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3-MANIFOLDS WHICH ADMIT FINITE GROUP ACTIONS

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ABSTRACT. We prove several results which support the following conjectures: (1) Any smooth action of a finite group on a geometric 3-manifold can be conjugated to preserve the geometric structure. (2) Every irreducible closed 3-manifold M with infinite $\pi_1(M)$ is finitely covered by a Haken 3-manifold.

1. INTRODUCTION AND PRELIMINARY

The following Conjectures 1 and 2 in 3-manifold theory are important.

Conjecture 1. Suppose M is a closed geometric 3-manifold in the sense of Thurston. Then any smooth action of a finite group G can be conjugated to preserve the geometric structure.

For Conjecture 1, after the work of Thurston, Meeks and Scott, and many others, the remaining open cases are that M supports the geometry of S^3 and G acts freely or M supports the hyperbolic geometry and the fixed point set of G, $\text{fix}(G)$, is a finite set (see [T2], [MS] or [E]).

For hyperbolic 3-manifolds, the conjecture below is closely related to Conjecture 1.

Conjecture 1a. Suppose M is a closed hyperbolic 3-manifold and two smooth group actions G_0 and G_1 are realizations of a subgroup $G \subset \text{Out}\,\pi_1(M)$. Then G_1 and G_0 are conjugate.

Conjecture 1a is true if the dimensions of the fixed point sets of both G_0 and G_1 are at least 1 by Thurston and Mostow (see the arguments in §2), or the first Betti number of M, $\beta_1(M)$, is positive and $G = Z_2$ by Tollefson.

Conjecture 2. Every irreducible rational homology 3-sphere M with $|\pi_1(M)| = \infty$ is finitely covered by a Haken 3-manifold.

Conjecture 2 naturally divides into two parts.

Conjecture 2a. Every 3-manifold M with $|\pi_1(M)| \neq 1$ has a nontrivial finite cover.

Conjecture 2a holds for every closed 3-manifold which is homotopy equivalent to a 3-manifold with Thurston's geometric decomposition or is not an integer homology sphere.

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Conjecture 2b. Every irreducible rational homology 3-sphere M with $|\pi_1(M)| = \infty$ and a nontrivial finite cover is finitely covered by a Haken 3-manifold.

On Conjecture 2, J. Hempel did some pioneering work. Many interesting facts have been proved by him and others, but this conjecture is still mainly open.

In this paper, several facts supporting the conjectures above are proved. (The author realizes some connections between Theorem 3 below and a main result of recently published work of J. Hempel (4.2 of [H3]).)

Theorem 1. *Suppose M is a closed hyperbolic 3-manifold, and smooth group actions G_0 and G_1 are two realizations of a subgroup $G \subset \text{Out}\ \pi_1(M)$. Then $\text{fix}(G_0)$ and $\text{fix}(G_1)$ are homeomorphic.*

Corollary of Theorem 1. *Suppose M, G, G_0 and G_1 are as in Theorem 1. Then G_0 and G_1 are conjugate if there is an i, $i = 0$ or $i = 1$, such that either $\dim \text{fix}(G_i) > 0$, or G_i is fixed point free and M/G_i is nonorientable.*

Theorem 2. *Suppose M is a closed hyperbolic 3-manifold. Then any smooth action of a finite group G on M can be conjugated to preserve the hyperbolic structure if either*

(1) *$\dim \text{fix}(G) > 0$, or G is a free action and M/G is nonorientable; or*

(2) *M is nonorientable and G is cyclic; or*

(3) *$\beta_1(M)$ is odd and G is cyclic; or*

(4) *M is nonorientable and $\beta_1(\widetilde{M})$ is even for the orientable double cover $\widetilde{M}$ of M.*

Theorem 3. *Suppose M is an irreducible rational homology 3-sphere with $|\pi_1(M)| = \infty$, and M admits an orientation reversing periodic map of order $2^n q$, where q is an odd number and $n > 0$. Then M is finitely covered by a Haken 3-manifold (actually Z-presentable 3-manifold) if either*

(1) *$n = 1$ and M has a nontrivial finite cover; or*

(2) *$q > 1$ and $|H_1(M, Z)|$ is odd (in particular, if M is an integer homology sphere); or*

(3) *M is not a cyclic branched cover of a homotopy 3-sphere branched over a strongly amphicheiral hyperbolic link and M has a nontrivial finite cover.*

The next corollary gives a minor support of Thurston's geometrization conjecture.

Corollary of Theorem 3. *Suppose an irreducible closed orientable 3-manifold M admits an orientation reversing periodic map h of order $2k$. Then M is homotopic to a 3-manifold with geometric decomposition in the sense of Thurston if either $k > 1$ or M has a nontrivial finite cover.*

Question. Suppose M is a cyclic branched cover of S^3 branched over a strongly amphicheiral hyperbolic link. Is M virtually Haken?

If the answer is positive, with (3) of Theorem 3, 3.3 of [T1] and 5.7.4 of [T2], an interesting corollary will be

Proposition. *Suppose the Poincaré Conjecture is true. Let M be a hyperbolic 3-manifold which admits an orientation reversing homeomorphism. Then M is virtually Haken.*

All terminologies about 3-manifold theory not defined in this section are standard and can be found in [BM, H1, R, S, T1, and T2]. For index of fixed point as well as covering space theory, see Appendices D and B of [Ki].

All manifolds and all group actions on manifolds are smooth. Suppose a finite group G acts on a manifold M. Then for any $g \in G$, $\operatorname{fix}(g) = \{x \in M | g(x) = x\}$; and $\operatorname{fix}(G) = \{x \in M | g(x) = x$ for some $g \in G, g \neq \mathrm{id}\}$. For any polyhedron X, the dimension of X is denoted by $\dim X$. When we say $\dim \operatorname{fix}(G) = 0$, we mean that $\operatorname{fix}(G)$ is a nonempty finite set. We say G is fixed point free or is a free action, if $\operatorname{fix}(g) = \varnothing$, $g \in G$, $g \neq \mathrm{id}$. For any set B, $|B|$ is the cardinality of B.

