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Δ-SETS

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ABSTRACT. A model of ZFC is constructed in which there exists a subset of the Moore plane that is countably paracompact but not normal. The method used in the construction is forcing using uncountable sets of finite partial functions. ω_1 and ω_2 are shown to be preserved using a fusion lemma.

1. INTRODUCTION

A topological space is said to be a Δ-set if and only if whenever $\{D_n : n \in \omega\}$ is a decreasing sequence of subsets of the space with empty intersection, there exists a decreasing sequence $\{E_n : n \in \omega\}$ of open subsets of the space, also with empty intersection, such that for every n, $D_n \subseteq E_n$. A G_δ-subset of a space is an intersection of countably many open sets. An uncountable subset X of $\mathbb{R}$ is a Q-set if and only if every subset of X is a G_δ-subset of X.

The definition of a Δ-set is due to G. M. Reed and E. K. van Douwen; see [R]. G. M. Reed proved that the tangent-disc space over a subset X of the reals is countably paracompact if and only if X is a Δ-set. T. C. Przymusiński showed in [P] that there exists a separable countably paracompact Moore space that is not normal if and only if there exists a Δ-subset of $\mathbb{R}$. A slightly different question concerns the properties of Pixley-Roy hyperspaces. It is shown in [Lu] that the Pixley-Roy hyperspace of a subset X of, for example, the reals is countably paracompact if and only if every finite power of X is a Δ-set; the relationship between such sets and the Pixley-Roy hyperspace is also mentioned in [T].

A natural question is whether there is a tangent-disc space that is countably paracompact and not normal, or equivalently, whether there is a subset of the line that is a Δ-set but not a Q-set. (This question is asked in several of the papers just mentioned; for a survey of questions relating to Q-sets and other similar subsets of the real line, see [F].)

It was proved in [P] that no second countable Δ-set can have cardinality $\mathfrak{c}$. Hence, under MA, every subset of the line that is a Δ-set is also a Q-set.

So, since neither V=L nor MA will separate real Δ-sets from Q-sets, the next obvious approach is to try to distinguish them by forcing.

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It is easy enough to force to directly turn a subset X of $\mathbb{R}$ into a Δ-set, using finite partial functions. All one has to do is to iterate c.c.c. forcing using finite supports, and for every function f in any intermediate model from X to ω, to construct a function g from X to ω such that for all x, $g(x) \geq f(x)$, and such that, for each n, $\{x\colon g(x) \geq n\}$ is open; a decreasing sequence $\langle D_n : n \in \omega \rangle$ of subsets of a set X with empty intersection corresponds with the function $f(x) = \min\{n : x \notin D_n\}$.

One specific way of doing this is to iterate using partial orders $\mathbb{K}_f$, for $f \in {}^{\omega_1}\omega$, elements of $\mathbb{K}_f$ being pairs (q, r) such that:

(1) $q \in \mathrm{Fn}(X, \omega)$;
(2) if $x \in \mathrm{dom}(q)$, then $q(x) \geq f(x)$;
(3) $r \in \mathrm{Fn}(\omega \times \omega, \mathscr{B})$ ($\mathscr{B}$ being a countable base of open intervals for the reals);
(4) for all $m > q(x)$, for all $i \in \omega$, $x \notin r(m, i)$;

then if G is $\mathbb{K}_f$-generic, the g mentioned above will be the generic function $\bigcup_{(q,r)\in G} q$, and if R_f is the generic function $\bigcup_{(q,r)\in G} r$, it will be the case, because the set $D(x, n) = \{(q, r)\colon x \in \mathrm{dom}(q)$ and either $q(x) < n$ or $(\exists i)(x \in r(n, i))\}$ is dense, that

$$\{x : g(x) \geq n\} = \bigcup_{i\in\omega} \bigcup_{m\geq n} R_f(m, i)$$

and this set will thus be open.

However this seems not to be sufficient to prevent X from also being a Q-set. The reason for this seems to be that one needs an infinite amount of information to describe a G_δ-set, so that one cannot exert sufficient control on the G_δ-subsets of X using finite partial functions. Countably closed forcing is, of course, completely useless for our purpose.

The development of the forcing poset relied on ideas borrowed from Laver forcing (see [L and J, p. 19]) and from the poset used in [FM] to create a Q-set concentrated around a countable set. The author would like to take the opportunity to thank G. M. Reed, under whose supervision much of this work was done, and A. Dow, W. G. Fleissner, A. J. Ostaszewski, W. S. Watson and the members of the Toronto Set-Theoretic Topology group for useful conversations and comments.

2. Prefatory remarks

The method used is based on the method which the author used in [K] to show it consistent that there existed a Δ-set that was not a Q-set. That method arose from conversations that the author had with A. Dow and W. G. Fleissner at the 1989 Spring Topology Conference, and consisted of forcing with countable trees of finite partial functions.

The version in this paper uses refinements made in this other method by W. S. Watson and the author in an attempt to solve a different problem, which remains unsolved at the time of writing.

The most important part of the method used is the notion of description, described in §7. To say that an element of the forcing partial order describes a function is essentially to say that it imposes rigorous limits on its behaviour, limiting its value at each element of its domain to a countable ground model set. In essence, a descriptor, or desribing condition, "knows about" the function it describes.

3. The finite partial functions

We assume ZFC+GCH in the ground model.

Let $\mathscr{B}$ be a countable basis for the reals, consisting of all open intervals with rational endpoints. We abuse notation by pretending that $\mathscr{B}$ and all its elements are absolute; we could imagine the elements of $\mathscr{B}$ as being coded by pairs of rationals. Let $\langle v_n : n \in \omega\rangle$ be a sequence of natural numbers mentioning each natural number infinitely often.

We define $\mathbb{Q}$ to be the set of all $x = \langle p_x, q_x, r_x\rangle$ for which there exist $F \in [\omega_1]^{<\omega}$, and $G \in [\omega_2]^{<\omega}$ such that

(1) $p_x : F \to \mathscr{B}$,
(2) $r_x : G \to {}^{<\omega}\mathscr{B}$, (though we will write $r_x(\gamma, i)$ for $r_x(\gamma)(i)$),
(3) $q_x \in \mathrm{Fn}(\mathrm{dom}(p_x) \times \mathrm{dom}(r_x), \omega)$,
(4) $q_x(\alpha, \gamma) < v_k \to p_x(\alpha) \cap r_x(\gamma, k) = \varnothing$.

We order $\mathbb{Q}$ so that $x \leq y$ if and only if $q_x \supseteq q_y$, $(\forall\gamma \in \mathrm{dom}(r_y))(\gamma \in \mathrm{dom}(r_x) \wedge r_y(\gamma) \subseteq r_x(\gamma))$, and $(\forall\alpha \in \mathrm{dom}(p_y))(\alpha \in \mathrm{dom}(p_x) \wedge p_x(\alpha) \subseteq p_y(\alpha))$.

ω_2 enumerates sequences $\langle D_n : n \in \omega\rangle$ of subsets of some set, the $p(\alpha)$ are approximations to the elements of that set, and the $r(\gamma)(i)$ go to make up open sets U_n containing the D_n; the $q(\alpha, \gamma)$ ensure that these U_n have empty intersection.

4. The partial order

If S is a subset of $\mathbb{Q}$, and s is an element of S, then we define $S \restriction s$ to be $\{t \in S : t \leq s\}$.

We define the partial order $\mathbb{P}$ to be the set of all pairs $\langle G, T\rangle$ for which there exists $\mathscr{C}(\langle G, T\rangle)$ such that

(P1) $G \in [\omega_2]^{<\omega_2}$,
(P2) $T \in [\mathbb{Q}]^{\omega_1}$,
(P3) For all t in T, $\mathrm{dom}(r_t) \subseteq G$,
(P4) Every finite compatible subset of T has a lower bound in T.
(P5) If $x \leq s$ and $s \in T$, then for each $\gamma \in \mathrm{dom}(r_s)$ there exists $y \in T$ below s such that $p_y = p_x$ and $r_y(\gamma) = r_x(\gamma)$; in addition, $\langle p_x, q_s, r_s\rangle \in T$.
(P6) $\mathscr{C}(\langle G, T\rangle)$ is a club subset of $\{\alpha \times C : \alpha \in \omega_1,\ C \in [G]^{<\omega_1}\}$.
(P7) If $x \in T$ and $A \times C \in \mathscr{C}(\langle G, T\rangle)$, then $x \restriction (A \times C)$, which we define to be equal to $\langle p_x \restriction A, q_x \restriction (A \times C), r_x \restriction C\rangle$, is also in T.
(P8) If $x \in T$, $w \in T$, $A \times C \in \mathscr{C}(\langle G, T\rangle)$, $w \leq x \restriction (A \times C)$, $\mathrm{dom}(p_w) \subseteq A$ and $\mathrm{dom}(r_w) \subseteq C$, then there exists $y \in T$ below x such that $y \restriction (A \times C) = w$ and $p_y \restriction (\omega_1 \setminus A) = p_x \restriction (\omega_1 \setminus A)$.

