$ is also Σ_2^1. We show below that $C\uparrow$ is, in general, no simpler than Σ_2^1, even when C is an open set. (If C is closed then $C\uparrow = C$.)

We work with the spaces 2^ω and $\mathscr{K}(2^\omega)$. Let $p_n\colon 2^\omega \to 2$ denote the nth projection function. Consider the following pointset in the space $\mathscr{K}(2^\omega)$.

$$Z = \{K\colon \text{ Some subsequence of } (p_n) \text{ converges to 0 pointwise on } K\} .$$

Theorem 3.1. *Z is complete $\mathbf{\Sigma}^1_2$.*

Proof. Z is obviously $\mathbf{\Sigma}^1_2$. Let $Q \subset 2^\omega$ be an arbitrary $\mathbf{\Sigma}^1_2$ set. To prove completeness, we define a continuous $h\colon 2^\omega \to \mathscr{K}(2^\omega)$ which reduces Q to Z. Let (g_n) satisfy Lemma 2.2 for this Q. For any $w \in 2^\omega$, let

$$K_w = \{y \in 2^\omega\colon \text{ There is an } x \in 2^\omega \text{ such that for all } n,\ y(n) = g_n(w, x)\} .$$

Then K_w is compact and the function $h\colon w \mapsto K_w$ is continuous. If $w \in Q$, then by Lemma 2.2, there is a strictly increasing $(n_i) \in \omega^\omega$ such that for all $x \in 2^\omega$, $g_{n_i}(w, x) \to 0$; hence for each $y \in K_w$, $y(n_i) \to 0$, that is, (p_{n_i}) converges to 0 pointwise on K_w; so $K_w \in Z$. Conversely, if (p_{n_i}) converges to 0 pointwise on K_w, then for all x, $g_{n_i}(w, x) \to 0$, which shows that $w \in Q$. Thus h reduces Q to Z. □

Let

$$Y = \{K \in \mathscr{K}(2^\omega)\colon \text{ For all } m, \text{ there is an } n > m \text{ such that } p_n \text{ is identically 0 on } K\}.$$

Then Y is a hereditary G_δ and $Z = Y\uparrow$, so sets of this form can be complete $\mathbf{\Sigma}^1_2$. But Y is not open.

Theorem 3.2. *There is a hereditary open $C \subset \mathscr{K}(2^\omega)$ such that $C\uparrow$ is complete $\mathbf{\Sigma}^1_2$.*

Proof. Let

$$C = \{K\colon \text{ There is an } n \text{ such that } p_n \text{ is identically 0 on } K\}.$$

Clearly C is hereditary and open and $C\uparrow = Z \cup C$. So it will suffice to show that there is a continuous $h\colon \mathscr{K}(2^\omega) \to \mathscr{K}(2^\omega)$ which reduces Z to $Z \cup C$. Let

$$F = \{x \in 2^\omega : \text{ There is at most one } n \text{ such that } x(n) = 1\} .$$

Let $h(K) = K \cup F$. □

If $C \subset \mathscr{K}(X)$ is a hereditary $\mathbf{\Sigma}^1_2$ set, then $(C\uparrow)\uparrow$ is again $\mathbf{\Sigma}^1_2$. We do not know whether or not there exists a compact Polish space X and a hereditary Borel $C \subset \mathscr{K}(X)$ for which $(C\uparrow)\uparrow$ is true $\mathbf{\Sigma}^1_2$.

4. Thin sets in harmonic analysis

Let SI denote the set of all strictly increasing sequences of natural numbers. Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the unit circle. Consider the following four classes of closed subsets of $\mathbf{T}$, that is, the following four pointsets in the space $\mathscr{K}(\mathbf{T})$.

$$D = \{K\colon \text{ For some } (n_i) \in \mathrm{SI}, \text{ the sequence } (\sin n_i t) \text{ converges to 0 uniformly on } K\},$$

$$N_0 = \left\{K\colon \text{ For some } (n_i) \in \mathrm{SI}, \text{ the series } \sum_{i=0}^{\infty} \sin n_i t \text{ converges absolutely on } K\right\},$$

$$A = \{K\colon \text{ For some } (n_i) \in \mathrm{SI}, \text{ the sequence } (n_i t) \text{ converges to 0 pointwise on } K\},$$

$$H = \{K\colon \text{ For some } (n_i) \in \mathrm{SI} \text{ and some interval } I \text{ of } \mathbf{T}, \text{ for all } i,\ n_i K \cap I = \varnothing\}.$$

These four hereditary classes of closed sets constitute four types of *exceptional sets*, or *thin sets*, which occur in harmonic analysis and which have been studied extensively. N_0-sets were introduced by Salem, in an attempt to simplify the definition of a class of thin sets called N-sets. A-sets were introduced by Arbault [1], who proved that A-sets, N-sets, and N_0-sets are different notions. The letter A stands for Arbault, and N stands for Nemytzkii [2]. H-sets were introduced by Rajchman [22]; H is for Hardy-Littlewood. We do not know where the concept of a D-set originates. D is for Dirichlet, who proved that finite sets are D-sets. Information on these (and many other) classes of thin sets can be found in the following references: Bary [21], Kahane [8], Kahane-Salem [9], Körner [17], and Lindahl-Poulsen [19]. Increasing unions of such classes, e.g., $D\uparrow$, are studied in Kahane [10, 11].