A 3-manifold M is virtually Haken, if M is finitely covered by a Haken 3-manifold. A 3-manifold M is Z-presentable, if the first Betti number of M, $\beta_1(M)$, is positive. A 3-manifold M is virtually Z-presentable, if M is finitely covered by a Z-presentable 3-manifold. A 3-manifold M is a rational homology sphere, if $H_*(M, Q) = H_*(S^3, Q)$. Here Q is the field of rational numbers and Z is the ring of integers.

The term hyperbolic 3-manifold (structure) always means complete hyperbolic 3-manifold (structure) of finite volume. A closed 3-manifold M is geometric, if M supports one of the eight geometries described in [T1] or [S]. A closed irreducible 3-manifold M has a geometric decomposition, if M is geometric or Haken. A link L is strongly amphicheiral, if L is invariant under an orientation reversing involution on M.

Let B^3 be the unit ball in R^3. Then the Poincaré model of hyperbolic 3-space H^3 can be identified with $\operatorname{int} B^3$, the interior of B^3. ∂B^3 will be denoted by S^2_∞. $\operatorname{Iso} H^3$, the full isometry group of H^3, can be naturally identified with $GM(S^2_\infty)$, the general Möbius group or the full conformal group on two-sphere. For a hyperbolic 3-manifold M, $\operatorname{out} \pi_1(M)$ is finite (see 5.7.4 of [T2]).

In proving Theorems 1 and 2, when we say a hyperbolic 3-manifold M, we mean that the 3-manifold M admits a hyperbolic structure, but no hyperbolic metric is specified; when we say a hyperbolic 3-manifold M_ρ, we mean the 3-manifold M with the given hyperbolic metric ρ.

For any $x, y \in M$, $\rho(x, y)$ denotes the distance between x, y under the given metric ρ. Let M be a closed hyperbolic 3-manifold, two hyperbolic metrics ρ and ρ_1 are equivalent if there exists a homeomorphism $h: M \to M$ such that $\rho_1(x, y) = \rho(h(x), h(y))$. We denote ρ_1 as $h^*\rho$. For a covering map $p: \widetilde{M} \to M_\rho$, we have the pullback metric $p^*\rho$ on $\widetilde{M}$. The deck transformation group acts on $\widetilde{M}_{p^*\rho}$ as a group of isometries.

The following two results will be repeatedly used in this paper.

Theorem A ([T3] or [MS]). *Suppose that M is a closed irreducible 3-manifold and a finite group G acts on M smoothly with* $\dim \operatorname{fix}(G) > 0$. *Then M has a geometric decomposition. Furthermore, if M admits a hyperbolic metric ρ, then G can be conjugated to be a group of isometries on M_ρ.*

Theorem B. *Suppose M is a closed hyperbolic 3-manifold with positive first Betti number. If f, g are two involutions on M such that the induced automorphisms f_*, g_* on $\pi_1(M)$ are in the same outer-automorphism class, then f and g are conjugate.*

Theorem B is a generalization of a special case of Theorem 7.1 of [To] which is only stated for the orientable case. We make a comment below for the verification of Theorem B on the nonorientable case. The proof of 7.1 in [To] need only to use 3.7, 4.2 and 6.1 of [To] where the condition "orientable" is posed for each of them. Since 3.3, 3.4, and 3.6 of [To] are actually proved for nonorientable case, so is 3.7. Since 4.1 is also true for nonorientable case (which is implicitly contained in Chapters 10 and 13 of [H1]), so is 4.2. Since $\chi(F) < 0$ is the only possible case in the verification of Theorem B, 6.1 is also true for nonorientable case by the uniqueness of the Nielsen Realization Theorem (see p. 342 of [E]). Finally, since the fundamental groups of hyperbolic 3-manifolds are center free, and since we need only to consider the case both $\dim \mathrm{fix}(f)$ and $\dim \mathrm{fix}(g)$ are zero by the Corollary of Theorem 1, many long and complicated arguments in the proof of 7.1 in [To] can be ignored in the verification.

2. The proof of Theorem 1

Lemma 1. *Let G be a finite group acting on a hyperbolic 3-manifold M_ρ. Then the following two statements are equivalent:*

(1) *G can be conjugated to be a group of isometries on M_ρ.*

(2) *There exists a hyperbolic metric ρ_1 on M under which G becomes a group of isometries.*

Proof. (2) $\Rightarrow$ (1) By the Mostow Rigidity Theorem (5.7.2 of [T2]), id: $\pi_1(M_\rho) \to \pi_1(M_{\rho_1})$ can be realized by the unique isometry $h\colon M_\rho \to M_{\rho_1}$. Then for any $g \in G$, we have

$$\rho(x, y) = \rho_1(h(x), h(y)) = \rho_1(gh(x), gh(y)) = \rho(h^{-1}gh(x), h^{-1}gh(y))$$

i.e. $h^{-1}gh$ is an isometry of M_ρ.

(1) $\Rightarrow$ (2) is a direct verification from the definition. □

Lemma 2. *A closed p^2-irreducible nonorientable 3-manifold M supports a hyperbolic structure if and only if $\pi_1(M)$ contains no $Z \oplus Z$ subgroup.*

Proof. Since the first Betti number of any nonorientable closed 3-manifold is positive, any P^2-irreducible nonorientable closed 3-manifold is sufficiently large (6.7 of [H1]). Hence the orientable double cover $\widetilde{M}$ of M is Haken. If $\pi_1(M)$ contains no $Z \oplus Z$ subgroup, then $\pi_1(\widetilde{M})$ contains no $Z \oplus Z$ subgroup. So $\widetilde{M}$ supports a hyperbolic structure by 2.5 of [T1]. The fact M is covered by hyperbolic 3-manifold implies that M is homotopy equivalent to a hyperbolic 3-manifold by 6.7.3 of [T2]. Finally since M is p^2-irreducible and sufficiently large, M itself is hyperbolic by 13.6 of [H1] and its proof.