We will often abuse notation by identifying an element of $\mathbb{P}$ with the corresponding subset of $\mathbb{Q}$, and write T and $\langle G(T), T\rangle$ interchangeably. If S is an element of $\mathbb{P}$, then we define $G(S \restriction s)$ to be equal to $G(S)$ for any s.

(P4) is technically extremely useful and helps to ensure, amongst other things, that ω_1 is preserved, by ensuring that forcing conditions have the c.c.c.

(P5) ensures that the generic functions built out of the p's and the r's are sufficiently generic.

(P6)–(P8) are used in the proof that the constructed space is not a Q-set; one can think of them as saying that $\{x \restriction (A \times C) : x \in T\}$ is a subset of T and is completely embedded in T for every $A \times C$ in $\mathscr{C}(T)$.

We order $\mathbb{P}$ by an order-relation $\leq$ defined so that $\langle G, S\rangle \leq \langle H, T\rangle$ if and only if there exist a function π and a $\mathscr{C}(S)$ as above such that

(O1) $H \subseteq G$,

(O2) $\mathrm{dom}(\pi) = S$ and $\mathrm{ran}(\pi) \subseteq T$,

(O3) If $\pi(s) = t$ and $t' \in T \restriction t$, then there exists $s' \in S \restriction s$ such that $\pi(s') = t'$,

(O4) If $\pi(s) = t$, then $t = \langle p_s, q_s \restriction (\omega_1 \times H), r_s \restriction H\rangle$,

(O5) Every finite compatible subset of $S \cup T$ has a lower bound in $S \cup T$.

(O6) Suppose that $s \in S$, $t \in T$, $A \times C \in \mathscr{C}(S)$, $s \restriction (A \times C) = s$ and $\pi(s) = t \restriction (A \times C)$. Then there exists $s' \in S$ such that $s' \restriction (A \times C) = s$ and $\pi(s') = t$.

The motivation of (O5) is that when we amalgamate conditions in the proof of the Fusion Lemma, we shall be taking the set-theoretic union of the corresponding subsets of $\mathbb{Q}$, and we want to make sure that the result satisfies (P4); we use (O6) to make sure that it satisfies (P8).

5. Some Lemmas

In this section we prove some technical lemmas which we shall use (usually implicitly) throughout the rest of the paper, and we also begin to set up the machinery that we will use in the Fusion Lemma.

5.1. **Lemma.** *$\mathbb{Q}$ has the c.c.c.*

Proof. This is a standard application of the Δ-system lemma and the Pigeonhole Principle. □

5.2. **Lemma.** *$\mathbb{P}$ is nonempty.*

Proof. The pair $T = \langle\{s \in \mathbb{Q} : \mathrm{dom}(r_s) = \varnothing\}, \{0\}\rangle$ is an element of $\mathbb{P}$; we define $\mathscr{C}(T)$ to be equal to $\{\alpha \times \{0\} : \alpha \in \omega_1\}$. □

5.3. **Lemma.** *The partial order is transitive; moreover, if $S \leq T \leq U$, the composition of the functions witnessing $S \leq T$ and $T \leq U$, witnesses $S \leq U$.*

Proof. Suppose that $S \leq T \leq U$, that π_0 witnesses $S \leq T$ and that π_1 witnesses $T \leq U$.

Define π to be $\pi_1 \circ \pi_0$. We show that π witnesses $S \leq U$.

Firstly, $G(U) \subseteq G(T) \subseteq G(S)$, so (O1) is satisfied.

Secondly, the domain of π is trivially the whole of S, and the range of π is contained in U.

Suppose that $\pi(s) = u$. Then there exists $t \in T$ such that $\pi_0(s) = t$ and $\pi_1(t) = u$. Suppose that $u' \in U \restriction u$. Then by condition (O3) on π_1, there exists $t' \in T \restriction t$ such that $\pi_1(t') = u'$. Now by condition (O3) on π_0, there exists $s' \in S \restriction s$ such that $\pi_0(s') = t'$. But then $\pi(s') = u'$.

Suppose that $\pi(s) = u$. Then there exists $t \in T$ such that $\pi_0(s) = t$ and $\pi_1(t) = u$. Then $u = \langle p_t, q_t \restriction (\omega_1 \times G(U)), r_t \restriction G(U)\rangle$ and $t = \langle p_s, q_s \restriction (\omega_1 \times G(T)), r_s \restriction G(T)\rangle$ by successive applications of (O4) to π_1 and π_0, and since $G(U) \subseteq G(T)$, we have that $u = \langle p_s, q_s \restriction (\omega_1 \times G(U)), r_s \restriction G(U)\rangle$ as required.

Now suppose that K is a finite subset of S and L is a finite subset of U, and that $K \cup L$ is compatible. Suppose that z is a lower bound for $K \cup L$, and let $K' = \{\langle p_z, q, r\rangle : \langle p, q, r\rangle \in K\}$. Then $K' \cup L$ is compatible, having z as a lower bound, and by (P5), K' is a subset of S.

Let L' be the image of K' under π_0. Then, by (O4), $L' \cup L$ is compatible with lower bound z. So, by (O5) applied to π_1, it has a lower bound w in T.

Then $K' \cup \{w\}$ is compatible.

For, let $y = \langle p_w, q_w \cup \bigcup_{s\in K} q_s, r\rangle$ where for every γ, $r(\gamma) = r_w(\gamma) \cup \bigcup_{s\in K} r_s(\gamma)$. $r_y(\gamma)$ is a well-defined function for each γ, because for any $\gamma \in \mathrm{dom}(r_y)$, either $\gamma \in G(T)$, when $r_y(\gamma) = r_w(\gamma) \cup \bigcup_{s\in K} r_s(\gamma) = r_w(\gamma)$, or $\gamma \notin G(T)$, when $\gamma \notin \mathrm{dom}(r_w)$, w being an element of T, and so $r_y(\gamma) = \bigcup_{s\in K} r_s(\gamma)$, and this is a function because K is compatible. q_y is a well-defined function for similar reasons.

The only condition for membership of $\mathbb{Q}$ which might present problems is condition (4). So suppose that $\langle\alpha, \gamma\rangle \in \mathrm{dom}(q_y)$. If $\gamma \in G(T)$, then $\langle\alpha, \gamma\rangle \in \mathrm{dom}(q_w)$. Since in that case $p_y = p_w$ and $r_y(\gamma) = r_w(\gamma)$, condition (4) holds because w is in $\mathbb{Q}$. If, on the other hand $\gamma \notin G(T)$, the $\langle\alpha, \gamma\rangle \in \mathrm{dom}(q_s)$ for some s in K, and $r_y(\gamma) = \bigcup_{s\in K} r_s(\gamma)$. But K is compatible, with a lower bound z, so that $r_y(\gamma) \subseteq r_z(\gamma)$, $q_y(\alpha, \gamma) = q_z(\alpha, \gamma)$, and $p_y(\alpha) \subseteq p_z(\alpha)$ because of the way that K' was used in the construction of y. So since condition (4) holds for z, it holds also for y.

y is a lower bound for $K' \cup \{w\}$, so $K' \cup \{w\}$ is compatible. So by (O5) applied to π_0, it has a lower bound x in S. Then x is a lower bound for $K \cup L$ as required.

Finally suppose that we have chosen $\mathscr{C}(S)$ so that (O6) holds for $S \le T$, and that we have chosen $\mathscr{C}(T)$ so that (O6) holds for $T \le U$. Suppose without loss of generality that whenever $A \times C \in \mathscr{C}(S)$, $A \times (C \cap G(T)) \in \mathscr{C}(T)$.

Suppose that $A\times C \in \mathscr{C}(S)$, $s \in S$, $u \in U$, $\mathrm{dom}(p_s) \subseteq A$ and $\mathrm{dom}(r_s) \subseteq C$, and that $\pi(s) = u \restriction (A \times C)$. Then if $t = \pi_0(s)$, then by (O4), $\mathrm{dom}(p_t) \subseteq A$ and $\mathrm{dom}(r_t) \subseteq C \cap G(T)$ and $\pi_1(t) = u \restriction (A \times (C \cap G(T)))$. So by (O6) for $T \le U$, there exists $t' \in T \restriction t$ such that $t' \restriction (A\times(C\cap G(T))) = t$ and $\pi_1(t) = u$. Now we apply (O6) to $S \le T$ to find $s' \in S \restriction s$ such that $s' \restriction (A \times C) = s$ and $\pi_0(s') = t'$; then $\pi(s') = u$ as required. □

5.4. **Lemma.** *If $S \in \mathbb{P}$ and $s \in S$, then $S \restriction s$ is an element of $\mathbb{P}$ and $S \restriction s \le S$.*

Proof. We define $\mathscr{C}(S \restriction s)$ to be $\{A \times C \in \mathscr{C}(S) : \mathrm{dom}(p_s) \subseteq A \ \&\ \mathrm{dom}(r_s) \subseteq C\}$. The appropriate function π is the identity. □

5.5. **Lemma.** *The set of S in $\mathbb{P}$ with a single greatest element is dense.*

Proof. Let $s \in S$. Then $S \restriction s$ has a single greatest element. □

5.6. **Lemma.** *If π witnesses $S \le T$ and $\pi(s) = t$, then $S \restriction s \le T \restriction t$.*

Proof. The same function π will work, because condition (O4) tells us that if $s' \le s$ and $\pi(s') = t'$, then $t' \le t$. □

We define an *orderly sequence* of elements of $\mathbb{P}$ to be a sequence $\langle T_\alpha : \alpha \in \mu\rangle$, μ being some ordinal, such that

(1) Whenever $\alpha < \beta$, there is a function $\pi_{\beta,\alpha}$ witnessing $T_\beta \leq T_\alpha$, this function including the identity function on T_α as a subset, such that whenever $\alpha < \beta < \gamma$, $\pi_{\gamma,\alpha} = \pi_{\beta,\alpha} \circ \pi_{\gamma,\beta}$,
(2) Whenever λ is a limit, $T_\lambda = \bigcup_{\alpha<\lambda}$, and for each $\alpha < \lambda$, $\pi_{\lambda,\alpha} = \bigcup_{\beta\in(\alpha,\lambda)} \pi_{\beta,\alpha}$,
(3) There exists a choice of $\mathscr{C}(T_\alpha)$ for each α such that if $\alpha < \beta$, then $\mathscr{C}(T_\alpha) = \{A \times (C \cap G(T_\alpha)) : A \times C \in \mathscr{C}(T_\beta)\}$.

