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KAC-MOODY LIE ALGEBRAS, SPECTRAL SEQUENCES, AND THE WITT FORMULA

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ABSTRACT. In this work, we develop a homological theory for the graded Lie algebras, which gives new information on the structure of the Lorentzian Kac-Moody Lie algebras. The technique of the Hochschild-Serre spectral sequences offers a uniform method of studying the higher level root multiplicities and the principally specialized affine characters of Lorentzian Kac-Moody Lie algebras.

INTRODUCTION

In the past 20 years, the theory of Kac-Moody Lie algebras has developed rapidly and with great success. Surprising connections to areas such as combinatorics, modular forms, and mathematical physics have shown Kac-Moody Lie algebras to be of uncommon interest. The discovery of the Macdonald identities gave rise to an intensive study of the class of affine Kac-Moody Lie algebras and their representations [Mcd]. The structure of such Lie algebras and their connections with other branches of mathematics and mathematical physics have been well-established and are being extensively investigated.

The next natural step after the affine case is that of the hyperbolic Kac-Moody Lie algebras. One of the most ambitious goals of current research activity in infinite dimensional Lie algebras may be to construct geometric realizations of the hyperbolic Kac-Moody Lie algebras. Once that is accomplished, we will have a much deeper understanding of the structure of Kac-Moody Lie algebras and their connections with number theory. Unfortunately, many basic questions regarding the hyperbolic case are still unresolved. For example, the behavior of the root multiplicities is not well-understood. Feingold-Frenkel [F-F] and Kac-Moody-Wakimoto [K-M-W] made some progress in this area. They computed the level 2 root multiplicities for the hyperbolic Kac-Moody Lie algebras $HA_1^{(1)}$ and $HE_8^{(1)}$. Other important works on the hyperbolic Kac-Moody Lie algebras include [Fe2, Fr, LM, and M2]. Recently, V. Kac has informed the author that he also discovered a level 3 root multiplicity formula for $HA_1^{(1)}$ (unpublished).

In this work, we develop a homological theory for the graded Lie algebras. Combining with the representation theory of affine Kac-Moody Lie algebras, we obtain new information on the structure of the Lorentzian Kac-Moody Lie algebras; i.e., Kac-Moody Lie algebras whose Cartan matrix has a Lorentzian

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signature. The technique of the Hochschild-Serre spectral sequences offers a uniform method of studying the higher level root multiplicities and the principally specialized affine characters of Lorentzian Kac-Moody Lie algebras.

More precisely, we start with the observation that any symmetrizable Kac-Moody Lie algebra can be realized as the minimal graded Lie algebra $L = \bigoplus_{n\in Z} L_n$ with the local part $V \oplus L_0 \oplus V^*$, where L_0 is a "smaller" Kac-Moody Lie algebra, V is an integrable irreducible highest weight module over L_0, and V^* is the contragredient of V in a suitable sense [Kang]. So $L = G/I$, where $G = \bigoplus_{n\in Z} G_n$ is the maximal graded Lie algebra with the local part $V \oplus L_0 \oplus V^*$, and $I = \bigoplus_{n\in Z} I_n$ is the maximal graded ideal of G intersecting the local part trivially. In particular, when L_0 is an affine Kac-Moody Lie algebra and V is the basic representation of L_0, we obtain a series of Lorentzian Kac-Moody Lie algebras (e.g., [F-F, K-M-W, Kang]). The idea is to study the structure of each homogeneous space L_n as a module over the affine Kac-Moody Lie algebra L_0. We note that $G_\pm = \bigoplus_{n\geq 1} G_{\pm n}$ is the free Lie algebra generated by $G_1 = V^*$ (respectively, $G_{-1} = V$). Thus we divide our study into two parts: the study of the free Lie algebra and the study of the maximal graded ideal.

For the free Lie algebra, by a direct generalization of the proof of the classical Witt formula given in [Se], we obtain a character formula for the free Lie algebra, which we also call the Witt formula. The main ingredients of the proof are the Poincaré-Birkhoff-Witt theorem and the Möbius inversion. For the maximal graded ideal, as O. Mathieu pointed out (in private communication), one can use the exact sequences in [Kac2] to understand the structure of I. The main result of this work is the following reduction theorem:

Let I be the graded ideal of the free Lie algebra G generated by the subspace I_m of G_m for $m \geq 2$. Let $I^{(j)} = \sum_{n\geq j\geq m} I_n$. Since I_m generates I, $I^{(j)}$ is also a graded ideal of G generated by the subspace I_j. Consider the quotient Lie algebra $L^{(j)} = G/I^{(j)}$. Then we have

$$I_{j+1} \cong (V \otimes I_j)/H_3(L^{(j)})_{j+1}.$$

Thus we reduce the problem to computing $H_3(L^{(j)})$. When j is the first nontrivial index, we can compute the homology modules using the Kostant formula [Ko, G-L, Liu]. For the higher levels, we invoke the technique of Hochschild-Serre spectral sequences and the five term exact sequences. Combining with the representation theory of affine Kac-Moody Lie algebras, we determine some of the boundary homomorphisms, and deduce new structural information on the maximal graded ideal. Applying this to certain Lorentzian Kac-Moody Lie algebras, we compute the principally specialized affine characters for certain higher levels. Furthermore, we compute the root multiplicities of the hyperbolic Kac-Moody Lie algebras $HA_1^{(1)}$ and $HA_2^{(2)}$ up to level 3. All the formulas obtained here are new. Comparing with Kac's result, we obtain a combinatorial identity. As far as we know, the formulas for levels higher than 3 are first computed here.

1. Preliminaries

An $n \times n$ matrix $A = (a_{ij})$ is called a *generalized Cartan matrix* if it satisfies the following conditions: (i) $a_{ii} = 2$ for $i = 1, \ldots, n$, (ii) a_{ij} are nonpositive integers for $i \neq j$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. A is called *symmetrizable* if DA is symmetric for some diagonal matrix $D = \operatorname{diag}(q_1, \ldots, q_n)$ with $q_i > 0$,

$q_i \in \mathbf{Q}$. A *realization* of an $n \times n$ matrix A of rank l is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a $(2n-l)$-dimensional complex vector space, $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ are linearly independent indexed subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$ for $i, j = 1, \dots, n$. The *Kac-Moody Lie algebra* $\mathfrak{g}(A)$ associated with a generalized Cartan matrix A is the Lie algebra generated by the elements e_i, f_i $(i = 1, \dots, n)$ and $\mathfrak{h}$ with the following defining relations:

$$
\begin{aligned}
&[h, h'] = 0 \quad \text{for } h, h' \in \mathfrak{h},\\
&[e_i, f_j] = \delta_{ij}\alpha_i^\vee \quad \text{for } i, j = 1, \dots, n,\\
&[h, e_j] = \alpha_j(h)e_j, \quad [h, f_j] = -\alpha_j(h)f_j \qquad \text{for } j = 1, \dots, n,\\
&(ade_i)^{1-a_{ij}}(e_j) = 0 \quad \text{for } i, j = 1, \dots, n \text{ with } i \neq j,\\
&(adf_i)^{1-a_{ij}}(f_j) = 0 \quad \text{for } i, j = 1, \dots, n \text{ with } i \neq j.
\end{aligned}
\tag{1.1}
$$

The Kac-Moody Lie algebra $\mathfrak{g}(A)$ has the *root space decomposition*

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) | [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

An element $\alpha \in \mathfrak{h}^*$ is called a *root* if $\mathfrak{g}_\alpha \neq 0$. The space $\mathfrak{g}_\alpha$ is called the α-*root space* and $\dim \mathfrak{g}_\alpha$ is called the *multiplicity* of α. Denote by $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) the subalgebra of $\mathfrak{g}(A)$ generated by the elements e_i (respectively, f_i) for $i = 1, \dots, n$. Then $\mathfrak{g}(A)$ has the *triangular decomposition* $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

A $\mathfrak{g}(A)$-module V is called a *highest weight module with highest weight* $\Lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v \in V$ such that (i) $\mathfrak{n}^+ \cdot v = 0$, (ii) $h \cdot v = \Lambda(h)v$ for all $h \in \mathfrak{h}$, (iii) $U(\mathfrak{g}(A)) \cdot v = V$, where $U(\mathfrak{g}(A))$ denotes the universal enveloping algebra of $\mathfrak{g}(A)$. A highest weight module V with highest weight Λ has the *weight space decomposition*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where

$$V_\lambda = \{v \in V | h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

It is easy to see that $\dim V_\lambda < \infty$. We define the *formal character* of V to be

$$\operatorname{ch} V = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e(\lambda),$$

where $e(\lambda)$ are the elements of the group algebra $\mathbf{C}[\mathfrak{h}^*]$ with the multiplication

$$e(\lambda)e(\mu) = e(\lambda + \mu) \quad \text{for } \lambda, \mu \in \mathfrak{h}^*.$$

For a weight $\lambda = \Lambda - \sum_i k_i \alpha_i$ of V, we define $\deg(\lambda) = \sum_i k_i$. Then setting

$$V_j = \bigoplus_{\lambda : \deg(\lambda) = j} V_\lambda$$

defines the *principal gradation* of V:

$$V = \bigoplus_{j \geq 0} V_j.$$

We define the *principally specialized character* of V to be

$$\mathrm{ch}_q = \sum_{j \geq 0} (\dim V_j) q^j .$$

For each $\Lambda \in \mathfrak{h}^*$, there exists a unique irreducible highest weight module $V(\Lambda)$ with highest weight Λ. If Λ is dominant integral, $V(\Lambda)$ is *integrable*; i.e., all the e_i and f_i $(i = 1, \dots, n)$ are locally nilpotent on $V(\Lambda)$.