The "only if" statement is well known. □

Lemma 3. *Let M be a closed hyperbolic 3-manifold. Suppose a smooth action of a finite group G on M is fixed point free. If the quotient manifold M/G is nonorientable, then there is a hyperbolic metric on M such that the action of G becomes a group of isometries under this metric.*

Proof. We have the covering map $p\colon M \to M/G$. Since any hyperbolic 3-manifold is p^2-irreducible, by the strong version of the equivariant sphere theorem [MSY], M/G contains no fake ball. So M/G is p^2-irreducible. Since $\pi_1(M)$ is a finite index subgroup of $\pi_1(M/G)$, from algebra, $\pi_1(M)$ contains no $Z \oplus Z$ subgroup implies that $\pi_1(M/G)$ contains no $Z \oplus Z$ subgroup. By

Lemma 2, we know that the quotient manifold M/G has a hyperbolic metric ρ. Lifting this metric to M, we get a hyperbolic 3-manifold $M_{p^*\rho}$. The action of G, as the action of deck transformation group, becomes a group of isometries on $M_{p^*\rho}$. □

Proof of Theorem 1. Suppose M_ρ is a closed hyperbolic 3-manifold and $g\colon M_\rho \to M_\rho$ is a homeomorphism. Then any lift $\tilde{g}\colon H^3 \to H^3$ of g has a unique continuous extension $\tilde{g}^\infty$ on $B^3 = S^2_\infty \cup H^3$ (see 5.9 of [T2]). Moreover, since $\tilde{g}$ has an inverse, $\tilde{g}^\infty$ is also a homeomorphism. If g is an isometry, then $\tilde{g} \in \operatorname{Iso} H^3$ and $\tilde{g}^\infty|_{S^2_\infty}$ is conformal.

Suppose G, G_0 and G_1 are as in Theorem 1. Let $\widetilde{G}_i$ be the group of all lifts on H^3 of all elements in G_i and $\widetilde{G}_i^\infty$ be the extension of $\widetilde{G}_i$ on $B^3 = S^2_\infty \cup H^3$, $i = 1, 2$.

Let the homomorphism $\gamma_i\colon \widetilde{G}_i \to \widetilde{G}_i^\infty \to \widetilde{G}_i^\infty|_{S^2_\infty}$ be the extension followed by the restriction.

Claim. $\gamma_i\colon \widetilde{G}_i \to \widetilde{G}_i^\infty|_{S^2}$ is an isomorphism, $i = 0, 1$.

Proof. We prove the claim for G_1. If we choose G_0 to be a subgroup of $\operatorname{Iso} M_\rho$, by Mostow Rigidity Theorem, then $\widetilde{G}_0$ is a subgroup of $\operatorname{Iso} H^3$ and the restriction of $\widetilde{G}_0^\infty|_{S^2_\infty}$ is a subgroup of $GM(S^2_\infty)$. Also $\gamma_0\colon \widetilde{G}_0 \to \widetilde{G}_0|_{S^2_\infty}$ is an isomorphism. Since M is aspherical, for any $g \in G$, if $g_0 \in G_0$ and $g_1 \in G_1$ are the corresponding elements, then g_0 and g_1 are homotopic. So there is $H\colon M \times I \to M$ such that $H(*, 0) = g_0$ and $H(*, 1) = g_1$. Let $\widetilde{H}\colon H^3 \times I \to H^3$ be a lift of H with $\widetilde{H}(*, 0) = \tilde{g}_0$ and $\widetilde{H}(*, 1) = \tilde{g}_1$. Then $\{\tau\widetilde{H}, \tau \in \pi_1(M \times I)\}$ are the lifts of H and we have an 1-1 correspondence

(I) $$\Phi = H_\#\colon \{\text{all lifts of } g_0\} \to \{\text{all lifts of } g_1\}$$

which is given by $\tau\tilde{g}_0 \mapsto \tau\tilde{g}_1$, $\tau \in \pi_1(M \times I) = \pi_1(M)$.

If H' is another homotopy from g_0 to g_1, let H'' be a cyclic homotopy which begins and ends at g_0 defined by $H''(*, t) = H(*, 2t)$ for $0 \le 2t \le 1$ and $H''(*, t) = H'(*, 2 - 2t)$ for $1 \le 2t \le 2$. For any $x \in M$, it is easy to see that the trace $H''(x, t)$, $0 \le t \le 1$, is a loop lying in the center of $\pi_1(M, g_0(x))$. Since $\pi_1(M)$ is center-free, $H''(x, t)$, $0 \le t \le 1$, is trivial. By covering space theory, any lift of the loop $H''(x, t)$, $0 \le t \le 1$ on H^3 is also a loop. Then one can verify by definition that the homotopy H' gives rise to the same correspondence Φ in (I), so Φ in (I) is independent the choice of the homotopy from g_0 to g_1 and yields an 1-1 correspondence

(II) $$\Phi\colon \widetilde{G}_0 \to \widetilde{G}_1.$$

Obviously the corresponding lifts $\tau\tilde{g}_0$ and $\tau\tilde{g}_1$ have a bounded distance given by H, i.e., $\operatorname{dist}(\tau\tilde{g}_0(\tilde{x}), \tau\tilde{g}_1(\tilde{x})) < A$ for a constant A and all $\tilde{x} \in H^3$. So the extensions $\tau\tilde{g}_0^\infty$ and $\tau\tilde{g}_1^\infty$ coincide on S^2_∞ (see 2.1 of [T2]). Hence the actions of $\widetilde{G}_0^\infty$ and $\widetilde{G}_1^\infty$ coincide on S^2_∞.