5.7. **Lemma.** *Suppose that $\langle T_\alpha : \alpha < \lambda\rangle$ is an orderly sequence, and that λ is some limit ordinal less than ω_2. If we define T_λ to be $\bigcup_{\alpha<\lambda} T_\alpha$, and $G(T_\lambda)$ to be $\bigcup_{\alpha<\lambda} G(T_\alpha)$, then $\langle T_\alpha : \alpha \leq \lambda\rangle$ is an orderly sequence.*

Proof. We first have to show that T_λ is an element of the partial order. But T_λ obviously has cardinality no greater than ω_1, and conditions (P1)–(P5) for being an element of $\mathbb{P}$ are properties of finite character.

Next, we have to construct $\mathscr{C}(T_\lambda)$. We define this to be the set of all increasing countable unions of members of $\bigcup_{\alpha<\lambda} \mathscr{C}(T_\alpha)$. Then $\mathscr{C}(T_\lambda)$ is indeed club, and if $A \times C \in \mathscr{C}(T_\lambda)$, then for each $\alpha < \lambda$, $A \times (C \cap G(T_\alpha))$ is a countable union of elements of $\mathscr{C}(T_\alpha)$, and is therefore itself in $\mathscr{C}(T_\alpha)$. The other properties of $\mathscr{C}(T_\lambda)$ are inherited from the $\mathscr{C}(T_\alpha)$.

Now define $\pi_{\lambda,\alpha}$ to be $\bigcup_{\beta\in(\alpha,\lambda)} \pi_{\beta,\alpha}$. Then $\pi_{\lambda,\alpha}$ inherits all the required properties from the $\pi_{\beta,\alpha}$. □

6. Extending domains

In this section we devote ourselves to setting up the technology that we will use to prove the Fusion Lemma in §7. The notion of amalgamation of two conditions that we will use in the Fusion Lemma is set-theoretic union; this allows us to amalgamate $\aleph_1$-many conditions without the difficulties attendant on pruning a tree-like condition uncountably many times. All the results in this section are essential, but Lemmas 6.3 and 6.5 are perhaps the high points.

If s is an element of $\mathbb{Q}$, and γ is an element of ω_2, define $s - \gamma$ to be $\langle p_s, q_s \restriction (\omega_1 \times (\omega_2 \setminus \{\gamma\})), r_s \restriction (\omega_2 \setminus \{\gamma\})\rangle$.

Suppose that S is an element of the partial order. Suppose that γ is not in $G(S)$. We define S^γ to be $\{x \in \mathbb{Q} : (\exists t \in S)(t = x - \gamma \;\&\; q_x = q_t \;\&\; \gamma \in \mathrm{dom}(r_x)\}$, with $G(S^\gamma)$ being $G(S) \cup \{\gamma\}$.

6.1. **Lemma.** *S^γ is an element of $\mathbb{P}$ and lies below S.*

Proof. First we have to show that S^γ is an element of the partial order.

Suppose that K is a finite compatible subset of S^γ. Then $\{s - \gamma : s \in K\}$ is a finite compatible subset of S, and thus has a lower bound w in S.

So, let x be defined to be $\langle p_w, \bigcup_{s\in K} q_s, r\rangle$ where $r \restriction G(S) = r_w$, and $r(\gamma) = \bigcup_{s\in K} r_s(\gamma)$. Then x is a lower bound for K and is an element of S^γ.

Suppose that $s \in S^\gamma$, and that $x \leq s$. If $\gamma' \in \mathrm{dom}(r_s)$ and $\gamma' \neq \gamma$, then by (P5) on S, we can find y in S below $s - \gamma$ such that $p_y = p_x$ and $r_y(\gamma') = r_x(\gamma')$; we can then extend this to an element of S^γ below s. And, of course, one can easily find $z \leq s$ in S^γ satisfying $p_z = p_x$ and $r_z(\gamma) = r_x(\gamma)$; $z = \langle p_y, q_y, r\rangle$ will do, where $r \restriction (\omega_2 \setminus \{\gamma\}) = r_y$, and $r(\gamma) = r_x(\gamma)$.

We define $\mathscr{C}(S^\gamma)$ to be $\{A \times (C \cup \{\gamma\}) : A \times C \in \mathscr{C}(S)\}$. $\mathscr{C}(S^\gamma)$, thus defined, has all the right properties.

We define π so that for every $s \in S^\gamma$, $\pi(s) = s - \gamma$. We prove (O5) in a way similar to the way we proved (P4). □

6.2. **Lemma.** *If $s \in S$, then $\{t \in S^\gamma : t - \gamma = s\}$ is predense below s in $\mathbb{Q}$.*

Proof. Suppose that x is an element of $\mathbb{Q}$ below s; without loss of generality $\gamma \in \mathrm{dom}(r_x)$. We then define t so that $t - \gamma = s$, and $r_t(\gamma) = r_x(\gamma)$.

Then $t \in S^\gamma$ and $x \leq t$. □

Suppose that $S \leq T$ and that $\gamma \notin G(S)$. Then we define $S^\gamma \oplus T$ to be $T \cup S^\gamma$, with $G(S^\gamma \oplus T) = G(S^\gamma)$; we define $\mathscr{C}(S^\gamma \oplus T)$ to be $\mathscr{C}(T) \cup \{A \times C : A \times C \in \mathscr{C}(S^\gamma)\ \&\ A \times (C \cap G(T)) \in \mathscr{C}(T)\}$.

6.3. **Lemma.** *If it exists, then $S^\gamma \oplus T$ is an element of $\mathbb{P}$ and lies below T.*

Proof. Suppose that π^* witnesses $S^\gamma \leq T$.

We show first that $S^\gamma \oplus T$ is an element of the partial order.

Suppose that K is a finite compatible subset of $S^\gamma \oplus T$. Then, since $S^\gamma \leq T$, K has a lower bound in $S^\gamma \oplus T$ by condition (O5).

When it comes to checking the properties of $\mathscr{C}(S^\gamma \oplus T)$, we notice that it is certainly club in $\{A \times C : A \in \omega_1\ \&\ C \in [G(S^\gamma \oplus T)]^{<\omega_1}\}$.

To prove (P7), we suppose that $x \in S^\gamma \oplus T$ and that $A \times C \in \mathscr{C}(S^\gamma \oplus T)$. If $x \in T$ and $A \times C \in \mathscr{C}(S^\gamma)$, then since $A \times C \in \mathscr{C}(S^\gamma \oplus T)$, $A \times (C \cap G(T)) \in \mathscr{C}(T)$, and $x \restriction (A \times C) = x \restriction (A \times (C \cap G(T)))$, which is in T by (P7) on T. So now suppose that $x \in S^\gamma$, and that $A \times C \in \mathscr{C}(T)$. Then $x \restriction (A \times C) = \pi^*(x) \restriction (A \times C)$, because $C \subseteq G(T)$. Now $\pi^*(x) \in T$, so by (P7) on T, $x \restriction (A \times C) \in T$.

The difficult case of (P8) is when $x \in S^\gamma$, $A \times C \in \mathscr{C}(T)$, $w \in S^\gamma \oplus T$, $\mathrm{dom}(p_w) \subseteq A$ and $\mathrm{dom}(r_w) \subseteq C$. Then, since $C \subseteq G(T)$, $\gamma \notin G(T)$ and $\gamma \in \mathrm{dom}(r_s)$ for every $s \in S^\gamma$, $w \in T$. Similarly, $x \restriction (A \times C) \in T$. Also, $x \restriction (A \times C) = \pi^*(x) \restriction (A \times C)$, and so by (P8) on T, there exists $y \in T \restriction \pi^*(x)$ such that $y \restriction (A \times C) = w$ and $p_y \restriction (\omega_1 \backslash A) = p_{\pi^*(x)} \restriction (\omega_1 \backslash A)$. Finally, by (O3), there exists $z \in S^\gamma \restriction x$ such that $\pi^*(z) = y$; then using (O4), $z \restriction (A \times C) = y \restriction (A \times C) = w$, and $p_z \restriction (\omega_1 \backslash A) = p_y \restriction (\omega_1 \backslash A) = p_{\pi^*(x)} \restriction (\omega_1 \backslash A) = p_x \restriction (\omega_1 \backslash A)$.