There has been a considerable amount of work done on connections between descriptive set theory and various types of thin sets in harmonic analysis. See Kechris-Louveau [14] for details. In particular, several important classes of thin sets have been proved to be complete $\mathbf{\Pi}_1^1$. In this paper we are concerned with classes of thin sets which, from the point of view of definability, are much more complicated: complete $\mathbf{\Sigma}_2^1$.

It is easy to see that

$$D\uparrow \subset N_0 \subset A \subset H\uparrow . \tag{4.1}$$

It is also easy to see that (when considered as pointsets in the space $\mathscr{K}(\mathbf{T})$) N_0 and A are $\mathbf{\Sigma}_2^1$ and D and H are Borel, hence $D\uparrow$ and $H\uparrow$ are $\mathbf{\Sigma}_2^1$.

Theorem 4.2. *$D\uparrow$, N_0, A, and $H\uparrow$ are all complete $\mathbf{\Sigma}_2^1$.*

To prove Theorem 4.2 we need another lemma.

Lemma 4.3. *There is a continuous $h\colon \mathscr{K}(2^\omega) \to \mathscr{K}(\mathbf{T})$ such that for all $K \in \mathscr{K}(2^\omega)$:*

(a) *If $K \in Z$ then $h(K) \in D\uparrow$.*

(b) *If $K \notin Z$ then $h(K) \notin H\uparrow$.*

Lemma 4.3 will be proved in §8. Note that Theorem 4.2 follows immediately from Theorem 3.1, (4.1), and Lemma 4.3.

A set K is an N-set if for some sequence (a_n) of positive numbers with $\sum a_n = \infty$, the series $\sum a_n \sin nt$ converges absolutely on K. Although the concept of an N_0-set was originally thought to be simpler than that of an N-set, it is actually more complicated. Using work of Björk and Kaufman (see Lindahl and Poulsen [19]), it can be shown that N is a G_δ-set in $\mathscr{K}(\mathbf{T})$, whereas N_0 is complete $\mathbf{\Sigma}^1_2$.

Let X be the space of all probability measures on $\mathbf{T}$ with the weak*- topology. If $C \subset \mathscr{K}(\mathbf{T})$, let $C^\perp$ denote those measures which annihilate all sets in C. Kechris and Lyons [16] and Kaufman [12] have shown that $D^\perp$ and $H^\perp$ (hence $(D\uparrow)^\perp$ and $(H\uparrow)^\perp$) are both true $\mathbf{\Pi}^1_1$ sets in the space X. Since N_0 and A are true $\mathbf{\Sigma}^1_2$, $N_0^\perp$ and $A^\perp$ may seem, at first glance, to be good candidates for true $\mathbf{\Pi}^1_2$ sets—but they are not. In fact, $D^\perp \subset A^\perp$; this follows from Egorov's theorem that a pointwise convergent sequence converges uniformly on a set of positive measure. Hence $A^\perp = N_0^\perp = D^\perp$, and it is a $\mathbf{\Pi}^1_1$ set. We thank Robert Kaufman for his helpful comments on these matters.

5. On the complexity of uniformizing functions

Let $R \subset X \times Y$ be a relation in some product space with the property that for every $x \in X$ there exists a $y \in Y$ such that $(x, y) \in R$. A function $f: X \to Y$ is called a *uniformizing function* for R if for all $x \in X$, $(x, f(x)) \in R$. Obviously a uniformizing function exists. But there may be no "nice" uniformizing function.

All the theorems of §§2–4 can be turned into theorems about the nonexistence of nice uniformizing functions. We will explicitly state these nonuniformization theorems for the case of N_0-sets. Similar theorems hold for the other complete $\mathbf{\Sigma}^1_2$ sets of §§2–4.