An indecomposable generalized Cartan matrix A is said to be of *finite* type if all its principal minors are positive, of *affine* type if all its proper principal minors are positive and $\det A = 0$, and of *indefinite* type if A is of neither finite nor affine type. A is of *hyperbolic* type if it is of indefinite type and all its proper principal submatrices are of finite or affine type.

Let $A = (a_{ij})_{i,j=0,1,\dots,l}$ be a generalized Cartan matrix of affine type and let $\widetilde{A} = (\tilde{a}_{ij})_{i,j=-1,0,1,\dots,l}$ be the generalized Cartan matrix defined by

$$(1.2) \qquad \begin{gathered} \tilde{a}_{-1,-1} = 2, \qquad \tilde{a}_{-1,0} = \tilde{a}_{0,-1} = -1, \\ \tilde{a}_{-1,i} = \tilde{a}_{i,-1} = 0, \quad \text{for } i = 1, \dots, l, \\ \tilde{a}_{i,j} = a_{i,j}, \quad \text{for } i, j = 0, 1, \dots, l. \end{gathered}$$

Then $\widetilde{A}$ has a Lorentzian signature. The Kac-Moody Lie algebras whose Cartan matrix has a Lorentzian signature are said to be of *Lorentzian type.* Note that the matrix $\widetilde{A}$ is of hyperbolic type for small values of n. In this work, we study the structure of Kac-Moody Lie algebras $\mathfrak{g}(\widetilde{A})$ of Lorentzian type, where $\widetilde{A}$ has the form (1.2).

Let $V = V(\Lambda_0)$ be the basic representation of the affine Kac-Moody Lie algebra $\mathfrak{g}(A)$ and let V^* be the contragredient of V in the sense that V^* is the space of all linear functionals on V which vanish on all but finitely many weight spaces of V. Hence V^* is the irreducible lowest representation of $\mathfrak{g}(A)$ with lowest weight $-\Lambda_0$. Let $\langle \ , \ \rangle$ denote the canonical bilinear pairing between V^* and V. For $x \in \mathfrak{g}(A)$, $v \in V$, $v^* \in V^*$, we simply write $\langle v^* x v \rangle$ for

$$\langle v^* \cdot x , v \rangle = -\langle x \cdot v^* , v \rangle = \langle v^* , x \cdot v \rangle .$$

Choose a pair of dual bases $\{x_i | i \in J\}$ and $\{y_i | i \in J\}$ of $\mathfrak{g}(A)$ with respect to the standard invariant symmetric bilinear form defined on $\mathfrak{g}(A)$ [Kac2, Chapter 2]. Define a bilinear map $\phi \colon V^* \times V \to \mathfrak{g}(A)$ by

$$(1.3) \qquad \phi(v^*, v) = -\sum_{i \in J} \langle v^* x_i v \rangle y_i .$$

It is clear that the map ϕ is well-defined. The space $V \oplus \mathfrak{g}(A) \oplus V^*$ has a local Lie algebra structure with the Lie bracket defined as follows.

(i) The Lie bracket in $\mathfrak{g}(A)$ is the obvious one.

(ii) The Lie bracket between $\mathfrak{g}(A)$ and V (respectively, V^*) is given by the $\mathfrak{g}(A)$-module action:

$$[x, v] = x \cdot v , \quad [x, v^*] = x \cdot v^* , \qquad \text{for } v \in V ,\ v^* \in V^* , \text{ and } x \in \mathfrak{g}(A) .$$

(iii) The bracket between V and V^* is given by the map $\phi \colon V^* \times V \to \mathfrak{g}(A)$:

$$[v^*, v] = \phi(v^*, v) \quad \text{for } v^* \in V^* \text{ and } v \in V .$$

Thus there exist the maximal graded Lie algebra $G = \bigoplus_{n\in Z} G_n$ and the minimal graded Lie algebra $L = \bigoplus_{n\in Z} L_n$ with the local part $V \oplus \mathfrak{g}(A) \oplus V^*$ [Kac1]. Note that $G_0 = L_0 = \mathfrak{g}(A)$. For $n \geq 1$, the homogeneous subspace G_n (respectively, G_{-n}) is spanned by all the brackets of n vectors from V^* (respectively, V), and the grading on L is induced by that of G. We define a graded ideal I as follows. For $n \geq 2$, let

$$I_{\pm n} = \{x \in G_{\pm n} | (\operatorname{ad} G_{\mp 1})^{n-1} x = 0\},$$

and define $I = \bigoplus_{n\in Z} I_n$. Set $I_\pm = \bigoplus_{n\geq 1} I_{\pm n}$. Then the subspaces I and $I_\pm$ are all graded ideals of G, and I is the maximal graded ideal of G intersecting the local part trivially [B-K-M, F-F, Kac1, Kang]. Hence $L = G/I$. We write $G_\pm = \bigoplus_{n\geq 1} G_{\pm n}$ and $L_\pm = \bigoplus_{n\geq 1} L_{\pm n}$. Then G_+ (respectively, G_-) is the free Lie algebra generated by V^* (respectively, V).

Using the Gabber-Kac Theorem [G-K], we can prove that L is isomorphic to the Kac-Moody Lie algebra $\mathfrak{g}(\widetilde{A})$ [F-F, K-M-W, Kang]. In particular, the ideal I_+ (respectively, I_-) is generated by the elements $(\operatorname{ad} v_0^*)^2(e_i)$ (respectively, $(\operatorname{ad} v_0)^2(f_i)$) for $i = 0, 1, \dots, l$, where v_0 and v_0^* are highest and lowest weight vectors of V and V^*, respectively. Thus the ideal I_+ (respectively, I_-) is generated by the space I_2 (respectively, I_{-2}). We denote by $\alpha_{-1}, \alpha_0, \dots, \alpha_l$ the simple roots of $\mathfrak{g}(A)$. Thus V is the irreducible highest weight module over $\mathfrak{g}(A)$ with highest weight $-\alpha_{-1}$ and V^* is the irreducible lowest weight module over $\mathfrak{g}(A)$ with lowest weight α_{-1}. We will study the structure of the homogeneous subspaces $L_n = G_n/I_n$ as modules over the affine Kac-Moody Lie algebra $\mathfrak{g}(A)$.

2. The Witt formula

Let $G = \bigoplus_{n\geq 1} G_n$ be the free Lie algebra with a finite set of (free) generators $\{x_1, \dots, x_r\}$. Then we have the following well-known *Witt formula* [J2]:

$$\dim G_n = \frac{1}{n} \sum_{d|n} \mu(d) r^{n/d},$$

where μ denotes the classical Möbius function. A nice proof is given in [Se]. The main ingredients of the proof are the Poincaré-Birkhoff-Witt theorem and Möbius inversion.

More generally, let $X = \{x_i | i = 1, 2, 3, \dots\}$ be a totally ordered set (possibly countably infinite) and let R be an (additive) partially ordered abelian semigroup with a countable basis such that each element α of R can be expressed as a sum of elements of R which are less than or equal to α in only finitely many ways. Let G be the free Lie algebra on the set X. We make G an R-graded Lie algebra as follows. Define $\deg(x_i) = \mu_i$ for $\mu_i \in R$ such that $\mu_i \leq \mu_j$ for $i < j$, and

$$\deg([[\cdots[x_{i_1}, x_{i_r}]\cdots]x_{i_r}]) = \mu_{i_1} + \cdots + \mu_{i_r}.$$

Let $G_\lambda = \{x \in G | \deg(x) = \lambda\}$ for $\lambda \in R$. Then G has the decomposition $G = \bigoplus_{\lambda\in R} G_\lambda$, and $[G_\lambda, G_\mu] \subset G_{\lambda+\mu}$ for $\lambda, \mu \in R$. If $G_\lambda \neq 0$, we call λ a *root* of G and G_λ the *λ-root space* of G. When all the root spaces are finite dimensional, we define the *formal character* of G to be

$$\operatorname{ch} G = \sum_{\lambda\in R} \dim G_\lambda e(\lambda),$$

where $e(\lambda)$ are the elements of the semigroup algebra $\mathbf{C}[R]$ with the multiplication $e(\lambda)e(\mu) = e(\lambda+\mu)$. Similarly, we can define the *roots, root spaces*, and *formal character* for the universal enveloping algebra $U(G)$ of G. Then by generalizing the proof of the Witt formula given in [Se], we obtain the following generalization:

Theorem 2.1. *Let $S = \{\tau_i | i = 1, 2, 3, \dots\}$ be the set of distinct degrees of the elements of X. Then the vector space V spanned by the elements of X has the following decomposition: $V = \bigoplus_{i=1}^{\infty} V_{\tau_i}$. For $\tau \in R$, set*

$$T(\tau) = \left\{(n) = (n_1, n_2, n_3, \dots) | n_i \in \mathbf{Z}_{\geq 0}, \ \sum n_i \tau_i = \tau\right\}, \tag{2.1}$$

and define

$$B(\tau) = \sum_{(n)\in T(\tau)} \frac{((\sum n_i) - 1)!}{\prod(n_i!)} \prod (\dim V_{\tau_i})^{n_i}. \tag{2.2}$$

Then

$$\dim G_\lambda = \sum_{\tau|\lambda} \mu\left(\frac{\lambda}{\tau}\right) \frac{\tau}{\lambda} B(\tau), \tag{2.3}$$

where $\tau|\lambda$ if $\lambda = k\tau$ for some positive integer k, in which case $\lambda/\tau = k$ and $\tau/\lambda = 1/k$.

Proof. Let $\gamma_1, \gamma_2, \gamma_3, \dots$ be an enumeration of elements of R and let $m_s = \dim G_{\gamma_s}$. For each $s = 1, 2, 3, \dots$, let $\{e_{\gamma_s, i_s} | i_s = 1, \dots, m_s\}$ be a basis of G_{γ_s}. Then by the Poincaré-Birkhoff-Witt theorem the vectors of the form

$$e_{\gamma_1,1}^{n_{1,1}} \cdots e_{\gamma_1,m_1}^{n_{1,m_1}} e_{\gamma_2,1}^{n_{2,1}} \cdots e_{\gamma_2,m_2}^{n_{2,m_2}} \cdots e_{\gamma_t,1}^{n_{t,1}} \cdots e_{\gamma_t,m_t}^{n_{t,m_t}}$$

are a basis of $U(G)$. Thus $\dim U(G)_\lambda$ is the cardinality of the family $\{n_{i,j}\}$ such that

$$(n_{1,1} + \cdots + n_{1,m_1})\gamma_1 + (n_{2,1} + \cdots + n_{2,m_2})\gamma_2 + \cdots = \lambda.$$

But this is the coefficient of $e(\lambda)$ in the following formal expression:

$$\prod_{s=1}^{\infty} (1 + e(\gamma_s) + e(2\gamma_s) + e(3\gamma_s) + \cdots)^{m_s}.$$

Therefore

$$\begin{aligned} \operatorname{ch} U(G) &= \sum_{\lambda\in R} (\dim U(G)_\lambda) e(\lambda) \\ &= \prod_{s=1}^{\infty} (1 + e(\gamma_s) + e(2\gamma_s) + e(3\gamma_s) + \cdots)^{m_s} \\ &= \prod_{s=1}^{\infty} \frac{1}{(1 - e(\gamma_s))^{m_s}} = \prod_{\lambda\in R} (1 - e(\lambda))^{-\dim G_\lambda}. \end{aligned}$$

On the other hand, let V be the vector space spanned by the set X. Then $U(G)$ is the tensor algebra $T(V) = \mathbf{C} \oplus V \oplus (V \otimes V) \oplus \cdots$ on V. Thus we have

$$\operatorname{ch} U(G) = 1 + \operatorname{ch} V + (\operatorname{ch} V)^2 + \cdots = \frac{1}{1 - \operatorname{ch} V} = \frac{1}{1 - \sum_{i=1}^{\infty} (\dim V_{\tau_i}) e(\tau_i)}.$$

Therefore we obtain

$$\operatorname{ch} U(G) = \frac{1}{1 - \sum_{i=1}^{\infty} (\dim V_{\tau_i}) e(\tau_i)} = \prod_{\lambda \in R} (1 - e(\lambda))^{-\dim G_\lambda}.$$

Using the formal power series $\log(1-t) = -\sum_{m=1}^{\infty} t^m/m$, we get from the left-hand side

$$\begin{aligned}
\log\left(\frac{1}{1-\operatorname{ch} V}\right) &= -\log(1-\operatorname{ch} V) = \sum_{m=1}^{\infty} \frac{(\operatorname{ch} V)^m}{m} \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{i=1}^{\infty} (\dim V_{\tau_i}) e(\tau_i)\right)^m \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{(n)=(n_i) \\ \sum n_i = m}} \frac{(\sum n_i)!}{\prod (n_i!)} \prod (\dim V_{\tau_i})^{n_i} e\left(\sum n_i \tau_i\right) \\
&= \sum_{\tau \in R} \sum_{(n) \in T(\tau)} \frac{((\sum n_i) - 1)!}{\prod (n_i!)} \prod (\dim V_{\tau_i})^{n_i} e(\tau) \\
&= \sum_{\tau \in R} B(\tau) e(\tau).
\end{aligned}$$

From the right-hand side, we have

$$\begin{aligned}
\log\left(\prod_{\lambda \in R} (1 - e(\lambda))^{-\dim G_\lambda}\right) &= \sum_{\lambda \in R} (\dim G_\lambda) \log\left(\frac{1}{1-e(\lambda)}\right) \\
= \sum_{\lambda \in R} (\dim G_\lambda) \sum_{k=1}^{\infty} \frac{e(k\lambda)}{k} &= \sum_{\lambda \in R} \left(\sum_{k=1}^{\infty} \frac{1}{k} (\dim G_\lambda) e(k\lambda)\right).
\end{aligned}$$

Therefore

$$B(\tau) = \sum_{\lambda \,:\, \tau = k\lambda} \frac{1}{k} \dim G_\lambda = \sum_{\lambda | \tau} \left(\frac{\lambda}{\tau}\right) \dim G_\lambda .$$

Hence by Möbius inversion

$$\dim G_\lambda = \sum_{\tau | \lambda} \mu\left(\frac{\lambda}{\tau}\right) \frac{\tau}{\lambda} B(\tau). \quad \square$$

Remark 2.2. We call the function $B(\tau)$ the *Witt partition function* on V. The formula (2.3) will also be called the *Witt formula.*

Let $V = V(\Lambda)$ be an integrable irreducible highest weight module over a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, and let $G = \bigoplus_{n \geq 1} G_n$ be the free Lie algebra generated by V. Then G is also an integrable module over $\mathfrak{g}$, and G has the root space decomposition induced by the weight space decomposition of V. Let $R = \mathfrak{h}^*$ with the usual partial ordering [Kac2]. Let $S = \{\tau_i | i = 1, 2, 3, \ldots\}$ be an enumeration of all the weights of V. Then by the Witt formula,

$$\dim G_\lambda = \sum_{\tau | \lambda} \mu\left(\frac{\lambda}{\tau}\right) \frac{\tau}{\lambda} B(\tau),$$

for $\lambda \in \mathfrak{h}^*$.

Consider the principal gradation on $V = \bigoplus_{m\geq 0} V_m$. Then each homogeneous space G_n is also principally graded induced by the principal gradation of V: $G_n = \bigoplus_{m\geq 0} G_{(n,m)}$. We give a new gradation to V by setting

$$\widehat{V}_{m+1} = V_m \quad \text{for } m \geq 0. \tag{2.4}$$

Thus we have $V = \bigoplus_{j\geq 1} \widehat{V}_j$. Then this induces a new gradation on G_n:

$$G_n = \bigoplus_{j\geq n} \widehat{G}_{(n,j)},$$

where $\widehat{G}_{(n,j)}$ are given by

$$\widehat{G}_{(n,n+m)} = G_{(n,m)} \quad \text{for } m \geq 0. \tag{2.5}$$

Let $R = \{(r, s) \in \mathbf{Z} \times \mathbf{Z} | r \geq 1,\ s \geq 1\}$ with the lexicographic ordering. Let $S = \{(1, j) | j \in \mathbf{Z}_{\geq 1}\}$ and for a pair of positive integers r, s, let

$$T(r, s) = \left\{(n) = (n_1, n_2, n_3, \dots) | n_j \in \mathbf{Z}_{\geq 0},\ \sum n_j = r,\ \sum j n_j = s\right\}.$$

Note that the set $T(r, s)$ corresponds to the set of partitions of s into r parts. Then by the Witt formula, we obtain

$$\begin{aligned} \dim G_{(n,m)} &= \dim \widehat{G}_{(n,n+m)} \\ &= \sum_{(r,s)|(n,n+m)} \mu\left(\frac{(n, n+m)}{(r, s)}\right) \frac{(r, s)}{(n, n+m)} \widehat{B}(r, s), \end{aligned} \tag{2.6}$$

where $\widehat{B}(r, s)$ is defined by

$$\begin{aligned} \widehat{B}(r, s) &= \sum_{(n)\in T(r,s)} \frac{((\sum n_j) - 1)!}{\prod(n_j!)} \prod(\dim \widehat{V}_j)^{n_j}, \\ &= \sum_{(n)\in T(r,s)} \frac{((\sum n_j) - 1)!}{\prod(n_j!)} \prod(\dim V_{j-1})^{n_j}. \end{aligned} \tag{2.7}$$