If $x \in T$, $A \times C \in \mathscr{C}(S^\gamma)$ and $w \in S^\gamma$, with $w \restriction (A \times C) = w$ and $w \leq x \restriction (A \times C)$, then also $\pi^*(w) \leq x \restriction (A \times C)$. We find $y \leq x$ such that $y \in T$, $y \restriction (A \times C) = \pi^*(w)$, and $p_y \restriction (\omega_1 \setminus A) = p_x \restriction (\omega_1 \setminus A)$. Now we use (O6) to find $z \in S^\gamma$ such that $z \restriction (A \times C) = w$ and $\pi^*(z) = y$. Then by (O4), $p_z \restriction (\omega_1 \setminus A) = p_x \restriction (\omega_1 \setminus A)$. So z is the element of $S^\gamma \oplus T$ below x that we were looking for.

We now observe that $S^\gamma \oplus T \leq T$. For, let π be defined so that $\pi(s) = t$ if and only if either $s = t \in T$ or $\pi^*(s) = t$.

Then (O4) holds because if $\pi(s) = t$, and $s \notin T$, $t = \langle p_s, q_s \restriction (\omega_1 \times G(T)), r_s \restriction G(T)\rangle$ by (O4) for S^γ and T; whereas if $s \in T$, then $t = s = \langle p_s, q_s \restriction (\omega_1 \times G(T)), r_s \restriction G(T)\rangle$, as required.

(O5) holds because of (P4) for T and (O5) for $S^\gamma \leq T$, and (O6) holds because of (O6) for $S^\gamma \leq T$. □

6.4. **Lemma.** *Suppose that $\langle T_\alpha : \alpha \leq \beta\rangle$ is an orderly sequence, and that $S \leq T_\beta$. Then if $\gamma \notin G(S) \cup G(T_\beta)$, and $T_{\beta+1}$ is defined to be $S^\gamma \oplus T_\beta$, then $\langle T_\alpha : \alpha \leq \beta + 1\rangle$ is also an orderly sequence.*

Proof. We use the π defined in the proof of Lemma 6.3, and the $\mathscr{C}(S^\gamma \oplus T)$ defined above. □

6.5. **Lemma.** *Suppose that* $s \in S^\gamma$. *Then* $S^\gamma \oplus T \restriction s = S^\gamma \restriction s$.

Proof. γ is in $\mathrm{dom}(r_s)$ for every s in S^γ, but is not in $\mathrm{dom}(r_t)$ for any $t \in T$. So, if $t \in T$ and $s \in S^\gamma$, then $t \not\leq s$. □

7. The Fusion Lemma

In this section, we prove that the partial order preserves ω_1 and ω_2. It turns out that it preserves all cardinals because it has cardinality ω_2, but this is unimportant for the purposes of this paper.

Suppose that $\dot{f}$ is a name for a function from the ground model to itself. Then we say that a condition T *describes* $\dot{f}$, or is a *descriptor* of $\dot{f}$, if whenever π witnesses $U \leq T$, and $U \Vdash \dot{f}(a) = b$, there exists t in the range of π such that $T \restriction t \Vdash \dot{f}(a) = b$.

Descriptors enable us to prevent functions from collapsing ω_1 and ω_2; they will also be useful in proving that the space X we will construct in the generic extension is a Δ-set and not a Q-set; to prove it a Δ-set we will use a descriptor of a function coding a countable collection of subsets of X with empty intersection; to prove that it is not a Q-set we will use a descriptor of a function coding a G_δ-set.

We construct the descriptor of a function by amalgamating into an element of the partial order other elements which decide values that the function takes; we use the methods of §6.

7.1. **Lemma.** *Suppose that* $T \Vdash \dot{f}\colon \mu \to V$, *where* $\mu \in \omega_2$. *Then there exists* $T' \leq T$ *such that* T' *describes* $\dot{f}$.

Proof. We construct T' as the last element of an orderly sequence $\langle T_\alpha : \alpha < \mu + 1\rangle$, with $T_0 = T$. We construct $T_{\alpha+1}$ by constructing an orderly sequence $\langle T_{\alpha,\beta} : \beta \in \lambda_\alpha\rangle$, with λ_α being some countable ordinal and $T_{\alpha,0}$ being equal to T_α, as follows:

For each $\beta < \lambda_\alpha$, we look for some $S_{\alpha,\beta} \leq T_{\alpha,\beta}$ with greatest element $s_{\alpha,\beta}$ which is incompatible with $s_{\alpha,\delta}$ for all $\delta < \beta$, such that $S_{\alpha,\beta}$ decides the value of $\dot{f}(\alpha)$. If such an $S_{\alpha,\beta}$ exists, choose some ordinal $\gamma_{\alpha,\beta} \in \omega_2 \setminus G[S_{\alpha,\beta}] \cup G[T_{\alpha,\beta}]$, and define $T_{\alpha,\beta+1}$ to be $S_{\alpha,\beta}{}^{\gamma_{\alpha,\beta}} \oplus T_{\alpha,\beta}$. Lemma 6.5 tells us that we do obtain an orderly sequence this way. If it is not possible to find such an $S_{\alpha,\beta}$, put $\lambda_\alpha = \beta + 1$ and $T_{\alpha+1} = T_{\alpha,\beta}$.

Having constructed T', we show that it describes $\dot{f}$.

Suppose that $U \leq T'$, that π witnesses this. Suppose that $U \Vdash \dot{f}(\alpha) = b$, and that U has greatest element u which is compatible with $s_{\alpha,\beta}$. (For, if it were the case that u was incompatible with all of them, we would have a contradiction to the definition of λ_α.)

Suppose that π^* witnesses $T' \leq T_{\alpha,\beta+1}$, and that $\pi^\dagger$ is equal to $\pi \circ \pi^*$.

Now u and $s_{\alpha,\beta}$ are compatible. So, by Lemma 6.2, there exists $\hat{s}$ in $S_{\alpha,\beta}{}^{\gamma_{\alpha,\beta}}$ which is compatible with u, and so by (O5) there exists $w \in S_{\alpha,\beta}{}^{\gamma_{\alpha,\beta}} \cup U$ which is below both. Without loss of generality $w \in U$. For if $w \in S_{\alpha,\beta}{}^{\gamma_{\alpha,\beta}}$, then because $w \le u$, $w \le \pi^\dagger(u)$, so that there exists $w' \in U \restriction u$ such that $\pi^\dagger(w') = w$. So use w' instead of w.

Now suppose that $\pi(w) = x$ and $\pi^*(x) = y$. Then $\pi^\dagger(w) = y$.

Now, $U \restriction w \le T' \restriction x \le T_{\alpha,\beta+1} \restriction y$. But $y \le \hat{s}$. Hence $\gamma_{\alpha,\beta} \in \mathrm{dom}(r_y)$, so that $y \in {S_{\alpha,\beta}}^{\gamma_{\alpha,\beta}}$, and by Lemma 6.5, $T_{\alpha,\beta+1} \restriction y = {S_{\alpha,\beta}}^{\gamma_{\alpha,\beta}} \restriction y$, and this in its turn is $\le S_{\alpha,\beta}$.

Now $S_{\alpha,\beta}$ decides the value of $\dot{f}(\alpha)$. But $U \restriction w$ is a common lower bound for $S_{\alpha,\beta}$ and U, and $U \Vdash \dot{f}(\alpha) = b$. So $S_{\alpha,\beta} \Vdash \dot{f}(\alpha) = b$.

Finally we observe that x is in the range of π, and $T' \restriction x \le S_{\alpha,\beta}$, and so forces $\dot{f}(\alpha) = b$ as required. □

7.2. **Lemma.** $\mathbb{P}$ *preserves* ω_1 *and* ω_2.

Proof. Suppose that μ is one of ω and ω_1 and that $\dot{f}$ is a name for a function from μ to μ^+. We show that $\dot{f}$ is not a name for a collapsing function.

For, $\dot{f}$ has a dense set of descriptors, and if T describes $\dot{f}$, then because T has the countable chain condition, for every element α of μ, there exists a countable ground model set A_α such that $T \Vdash \dot{f}(\alpha) \in A_\alpha$. Then, $T \Vdash \dot{f}\colon \mu \to \bigcup_{\alpha\in\mu} A_\alpha$; that is to say, T forces the range of $\dot{f}$ to lie in some ground model set of cardinality no greater than μ. So $\dot{f}$ does not collapse μ^+. □

8. The space

In this section, we construct the space X which we shall prove is a Δ-set and not a Q-set. If S is an element of the partial order with greatest element s, the $p_s(\alpha)$ is an approximation to the αth point in the space.