Let $W \subset (\mathscr{K}(\mathbf{T}) \times \omega^\omega)$ be the following relation:

$$W = \left\{ (K, (n_i)) : (n_i) \in \mathrm{SI} \text{ and the series } \sum_{i=0}^{\infty} \sin n_i t \text{ converges absolutely on } K \right\}.$$

(Thus W is a $\mathbf{\Pi}^1_1$ set whose projection is the $\mathbf{\Sigma}^1_2$ set N_0. The pair $(K, (n_i))$ is in W if (n_i) is a *witness* that K is an N_0-set.) Let E be a subset of N_0 which is closed in $\mathscr{K}(\mathbf{T})$; then W^E denotes $W \cap (E \times \omega^\omega)$. Clearly $E \times \omega^\omega$ is Polish and W^E is a $\mathbf{\Pi}^1_1$ set in $E \times \omega^\omega$.

Since $E \subset N_0$, for every $K \in E$ there exists an (n_i) in ω^ω such that $(K, (n_i)) \in W^E$. So uniformizing functions for W^E exist. But, in general, there does not exist a Borel uniformizing function. In fact a much stronger nonuniformization theorem holds.

Theorem 5.1. *Let $\mathscr{F}$ be a family of functions (with domain and range various Polish spaces) with the following closure property: If $f \in \mathscr{F}$, b is a Borel function, and h is a homeomorphism then $b \circ f \circ h$ is in $\mathscr{F}$. Suppose that there exists a $\mathbf{\Pi}^1_1$ set $P \subset 2^\omega \times 2^\omega$ such that for all $x \in 2^\omega$ there is a $y \in 2^\omega$ with $(x, y) \in P$, but there is no uniformizing function for P in $\mathscr{F}$. Then there is an uncountable closed $E \subset N_0$ such that there is no uniformizing function for W^E in $\mathscr{F}$.*

What Theorem 5.1 says is that uniformizing the W^E's is no simpler than uniformizing an arbitrary $\mathbf{\Pi}_1^1$ relation. So all the $\mathbf{\Pi}_1^1$ nonuniformization theorems of descriptive set theory are applicable to relations of the form W^E. There is a $\mathbf{\Pi}_1^1$ $P \subset 2^\omega \times 2^\omega$ with no Borel uniformization. So taking $\mathscr{F}$ to be the Borel functions, Theorem 5.1 implies that for some E, W^E has no Borel uniformization. (This E can be taken to be hereditary; if not, take its hereditary closure.) Similarly, if $\mathscr{F}$ is the family of C-measurable functions (see Burgess [5]), or one of the other classical families of measurable functions, again there is an E such that W^E has no uniformization in $\mathscr{F}$. Kondo's Theorem states that W^E always has a $\mathbf{\Delta}_2^1$ uniformization. Theorem 5.1 actually says that $\mathbf{\Delta}_2^1$ is best possible for uniformizing relations of the form W^E. Or in the language of *effective* descriptive set theory, Δ_2^1 is the best possible basis for picking a witness that a given set is an N_0-set. For information about uniformization and basis theorems, and about nonuniformization and nonbasis theorems, see Moschovakis [20].

Instead of considering the existence of simply definable uniformizing functions, one could consider functions which are nice in a different sense: measurable. This leads us to a question which cannot be answered in ZFC.

Theorem 5.2. *If ZFC is consistent, then so is each of the following two theories.*

(a) *ZFC + there is a universally measurable $f\colon \mathscr{K}(\mathbf{T}) \to \omega^\omega$ such that for all $K \in N_0$, $(K, f(K)) \in W$.*

(b) *ZFC + there exists an uncountable closed $E \subset N_0$ such that: If f is any uniformizing function for W^E and μ is any nonzero measure on E which gives points measure* 0*, then f is not measurable with respect to μ.*

Part (a) of Theorem 5.2 follows trivially from the fact that it is relatively consistent with ZFC that every $\mathbf{\Sigma}_2^1$ relation has a universally measurable uniformization. Part (b) of Theorem 5.2 follows from Theorem 5.1, by taking $\mathscr{F}$ to be the family of functions which are measurable with respect to some such μ; it is consistent that there is a $\mathbf{\Pi}_1^1 P \subset 2^\omega \times 2^\omega$ which is the graph of a function that is not measurable with respect to any such μ (see Moschovakis [20]). Large cardinal axioms imply that it is *true* (as opposed to merely consistent) that $\mathbf{\Sigma}_2^1$ relations have measurable uniformizations; hence these axioms imply that the W^E's do. For information on consistency proofs, large cardinals, etc., and their relationship to descriptive set theory, see Jech [7] and Moschovakis [20].