Hence the principally specialized character of G_n is given by

$$\begin{aligned} \mathrm{ch}_q\, G_n &= \sum_{m\geq 0} (\dim G_{(n,m)}) q^m \\ &= \sum_{m\geq 0} \left(\sum_{(r,s)|(n,n+m)} \mu\left(\frac{(n, n+m)}{(r, s)}\right) \frac{(r, s)}{(n, n+m)} \widehat{B}(r, s) \right) q^m. \end{aligned} \tag{2.8}$$

3. Hochschild-Serre spectral sequences

Let G be a Lie algebra and V a module over G. We define the space $C_q(G, V)$ of *q-dimensional chains* of the Lie algebra G with coefficients in V to be $\Lambda^q(G) \otimes V$. The *differential* $d_q\colon C_q(G, V) \to C_{q-1}(G, V)$ is defined by the formula

$$\begin{aligned} &d_q(g_1 \wedge \cdots \wedge g_q \otimes v) \\ &\quad = \sum_{1\leq s<t\leq q} (-1)^{s+t-1}([g_s, g_t] \wedge g_1 \wedge \cdots \wedge \hat{g}_s \wedge \cdots \wedge \hat{g}_t \wedge \cdots \wedge g_q) \otimes v \\ &\quad\quad + \sum_{1\leq s\leq q} (-1)^s (g_1 \wedge \cdots \wedge \hat{g}_s \wedge \cdots \wedge g_q) \otimes g_s \cdot v, \end{aligned} \tag{3.1}$$

where $v \in V$, $g_1, \dots, g_q \in G$. For $q < 0$, we define $C_q(G, V) = 0$ and $d_q = 0$. Then we have $d_q \circ d_{q+1} = 0$. The homology of the complex $(C, d) = \{C_q(G, V), d_q\}$ is called the *homology of the Lie algebra G with coefficients in V*, and is denoted by $H_q(G, V)$. When $V = \mathbf{C}$, we write $H_q(G)$ for $H_q(G, \mathbf{C})$.

We shall be interested in the cases where G, V, and $C_q(G, V)$ are completely reducible modules in category $\mathcal{O}$ over a Kac-Moody Lie algebra $\mathfrak{g}(A)$, with d_q being $\mathfrak{g}(A)$-module homomorphisms, so that $H_q(G, V)$ are modules over $\mathfrak{g}(A)$.

Let I be an ideal of G and $L = G/I$. We define a filtration $\{K_p = K_pC\}$ of the complex (C, d) by

$$K_pC_{p+q} = \{g_1 \wedge g_2 \wedge \cdots \wedge g_{p+q} \otimes v | g_i \in I \text{ for } p+1 \le i \le p+q\}. \tag{3.2}$$

This gives rise to a spectral sequence $\{E^r_{p,q}, d_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ such that $E^2_{p,q} \cong H_p(L, H_q(I, V))$ [H-S, Mc, Mo-T]. The terms $E^r_{p,q}$ are determined by

$$E^{r+1}_{p,q} = \operatorname{Ker}(d_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}) / \operatorname{Im}(d_r\colon E^r_{p+r,q-r+1} \to E^r_{p,q}) \tag{3.3}$$

with boundary homomorphism $d_{r+1}\colon E^r_{p,q} \to E^r_{p-r-1,q+r}$. For each p, q, the modules $E^r_{p,q}$ become stable for $r > \max(p, q+1)$, and the stable module is denoted by $E^\infty_{p,q}$. The spectral sequence $\{E^r_{p,q}, d_r\}$ *converges* to $H(G, V)$ in the following sense:

$$H_n(G, V) = \bigoplus_{p+q=n} E^\infty_{p,q}. \tag{3.4}$$

When we have a convergent spectral sequence $\{E^r_{p,q}, d_r\}$ as above, we have the following exact sequence [C-E]:

$$H_2 \to E^2_{2,0} \to E^2_{0,1} \to H_1 \to E^2_{1,0} \to 0.$$

In the above case, the following sequence is exact:

$$\begin{aligned} H_2(G, V) &\to H_2(L, H_0(I, V)) \to H_0(L, H_1(I, V)) \\ &\to H_1(G, V) \to H_1(L, H_0(I, V)) \to 0. \end{aligned}$$

This sequence is called the *Hochschild-Serre five term exact sequence.* To summarize:

Theorem 3.1. *Let G be a Lie algebra and V be a module over G. Let I be an ideal of G and let $L = G/I$. Then there exists a spectral sequence $\{E^r_{p,q}, d_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ converging to $H_*(G, V)$ such that*

$$E^2_{p,q} \cong H_p(L, H_q(I, V)). \tag{3.5}$$

Moreover, the following sequence is exact:

$$\begin{aligned} H_2(G, V) &\to H_2(L, H_0(I, V)) \to H_0(L, H_1(I, V)) \\ &\to H_1(G, V) \to H_1(L, H_0(I, V)) \to 0. \end{aligned} \tag{3.6}$$

As an application of Theorem 3.1, we have the following lemma.

Lemma 3.2. *Let $\mathfrak{g}(A)$ be a Kac-Moody Lie algebra and let $G = \bigoplus_{n\geq 1} G_n$ be the free Lie algebra generated by a $\mathfrak{g}(A)$-module $G_1 = V$. Let $I = \bigoplus_{n\geq m} I_n$ be the graded ideal of G generated by the $\mathfrak{g}(A)$-submodule I_m of G_m for $m \geq 2$ and let $L = G/I$. Then we have an isomorphism of $\mathfrak{g}(A)$-modules*

$$H_2(L) \cong I_m. \tag{3.7}$$

Proof. Note that $L = \bigoplus_{n\geq 1} L_n$ is also a graded Lie algebra generated by the subspace $L_1 = G_1$. For the trivial module $\mathbf{C}$, the Hochschild-Serre five term exact sequence becomes

$$H_2(G) \to H_2(L) \to H_0(L, H_1(I)) \to H_1(G) \to H_1(L) \to 0.$$

Since

$$H_1(L) \cong \mathbf{C} \otimes L/[L, L] \cong L_1 \quad \text{and} \quad H_1(G) \cong \mathbf{C} \otimes G/[G, G] \cong G_1,$$

the homomorphism $H_1(G) \to H_1(L)$ is an isomorphism. Since G is free, by definition, $H_2(G) = 0$. Therefore we have

$$\begin{aligned} H_2(L) &\cong H_0(L, H_1(I)) \cong H_0(L, I/[I, I]) \cong (I/[I, I])/L \cdot (I/[I, I]) \\ &\cong (I/[I, I])_m \cong I_m. \quad \square \end{aligned}$$

4. The Kostant formula for Kac-Moody Lie algebras

Let A be an $n \times n$ symmetrizable generalized Cartan matrix and let $\mathfrak{g}(A)$ be the Kac-Moody Lie algebra with Cartan matrix A. Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}(A)$ and denote by Δ^+ (respectively, Δ^-) the set of positive (respectively, negative) roots of $\mathfrak{g}(A)$. Then the triangular decomposition becomes $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^\pm = \bigoplus_{\alpha\in\Delta^\pm} \mathfrak{g}_\alpha$. Let r_i be the reflection on $\mathfrak{h}^*$ determined by α_i for $i = 1, \dots, n$, and let W be the Weyl group of $\mathfrak{g}(A)$ generated by the $r_1, \dots, r_n$. The expression $w = r_{i_1} \cdots r_{i_s} \in W$ is called *reduced* if s is minimal possible among all representations of $w \in W$ as a product of the r_i. In this case, s is called the *length* of w and is denoted by $l(w)$.