Let H be $\mathbb{P}$-generic. Let $\hat{H}$ be the set of all x for which there exists $T \in H$ such that x is the greatest element of T. (Lemma 5.5 tells us that the set of T with a single greatest element is dense.) Let $\hat{P}\colon \omega_1 \to \mathscr{P}(\mathbb{R})$ be defined so that for each α, if there exists $x \in \hat{H}$ such that $\alpha \in \mathrm{dom}(p_x)$, then $\hat{P}(\alpha) = \bigcap_{x\in\hat{H}} p_x(\alpha)$.

8.1. **Lemma.** $\hat{P}(\alpha)$ *is defined for each* α.

Proof. Let $T \in \mathbb{P}$ and let $t \in T$. Construct $s \le t$ with $\alpha \in \mathrm{dom}(p_s)$. Then there exists $z \in T$ such that $z \le t$ and $p_z = p_s$. Then $T \restriction z$ is an element of $\mathbb{P}$ with a greatest element z for which $\alpha \in \mathrm{dom}(p_z)$.

So we see that the set of U, with a single greatest element u with $\alpha \in \mathrm{dom}(p_u)$, is dense. So $\alpha \in \mathrm{dom}(\hat{P})$. □

8.2. **Lemma.** *If* T *and* U *are compatible elements of* $\mathbb{P}$ *with greatest elements* t *and* u *respectively, then* t *and* u *are compatible.*

Proof. Suppose that S is below both. Then, if $s \in S$, by using order condition (O4) we can easily show that $s \le t$ and $s \le u$. □

8.3. **Lemma.** *Suppose that* T *has greatest element* t *and that* $\alpha \in \mathrm{dom}(p_t)$. *Then the set* $D(T, \alpha)$ *of all* U, *with a greatest element* u *having the properties that* $\overline{p_u(\alpha)} \subseteq p_t(\alpha)$ *and that the measure of* $p_u(\alpha)$ *is less than half the measure of* $p_t(\alpha)$, *is dense below* T.

Proof. Suppose that S is below T. Without loss of generality, S has a greatest element s. Construct $x \leq s$ such that $\overline{p_x(\alpha)} \subseteq p_t(\alpha)$ and that the measure of $p_x(\alpha)$ is less than half the measure of $p_t(\alpha)$.

Then there exists $z \in S$ such that $p_z = p_x$. Then $S \restriction z \in D(T, \alpha)$. □

8.4. **Lemma.** *For each* α, $\hat{P}(\alpha)$ *has a unique element.*

Proof. Let T be an element of the generic filter H, with a greatest element t with $\alpha \in \mathrm{dom}(p_t)$. We construct a sequence $\langle T_n : n \in \omega \rangle$, such that $T = T_0$, and for each n, T_{n+1} is an element of $D(T_n, \alpha)$ in H. Let $I_n = p_n(\alpha)$; then $\langle I_n : n \in \omega \rangle$ is a sequence of intervals with widths tending to zero, the closure of each being contained in its predecessor. Clearly, the intersection of the I_n is a single point. (We appear to have fudged the issue of the absoluteness of the closure operation; in this particular case this does not matter.) Finally, if S is an element of H, and if S has a greatest element s with $\alpha \in \mathrm{dom}(p_s)$, then because S is compatible with each of the T_n, for each n, $p_s(\alpha) \cap I_n \neq \varnothing$. Now, the only way that $p_s(\alpha)$ can fail to contain the single point in $\bigcap_{n \in \omega} I_n$ is if that point is an endpoint of $p_s(\alpha)$. But in that case there exists some element S' of $D(S, \alpha)$ in H, and if s' is the greatest element of S', then that single point is not an endpoint of $p_{s'}(\alpha)$, so that for some n, $p_{s'}(\alpha) \cap I_n = \varnothing$, which is impossible. □

We denote this unique element by either $P(\alpha)$ or x_α, and let X be equal to $\{x_\alpha : \alpha \in \omega_1\}$. X is our example.

8.5. **Lemma.** *If* $\alpha \neq \alpha'$, *then* $P(\alpha) \neq P(\alpha')$, *and for any* α, $P(\alpha)$ *is not in the ground model.*

Proof. Suppose that $\alpha \neq \alpha'$, and that y is a ground model real. Let T be an element of $\mathbb{P}$ and t be an element of T. Then we construct $x \leq t$ such that $\alpha, \alpha' \in \mathrm{dom}(p_x)$, $p_x(\alpha) \cap p_x(\alpha') = \varnothing$, and $y \notin p_x(\alpha)$.

Then there exists $z \in T$ such that $p_z = p_x$. Then $T \restriction z \Vdash y \neq \dot{x}_\alpha \neq \dot{x}_{\alpha'}$. □

9. X IS A Δ-SET

In this section we show that X is a Δ-set. We do this by taking descriptor of a function coding a decreasing sequence $\langle D_n : n \in \omega \rangle$ of subsets of X with empty intersection. Choosing some $\gamma \notin G(T)$, we construct a new condition S out of T in such a way that the various $r_s(\gamma, i)$, for $s \in S$, are forced to union up to form a sequence $\langle U_n : n \in \omega \rangle$ of open subsets of X; the various q_s compel this sequence to have empty intersection, and $D_n \subseteq U_n$ by genericity.

Suppose that $\langle D_n : n \in \omega \rangle$ is a decreasing sequence of subsets of X with empty intersection. We define g so that $g(\alpha) = \min\{n : \alpha \notin D_n\}$.

Now let T be an element of $\mathbb{P}$ describing $\dot{g}$, and let $\gamma \in \omega_2 \setminus G[T]$. For each $\alpha \in \omega_1$, let A_α be a maximal antichain in T, having the property that if $a \in A_\alpha$, then $T \restriction a$ decides the value of $\dot{g}(\alpha)$. Each A_α is countable.

Define S to have as elements all elements x of $\mathbb{Q}$ for which $x - \gamma \in T$, and such that if $\langle \alpha, \gamma \rangle \in \mathrm{dom}(q_x)$, then for some $a \in T$ above $x - \gamma$, $a \in A_\alpha$ and $T \restriction a \Vdash \dot{g}(\alpha) = k$ for some $k \leq q_x(\alpha, \gamma)$. We define $G(S)$ to be $G(T) \cup \{\gamma\}$.

9.1. **Lemma.** $S \in \mathbb{P}$.

Proof. Suppose that K is a finite compatible subset of S. Let x be a lower bound in $\mathbb{P}$ for K. Let $K' = \{\langle p_x, q, r\rangle : \langle p, q, r\rangle \in K\}$. Let $L = \{s - \gamma : s \in K'\}$. Then L is a finite compatible subset of T, and thus has a lower bound y in T.

Now define z to be equal to $\langle p_y, q_y \cup \bigcup_{s\in K'} q_s, r\rangle$, where $r \restriction (\omega_2 \setminus \{\gamma\}) = r_y$, $r(\gamma) = \bigcup_{s\in K'} r_s(\gamma)$.

Then $z \in \mathbb{Q}$.

For, suppose that $\langle\alpha, \gamma\rangle \in \operatorname{dom}(q_z)$, and that $q_z(\alpha, \gamma) = k$. Then $\langle\alpha, \gamma\rangle \in \bigcup_{s\in K'} \operatorname{dom}(q_s)$, because $\langle\alpha, \gamma\rangle$ cannot possibly have been inherited from $\operatorname{dom}(q_y)$.

Because y is a lower bound for L, $p_y(\alpha) \subseteq p_t(\alpha)$ for all t in L. But for each $t \in L$, $p_t(\alpha) = p_x(\alpha)$, and for every $t \in K$, $p_x(\alpha) \subseteq p_t(\alpha)$. Also $r_z(\gamma) = \bigcup_{s\in K'} r_s(\gamma)$. Now K' is compatible, with lower bound x. So if $\langle\alpha, \gamma\rangle \in \operatorname{dom}(q_z)$, then for some $s \in K'$, $\langle\alpha, \gamma\rangle \in \operatorname{dom}(q_s)$, and so $\langle\alpha, \gamma\rangle \in \operatorname{dom}(q_x)$, and $q_x(\alpha, \gamma) = q_z(\alpha, \gamma) = k$. Now if $i \in \operatorname{dom}(r_z(\gamma))$ and $v_i > k$, then $i \in \operatorname{dom}(r_x(\gamma))$, and $p_x(\alpha) \cap r_x(\gamma, i) = \varnothing$. Now $p_y(\alpha) \subseteq p_x(\alpha)$ and $r_z(\gamma, i) = r_x(\gamma, i)$, so also $p_z(\alpha) \cap r_z(\gamma, i) = \varnothing$.

Also $z \in S$, because $z - \gamma = y \in T$, and if $\langle\alpha, \gamma\rangle \in \operatorname{dom}(q_z)$, then for some $t \in K$, $\langle\alpha, \gamma\rangle \in \operatorname{dom}(q_t)$, and $T \restriction a \Vdash \dot{g}(\alpha) = k$ for some $k < q_t(\alpha, \gamma)$ and some $a \in A_\alpha$ above $t - \gamma$. Now $y = z - \gamma \leq t - \gamma$, and $q_z(\alpha, \gamma) = q_t(\alpha, \gamma)$. So $T \restriction a \Vdash \dot{g}(\alpha) = k$ for some $k < q_z(\alpha, \gamma)$ and some $a \in A_\alpha$ above $z - \gamma$, as required.