As previously mentioned, similar theorems hold for the other complete $\mathbf{\Sigma}_2^1$ sets of §§2–4. For example, for S_1: There is no simply definable function (and consistently, no measurable function) $f\colon E \to (C[0, 1])^\omega$, for $E \subset S_1$, such that $f((f_n))$ is a pointwise convergent subsequence of (f_n). For $D\uparrow$: There is no nice function $f\colon E \to (\mathscr{K}(\mathbf{T}))^\omega$ such that $f(K)$ is an increasing sequence of D-sets whose union covers K. This situation for increasing countable unions is different from the case of arbitrary countable unions. If $C \subset \mathscr{K}(X)$ is hereditary and Borel then the Cantor-Bendixson derivation (see Kechris and Louveau [14]) gives a function $f\colon C^\sigma \to C^\omega$ such that for all $K \in C^\sigma$, $\bigcup f(K) = K$; moreover, this function is $\mathbf{\Delta}_1^1$ (on its domain). This means that in the case of *increasing* countable unions, there is nothing similar to the Cantor-Bendixson analysis of countable unions. For other, more combinatorial, differences between the two operations, see Kahane [11].

Theorem 5.1 can be proved by the same method used to prove that N_0 is complete $\mathbf{\Sigma}^1_2$, so we merely note the changes necessary and leave the details to the reader. Saying N_0 is *complete* $\mathbf{\Sigma}^1_2$ means:

(5.3) For any $\mathbf{\Sigma}^1_2$ set $Q \subset 2^\omega$ there is a continuous $h: 2^\omega \to \mathscr{K}(\mathbf{T})$ such that for all $w \in 2^\omega$, the following two conditions are satisfied.

(a) If $w \in Q$ then there exists an (n_i) such that $(h(w), (n_i)) \in W$.

(b) If there exists an (n_i) such that $(h(w), (n_i)) \in W$, then $w \in Q$.

There is a stronger version of (5.3) which can be proved, namely

Theorem 5.4. *For any* $\mathbf{\Pi}^1_1$ *set* $P \subset 2^\omega \times 2^\omega$, *there is a one-to-one continuous* $h: 2^\omega \to \mathscr{K}(\mathbf{T})$ *and a Borel function* $b: \omega^\omega \to 2^\omega$, *such that for all* $w \in 2^\omega$, *the following two conditions are satisfied.*

(a) *If there exists a* $y \in 2^\omega$ *such that* $(w, y) \in P$, *then there exists an* (n_i) *such that* $(h(w), (n_i)) \in W$.

(b) *For all* $(n_i) \in \omega^\omega$, *if* $(h(w), (n_i)) \in W$, *then* $(w, b((n_i))) \in P$.

Note that (5.3) follows immediately from Theorem 5.4, by taking P to be a $\mathbf{\Pi}^1_1$ set whose projection is Q. The proof of Theorem 5.4 is implicit in the proof of (5.3) given in this paper. The various reducing functions defined in this paper are in fact not one-to-one. But by minor changes in the definition, we can make them one-to-one without harming anything else. The one place where this is not obvious is in the proof of Lemma 4.3. In §8, after proving Lemma 4.3, we indicate the modification needed to get a one-to-one reducing function.

We now prove Theorem 5.1 from Theorem 5.4. Let $P \subset 2^\omega \times 2^\omega$ be a $\mathbf{\Pi}^1_1$ relation with no uniformization in $\mathscr{F}$ such that for all x there is a y with $(x, y) \in P$. Let h and b satisfy Theorem 5.4 and let E be the image of h. Then there can be no uniformizing function f for W^E which is in $\mathscr{F}$; for if such an f existed, then $b \circ f \circ h$ would uniformize P.

6. Set theory and N_0-sets

There are various types of pathological pointsets, e.g., nonmeasurable, which can be produced using the axiom of choice. It is consistent with ZFC that these pathologies occur at the level of $\mathbf{\Sigma}^1_2$ sets (see Moschovakis [20]). It is also consistent that counterexamples to the continuum hypothesis (CH) occur at the level of $\mathbf{\Sigma}^1_2$ sets (see Jech [7]).

Theorem 6.1. *If ZFC is consistent, then so is each of the following two theories.*

(a) *ZFC + there exists an uncountable closed* $E \subset \mathscr{K}(\mathbf{T})$ *such that the pointset* $(N_0 \cap E)$:

(i) *does not have the property of Baire* (*with respect to the space* E);

(ii) *is not measurable with respect to any nonzero measure on* E *which gives points measure* 0.

(b) *ZFC* $+ \neg$ *CH + there is a closed* $E \subset \mathscr{K}(\mathbf{T})$ *such that* $\operatorname{card}(N_0 \cap E) = \aleph_1$.