Suppose $\Phi(\tilde{g}_0) = \tilde{g}_1$ and $\Phi(\tilde{g}_0') = \tilde{g}_1'$ respectively, here $\tilde{g}_i$ and $\tilde{g}_i'$ are the lifts of g_i and g_i'. By the definition of Φ, there is a homotopy $\widetilde{H}$ from $\tilde{g}_0$ to $\tilde{g}_1$ which covers a homotopy H from g_0 to g_1; and a homotopy $\widetilde{H}'$ from $\tilde{g}_0'$ to $\tilde{g}_1'$ which covers a homotopy H' from g_0' to g_1'. Let $\widetilde{H}''$ be a homotopy from $\tilde{g}_0'\tilde{g}_0$ to $\tilde{g}_1'\tilde{g}_1$ which covers a homotopy from $g_0'g_0$ to $g_1'g_1$ defined by $\widetilde{H}''(*, t) = \tilde{g}_0'(*)\widetilde{H}(*, 2t)$ for $0 \le 2t \le 1$ and $\widetilde{H}''(*, t) = \widetilde{H}'(*, 2-2t)\tilde{g}_1(*)$ for $1 \le 2t \le 2$. By definition, $\Phi(g_0'g_0) = g_1'g_1$. So Φ is a homomorphism, and therefore is an isomorphism. So

$$\gamma_1 \colon \widetilde{G}_1 \xrightarrow{\Phi} \widetilde{G}_0 \xrightarrow{\gamma_0} \widetilde{G}_0^{\infty}|_{S^2_\infty} = \widetilde{G}_1^{\infty}|_{S^2_\infty}$$

is also an isomorphism. □

Now it is time to prove Theorem 1. By definition,

$$\operatorname{fix}(G_i) = \bigcup_{\substack{g_i \in G_i \\ g_i \neq \mathrm{id}}} \operatorname{fix}(g_i).$$

Since both $\widetilde{G}_0^{\infty}$ and $\widetilde{G}_1^{\infty}$ coincide on S^2_∞, and both γ_0 and γ_1 are isomorphisms, for any finite order element $\tilde{g}_0 \in \widetilde{G}_0$, there is a $\tilde{g}_1 \in \widetilde{G}_1$ such that

(III) $\quad \tilde{g}_0^{\infty}|_{S^2_\infty} = \tilde{g}_1^{\infty}|_{S^2_\infty}$, and the order of $\tilde{g}_1$ = the order of $\tilde{g}_0$.

Let h be a periodic map on B^3. By the classic Smith theory and the fact that an i-homology ball (sphere) is an i-ball (sphere) for $i = 0, 1, 2$, we see that h is either a rotation about a proper arc (in this case $\operatorname{fix}(h|_{\partial B^3})$ is two points), or a reflection about a proper disc (in this case $\operatorname{fix}(h|_{\partial B^3})$ is a topological circle), or h is fixed point free on ∂B^3.

Without loss of generality, we may assume that $\dim \operatorname{fix}(G_0) \ge \dim \operatorname{fix}(G_1)$.

If $\dim \operatorname{fix}(G_0) > 0$, we may assume that G_0 has been conjugated to be a group of isometry on M_ρ by Theorem A.

Case 1. $\dim \operatorname{fix}(G_0) = 2$. In this case, there must be a $g_0 \in G_0 \subset \operatorname{Iso} M_\rho$ such that $\operatorname{fix}(g_0)$ has a geodesic surface component. Then there is a lift $\tilde{g}_0 \in \operatorname{Iso} H^3$ of g_0 which is a reflection about a geodesic plane in H^3; therefore $\tilde{g}_0^{\infty}$ is a reflection about a proper disc on B^3. By (III), there is a periodic map $\tilde{g}_1 \in \widetilde{G}_1$ such that $\tilde{g}_1^{\infty}|_{S^2_\infty} = \tilde{g}_0^{\infty}|_{S^2_\infty}$; therefore $\operatorname{fix}(\tilde{g}_1^{\infty}|_{S^2_\infty})$ has a circle component which forces $\tilde{g}_1^{\infty}$ to be a reflection about a proper disc. Hence $\dim \operatorname{fix}(\tilde{g}_1) = 2$. It follows that $\dim \operatorname{fix}(G_1) = 2$.

Case 2. $\dim \operatorname{fix}(G_0) = 1$. Now there must be a $g_0 \in G_0 \subset \operatorname{Iso} M_\rho$ such that $\operatorname{fix}(g_0)$ has a geodesic circle component. Then there is a lift $\tilde{g}_0 \in \operatorname{Iso} H^3$ of g_0 which is a rotation about a geodesic axis in H^3; therefore $\tilde{g}_0^{\infty}$ is a rotation about a proper arc on B^3. Similarly by (III) there is a periodic map $\tilde{g}_1 \in \widetilde{G}_1$ such that $\operatorname{fix}(\tilde{g}_1^{\infty}|_{S^2_\infty})$ has two points which forces $\tilde{g}_1^{\infty}$ to be a rotation about a proper arc. It follows that $\dim \operatorname{fix}(G_1) = 1$.

From Cases 1 and 2, we see that if $\dim \operatorname{fix}(G_0) > 0$, then $\dim \operatorname{fix}(G_1) > 0$. By Theorem A, G_i conjugates to a $G_i' \subset \operatorname{Iso} M_\rho$. By the uniqueness part of Mostow Rigidity Theorem, we have $G_0' = G_1'$. Hence $\operatorname{fix}(G_0)$ and $\operatorname{fix}(G_1)$ are homeomorphic.

Case 3. $\dim \operatorname{fix}(G_0) = 0$. We may assume that G_0 is a group of isometry under some metric (not hyperbolic in general) on M. Suppose $\operatorname{fix}(g_0) \neq \varnothing$. Then for any $x \in \operatorname{fix}(g_0)$, there is a g_0-invariant ball $B(x)$ centered at x on which g_0 is orientation reversing. g_0 must be order 2, otherwise $\dim \operatorname{fix}(g_0) > 0$. It is not difficult to see the local index of g_0 on fixed point x is 1. So the Lefschetz number $L(g_0) = \#\operatorname{fix}(g_0)$. For $g_1 \in G_1$ which is homotopic to g_0, we have $L(g_0) = L(g_1)$; therefore $\#\operatorname{fix}(g_0) = \#\operatorname{fix}(g_1)$. Furthermore, if g_i, $g_i' \in G_i$, $g_i \neq g_i'$ and $\operatorname{fix}(g_i) \neq \varnothing \neq \operatorname{fix}(g_i')$, then $\operatorname{fix}(g_i) \cap \operatorname{fix}(g_i') = \varnothing$. (Otherwise, pick $x \in \operatorname{fix}(g_i) \cap \operatorname{fix}(g_i')$, there is a g_i- and g_i'-invariant ball $B(x)$ centered at x on which both g_i and g_i' are orientation reversing. As an orientation preserving periodic map on $B(x)$, $g_i g_i'$ has fixed point set of dimension at least 1. By our assumption on Case 3, we must have $g_i g_i' = \mathrm{id}$. Since both g_i and g_i' are order 2, we have $g_i = g_i'$.)