To prove (P5), suppose that $s \in S$ and that $x \leq s$. Then $x \leq s - \gamma$, and so since $s - \gamma \in T$, $\langle p_x, q_{s-\gamma}, r_{s-\gamma}\rangle \in T$. But now, $\langle p_x, q_s, r_s\rangle$ is a permitted element of S; the only thing that would need to be checked is that it is an element of $\mathbb{Q}$, and that is straightforward enough.

In addition, if $\gamma' \in \operatorname{dom}(r_s) \setminus \{\gamma\}$, we can find $t \in T$ below s_y such that $p_t = p_x$ and $r_t(\gamma') = r_x(\gamma')$; we then observe that $\langle p_x, q_s \cup q_t, r\rangle$ is in S, where $r \restriction \omega_2 \setminus \{\gamma\} = r_t$, and $r(\gamma) = r_s(\gamma)$.

We now need to find an element t of S such that $p_x = p_t$, $r_x(\gamma) = r_t(\gamma)$, and $t \leq s$. Well, let $t = \langle p_x, q_s, r\rangle$, where $r \restriction \omega_2 \setminus \{\gamma\} = r_s$, and $r_t(\gamma) = r_x(\gamma)$. Then t is an element of $\mathbb{Q}$, because s and x are. Now all we need to observe is that it is a permitted element of S.

Next, we construct $\mathscr{C}(S)$. Let $\mathscr{C}^* = \{A \times (C \cup \{\gamma\}) : A \times C \in \mathscr{C}(T)\}$. Then $\mathscr{C}^*$ is club in $[\omega_1 \times G(S)]^\omega$. We define $\mathscr{C}(S)$ to be the set of all $A \times (C \cup \{\gamma\})$ in $\mathscr{C}^*$ such that if $\alpha \in A$, then for all a in A_α, $\operatorname{dom}(p_a) \subseteq A$ and $\operatorname{dom}(r_a) \subseteq C$. Clearly, $\mathscr{C}(S)$ is club.

Now suppose that $x \in S$ and $A \times (C \cup \{\gamma\})$ is in $\mathscr{C}(S)$. Then we have to show that $x \restriction (A \times (C \cup \{\gamma\})) \in S$. Suppose that $\langle\alpha, \gamma\rangle \in \operatorname{dom}(q_{x\restriction(A\times(C\cup\{\gamma\}))})$. Then for some $a \in A_\alpha$ above x, $T \restriction a \Vdash \dot{g}(\alpha) = k$ for some k less than or equal to $q_x(\alpha, \gamma)$. But since $a \in A_\alpha$, $\operatorname{dom}(p_a) \subseteq A$ and $\operatorname{dom}(r_a) \subseteq C$. So, a is also above $x \restriction (A \times (C \cup \{\gamma\}))$. Since $x \restriction (A \times (C \cup \{\gamma\})) - \gamma = (x - \gamma) \restriction (A \times C)$, and $A \times C \in \mathscr{C}(T)$, $x \restriction (A \times (C \cup \{\gamma\})) - \gamma$ is in T, and so $x \restriction (A \times (C \cup \{\gamma\}))$ is in S as required.

Suppose that $w \leq x \restriction (A \times (C \cup \{\gamma\}))$, and that $\operatorname{dom}(p_w) \subseteq A$ and $\operatorname{dom}(r_w) \subseteq C \cup \{\gamma\}$. Then $w - \gamma \leq x \restriction (A \times (C \cup \{\gamma\})) - \gamma = x - \gamma \restriction (A \times C)$, so there exists $y' \in T$ such that $y' \restriction (A \times C) = w - \gamma$, $p_{y'} \restriction (\omega_1 \setminus A) = p_x \restriction (\omega_1 \setminus A)$, and $y' \leq x - \gamma$. So let $y = \langle p_{y'}, q_w \cup q_x \cup q_{y'}, r\rangle$ where $r(\gamma') = r_{y'}(\gamma')$

for $\gamma' \neq \gamma$ and $r(\gamma) = r_w(\gamma)$. Then y is an element of S, is below x, $p_y \restriction (\omega_1 \setminus A) = p_x \restriction (\omega_1 \setminus A)$, and $w = y \restriction (A \times (C \cup \{\gamma\}))$. □

9.2. **Lemma.** $S \leq T$.

Proof. Suppose that $x \in S$; then we say that $\pi(x) = y$ if and only if $y = x - \gamma$. Clearly, in this case, $y \in T$.

We now show that π has the right properties.

To prove (O3), suppose that $\pi(s) = t$. Suppose that $t' \in T \restriction t$. We exhibit $s' \in S \restriction s$ such that $\pi(s') = s$. We define s' to be $\langle p_{t'}, q_{t'} \cup q_s, r\rangle$, where $r \restriction \omega_2 \setminus \{\gamma\} = r_{t'}$, and $r(\gamma) = r_s(\gamma)$. Then $s' \in S$, is below s, and is sent to t' by π.

To prove (O5), suppose that K is a finite subset of $S \cup T$. Let x be a lower bound for K, and let $K' = (K \cap S) \cup \{\langle p, q, r\rangle : \langle p, q, r \restriction (\omega_2 \setminus \{\gamma\})\rangle \in K \cap T \ \&\ r(\gamma) \subseteq r_x(\gamma)\}$. Then K' is a finite subset of S, and is compatible, since it has x for a lower bound. So, by (P4) for S, it has a lower bound in S.

To prove (O6), suppose that $A \times C \in \mathscr{C}(S)$, that $s \in S$, $\mathrm{dom}(p_s) \subseteq A$ and $\mathrm{dom}(r_s) \subseteq C$, and that $t \in T$ satisfies $\pi(s) = t \restriction (A \times C)$. Then we define s' so that $s' - \gamma = t$, $r_{s'}(\gamma) = r_s(\gamma)$, and $q_{s'} \restriction (\omega_1 \times \{\gamma\}) = q_s \restriction (\omega_1 \times \{\gamma\})$. Then $s' \in S$, and is below s, and $\pi(s') = t$. □

Given a generic filter H, we define a function Q in the corresponding generic extension so that $Q(\alpha) = k$ if and only if, for some element U of H with top element u, $q_u(\alpha, \gamma) = k$.

9.3. **Lemma.** $S \Vdash \dot{Q}\colon \omega_1 \to \omega$, *and* $S \Vdash (\forall\alpha)(\dot{Q}(\alpha) \geq \dot{g}(\alpha))$.

Proof. For suppose that $U \leq S$ decides the value of $\dot{g}(\alpha)$, and either decides the value of $\dot{Q}(\alpha)$ or forces it to be undefined.

Let π_0 witness $U \leq S$, π_1 witness $S \leq T$, and let π be their composition; π witnesses $U \leq T$.

Since T describes $\dot{g}$, there exists $t \in \mathrm{ran}(\pi)$ such that $T \restriction t$ decides the value of $\dot{g}(\alpha)$; without loss of generality (using (O3)), $t \leq a$ for some a for which $T \restriction a \Vdash \dot{g}(\alpha) = k$, so that U also forces $\dot{g}(\alpha) = k$. Suppose that $\pi(u) = t$, and that $\pi_0(u) = s$ and $\pi_1(s) = t$.

Then there exists $s' \in S \restriction s$ such that $\langle \alpha, \gamma\rangle \in \mathrm{dom}(q_{s'})$; if $\langle \alpha, \gamma\rangle \notin \mathrm{dom}(q_s)$, then a suitable s' can be defined just by adding $\langle \alpha, \gamma\rangle$ to the domain of $\mathrm{dom}(q_s)$, and setting $q_{s'}(\alpha, \gamma) = l$ for some sufficiently great l for which $l \geq k$. So, $S \restriction s' \Vdash \dot{Q}(\alpha) = l$.

Now we find $u' \in U \restriction u$ such that $\pi_0(u') = s'$; then $U \restriction u' \leq S \restriction s'$.

We said that U either decides the value of $\dot{Q}(\alpha)$ or forces it to be undefined; clearly it forces $\dot{Q}(\alpha) = l$. So, $U \Vdash \dot{Q}(\alpha) = l \geq k = \dot{g}(\alpha)$. □

Now let $E_n = \{x \in X : Q(x) \geq n\}$, and let $U_n = \bigcup\{R(i) : v_i \geq n\}$, where R is the generic function constructed from the $r_x(\gamma)$.

9.4. **Lemma.** (1) $S \Vdash \dot{D}_n \subseteq \dot{E}_n$.

(2) $S \Vdash \dot{E}_n = \dot{U}_n$, *so that* S *forces* $\dot{E}_n$ *to be open.*

(3) $S \Vdash \bigcap_{n\in\omega} \dot{U}_n = \varnothing$.

It is in the proof of this lemma that we use the full power of (P5).

Proof. That $S \Vdash \dot{D}_n \subseteq \dot{E}_n$ is a trivial consequence of Lemma 9.3.

We now prove that S forces $\dot{E}_n$ and $\dot{U}_n$ to be equal.