Note that (a)(ii) of Theorem 6.1 implies that in the space $\mathscr{K}(\mathbf{T})$, N_0 is not a universally measurable set. Theorem 6.1 follows from the fact that N_0 is complete $\mathbf{\Sigma}^1_2$ via one-to-one reduction functions (see Theorem 5.4ff.); therefore the existence of $\mathbf{\Sigma}^1_2$ subsets of 2^ω satisfying Theorem 6.1(a), (b), implies that $\mathbf{\Sigma}^1_2$ sets of the form $N_0 \cap E$ satisfy it. It is also consistent that no $\mathbf{\Sigma}^1_2$ set exhibits the pathologies of Theorem 6.1, and large cardinal axioms imply that $\mathbf{\Sigma}_2^1$ sets truly do not.

This method of proof also gives the next theorem, which set theorists may find amusing. There is a fixed uncountable closed set $E_0 \subset \mathscr{K}(\mathbf{T})$, with recursive code which can be explicitly defined, for which the following is provable in ZFC.

Theorem 6.2. *Let $K \in E_0$. Then K is an N_0-set iff K is constructible.*

Constructible means in the smallest transitive model of ZFC containing all ordinals; see Jech [7] or Moschovakis [20].

We have, as in §5, stated the theorems for the case of N_0-sets. But again, similar theorems hold for the other complete $\mathbf{\Sigma}_2^1$ sets of §§2–4.

7. Proof of Lemma 2.2

Our proof of Lemma 2.2 is based on some ideas in Becker [3].

We first establish some notation. $X^{<\omega}$ is the set of all finite sequences from X. Finite sequences are denoted by lowercase Greek letters. $\sigma \prec \tau$ means that σ is an initial segment of τ, and similarly if x is an infinite sequence, $\sigma \prec x$ means that σ is an initial segment of x. A *tree* T on X is a subset of $X^{<\omega}$ such that if $\sigma \prec \tau$ and $\tau \in T$ then $\sigma \in T$. The proof of Lemma 2.2 will involve trees on $X = (2 \times 2 \times 2)$; we identify a sequence in $(2 \times 2 \times 2)^{<\omega}$ of length n with three length n sequences in $2^{<\omega}$, and similarly for infinite sequences. Let T be a tree on $(2 \times 2 \times 2)$; then $[T]$ denotes the set of infinite branches through T, that is,

$$[T] = \{(w, y, z) \in 2^\omega \times 2^\omega \times 2^\omega : \text{For all } n \in \omega,\ (w \restriction n,\ y \restriction n,\ z \restriction n) \in T\}.$$

Lemma 7.1. *Let $S \subset 2^\omega \times 2^\omega$ be $\mathbf{\Sigma}_1^1$. There is a tree T on $2 \times 2 \times 2$ such that for all $(w, y) \in 2^\omega \times 2^\omega$, $(w, y) \in S$ iff: There is a $z \in 2^\omega$ which is not eventually* 0 *and $(w, y, z) \in [T]$.*

The usual representation theorem for $\mathbf{\Sigma}_1^1$ (see Moschovakis [20]) gives a tree T' on $2 \times 2 \times \omega$ such that S is the projection of $[T']$. This is another way of saying that S is the projection of a closed set in $2^\omega \times 2^\omega \times \omega^\omega$. Lemma 7.1 follows from this by identifying $x \in \omega^\omega$ with $0^{x(0)\frown}1^\frown 0^{x(1)\frown}1^\frown 0^{x(2)\frown}1^\frown \cdots \in 2^\omega$.

Fix a family $\{N_\sigma : \sigma \in (2 \times 2 \times \omega \times 2)^{<\omega}\}$ of clopen subsets of 2^ω with the following properties.

(i) If $\sigma \prec \tau$ then $N_\tau \subset N_\sigma$.

(ii) If σ and τ are incompatible then $N_\sigma \cap N_\tau = \varnothing$.

Let $n \mapsto \tau_n$ be an enumeration of $2^{<\omega}$ such that if $\tau_n \prec \tau_m$ then $n \le m$. Let t_n be the length of τ_n.

Let $\sigma = (\alpha, \beta, \gamma, \delta) \in (2 \times 2 \times \omega \times 2)^{<\omega}$ be a sequence, let $k+1$ be its length, and let $n \in \omega$. Call σ *n-good* if all three of the following conditions hold.

(i) $\gamma \in \omega^{<\omega}$ is strictly increasing and $\gamma(k) = t_n$.

(ii) $\beta(k) = \delta(k) = 1$.

(iii) If $p = \operatorname{card}\{m : \delta(m) = 1\}$ then $\alpha \restriction p = \tau_n \restriction p$.