Let $S = \{1, \dots, s\}$ be a subset of $N = \{1, \dots, n\}$. Consider the subalgebra $\mathfrak{g}_S$ of $\mathfrak{g}(A)$ generated by the elements e_i, f_i $(i = 1, \dots, s)$ and $\mathfrak{h}$. Denote by Δ_S^+ the set of positive roots generated by $\alpha_1, \dots, \alpha_s$, and let $\Delta_S^- = -\Delta_S^+$. Then $\mathfrak{g}_S$ has a triangular decomposition $\mathfrak{g}_S = \mathfrak{n}_S^- \oplus \mathfrak{h} \oplus \mathfrak{n}_S^+$, where $\mathfrak{n}_S^\pm = \bigoplus_{\alpha\in\Delta_S^\pm} \mathfrak{g}_\alpha$, and $\Delta_S = \Delta_S^+ \cup \Delta_S^-$ is the root system of $\mathfrak{g}_S$. Let $\Delta^\pm(S) = \Delta^\pm \backslash \Delta_S^\pm$ and let $\mathfrak{n}^\pm(S) = \bigoplus_{\alpha\in\Delta^\pm(S)} \mathfrak{g}_\alpha$. Then

$$\mathfrak{g}(A) = \mathfrak{n}^-(S) \oplus \mathfrak{g}_S \oplus \mathfrak{n}^+(S).$$

Note that $\mathfrak{n}^\pm(S)$ are $\mathfrak{g}_S$-modules via adjoint action. Let W_S be the Weyl group of $\mathfrak{g}_S$ generated by $r_1, \dots, r_s$, and let

$$W(S) = \{w \in W | w\Delta^- \cap \Delta^+ \subset \Delta^+(S)\} = \{w \in W | \Phi_w \subset \Delta^+(S)\}, \tag{4.1}$$

where $\Phi_w = \{\alpha \in \Delta^+ | w^{-1}(\alpha) < 0\}$.

For $\lambda \in \mathfrak{h}^*$, we denote by $\widetilde{V}(\lambda)$ the irreducible highest weight module over $\mathfrak{g}(A)$ and $V(\lambda)$ the irreducible highest weight module over $\mathfrak{g}_S$. Then the structure of the homology modules $H_*(\mathfrak{n}^-(S), \widetilde{V}(\lambda))$ is determined by the following theorem.

Theorem 4.1 [Liu].

$$H_j(\mathfrak{n}^-(S), \widetilde{V}(\lambda)) \cong \bigoplus_{\substack{w \in W(S) \\ l(w)=j}} V(w(\lambda+\rho)-\rho). \tag{4.2}$$

Remark 4.2. This formula was first introduced by Kostant for finite dimensional complex semisimple Lie algebras [Ko]. In [G-L], Garland and Lepowsky proved this formula for symmetrizable Kac-Moody Lie algebras under the condition that S is of finite type. In [Liu], Liu proved this formula for symmetrizable Kac-Moody Lie algebras without assuming that S is of finite type. We will call the formula (4.2) the *Kostant formula.*

The following lemma is very useful in the actual computation.

Lemma 4.3. *Suppose* $w = w'r_j$ *and* $l(w) = l(w') + 1$. *Then* $w \in W(S)$ *if and only if* $w' \in W(S)$ *and* $w'(\alpha_j) \in \Delta^+(S)$.

Proof. By definition, $w \in W(S)$ if and only if $\Phi_w = \{\alpha \in \Delta^+ | w^{-1}(\alpha) < 0\} \subset \Delta^+(S)$. But $\Phi_w = \Phi_{w'} \cup \{w'(\alpha_j)\}$. Thus $\Phi_w \subset \Delta^+(S)$ if and only if $\Phi_{w'} \subset \Delta^+(S)$ and $w'(\alpha_j) \in \Delta^+(S)$, which is equivalent to saying that $w' \in W(S)$ and $w'(\alpha_j) \in \Delta^+(S)$. □

Now let $A = (a_{ij})_{i,j=0,1,\dots,l}$ be a generalized Cartan matrix of affine type and let $\widetilde{A} = (a_{ij})_{i,j=-1,0,1,\dots,l}$ be a generalized Cartan matrix of Lorentzian type as defined in §1. We have seen that the Kac-Moody Lie algebra $\mathfrak{g}(\widetilde{A})$ can be realized as the minimal graded Lie algebra $L = \bigoplus_{n \in Z} L_n$ with the local part $V + \mathfrak{g}(A) + V^*$, where $V = V(-\alpha_{-1})$ is the basic representation of the affine Kac-Moody Lie algebra $\mathfrak{g}(A)$. Let $L_\pm = \bigoplus_{n \geq 1} L_{\pm n}$. Then $L_\pm$ coincides with $\mathfrak{n}^\pm(S)$ for the set $S = \{0, 1, \dots, l\}$. Thus the Kostant formula enables us to compute the homology modules of the Lie algebra $L_\pm$ with coefficients in the trivial module $\mathbf{C}$:

$$H_j(L_-) \cong \bigoplus_{\substack{w \in W(S) \\ l(w)=j}} V(w\rho - \rho). \tag{4.3}$$

Example 4.4. Let

$$\widetilde{A} = (a_{ij})_{i,j=-1,0,1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

be a generalized Cartan matrix of hyperbolic type. We realize the corresponding hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$ as the minimal graded Lie algebra $L = \bigoplus_{n \in Z} L_n$ with local part $V + \mathfrak{g}(A) + V^*$, where $V = V(-\alpha_{-1})$ is the basic representation of the affine Kac-Moody Lie algebra $A_1^{(1)}$. Let $L_\pm = \bigoplus_{n \geq 1} L_{\pm n}$ and let $S = \{0, 1\}$. We will compute some of the homology modules of the Lie algebra $L_\pm$ with coefficients in the trivial module $\mathbf{C}$ using the Kostant formula.

For $j = 1$, the only element in $W(S)$ of length 1 is r_{-1}, and $r_{-1}\rho - \rho = -\alpha_{-1}$. Thus $H_1(L_-) \cong V(-\alpha_{-1})$. For $j = 2$, by Lemma 4.3, we have only to consider the element $r_{-1}r_0$. Since $r_{-1}(\alpha_0) = \alpha_{-1} + \alpha_0 \in \Delta^+(S)$, $r_{-1}r_0 \in W(S)$ by Lemma 4.3. We have $r_{-1}r_0\rho - \rho = -2\alpha_{-1} - \alpha_0$, and hence $H_2(L_-) \cong$

$V(-2\alpha_{-1}-\alpha_0)$. For $j=3$, by Lemma 4.3, we need to check the elements $r_{-1}r_0r_{-1}$ and $r_{-1}r_0r_1$. Since

$$r_{-1}r_0(\alpha_1)=2\alpha_{-1}+2\alpha_0+\alpha_1\in\Delta^+(S),$$

we have $r_{-1}r_0r_1\in W(S)$. But $r_{-1}r_0(\alpha_{-1})=\alpha_0\notin\Delta^+(S)$, hence $r_{-1}r_0r_{-1}\notin W(S)$. By an easy calculation, we get

$$r_{-1}r_0r_1\rho-\rho=-4\alpha_{-1}-3\alpha_0-\alpha_1.$$

Therefore

$$H_3(L_-)\cong V(-4\alpha_{-1}-3\alpha_0-\alpha_1).$$

Continuing this process,we obtain

$$H_4(L_-)\cong V(-7\alpha_{-1}-6\alpha_0-3\alpha_1),$$
$$H_5(L_-)\cong V(-10\alpha_{-1}-10\alpha_0-5\alpha_1)\oplus V(-11\alpha_{-1}-10\alpha_0-6\alpha_1),\ \text{etc.}$$

5. Homological study of the graded Lie algebras

Let $G=\bigoplus_{n\geq 1}G_n$ be the free Lie algebra generated by the subspace G_1, and let $I=\bigoplus_{n\geq m}I_n$ be the graded ideal of G generated by the subspace I_m for $m\geq 2$. Consider the quotient Lie algebra $L=G/I$. Then $L=\bigoplus_{n\geq 1}L_n$ is also a graded Lie algebra generated by the subspace $L_1=G_1$. Let $J=I/[I,I]$. Since I is an ideal of G, L acts on $J=I/[I,I]$ via adjoint action, and thus J becomes an L-module generated by the subspace J_m. Note that, as vector spaces, $J_n\cong I_n$ for $m\leq n<2m$.

Suppose that G_1 and I_m are modules over a Kac-Moody Lie algebra $\mathfrak{g}(A)$. Then each homogeneous subspace G_n has a $\mathfrak{g}(A)$-module structure such that $x\cdot[v,w]=[x\cdot v,w]+[v,x\cdot w]$ for $x\in\mathfrak{g}(A)$, $v\in G_1$, $w\in G_{n-1}$. Then $\mathfrak{g}(A)$-module structure on the homogeneous spaces I_n is given similarly. We also have the induced $\mathfrak{g}(A)$-module structure on the homogeneous subspaces L_n and J_n.

Now consider the exact sequence of L-modules

$$0\to K\to U(L)\otimes J_m\xrightarrow{\psi}J\to 0, \tag{5.1}$$

where ψ is the usual bracket mapping, and K is the kernel of ψ. This produces a long exact sequence

$$\begin{aligned}\cdots&\to H_1(L,K)\to H_1(L,U(L)\otimes J_m)\to H_1(L,J)\to H_0(L,K)\\&\to H_0(L,U(L)\otimes J_m)\to H_0(L,J)\to 0.\end{aligned}$$

Since $U(L)\otimes J_m$ is free over L, i.e., over $U(L)$, we have

$$H_j(L,U(L)\otimes J_m)=0\quad\text{for } j\geq 1$$

[J1, J2]. By definition, $H_0(L,J)\cong J/[L,J]\cong J_m$ and

$$H_0(L,U(L)\otimes J_m)\cong U(L)\otimes J_m/L(U(L)\otimes J_m)\cong(U(L)\otimes J_m)_m\cong J_m.$$

Thus from the long exact sequence we get a $\mathfrak{g}(A)$-module isomorphism

$$H_1(L,J)\cong H_0(L,K)\cong K/L\cdot K. \tag{5.2}$$

On the other hand, using the Poincaré-Birkhoff-Witt theorem, we can prove the following lemma (e.g., [Kac2, Exercise 9.13; Kang, Lemma 4.5]).