Recall that $L(\mathrm{id}) = \chi(M) = 0$ for $\mathrm{id} \in G$. We have

$$\begin{aligned}\#\operatorname{fix}(G_0) &= \sum_{\substack{g_0 \in G_0,\\ g_0 \neq \mathrm{id}}} \#\operatorname{fix}(g_0) = \sum_{g_0 \in G_0} L(g_0)\\ &= \sum_{g_1 \in G_1} L(g_1) = \sum_{\substack{g_1 \in G_1,\\ g_1 \neq \mathrm{id}}} \#\operatorname{fix}(g_1) = \#\operatorname{fix}(G_1)\end{aligned}$$

which implies $\operatorname{fix}(G_0)$ is homeomorphic to $\operatorname{fix}(G_1)$.

Case 4. $\operatorname{fix}(G_0) = \varnothing$. Then $\operatorname{fix}(G_1) = \varnothing$. This is a trivial case. □

Proof of the Corollary of Theorem 1. The case of $\dim \operatorname{fix}(G_i) > 0$ for $i = 1$ or $i = 2$ has been verified in the proof of Theorem 1.

Suppose G_i is fixed point free and M/G is nonorientable for $i = 0$ or $i = 1$, then G_i is fixed point free and M/G is nonorientable for both $i = 0$ and $i = 1$ by Theorem 1. By Lemma 3, there is a hyperbolic metric ρ_i on M such that G_i is a group of isometry on M_{ρ_i}. By Lemma 1, G_1 conjugates to a subgroup $G_1' \subset \operatorname{Iso} M_{\rho_0}$. By the uniqueness part of Mostow Rigidity Theorem, as subgroups, $G_0 = G_1'$. □

3. The proof of Theorem 2

Lemma 4. *Suppose M_ρ is a hyperbolic 3-manifold with positive first Betti number, and f is an involution on M, then f conjugates to an isometry of M_ρ.*

Proof. By the Mostow Rigidity Theorem, $f_*: \pi_1(M_\rho) \to \pi_1(M_\rho)$ can be realized by the unique isometry $g: M_\rho \to M_\rho$. Then f and g are involutions such that the induced automorphisms f_*, g_* on $\pi_1(M)$ are in the same outer-automorphism class of $\pi_1(M)$. By Theorem B, f and g are conjugate. □

Proof of Theorem 2. By Lemma 1, to prove Theorem 2, we need only to prove that there is a hyperbolic metric ρ on M such that G become a group of isometries on M_ρ.

Proof of (1). If $\dim \operatorname{fix}(G) > 0$, then Theorem 2 follows from Theorem A. If G is fixed point free and M/G is nonorientable, then Theorem 2 follows from Lemma 3.

Proof of (2). Suppose M is a closed nonorientable hyperbolic 3-manifold and $G = Z_n = \langle h | h^n = \mathrm{id}\rangle$.

By (1), we may assume that $\dim \operatorname{fix}(G) = 0$.

Let q be the smallest positive integer such that $\operatorname{fix}(h^q) \neq \varnothing$. Then h^{2q} is an orientation preserving periodic map and therefore $\dim \operatorname{fix}(h^{2q}) \geq 1$. By our assumption, we must have $h^{2q} = \mathrm{id}$ and then $2q = n$. Let $q = 2^k l$, here l is an odd number. Then $n = 2^{k+1} l$. There are two subcases.

Subcase (1). $k = 0$, then $G = Z_{2l}$, $l > 1$ is an odd number. We may assume h^2 acts freely on M. (Otherwise as above we get $\dim \mathrm{fix}(G) = 1$.) So we get a nonorientable hyperbolic 3-manifold

$$\overline{M} = \frac{M}{\langle h^2 | (h^2)^l = \mathrm{id}\rangle}$$

and h induces an involution $\bar{h}$ on $\overline{M}$. By Lemmas 3 and 4, there is a hyperbolic metric ρ under which $\bar{h}$ becomes an isometry. Put the metric $p^*\rho$ on M, where p is the covering map from M to $\overline{M}$, then h becomes an isometry on M. Hence the action of G become a group of isometries.

Subcase (2). $k > 0$, let $\widetilde{M}$ be the orientable double cover of M and $\langle \eta | \eta^2 = \mathrm{id}\rangle$ be the deck transformation group, where η is an orientation reversing involution.

Since every homeomorphism sends orientation preserving loops to orientation preserving loops, the action of group G can be lifted to an action of a group $\widetilde{G}$ with $|\widetilde{G}| = 2^{k+2} l$. Let $\tilde{h}$ be an orientation reversing lift of h, then $\tilde{h}^{2^{k+1} l}$ is a lift of the identity in G. Since $\tilde{h}^{2^{k+1} l}$ is orientation preserving, it must be the identity of $\widetilde{G}$.

Let $G_1 = \langle \tilde{h} | \tilde{h}^{2^{k+1} l} = \mathrm{id}\rangle$. If $\tilde{h}^i$ is not fixed point free for some $i < 2^{k+1} l$, then h^i is not fixed point free. This implies that $i = q = 2^k l$. Since $k > 0$, $\tilde{h}^{2^k l}$ is orientation preserving, it must have the fixed point set of dimension 1. It is impossible by our assumption. So G_1 acts freely on $\widetilde{M}$. Now we get a nonorientable hyperbolic 3-manifold $M_1 = \widetilde{M}/G_1$ and η induces an involution $\bar{\eta}$ on M_1. By Lemmas 3 and 4, there is a hyperbolic metric ρ on $\overline{M}$ such that $\bar{\eta}$ becomes an isometry.