Lemma 7.2. *For any n, there are only finitely many sequences in $(2 \times 2 \times \omega \times 2)^{<\omega}$ which are n-good.*

Now to prove Lemma 2.2, let $Q \subset 2^\omega$ be an arbitrary $\mathbf{\Sigma}^1_2$ set. Let $P \subset 2^\omega \times 2^\omega$ be $\mathbf{\Pi}^1_1$ such that $w \in Q$ iff there exists a y such that $(w, y) \in P$. By Lemma 7.1, fix a tree T on $2 \times 2 \times 2$ such that $(w, y) \notin P$ iff:

(7.3) There is a $z \in 2^\omega$ which is not eventually 0 and $(w, y, z) \in [T]$.

We define, for each n, a function $g_n: 2^\omega \times 2^\omega \to 2$ as follows.

$$g_n(w, x) = \begin{cases} 1 & \text{if there exists a } k \in \omega \text{ and there exists an } n\text{-good} \\ & \sigma = (\alpha, \beta, \gamma, \delta) \in (2 \times 2 \times \omega \times 2)^{<\omega} \text{ of length } k+1 \\ & \text{such that } x \in N_\sigma \text{ and } (w \restriction k+1, \alpha, \beta) \in T, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 7.2, there are only finitely many N_σ's on which g_n is 1. So g_n is continuous.

We show below that these g_n's satisfy Lemma 2.2 for the given Q. Before giving a formal proof, let us point out the idea behind the definition of g_n. By (7.3) $w \notin Q$ iff for all y there is a z not eventually 0 with $(w, y, z) \in [T]$. The idea behind the definition of g_n is this: The fact that $g_{n_i}(w, x) = 1$ for $(n_i) \in \mathrm{SI}$ corresponds to the existence of such a branch (w, y, z). For each n_i, the fact that $g_{n_i}(w, x) = 1$ determines a finite approximation $(\alpha, \beta) \in (2 \times 2)^{<\omega}$ to (y, z). The γ controls the length of (α, β). The (α, β)'s are not initial segments of (y, z), but they converge to (y, z) as $n_i \to \infty$; the δ controls the rate of convergence.

(a) $\Rightarrow$ (b). Let $w \in Q$. Q is the projection of P, so choose a $y \in 2^\omega$ such that $(w, y) \in P$. Let $(n_i) \in \mathrm{SI}$ be such that $\tau_{n_i} = y \restriction i$. We show that for all $x \in 2^\omega$, $g_{n_i}(w, x) \to 0$, and thus prove that (b) holds.

Suppose this is not so. Then there is an $x \in 2^\omega$ and a subsequence (m_j) of (n_i) such that for all j, $g_{m_j}(w, x) = 1$. Consider the definition of g_{m_j}. Clearly for each j there is an m_j-good finite sequence σ_j which causes $g_{m_j}(w, x)$ to be 1. Since $x \in N_{\sigma_j}$, the σ_j's must all be compatible. So there is a $(y', z, u, v) \in 2^\omega \times 2^\omega \times \omega^\omega \times 2^\omega$, and for each j there is an $l_j \in \omega$ such that $\sigma_j = (y' \restriction l_j+1, z \restriction l_j+1, u \restriction l_j+1, v \restriction l_j+1)$. By definition of m_j-good and of the function g_{m_j}, this means:

(i) $u \restriction l_j + 1$ is strictly increasing and $u(l_j) = t_{m_j}$.

(ii) $z(l_j) = v(l_j) = 1$.

(iii) If $p_j = \mathrm{card}\{m \leq l_j: v(m) = 1\}$ then $y' \restriction p_j = \tau_{m_j} \restriction p_j$.

(iv) $(w \restriction l_j+1, y' \restriction l_j+1, z \restriction l_j+1) \in T$.

By definition of the sequence (n_i), $\mathrm{length}(\tau_{n_i}) = i$; as (m_j) is a subsequence of (n_i), $t_{m_j} = \mathrm{length}(\tau_{m_j}) \geq j$; hence (i) implies that $l_j \to \infty$. So (ii) implies that neither z nor v is eventually 0. Therefore $p_j \to \infty$; so by (iii) and the definition of (n_i), $y' = y$. But then, by (iv), $(w, y, z) \in [T]$ and z is not eventually 0. So by (7.3) $(w, y) \notin P$. But y was originally chosen so that $(w, y) \in P$.

(b) $\Rightarrow$ (a). Let $w \in 2^\omega \setminus Q$, and fix $(n_i) \in \mathrm{SI}$. We show that there is an $x \in 2^\omega$ such that the sequence $(g_{n_i}(w, x))$ does not converge to 0. By passing to a subsequence if necessary, we may assume that $t_{n_i} = \mathrm{length}(\tau_{n_i})$ is strictly increasing and that (τ_{n_i}) converges to some $y \in 2^\omega$.