Lemma 5.1. *There is an exact sequence of L-modules*

$$0 \to J \xrightarrow{\alpha} U(L) \otimes G_1 \xrightarrow{\beta} U(L) \xrightarrow{\gamma} \mathbf{C} \to 0. \tag{5.3}$$

Proof. Consider the homomorphisms

$$\alpha\colon J = I/[I, I] \to U_0(G)/IU_0(G) \quad \text{and} \quad \beta\colon U_0(G)/IU_0(G) \to U(L)$$

induced by the natural injection $I \to U_0(G)$ and by the canonical surjection $G \to L$, respectively. Define a homomorphism $\gamma\colon U(L) \to \mathbf{C}$ by $\gamma|_{\mathbf{C}} = \mathrm{id}$ and $\gamma(U_0(L)) = 0$.

By the Poincaré-Birkhoff-Witt theorem, we have $I \cap IU_0(G) = [I, I]$, and hence $\operatorname{Ker}\alpha = 0$. It is clear that $\operatorname{Im}\alpha = I + IU_0(G)/IU_0(G)$. Since $IU(G) \subset U_0(G)$,

$$\operatorname{Ker}\beta = U_0(G) \cap IU(G)/IU_0(G) = IU(G)/IU_0(G).$$

Since $U(G) = \mathbf{C} + U_0(G)$, $IU(G) = I + IU_0(G)$. Thus $\operatorname{Ker}\beta = I + IU_0(G) = \operatorname{Im}\alpha$. Finally, it is obvious that $\operatorname{Im}\beta = U_0(L) = \operatorname{Ker}\gamma$. Therefore we have an exact sequence

$$0 \to J \xrightarrow{\alpha} U_0(G)/IU_0(G) \xrightarrow{\beta} U(L) \xrightarrow{\gamma} \mathbf{C} \to 0.$$

Let $\mathfrak{B}_0 = \{u_i | i \in \Omega\}$ be a basis of I and let

$$\mathfrak{B} = \{u_i | i \in \Omega\} \cup \{u'_j | j \in \Omega'\}$$

be a basis of G extending $\mathfrak{B}_0$. By the Poincaré-Birkhoff-Witt theorem, we have a basis of $U(G)$ consisting of the elements of the form

$$u_{i_1} u_{i_2} \cdots u_{i_s} u'_{j_1} u'_{j_2} \cdots u'_{j_t} \tag{5.4}$$

with $i_1 \leq \cdots \leq i_s$ and $j_1 \leq \cdots \leq j_t$. Note that $U(L) \cong U(G)/IU(G)$. We define a linear map $\eta\colon U(L) \otimes_{U(G)} U_0(G) \to U_0(G)/IU_0(G)$ by

$$\eta((u + IU(G)) \otimes v) = uv + IU_0(G)$$

for $u \in U(G)$, $v \in U_0(G)$. Using the PBW basis (5.4), it is easy to show that η is an isomorphism. Hence we have the following isomorphisms

$$U_0(G)/IU_0(G) \cong U(L) \otimes_{U(G)} U_0(G) \cong U(L) \otimes_{U(G)} (U(G) \otimes G_1) \cong U(L) \otimes G_1.$$

Therefore we get the desired exact sequence. $\square$

Theorem 5.2. *There is an isomorphism of $\mathfrak{g}(A)$-modules*

$$H_j(L, J) \cong H_{j+2}(L) \quad \textit{for } j \geq 1. \tag{5.5}$$

Proof. We split the exact sequence (5.3) into the following exact sequences:

$$0 \to J \xrightarrow{\alpha} U(L) \otimes G_1 \xrightarrow{\beta} \operatorname{Im}\beta \to 0,$$

and

$$0 \to \operatorname{Im}\beta = \operatorname{Ker}\gamma \to U(L) \to \operatorname{Coker}\beta \to 0.$$

Then we get long exact sequences:

$$\begin{aligned} \cdots &\to H_{j+1}(L, J) \to H_{j+1}(L, U(L) \otimes G_1) \to H_{j+1}(L, \operatorname{Im}\beta) \\ &\to H_j(L, J) \to H_j(L, U(L) \otimes G_1) \to H_j(L, \operatorname{Im}\beta) \to \cdots, \end{aligned}$$

and

$$\cdots \to H_{j+2}(L, \operatorname{Im}\beta) \to H_{j+2}(L, U(L)) \to H_{j+2}(L, \operatorname{Coker}\beta)$$
$$\to H_{j+1}(L, \operatorname{Im}\beta) \to H_{j+1}(L, U(L)) \to H_{j+1}(L, \operatorname{Coker}\beta) \to \cdots.$$

Since $U(L)$ and $U(L)\otimes G_1$ are free over $U(L)$, we have

$$H_k(L, U(L)) = H_k(L, U(L)\otimes G_1) = 0 \quad \text{for } k \geq 1.$$

Thus we get

$$H_j(L, J) \cong H_{j+1}(L, \operatorname{Im}\beta) \cong H_{j+2}(L, \operatorname{Coker}\beta) \quad \text{for } j \geq 1.$$

But since γ is surjective, $\operatorname{Coker}\beta = U(L)/\operatorname{Ker}\gamma \cong \mathbf{C}$. Therefore we get the desired isomorphism. □

In particular, $H_1(L, J) \cong H_3(L)$. Therefore combining with (5.2) yields

$$K/L\cdot K \cong H_3(L). \tag{5.6}$$

Note that $H_3(L)$ has a graded module structure over $\mathfrak{g}(A)$ induced by that of L.

Theorem 5.3. *Let t be the smallest homogeneous degree of $H_3(L)$. Then $K_j = 0$ for $j < t$ and $K_t \cong H_3(L)_t$. In particular, we have*

$$K_{m+1} \cong H_3(L)_{m+1}. \tag{5.7}$$

Proof. Note that $K_m = 0$. Thus $H_3(L)_m = (K/L\cdot K)_m = 0$ and hence $t \geq m+1$. Now for $m \leq j < t$, assume inductively that $K_m = \cdots = K_{j-1} = 0$. Then

$$0 = H_3(L)_j \cong (K/L\cdot K)_j \cong K_j \Big/ \left(\sum_{i=m}^{j-1} L_{j-i}\cdot K_i\right) \cong K_j,$$

proving the first assertion. Similarly, since $K_m = \cdots = K_{t-1} = 0$, we have

$$H_3(L)_t \cong (K/L\cdot K)_t \cong K_t \Big/ \left(\sum_{i=m}^{t-1} L_{t-i}\cdot K_i\right) \cong K_t. \quad \square$$

Theorem 5.4. (a) *For $m \leq j < \min(2m, t)$, we have*

$$I_j \cong \underbrace{G\otimes\cdots\otimes G_1}_{(j-m)\ \text{times}} \otimes I_m. \tag{5.8}$$

(b) *If $t < 2m$, we have*

$$I_t \cong \underbrace{G_1\otimes\cdots\otimes G_1}_{(t-m)\ \text{times}} \otimes I_m/H_3(L)_t. \tag{5.9}$$

In particular,

$$I_{m+1} \cong G_1 \otimes I_m/H_3(L)_{m+1}. \tag{5.10}$$

Proof. (a) Since $K_j = 0$ for $j < t$, we have

$$I_j \cong J_j \cong (U(L)\otimes I_m)_j/K_j = (U(L)\otimes I_m)_j \cong U(L)_{j-m}\otimes I_m.$$

Since $m \leq j < \min(2m, t)$, we have $j - m < m$. Thus $L_k \cong G_k$ for $1 \leq k \leq j-m$, which implies $U(L)_{j-m} \cong U(G)_{j-m}$. Since G is the free Lie algebra generated by the subspace G_1, the universal enveloping algebra of G is the tensor algebra on G_1. Hence we get

$$U(G)_{j-m} \cong \underbrace{G_1 \otimes \cdots \otimes G_1}_{(j-m)\text{ times}} .$$

Therefore

$$I_j \cong \underbrace{G_1 \otimes \cdots \otimes G_1}_{(j-m)\text{ times}} \otimes I_m .$$

(b) Since $t < 2m$, we have

$$I_t \cong J_t \cong (U(L) \otimes I_m)_t / K_t \cong (U(L)_{t-m} \otimes I_m)/K_t .$$