Put the metric $p^*\rho$ on $\widetilde{M}$, where p is the covering map from $\widetilde{M}$ to M_1. G_1 and η becomes isometries on $\widetilde{M}_{p^*\rho}$. Hence the action $\widetilde{G}$ become a group of isometry on $\widetilde{M}_{p^*\rho}$. Since the generator of deck transformation group $\langle \eta | \eta^2 = \mathrm{id}\rangle$ is an isometry, the metric $p^*\rho$ on $\widetilde{M}$ induces a hyperbolic metric $\bar{\rho}$ on M under which the action of G is a group of isometries.

Proof of (3). Suppose M is a closed orientable hyperbolic 3-manifold with odd first Betti number. By (1), we may assume that $\dim \mathrm{fix}(G) \le 0$.

If the action of G is free, then the quotient manifold M/G is p^2-irreducible and sufficiently large by [MSY and Ha]. Moreover $\pi_1(M/G)$ contains no $Z \oplus Z$ subgroup; therefore it admits a hyperbolic metric ρ by Lemma 2. Then G becomes a group of isometries on $M_{p^*\rho}$.

If the action of G is not free, then the generator h is orientation reversing. Let q be the minimum positive integer such that $\dim \mathrm{fix}(h^q) = 0$. By the proof of (2), we have q is odd and $n = 2q$.

Let $G_1 = Z_q = \langle h^2 | (h^2)^q = \mathrm{id}\rangle$. Then G_1 acts freely on M. So we get $\overline{M} = M/G_1$.

Claim. The first Betti number of $\overline{M}$ is positive.

Proof. From linear algebra, the oddness of the first Betti number of M implies that any periodic automorphism on $H_2(M, Z)$ has a nonzero primitive invariant homology class α up to $\pm$. Then we have a h^2-equivariant two-sided incompressible surface F representing α by [Ha]. Now F is nonseparating. After assigning $+$ and $-$ to the two sides of F, by the fact that F and $h^{2i}(F)$ (with suitable orientations) are in the same homology class, there is a natural way to assign $+$ or $-$ to the two sides of $h^{2i}(F)$ for each i. The fact that q is odd implies that h always sends the positive sides to the positive sides. So the image $p(F)$ is still two-sided in $\overline{M}$, here p is the covering map from M to $\overline{M}$. The same argument shows that $p(F)$ is nonseparating in $\overline{M}$. Hence the first Betti number of $\overline{M}$ is positive.

Now (3) follows by the arguments used in the previous cases.

Proof of (4). Suppose M is nonorientable and the first Betti number of the orientable double cover $\widetilde{M}$ of M is even. By Theorem A and Lemma 3, we need only to consider the case $\dim \operatorname{fix}(G) = 0$. As we see before, if $\dim \operatorname{fix}(G) = 0$, there must be an involution with isolated fixed points only. This case is ruled out by the following Lemma 5.

Lemma 5. *Suppose M is a nonorientable closed 3-manifold, and g is an involution on M with isolated fixed points only, then the orientable double cover $\widetilde{M}$ of M has odd first Betti number.*

Proof. Since the Euler number of any closed 3-manifold is zero, $\#\operatorname{fix}(g)$ is equal to the absolute value of the Euler number of $M-\operatorname{fix}(g)$. Since $M-\operatorname{fix}(g)$ doubly covers $\{M-\operatorname{fix}(g)\}/\langle g\rangle$, $\#\operatorname{fix}(g)$ is even, denoted by $2k$.

Let $\tilde{g}$ and $\tilde{g}_1$ be the lifts of g on $\widetilde{M}$. Then we should have that $\#\operatorname{fix}(\tilde{g}) + \#\operatorname{fix}(\tilde{g}_1) = 4k$ (see 2.1 of [Ki]). Since the generator of deck transformation group is orientation reversing, one of those two lifts of g, say $\tilde{g}_1$, is orientation preserving. Hence $\tilde{g}_1$ acts freely on $\widetilde{M}$ and $\#\operatorname{fix}(\tilde{g}) = 4k$. The local index of an isolated fixed point of an orientation reversing involution is 1. So we get the Lefschetz number $L(\tilde{g}) = 4k$.

Let $\widetilde{g_{*i}}$ be the induced homomorphism on H_i and $\tilde{g}_i^*$ be the induced homomorphism on H^i. Let A_1 be the matrix of $\widetilde{g_{*1}}$ under some basis B of H_1 and A_2^* be the matrix of $\tilde{g}_2^*$ under the Poincaré dual basis B^* of B; A_2 be the matrix of $\widetilde{g_{*2}}$ under the algebraic dual basis of B^*.

Let a be a fundamental class of $H_3(\widetilde{M}, Z)$ and $b \in H^2(\widetilde{M}, Z)$. By Proposition 24.24 of [G], we have $\tilde{g}_{*1}[(a \cap \tilde{g}_2^*(b)] = \tilde{g}_{*3}(a) \cap b$.

From $\tilde{g}_{*1}^2 = \mathrm{id}$ and $\tilde{g}_{*3}(a) = -a$, we have $-a \cap \tilde{g}_2^*(b) = \tilde{g}_{*1}(a \cap b)$ which implies that under the Poincaré dual basis, $A_2^* = -A_1$. On the algebraic dual basis, we have $A_2^t = A_2^*$, here A_2^t is the transpose of A_2. Hence we get $A_2^t = -A_1$.

Set $\operatorname{trace} A_1 = l$, then $\operatorname{trace} A_2 = \operatorname{trace} A_2^t = -l$.