First we show that $S \Vdash \dot{U}_n \subseteq \dot{E}_n$. For, suppose that $U \le S$, and that $U \Vdash \dot{Q}(\alpha) = k < n$, so that $U \Vdash \dot{x}_\alpha \notin \dot{E}_n$. Suppose that $u \in U$ has the property that $\langle \alpha, \gamma \rangle \in \mathrm{dom}(q_u)$.

Then, whenever $v \le u$, $\langle \alpha, \gamma \rangle \in \mathrm{dom}(q_v)$ and $q_v(\alpha, \gamma) = k$. So, if $v_i > k$, $p_v(\alpha) \cap r_v(\gamma, i) = \varnothing$.

Hence, if $v_i > k$, $U \restriction u \Vdash \dot{x}_\alpha \notin \dot{R}(i)$, and so $U \restriction u \Vdash \dot{x}_\alpha \notin \dot{U}_n$. Because the set of such $U \restriction u$ is predense below U, $U \Vdash \dot{x}_\alpha \notin \dot{U}_n$. A density argument now allows us to see that $S \Vdash (\forall \alpha)(\dot{x}_\alpha \notin \dot{E}_n \to \dot{x}_\alpha \notin \dot{U}_n)$. Thus $S \Vdash \dot{U}_n \subseteq \dot{E}_n$.

Now we prove that $S \Vdash \dot{E}_n \subseteq \dot{U}_n$. For, suppose that $u \in U$, and that $q_u(\alpha, \gamma) = k \ge n$, so that $U \restriction u \Vdash \dot{x}_\alpha \in \dot{E}_n$. Then, construct $z \le u$ such that $(\forall \alpha' \in \mathrm{dom}(p_u) \setminus \{\alpha\})(p_z(\alpha) \cap p_z(\alpha') = \varnothing)$, and such that for some i for which $v_i = k$, $r_z(\gamma, i) = p_z(\alpha)$. Clearly such a z can be found.

We now use (P5) to find $x \in U \restriction u$ such that $p_x = p_z$, and such that $r_x(\gamma, i) = r_z(\gamma, i) = p_z(\alpha) = p_x(\alpha)$. Then $U \restriction x \Vdash \dot{x}_\alpha \in \dot{R}(i)$, so that $U \restriction x \Vdash \dot{x}_\alpha \in \dot{U}_n$.

So, $S \Vdash \dot{E}_n \subseteq \dot{U}_n$.

Finally we prove that S forces the $\dot{U}_n$ to have empty intersection.

But $\dot{U}_n$ is forced to be equal to $\dot{E}_n$, and if $Q(x) = n$, then $x \notin E_{n+1}$, by definition of the E_i. Since $\dot{Q}$ is forced to be a total function, $\bigcap U_n = \varnothing$ as required. □

9.5. **Theorem.** *X is a Δ-set.*

Proof. We have put around the D_n a sequence $\langle U_n : n \in \omega \rangle$ of open sets with empty intersection, as was required. □

10. X IS NOT A Q-SET

We prove this by starting with a particular subset L' of X, and proving that out of any descriptor of a function coding a G_δ-subset of X, we can construct a new condition which forces that G_δ-set to be different from L'. The way this works is that a G_δ-set is determined by a countable function, so that if T describes a G_δ-set, then only countably many elements of T are involved in the description. So, take two elements α_0 and α_1 of ω_1 not mentioned in any element of T which takes part in the description; one of x_{α_0} and x_{α_1} will be in L' and the other will not. Now construct a new condition U out of T in such a way that U treats α_0 and α_1 exactly the same; that is to say, there is a symmetry on U which interchanges the roles of α_0 and α_1, and which leaves the part of U which describes the G_δ-set unaffected. It turns out that the symmetry is so great that if U forces x_{α_0} to be in the G_δ-set, then it does not force x_{α_1} to be outside it; and so we show λ that the G_δ-set is different from L'.

Let ω_1 be divided, in the ground model, into two disjoint uncountable sets L and M. We show that in the generic extension, $L' = \{x_\alpha : \alpha \in L\}$ is not a G_δ-set. In particular, we show that if, in the generic extension, $\bigcap_{n\in\omega} U_n$ is a G_δ-set, then $L' \ne \bigcap_{n\in\omega} U_n$.

For, let $\dot{g}$ be some name for a function from $\omega \times \omega$ to $\mathscr{B}$, such that for each n, $\Vdash \bigcup_{k\in\omega} \dot{g}(n, k) = \dot{U}_n$.

Let $T \in \mathbb{P}$ describe $\dot{g}$ and have a greatest element t. For each n and k, let $A_{n,k}$ be a maximal antichain in T such that for each $a \in A_{n,k}$, $T \restriction a$ decides the value of $\dot{g}(n, k)$. Let α^* and $C \subseteq G(T)$ be such that for each $\alpha^* \times C \in \mathscr{C}(T)$, and for all a in any of the $A_{n,k}$, $\operatorname{dom}(p_a) \subseteq \alpha^*$, and $\operatorname{dom}(r_a) \subseteq C$. Let $\alpha_0 \in L$ and $\alpha_1 \in M$ be greater than α^*.

Let $\varpi_0 \colon \omega_1 \to \omega_1$ be defined to fix every element of ω_1 except α_0 and α_1, which it switches, and let ϖ_1 be a self-inverse bijection on ω_2, with the property that $\varpi_1[G[T]] \cap G[T] = \varnothing$. Let $\varpi^\sharp$ be the isomorphism on $\mathbb{Q}$, and $\varpi^{\times}$ be the isomorphism on $\mathbb{P}$, induced by $\varpi_0 \times \varpi_1$. Notice that $\varpi^\sharp$ and $\varpi^{\times}$ are self-inverse. ($\times$ is meant to suggest the double-sharp sign.)

If $s \in T$, $s' \in \varpi^{\times}(T)$ and $p_s = p_{s'}$, we define the *amalgamation* $[s, s']$ of s and s', to be $\langle p_s, q_s \cup q_{s'}, r_s \cup r_{s'} \rangle$.

Define U to be $\{[s, s'] : s \in T,\ s' \in \varpi^{\times}(T), s \restriction (\alpha^* \times C) = \varpi^\sharp(s') \restriction (\alpha^* \times C)\ \&\ p_s = p_{s'}\}$, with $G(U)$ being $G(T) \cup \varpi_1[G[T]]$.

We define $\mathscr{C}(U)$ to be $\{A \times (\bar{C} \cup \varpi_1[\bar{C}]) : (A \times \bar{C}) \in \mathscr{C}(T)\ \&\ \alpha^* \subseteq A\ \&\ C \subseteq \bar{C}\}$.

10.1. **Lemma.** *$U \in \mathbb{P}$, $U \leq T$, and $\varpi^{\times}(U) = U$.*

Proof. To prove (P4), suppose that K is a finite compatible subset of U. Let $K' = \{s : (\exists s')([s, s'] \in K)\}$. Then K' is a compatible subset of T. Let u be a lower bound in T for this set. Let $K_1 = \{\langle p_u, q, r\rangle : (\exists p)(\langle p, q, r\rangle \in K)\} = \{[\langle p_u, q, r\rangle, \langle p_u, q', r'\rangle] : (\exists p, p')([\langle p, q, r\rangle, \langle p', q', r'\rangle] \in K)\}$; then K_1 is a compatible subset of U, and $K_1' = \{s : (\exists s')([s, s'] \in K_1)\}$ has lower bound u. Now, since $\alpha^* \times C \in \mathscr{C}(T)$, $v = u \restriction (\alpha^* \times C) \in T$. But then, since $s \restriction (\alpha^* \times C) = \varpi^\sharp(s') \restriction (\alpha^* \times C)$ whenever $[s, s'] \in K$, if $K_1'' = \{s' : (\exists s)([s, s'] \in K_1)\}$, then $K_1'' \cup \{\varpi^\sharp(v)\}$ is compatible and so has a lower bound w in $\varpi^{\times}(T)$. Let $u' = \langle p_w, q_u, r_u\rangle$; then u' is in T and lies below u, and since $w \leq \varpi^\sharp(v)$, $\varpi^\sharp(w) \restriction (\alpha^* \times C) \leq u' \restriction (\alpha^* \times C) \leq v$. Finally, by (P8), there exists x in T such that $x \leq u'$ and $x \restriction (\alpha^* \times C) = \varpi^\sharp(w) \restriction (\alpha^* \times C)$. Since $x \leq u'$, p_x extends $p_{u'} = p_w$. Let $w' = \langle p_x, q_w, r_w\rangle$; this is an element of $\varpi^{\times}(T)$ and lies below w.

Then w' is a lower bound in $\varpi^{\times}(T)$ for K_1'', x is a lower bound in T for K_1', w' and x are compatible, $p_{w'} = p_x$ and $x \restriction (\alpha^* \times C) = \varpi^\sharp(w') \restriction (\alpha^* \times C)$, so $[x, w']$ exists as a member of U and is a suitable lower bound for $[s, s']$ and $[t, t']$.

Now we show that $U \leq T$. Define π so that $\pi([s, t]) = s$. We show that this works.

If $[s, t]$ exists, then certainly $s = \langle p_{[s,t]}, q_{[s,t]} \restriction (\omega_1 \times G(T)), r_{[s,t]} \restriction G(T)\rangle$.