Since Q is the projection of P and $w \notin Q$, clearly $(w, y) \notin P$. By (7.3) there is a $z \in 2^\omega$ not eventually 0 such that $(w, y, z) \in [T]$. Define $u \in \omega^\omega$ by $u(i) = t_{n_i}$. Define $v: \omega \to 2$ by induction.

$$v(i) = \begin{cases} 1 & \text{if } z(i) = 1 \text{ and if we set } p = 1 + \operatorname{card}\{m < i : v(m) = 1\}, \\ & \text{then for all } j \geq i,\ y \upharpoonright p = \tau_{n_j} \upharpoonright p, \\ 0 & \text{otherwise}. \end{cases}$$

Note that v is not eventually 0. For let i_0 be such that $v(i_0) = 1$ (or $i_0 = -1$), and let $p = 1 + \operatorname{card}\{m \leq i_0 : v(m) = 1\}$. Since $\tau_{n_i} \to y$, there is a least $j_0 > i_0$ so that if $j \geq j_0$ then $\tau_{n_j} \upharpoonright p = y \upharpoonright p$. And since z is not eventually 0, there is a least $i_1 \geq j_0$ with $z(i_1) = 1$. But then $v(i) = 0$ for $i_0 < i < i_1$, and $v(i_1) = 1$.

Let $x \in \bigcap\{N_\sigma : \sigma \prec (y, z, u, v)\}$. To finish the proof it will suffice to show that for any j with $v(j) = 1$, $g_{n_j}(w, x) = 1$. Fix such a j. Let $\sigma_j = (y \upharpoonright j+1, z \upharpoonright j+1, u \upharpoonright j+1, v \upharpoonright j+1)$. The following three facts are direct consequences of the definitions of u and v.

(i) $u \upharpoonright j+1$ is strictly increasing and $u(j) = t_{n_j}$.

(ii) $z(j) = v(j) = 1$.

(iii) If $p = \operatorname{card}\{m \leq j : v(m) = 1\}$ then $y \upharpoonright p = \tau_{n_j} \upharpoonright p$.

Now (i)–(iii) mean that σ_j is n_j-good. Clearly $x \in N_{\sigma_j}$. Since $(w, y, z) \in [T]$, clearly $(w \upharpoonright j+1, y \upharpoonright j+1, z \upharpoonright j+1) \in T$. So by definition of g_{n_j}, $g_{n_j}(w, x) = 1$. □

8. Proof of Lemma 4.3

We can view points t in $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ as being members of 2^ω, by writing the real number $t/2\pi$ in base 2. Formally, let $f: 2^\omega \to \mathbf{T}$ be the function

$$f(x) = 2\pi \cdot \sum_{i=0}^{\infty} x(i) \cdot 2^{-(i+1)}.$$

Thus multiplication by 2, in $\mathbf{T}$, corresponds to the operation of removing the first coordinate, in 2^ω. We use the following lemma, which is elementary.

Lemma 8.1. *Let $I \subset \mathbf{T}$ be an interval of length $2\pi/s$, and let m be a positive integer. Let p be the integer part of $\log_2(m)$ and let q be the integer part of $\log_2(s)$. For any $\sigma: p \to 2$ there exists a $\tau: (p+q+3) \to 2$ such that $\sigma \prec \tau$, and such that for any $y \in 2^\omega$, if $\tau \prec y$ then $m \cdot f(y) \in I$.*

Fix a sequence $[a(n), b(n)]$ of intervals in ω such that:

$$\text{(8.2)} \qquad \begin{array}{l} \text{(i) } a(n+1) > b(n). \\ \text{(ii) } b(n) - a(n) \to \infty. \\ \text{(iii) } a(n+1) - b(n) \to \infty. \end{array}$$

For $x \in 2^\omega$, let

$$E_x = \{y \in 2^\omega : \text{For all } n, \text{ if } x(n) = 0 \text{ then for all } c \in [a(n), b(n)],\ y(c) = 0\}.$$

For $K \in \mathscr{K}(2^\omega)$, let $\widehat{K} = \bigcup\{E_x : x \in K\}$.

To prove Lemma 4.3, let $h: \mathscr{K}(2^\omega) \to \mathscr{K}(\mathbf{T})$ be the following function: $h(K) = f[\widehat{K}]$. It is easy to see that $h(K)$ is compact and h is continuous.

Proof of Lemma 4.3(a). Let $K \in Z$. Then there is a sequence (p_{n_i}) of projection functions which converges to 0 pointwise on K. For $j \in \omega$, let $L_j = \{y \in 2^\omega : \text{For all } i \geq j \text{ and all } c \in [a(n_i), b(n_i)],\ y(c) = 0\}$, let $M_j = f[L_j]$ and let $M = \bigcup_j M_j$. If $x \in K$ then there is a j such that for all $i \geq j$, $p_{n_i}(x) = 0$; hence if $y \in \widehat{K}$, there is a j such that $y \in L_j$. That is, $h(K) \subset M$. So to prove Lemma 4.3(a), all that remains to be shown is that $M \in D\uparrow$.