By Theorem 5.3, $K_t \cong H_3(L)_t$, and the argument of (a) gives

$$U(L)_{t-m} \cong U(G)_{t-m} \cong \underbrace{G_1 \otimes \cdots \otimes G_1}_{(t-m)\text{ times}} .$$

Therefore

$$I_t \cong \underbrace{G_1 \otimes \cdots \otimes G_1}_{(t-m)\text{ times}} \otimes I_m / H_3(L)_t .$$

In particular,

$$I_{m+1} \cong (G_1 \otimes I_m)/H_3(L)_{m+1} . \quad \square \tag{5.11}$$

Now we study the homogeneous spaces I_j of I for arbitrary $j \geq m$. We repeat the above process under the following setting. For $j \geq m$, let $I^{(j)} = \sum_{n \geq j} I_n$. Since I_m generates the ideal I, we have

$$I_{m+1} = (\operatorname{ad} G_1)(I_m), \ldots, I_n = (\operatorname{ad} G_1)^{n-m}(I_m) = (\operatorname{ad} G_1)(I_{n-1}) .$$

Thus $I^{(j)}$ is an ideal of G generated by the subspace I_j. Consider the quotient Lie algebra $L^{(j)} = G/I^{(j)}$. Thus $L = L^{(m)}$ in this notation. Set $J^{(j)} = I^{(j)}/[I^{(j)}, I^{(j)}]$. Then $J^{(j)}$ is an $L^{(j)}$-module generated by the subspace $J_j^{(j)}$. Note that $J_n^{(j)} \cong I_n^{(j)} = I_n$ for $j \leq n < 2j$. Considering the exact sequences

$$0 \to K^{(j)} \to U(L^{(j)}) \otimes J_j^{(j)} \xrightarrow{\psi} J^{(j)} \to 0 ,$$

and

$$0 \to J^{(j)} \xrightarrow{\alpha} U(L^{(j)}) \otimes G_1 \xrightarrow{\beta} U(L^{(j)}) \xrightarrow{\gamma} \mathbf{C} \to 0 ,$$

the same homological argument shows that the structure of $H_3(L^{(j)})$ determines the $\mathfrak{g}(A)$-module structure of the homogeneous spaces I_n for $j+1 \leq n < \min(2j, t^{(j)})$, where $t^{(j)}$ is the smallest homogeneous degree of $H_3(L^{(j)})$. In particular,

$$I_{j+1} \cong (G_1 \otimes I_j)/H_3(L^{(j)})_{j+1} . \tag{5.12}$$

Therefore we now have an inductive algorithm to determine the $\mathfrak{g}(A)$-module structure of the homogeneous subspaces of the graded ideal I by studying the homology modules $H_3(L^{(j)})$ for $j \geq m$.

Suppose we have determined the structure of homology modules $H_*(L^{(j-1)})$. We define the subspace $N^{(j-1)}$ of $L^{(j)}$ by $N^{(j-1)} = I^{(j-1)}/I^{(j)}$. Then $N^{(j-1)}$ is an abelian ideal of $L^{(j)}$ and $L^{(j)}/N^{(j-1)}$ is isomorphic to $L^{(j-1)}$. Then from

Theorem 3.1, there exists a spectral sequence $\{E^r_{p,q}, d_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ converging to $H_*(L^{(j)})$ such that

$$E^2_{p,q} \cong H_p(L^{(j-1)}, H_q(N^{(j-1)})).$$

In particular, from (3.4), we have

$$H_3(L^{(j)}) \cong E^\infty_{3,0} \oplus E^\infty_{2,1} \oplus E^\infty_{1,2} \oplus E^\infty_{0,3}. \tag{5.13}$$

Thus to study the structure of $H_3(L^{(j)})$, we need to determine the boundary maps $d_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$. The maps d_r are $\mathfrak{g}(A)$-module homomorphisms since they are induced by the $\mathfrak{g}(A)$-module homomorphisms d_q defined by (3.1).

Since $L^{(j-1)}$ acts on $N^{(j-1)}$ trivially, we have

$$H_p(L^{(j-1)}, H_q(N^{(j-1)})) \cong H_p(L^{(j-1)}) \otimes H_q(N^{(j-1)}).$$

Note that $N^{(j-1)} \cong I_{j-1}$ as $\mathfrak{g}(A)$-modules and that $N^{(j-1)}$ is an abelian Lie algebra. Thus, as $\mathfrak{g}(A)$-modules, $H_q(N^{(j-1)}) \cong \Lambda^q(I_{j-1})$. Therefore we have

$$E^2_{p,q} \cong H_p(L^{(j-1)}) \otimes \Lambda^q(I_{j-1}). \tag{5.14}$$

In the next section, using the homological approach developed here, together with the representation theory of affine Kac-Moody Lie algebras, we will determine some of the boundary homomorphisms and deduce some new structural information on the Kac-Moody Lie algebras of Lorentzian type.

6. The structure of the maximal graded ideal

In this section, we study the structure of the maximal graded ideal I. We know that the ideal I_- of G_- is generated by the homogeneous subspace I_{-2} and hence we may write $I_- = I_-^{(2)}$ following the notation introduced in the previous section. Similarly, for $j \geq 2$, we write $I_-^{(j)} = \sum_{n \geq j} I_{-n}$, $L_-^{(j)} = G/I_-^{(j)}$, and $N_-^{(j)} = I_-^{(j)}/I_-^{(j+1)}$.

Lemma 6.1.

$$I_{-2} \cong V(-2\alpha_{-1} - \alpha_0). \tag{6.1}$$

Proof. Since G_- is free and I_- is generated by the subspace I_{-2}, from the Hochschild-Serre five term exact sequence, we see that $I_{-2} \cong H_2(L_-)$ (Lemma 3.2). By the Kostant formula (see the computations at the end of §4), we have

$$H_2(L_-) \cong \sum_{\substack{w \in W(S) \\ l(w)=2}} V(w\rho - \rho) \cong V(r_{-1}r_0\rho - \rho) = V(-2\alpha_{-1} - \alpha_0). \quad \square$$

By the homological theory developed in the previous section, we have in general that

$$I_{-(j+1)} \cong V \otimes I_{-j}/H_3(L_-^{(j)})_{-(j+1)} \quad \text{for } j \geq 2. \tag{6.2}$$

When $j = 2$, $L_-^{(2)}$ coincides with the subalgebra $\mathfrak{n}^-(S)$ for $S = \{0, 1, \ldots, l\}$, and therefore we can compute $H_3(L_-^{(2)})$ using the Kostant formula. For instance, in the case of $HA_1^{(1)}$, we have $H_3(L_-^{(2)}) \cong V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1)$. Thus $H_3(L_-^{(2)})_{-3} = 0$, and hence we obtain

$$I_{-3} \cong V \otimes I_{-2}/H_3(L_-^{(2)})_{-3} \cong V \otimes I_{-2}. \tag{6.3}$$

So we can easily determine the structure of the subspace I_{-3} using the Kostant formula. However, for $j \geq 3$, we cannot use the Kostant formula to compute the homology module $H_3(L_-^{(j)})$. We study the higher levels by developing some techniques from the Hochschild-Serre spectral sequences.

Theorem 6.2. *Let* $\mathfrak{g}(A)$ *be an affine Kac-Moody Lie algebra,* V *be its basic representation, and let* G, I, *and* L *be as defined in* §1. *Then we have the following information on the* $\mathfrak{g}(A)$*-module structure of the maximal graded ideal* I.

$HA_1^{(1)}$:
$$I_{-3} \cong V \otimes I_{-2},$$
$$I_{-4} \cong V \otimes I_{-3}/(V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2})),$$
$$I_{-5} \cong V \otimes I_{-4}/V \otimes \Lambda^2(I_{-2});$$

$HA_2^{(2)}$:
$$I_{-3} \cong V \otimes I_{-2},$$
$$I_{-4} \cong V \otimes I_{-3}/S^2(I_{-2}),$$
$$I_{-5} \cong V \otimes I_{-4}/V \otimes \Lambda^2(I_{-2});$$

$HA_l^{(1)}$, $l \geq 2$:
$$I_{-3} \cong V \otimes I_{-2}/(V(-3\alpha_{-1} - 2\alpha_0 - \alpha_1) \oplus V(-3\alpha_{-1} - 2\alpha_0 - \alpha_l)),$$
$$I_{-4} \cong V \otimes I_{-3}/S^2(I_{-2});$$

$HC_2^{(1)}$, $HD_4^{(3)}$:
$$I_{-3} \cong V \otimes I_{-2}/V(-3\alpha_{-1} - 2\alpha_0 - \alpha_1),$$
$$I_{-4} \cong V \otimes I_{-3}/S^2(I_{-2});$$

$HA_{2l}^{(2)}$, $HD_{l+1}^{(2)}$, $l \geq 2$:
$$I_{-3} \cong V \otimes I_{-2},$$
$$I_{-4} \cong V \otimes I_{-3}/(V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2}));$$

$HB_l^{(1)}$, $l \geq 3$, $HD_l^{(1)}$, $l \geq 4$: $\quad I_{-3} \cong V \otimes I_{-2}/V(-3\alpha_{-1} - 2\alpha_0 - \alpha_2);$

$HC_l^{(1)}$, $HA_{2l-1}^{(2)}$, $l \geq 3$, $HF_4^{(1)}$, $HG_2^{(1)}$, $HE_7^{(1)}$, $HE_8^{(1)}$, $HE_6^{(2)}$:
$$I_{-3} \cong V \otimes I_{-2}/V(-3\alpha_{-1} - 2\alpha_0 - \alpha_1);$$

$HE_6^{(1)}$: $\quad I_{-3} \cong V \otimes I_{-2}/V(-3\alpha_{-1} - 2\alpha_0 - \alpha_6).$

Proof. The Kostant formula determines the $\mathfrak{g}(A)$-module structure of the subspace I_{-3} for all cases. For higher levels, we consider the following four cases separately.