Now we have

$$\begin{aligned} 4k = L(\tilde{g}) &= 1 - \operatorname{trace} A_1 + \operatorname{trace} A_2 - (-1) \\ &= 1 - \operatorname{trace} A_1 - \operatorname{trace} A_1 - (-1) = 2 - 2l = 2(1-l). \end{aligned}$$

So $l-1$ should be even, therefore l should be odd. Eigenvalues of A_1 are 1 or -1 because $A_1^2 = \mathrm{id}$. This is possible only if the first Betti number of $\widetilde{M}$ is odd. □

4. The proof of Theorem 3

Lemma 6. *Suppose* $p: M \to \overline{M}$ *is a cyclic branched cover branched over a link* $\overline{L}$ *in* $\overline{M}$. *If* $\overline{M}$ *is virtually* Z*-presentable, then* M *is virtually* Z*-presentable.*

Proof. Suppose there is a finite cover $p': M' \to \overline{M}$ such that $\beta_1(M') > 0$. By passing to a finite cover, we may assume that p' is a regular cover. Let $N(\overline{L})$ be the regular neighborhood of $\overline{L}$. Then we have the commutative diagram of regular covering maps

$$\begin{array}{ccc} M_1 & \xrightarrow{p_1'} & M - p^{-1}(\operatorname{int} N(\overline{L})) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ M' - p'^{-1}(\operatorname{int} N(\overline{L})) & \xrightarrow{p'} & \overline{M} - \operatorname{int} N(\overline{L}) \end{array}$$

where M_1 is the regular covering space of $\overline{M} - \operatorname{int} N(\overline{L})$ corresponding to the subgroup $p_*(M - p^{-1}(\operatorname{int} N(\overline{L}))) \cap p'_*(M' - p'^{-1}(\operatorname{int} N(\overline{L})))$.

For any torus component T_1 of ∂M_1, $p_1'(T_1)$ is a boundary torus T of a component S of $p^{-1}(N(\overline{L}))$; $p_1(T_1)$ is a boundary torus T' of a component S' of $p'^{-1}(N(\overline{L}))$. We have $p'(S') = p(S) = \overline{S}$ for some component $\overline{S}$ of $N(\overline{L})$. Let $\overline{m}$ be the meridian of $\overline{S}$. Then $q\overline{m}$ can be lifted as the meridian of S for some q and $\overline{m}$ can be lifted as the meridian of S' (since $p': M' \to \overline{M}$ is a covering), therefore $q\overline{m}$ can be lifted as a simple closed curve m_1 on T_1 by the covering space theory and the definition of M_1. Note that $p_1'|: m_1 \to p_1'(m_1)$ is a covering map of degree 1 and $p_1|: m_1 \to p_1(m_1)$ is a covering map of degree q.

Now attaching solid torus $S^1 \times D^2$ to M_1 by identifying $(\partial S^1 \times D^2, x \times \partial D^2)$ with (T_1, m_1), then we can extend maps p_1 and p_1' from M_1 to $M_1 \cup S^1 \times D^2$ so that $p_1'|: S^1 \times D^2 \to S$ is a covering and $p_1|: S^1 \times D^2 \to S'$ is a branched covering with branching index q. Do the same thing for all components of ∂M_1, and denote $\widetilde{M} = M_1 \cup (\bigcup\{S^1 \times D^2\}'s)$, we get a commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{p_1'} & M \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ M' & \xrightarrow{p'} & \overline{M} \end{array}$$

where $p_1': \widetilde{M} \to M$ is a covering and $p_1: \widetilde{M} \to M'$ is a branched covering. Since the degree of p_1 is not 0, it is well known that $p'_{1*}(\pi_1(M_1))$ is a finite index subgroup of $\pi_1(M')$, then it follows that $p'_{1\#}(H_1(M_1, Z))$ is a finite index subgroup of $H_1(M', Z)$. Hence $\beta_1(M_1) \geq \beta_1(M') > 0$. □

Proof of Theorem 3. Let h be the orientation reversing periodic map of order $2^n q$.

Proof of (1). If $n = 1$ and M has a nontrivial finite cover. Since h^q is an orientation reversing involution on M, (1) follows from [W2].

From now on, we assume $n > 1$.

If the fixed point set of an orientation reversing periodic map is empty or is dimension 2, then it is not difficult to see that M virtually is Z-presentable (see [H2]). So we assume that h has isolated fixed points only. Then it is easy to see that the fixed point set of h^{2m} is dimension 1 for any $0 < 2m < 2^n q$.

By Lemma 6, to prove that M is virtually Z-presentable, we need only to prove that $M/\langle h^{2^i}\rangle$ is virtually Z-presentable for some i.

We have a branched covering $p\colon M \to M/\langle h^{2q}\rangle \to \overline{M} = M/\langle h^2\rangle$.

Now h induces the orientation reversing periodic maps h' and $\overline{h}$ on $M/\langle h^{2q}\rangle$ and $\overline{M}$ with isolated fixed points only respectively.

Claim 1. (a) If $|\pi_1(\overline{M})|$ is finite, then $|\pi_1(\overline{M})| = 1$.

(b) If $|\pi_1(M/\langle h^{2q}\rangle)|$ is finite, then $|\pi_1(M/\langle h^{2q}\rangle)| = 1$.

Proof. For (a). Suppose $|\pi_1(\overline{M})| = l$ for some integer l. Now $\overline{h}$ is an orientation reversing involution on $\overline{M}$ with isolated fixed point only. The fundamental group of nonorientable 3-manifold $(\overline{M} - \text{fix}(\overline{h}))/\langle\overline{h}\rangle$ is of order $2l$. By Epstein's Theorem (9.5 of [H]), $l = 1$.

For (b). Since h'^q is an orientation reversing involution on $M/\langle h^{2q}\rangle$ with isolated fixed point only, the proof of (b) is the same as that of (a).

Proof of (2). Since $q > 1$, so h'^2 has dimension 1 fixed point set. By Theorem A, $M/\langle h^{2q}\rangle$ admits a geometric decomposition. In particular, $\pi_1(M/\langle h^{2q}\rangle)$ is residually finite by Theorem 3.3 of [T1]; therefore M has a nontrivial finite cover. If $|\pi_1(M/\langle h^{2q}\rangle)| = \infty$, by the fact that h'^q is an orientation reversing involution on $M/\langle h^{2q}\rangle$ and [W2], $M/\langle h^{2q}\rangle$ is virtually Z-presentable.