Suppose that K is a finite subset of T and $\{[s_i, t_i] : i \in k\}$ is a finite subset of U, and that their union is compatible. Then if $t' = \varpi^\sharp(t \restriction (\alpha^* \times C))$ and $t'' = \langle p_t, q_{t'}, r_{t'}\rangle$, then by (P5) and (P7) on $\varpi^{\times}(T)$, $\{[t, \varpi^\sharp(t \restriction (\alpha^* \times C))] : t \in K\} \subseteq U$, and its union with the set of $[s_i, t_i]$ is compatible. So there is in U a lower bound for all of them.

Suppose that $[s, t] \in U$; then $\pi([s, t]) = s$. Suppose that $s'' \in T \restriction s$. Let $s' = \langle p_{s''}, q_s, r_s\rangle$, and let $t' = \langle p_{s''}, q_t, r_t\rangle$; then $[s', t']$ is an element of U below $[s, t]$. Then $s'' \restriction (\alpha^* \times C) \leq s' \restriction (\alpha^* \times C) = \varpi^\sharp(t') \restriction (\alpha^* \times C)$. So, by (P8) on T, there exists $\varpi^\sharp(t'') \leq \varpi^\sharp(t')$ such that $\varpi^\sharp(t'') \restriction (\alpha^* \times C) = s' \restriction (\alpha^* \times C)$ and $p_{t''} = p_{s''}$. Then $[s'', t''] \in U$, and $\pi([s'', t'']) = s''$.

To prove (O6), suppose that $[s, t] \in U$, and $A \times \bar{C} \in \mathscr{C}(U)$, that $\operatorname{dom}(p_{[s,t]})$

$\subseteq A$ and $\mathrm{dom}(r_{[s,t]}) \subseteq \bar{C}$, that $s' \in T$, and that $\pi([s,t]) = s = s' \restriction (A \times \bar{C})$. Then we use (P8) and (P5) on T to find $t' \in \varpi^{\times}(T) \restriction t$ such that $t' \restriction (\alpha^* \times \bar{C}) = \varpi^{\sharp}(s' \restriction (\alpha^* \times C))$, and such that $p_{t'} = p_{s'}$. Then $[s', t']$ is an element of U below $[s, t]$, and $\pi([s', t']) = s'$, as required.

We also observe that U is fixed by $\varpi^{\sharp}$. □

10.2. **Lemma.** *Suppose that $W \leq U$, and that $W \Vdash \dot{x}_{\alpha_0} \in \bigcap_{n\in\omega} \dot{U}_n$. Then $\varpi^{\times}(W) \Vdash \dot{x}_{\alpha_1} \in \bigcap_{n\in\omega} \dot{U}_n$.*

Proof. Suppose that $W \leq U$, and that $W \Vdash \dot{x}_{\alpha_0} \in \bigcap_{n\in\omega} \dot{U}_n$. Then for each n, there is a dense set of Y below W, for each of which there exists $I \in \mathscr{B}$ such that $Y \Vdash \dot{x}_{\alpha_0} \in I \subseteq \dot{U}_n$.

We show that for each such Y, there exists Z below $\varpi^{\times}(Y)$ such that $Z \Vdash \dot{x}_{\alpha_1} \in I \subseteq \dot{U}_n$.

Without loss of generality, Y has a greatest element y.

Suppose that π_0 witnesses $Y \leq U$, that π_1 is the function defined in the proof of Lemma 10.1 witnessing $U \leq T$, and that π is their composition.

We are given that $Y \Vdash \dot{x}_{\alpha_0} \in I$. But this can only mean that $\alpha_0 \in \mathrm{dom}(p_y)$, and that $p_y(\alpha_0) \subseteq I$. But then, $\varpi^{\sharp}(y)$ is the greatest element of $\varpi^{\times}(Y)$, and $p_{\varpi^{\sharp}(y)}(\alpha_1) = p_y(\alpha_0) \subseteq I$, so $\varpi^{\times}(Y) \Vdash \dot{x}_{\alpha_1} \in I$.

We are also given that $Y \Vdash I \subseteq \dot{U}_n$. Well, T describes $\dot{g}$, so there exists t in the range of π such that $T \restriction t \Vdash I \subseteq \dot{U}_n$. Suppose that $\pi_0(z) = u$ and that $\pi_1(u) = t$.

Then since $T \restriction t \Vdash I \subseteq \dot{U}_n$, by the definition of α^* and C, there exists $s \in T$ compatible with t such that $\mathrm{dom}(p_s) \subseteq \alpha^*$, $\mathrm{dom}(r_s) \subseteq C$, and $T \restriction s \Vdash I \subseteq \dot{U}_n$.

Now since $u \in U$ and $\pi_1(u) = t$, there exists t' such that $u = [t, t']$; t' necessarily satisfies $t \restriction (\alpha^* \times C) = \varpi^{\sharp}(t') \restriction (\alpha^* \times C)$. So $\varpi^{\sharp}(s)$ is compatible with t'. (Note that $[s, \varpi^{\sharp}(s)]$ is an element of U.) Hence $[s, \varpi^{\sharp}(s)]$ is compatible with u.

But, $[s, \varpi^{\sharp}(s)] = \varpi^{\sharp}([s, \varpi^{\sharp}(s)])$, and so is compatible with $\varpi^{\sharp}(u) = [\varpi^{\sharp}(t'), \varpi^{\sharp}(t)]$.

Now, if π_0^* is the function induced by π_0 under the map $\varpi^{\sharp}$, then π_0^* witnesses $\varpi^{\times}(Y) \leq \varpi^{\times}(U) = U$. Since $\pi_0^*(\varpi^{\sharp}(z)) = \varpi^{\sharp}(u)$, and $\varpi^{\sharp}(u)$ and $[s, \varpi^{\sharp}(s)]$ are compatible, $\varpi^{\sharp}(z)$ and $[s, \varpi^{\sharp}(s)]$ are compatible (using condition (O4)) and so there exists v in $\varpi^{\times}(Y)$ which is a lower bound for $[s, \varpi^{\sharp}(s)]$ and $\varpi^{\sharp}(z)$.

Then $\varpi^{\times}(Y) \restriction v \leq U \restriction [s, \varpi^{\sharp}(s)] \leq T \restriction s$, so $\varpi^{\times}(Y) \restriction v \Vdash I \subseteq \dot{U}_n$.

Since $\varpi^{\times}(Y) \restriction v \leq \varpi^{\times}(Y)$, $\varpi^{\times}(Y) \restriction v \Vdash \dot{x}_{\alpha_1} \in I \subseteq \dot{U}_n$.

$\varpi^{\times}(Y) \restriction v$ is therefore the element of $\mathbb{P}$ below $\varpi^{\times}(Y)$ that we were looking for. □

10.3. **Proposition.** $U \not\Vdash \dot{L}' = \bigcap_{n\in\omega} \dot{U}_n$.

Proof. By Lemma 10.2, if $U \Vdash \dot{x}_{\alpha_0} \in \bigcap_{n\in\omega} \dot{U}_n$, then $U \not\Vdash \dot{x}_{\alpha_1} \notin \bigcap_{n\in\omega} \dot{U}_n$. Hence, if $U \Vdash \dot{L}' \subseteq \bigcap_{n\in\omega} \dot{U}_n$, then $U \not\Vdash \bigcap_{n\in\omega} \dot{U}_n \subseteq \dot{L}'$. □

10.4. **Theorem.** *X is not a Q-set.*

Proof. We have shown in this section that if $\bigcap_{n\in\omega} \dot{U}_n$ is a name for a G_δ-set and T is a descriptor of it, then there exists $U \leq T$ which does not force the G_δ-set equal to $\dot{L}'$. Since the set of descriptors of a G_δ-set is dense, $\Vdash \dot{L}' \neq \bigcap_{n\in\omega} \dot{U}_n$. Since this is true for every name for a G_δ-set, L' itself is not G_δ. Since X has a subset which is not G_δ, it is not a Q-set. □

11. Conclusions

The proof that it is consistent that there exists a Δ-set that is not a Q-set is now complete. The method presented can be easily modified to yield a set whose every finite power is a Δ-set, but which is not a Q-set, thus answering D. J. Lutzer's question. In this model, $2^{\aleph_0} = 2^{\aleph_1}$. A model in which there was an uncountable Δ-set and in which the continuum was less than $2^{\aleph_1}$ would contain a subset of size $2^{\aleph_0}$ of the set of all functions from ω_1 to ω, which dominated under the product partial order, and hence under the partial order of eventual dominance; this is because if X is a Δ-set, then the set of functions from X to ω corresponding to decreasing countable sequences of open subsets of X with empty intersection, is a set of size 2^ω and is a dominating family in ${}^{\omega_1}\omega$. (The question of the existence of such a model was investigated in [S] and [JP], and certain restrictions were placed on models in which one existed, for example, that in such a model, the continuum must be greater than or equal to $\aleph_{\omega_1}$ and cannot be real-valued measurable.) However it seems to be difficult to modify the partial order used in this paper to simultaneously add a large number of reals and preserve cardinals greater than ω_2.

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