Clearly $M_j \subset \mathbf{T}$ is compact and $M_j \subset M_{j+1}$. Fix j. We show that M_j is a D-set. By definition of L_j, if $t \in M_j$ then for $i \geq j$:

$$|2^{a(n_i)}t| \leq 2^{-(b(n_i)-a(n_i))}.$$

Hence by (8.2)(ii), the sequence $(\sin 2^{a(n_i)}t)$ converges to 0 uniformly on M_j.

Proof of Lemma 4.3(b). Let $K \in \mathscr{K}(2^\omega) \setminus Z$ and let $M \in H\uparrow$. We show that $h(K) \neq M$. Suppose $M = \bigcup_i M_i$, where $M_i \in \mathscr{K}(\mathbf{T})$ is an H-set and $M_i \subset M_{i+1}$. Also suppose that I_i is an interval of $\mathbf{T}$ and v_i is a strictly increasing sequence of positive integers such that I_i and v_i witness that M_i is an H-set; that is, for all j, $v_i(j) \cdot M_i \cap I_i = \varnothing$. To show that $h(K) \neq M$, it will suffice to prove the following fact: There is a $t \in h(K)$ such that for infinitely many i's there is some $m_i = v_i(j_i)$ for which $m_i t \in I_i$.

Let $p(i, j)$ be the integer part of $\log_2(v_i(j))$ and let $q(i)$ be the integer part of $\log_2(s_i)$, where $2\pi/s_i$ is the length of I_i. We inductively define strictly increasing functions $i \mapsto j_i$ and $i \mapsto n_i$ such that the intervals

$$J_i = [p(i, j_i), p(i, j_i) + q(i) + 2]$$

satisfy the following two properties.

$$\text{(8.3)} \qquad \begin{aligned} &\text{(i) } \max(J_i) < \min(J_{i+1}). \\ &\text{(ii) } J_i \subset [b(n_{i-1}) + 1, a(n_{i+1}) - 1]. \end{aligned}$$

That is, J_i intersects at most one of the intervals of (8.2), namely $[a(n_i), b(n_i)]$. This is clearly possible by (8.2)(iii) and the fact that $\lim_{j\to\infty} p(i, j) = \infty$. Let $m_i = v_i(j_i)$. By Lemma 8.1, if $\sigma_i \in 2^{<\omega}$ is defined on numbers less than $\min(J_i)$, then there is a $\tau_i \in 2^{<\omega}$ extending σ_i and defined on numbers less than or equal to $\max(J_i)$, such that for any $y \in 2^\omega$ with $\tau_i \prec y$, $m_i \cdot f(y) \in I_i$. Hence for any $B \subset \omega$ there is a $y \in 2^\omega$ with the following property. For all $i \in B$, $m_i \cdot f(y) \in I_i$, and for all $c \notin \bigcup\{J_i : i \in B\}$, $y(c) = 0$.

Since $K \notin Z$, there is an $x \in K$ such that $B = \{i : x(n_i) = 1\}$ is infinite. Fix such an x and B. Choose a y with the above property, for this particular B. Let $t = f(y)$. Then for infinitely many i, $m_i t \in I_i$. So to complete the proof of Lemma 4.3, all we need to show is that $t \in h(K)$. If $c \notin \bigcup\{J_i : i \in B\}$, then $y(c) = 0$. By (8.3), if $n \notin \{n_i : i \in B\}$ then for all $c \in [a(n), b(n)]$, $y(c) = 0$. This ensures that $y \in E_x$. Hence $y \in \widehat{K}$ and $t \in h(K)$. □

The function h in the above proof is not one-to-one. In order to prove Theorem 5.4 we need to modify the proof of Lemma 4.3 to get a one-to-one

h'. This can be done as follows. Let $h_1 : \mathscr{K}(2^\omega) \to \mathscr{K}([0, \pi])$ be the function $h_1(K) = h(K) \cap [0, \pi]$. As is clear from the proof of Lemma 4.3(a), there is an $F \subset (\pi, 2\pi)$ such that F is homeomorphic to 2^ω and the sequence $(\sin 2^{a(n)} t)$ converges to 0 on F. $\mathscr{K}(2^\omega)$ is also homeomorphic to 2^ω. Let $h_2 : \mathscr{K}(2^\omega) \to F$ be a homeomorphism. Then let

$$h'(K) = h_1(K) \cup \{h_2(K)\}.$$

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