Otherwise $|\pi_1(M/\langle h^{2q}\rangle)| = 1$ by Claim 1. Since $M/\langle h^{2q}\rangle$ has geometric decomposition, $M/\langle h^{2q}\rangle$ must be S^3. Then the cyclic branched covering $M/\langle h^{2g}\rangle \to \overline{M}$ is the cyclic branched covering from S^3 to S^3 branched over an unknotted circle $\overline{C}$ in downstair S^3 by the positive answer of the Smith Conjecture (see [BM]).

Since $\text{fix}(h) \neq \varnothing$, we can pick a circle component C of $\text{fix}(h^2)$. Then C is a component of upstair branch set with branching index $2^{n-1}q$.

Claim 2. $\text{fix}(\langle h\rangle) = C$.

If there is another component C_1 lying $\text{fix}(\langle h\rangle)$, then C_1 belongs to $\text{fix}(h^{2^{n-1}q})$. So the involution $h^{2^{n-1}q}$ contains at least two circles. The $|H_1(M, Z)|$ is odd implies that M is a mod 2 homology 3-sphere. By the classical Smith theory (5.1 of [B]), the one dimensional closed manifold $\text{fix}(h^{2^{n-1}q})$ is a module 2 homology sphere. It is a contradiction.

So M is a cyclic branched cover $M/\langle h^2\rangle = S^3$ branched over an unknotted circle C. It follows that M itself is S^3. It is a contradiction.

Proof of (3). If $|\pi_1(\overline{M})| = \infty$, since we assume that $\overline{M}$ has a finite cover and $\overline{h}$ is an involution with isolated fixed point only, $\overline{M}$ is virtually Z-presentable by [W2].

So we assume that $\overline{M}$ is a homotopy 3-sphere by Claim 1. First the branch set of p in down-stair is a $\overline{h}$-strongly amphicheiral link in $\overline{M}$.

Since $n > 1$, M has a geometric decomposition. If M contains two-sided incompressible torus, then M is virtually Z-presentable by [K or L]. Otherwise M is a Seifert manifold or a hyperbolic 3-manifold. If M is a Seifert 3-manifold, then $\pi_1(M) = \infty$ implies that M is finitely covered by a circle bundle over F_g with $g > 0$ (see p. 438 of [S]); therefore M is virtually Z-presentable. If M is hyperbolic, by Corollary 2.1 on p. 142 of [BM], the branch set is a hyperbolic link. □

Proof of the Corollary of Theorem 3. We only sketch the proof, the details can be found in the proof of Theorem 3.

If $\text{fix}(h)$ is empty or $\dim \text{fix}(h) = 2$, then M itself is Z-presentable; therefore M is Haken. So we may assume that $\text{fix}(h)$ is a nonempty finite set. If $k > 1$, then $\dim \text{fix}(h^2) = 1$, so M itself has a geometric decomposition. If $k = 1$, then $|\pi_1(M)| = 1$ or $|\pi_1(M)| = \infty$. If $|\pi_1(M)| = 1$, then $M = S^3$; if $|\pi_1(M)| = \infty$, then M has a nontrivial finite cover implies that M is finitely covered by Haken manifold $\widetilde{M}$. If $\widetilde{M}$ is a 3-manifold with a nontrivial torus decomposition, or is a Seifert manifold, so is M by Theorems 8.6 and 2.1 of [MS]. If $\widetilde{M}$ is hyperbolic, then M is homotopy equivalent to a hyperbolic 3-manifold by 6.7.3 of [T2]. □

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References

[B] G. Bredon, *Introduction to compact transformation group*, Academic Press, 1972.

[BM] H. Bass and J. Morgon, *The Smith conjecture*, Academic Press, 1984.

[E] A. L. Edmonds, *Transformation groups and low-dimensional manifolds*, Contemp. Math., vol. 36, Amer. Math. Soc., Providence, R.I., 1985, pp. 339–366.

[G] M. J. Greenberg, *Lectures on algebraic topology*, Benjamin, New York, 1967.

[Ha] J. Hass, *Surfaces minimizing area in their homology class and group action on* 3-*manifolds*, Math. Z. **199** (1988), 501–509.

[H1] J. Hempel, 3-*manifolds*, Ann. of Math. Studies, no. 86, Princeton Univ. Press, Princeton, N.J., 1976.

[H2] ——, *Orientation reversing involutions and the first betti number for finite covering of* 3-*manifolds*, Invent. Math. **67** (1982), 133–143.

[H3] ——, *Branch cover over strongly amphicheiral links*, Topology **29** (1990), 247–255.

[Ki] T. H. Kiang, *The theory of fixed point classes*, Science Press, Beijing, 1979, 1986; English transl., Springer-Verlag, Berlin, Heidelberg, and New York, 1989.

[K] S. Kojima, *Finite covers of* 3-*manifolds containing essential surfaces of Euler characteristic* = 0, Proc. Amer. Math. Soc. **101** (1987), 743–748.

[L] J. Luecke, *Finite covers of* 3-*manifolds containing essential tori*, Trans. Amer. Math. Soc. **310** (1988), 371–381.

[MS] W. H. Meeks and P. Scott, *Finite group action on* 3-*manifolds*, Invent. Math. **86** (1986), 287–345.

[MSY] W. Meeks, L. Simon, and S. T. Yau, *Embedded minimal surfaces, exotic spheres and manifolds of positive Ricci curvature*, Ann. of Math. (2) **116** (1982), 621–659.

[R] D. Rolfsen, *Knots and links*, Publish or Perish, 1976.

[S] P. Scott, *The geometries of* 3-*manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

[To] J. L. Tollefson, *Involution of sufficiently large 3-manifolds*, Topology **20** (1981), 323–352.

[T1] W. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) **7** (1983), 357–382.

[T2] ——, *Geometry and topology on 3-manifolds*, preprint.

[T3] ——, *Three-manifolds with symmetry*, preprint.

[W1] S. Wang, *Several results in 2- and 3-manifolds*, Thesis, UCLA, 1988.

[W2] ——, *The virtual Z-representibility of 3-manifolds which admit orientation reversing involution*, Proc. Amer. Math. Soc. **110** (1990), 497–501.

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