Case 1. $HA_1^{(1)}$.

By (6.2), to determine the structure of I_{-4}, we need to determine the structure of $H_3(L_-^{(3)})_{-4}$. We consider the following short exact sequence

$$0 \to N_-^{(2)} \to L_-^{(3)} \to L_-^{(2)} \to 0$$

and the corresponding spectral sequence $\{E_{p,q}^r\}$ converging to $H_*(L_-^{(3)})$ such that

$$E_{p,q}^2 \cong H_p(L_-^{(2)}) \otimes \Lambda^q(I_{-2}).$$

We will compute $H_3(L_-^{(3)})_{-4}$ from this sequence.

Let us start with the sequence

$$0 \to E_{2,0}^2 \xrightarrow{d_2} E_{0,1}^2 \to 0.$$

Note that

$$H_1(L_-^{(3)}) \cong L_-^{(3)}/[L_-^{(3)}, L_-^{(3)}] \cong L_{-1} = V.$$

Since the spectral sequence converges to $H_*(L_-^{(3)})$, we have

$$H_1(L_-^{(3)}) \cong E_{1,0}^\infty \oplus E_{0,1}^\infty.$$

But

$$E_{1,0}^\infty = E_{1,0}^2 \cong H_1(L_-^{(2)}) \cong L_1^{(2)}/[L_-^{(2)}, L_-^{(2)}] \cong L_{-1} = V,$$

which implies $E_{0,1}^\infty = E_{0,1}^3 = 0$. Hence the homomorphism d_2 is surjective. Since $E_{2,0}^2 \cong I_{-2}$ and $E_{0,1}^2 \cong I_{-2}$, d_2 must be an isomorphism. Thus $E_{2,0}^3 = 0$, and hence $E_{2,0}^\infty = 0$.

Now consider the following sequence

$$0 \to E_{3,0}^2 \xrightarrow{d_2} E_{1,1}^2 \to 0.$$

By the Kostant formula, we have

$$E_{3,0}^2 \cong H_3(L_-^{(2)}) \cong V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1)$$

and

$$E_{1,1}^2 \cong H_1(L_-^{(2)}) \otimes I_{-2} \cong V \otimes I_{-2}.$$

Since $V \otimes I_{-2}$ is a direct sum of irreducible highest weight modules over $A_1^{(1)}$ of level 3, by comparing the levels of both terms, we see that $d_2 \colon E_{3,0}^2 \to E_{1,1}^2$ is trivial. So $E_{3,0}^3 = E_{3,0}^2$, and $E_{1,1}^\infty = E_{1,1}^3 = E_{1,1}^2 \cong V \otimes I_{-2}$. Since $I_-^{(3)}$ is generated by I_{-3}, by Lemma 3.2, and (6.3) give

$$H_2(L_-^{(3)}) \cong I_{-3} = V \otimes I_{-2}.$$

But we have

$$H_2(L_-^{(3)}) \cong E_{2,0}^\infty \oplus E_{1,1}^\infty \oplus E_{0,2}^\infty.$$

It follows that $E_{0,2}^\infty = E_{0,2}^4 = 0$. Hence we conclude either $E_{0,2}^3 = 0$ or the homomorphism $d_3 \colon E_{3,0}^3 \to E_{0,2}^3$ is surjective.

Assume first that $E_{0,2}^3 = 0$. This implies that $d_3 \colon E_{3,0}^3 \to E_{0,2}^3$ is trivial and that the homomorphism $d_2 \colon E_{2,1}^2 \to E_{0,2}^2$ is surjective in the sequence

$$0 \to E_{4,0}^2 \to E_{2,1}^2 \to E_{0,2}^2 \to 0.$$

Thus

$$\begin{aligned} E_{3,0}^\infty = E_{3,0}^4 &= \operatorname{Ker}(d_3 \colon E_{3,0}^3 \to E_{0,2}^3)/\operatorname{Im}(d_3 \colon 0 \to E_{3,0}^3) \\ &= E_{3,0}^3 = E_{3,0}^2 \cong V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1). \end{aligned}$$

By comparing levels, we see that $d_2 \colon E_{4,0}^2 \to E_{2,1}^2$ must be trivial. Note that $E_{0,2}^2 \cong \Lambda^2(I_{-2})$. Therefore $E_{4,0}^3 = E_{4,0}^2$ and

$$\begin{aligned} E_{2,1}^\infty = E_{2,1}^3 &= \operatorname{Ker}(d_2 \colon E_{2,1}^2 \to E_{0,2}^2)/\operatorname{Im}(d_2 \colon E_{4,0}^2 \to E_{2,1}^2) \\ &\cong \operatorname{Ker}(d_2 \colon E_{2,1}^2 \to E_{0,2}^2). \end{aligned}$$

Since $d_2\colon E^2_{2,1} \to E^2_{0,2}$ is surjective, we have

$$\Lambda^2(I_{-2}) \cong E^2_{0,2} \cong E^2_{2,1}/\operatorname{Ker} d_2 \cong I_{-2} \otimes I_{-2}/\operatorname{Ker} d_2 .$$

Therefore $\operatorname{Ker} d_2 \cong S^2(I_{-2})$. Hence $E^\infty_{2,1} \cong S^2(I_{-2})$.

If $E^3_{0,2}$ is nonzero and $d_3\colon E^3_{3,0} \to E^3_{0,2}$ is surjective, then since $E^3_{3,0} = E^2_{3,0}$ is irreducible, $d_3\colon E^3_{3,0} \to E^3_{0,2}$ is an isomorphism. Thus $E^\infty_{3,0} = E^4_{3,0} = 0$ and

$$\begin{aligned} V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) &\cong E^3_{3,0} \cong E^3_{0,2} \cong E^2_{0,2}/\operatorname{Im}(d_2\colon E^2_{2,1} \to E^2_{0,2}) \\ &\cong \Lambda^2(I_{-2})/\operatorname{Im}(d_2\colon E^2_{2,1} \to E^2_{0,2}). \end{aligned}$$

Since all the modules involved here are completely reducible over $A_1^{(1)}$, we have

$$\operatorname{Im}(d_2\colon E^2_{2,1} \to E^2_{0,2}) \cong \Lambda^2(I_{-2})/V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1).$$

We have seen that the homomorphism $d_2\colon E^2_{4,0} \to E^2_{2,1}$ is trivial. Thus

$$\begin{aligned} E^\infty_{2,1} = E^3_{2,1} &= \operatorname{Ker}(d_2\colon E^2_{2,1} \to E^2_{0,2})/\operatorname{Im}(d_2\colon E^2_{4,0} \to E^2_{2,1}) \\ &= \operatorname{Ker}(d_2\colon E^2_{2,1} \to E^2_{0,2}). \end{aligned}$$

Since

$$\begin{aligned} \operatorname{Im} d_2 &\cong \Lambda^2(I_{-2})/V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \\ &\cong E^2_{2,1}/\operatorname{Ker} d_2 \cong I_{-2} \otimes I_{-2}/\operatorname{Ker} d_2 , \end{aligned}$$

we have

$$\operatorname{Ker} d_2 \cong S^2(I_{-2}) \oplus V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1).$$

Therefore in either case, we have

$$E^\infty_{3,0} \oplus E^\infty_{2,1} \cong S^2(I_{-2}) \oplus V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1).$$

Now consider the sequence

$$0 \to E^2_{5,0} \to E^2_{3,1} \to E^2_{1,2} \to 0 .$$

By comparing levels, we see that the homomorphism $d_2\colon E^2_{3,1} \to E^2_{1,2}$ is trivial. Thus

$$E^3_{1,2} = E^2_{1,2} \cong V \otimes \Lambda^2(I_{-2}).$$

Again by comparing the levels of the terms in the sequence

$$0 \to E^3_{4,0} \xrightarrow{d_3} E^3_{1,2} \to 0 ,$$

we conclude that $d_3 = 0$. Therefore $E^\infty_{1,2} = E^4_{1,2} = E_{1,2} \cong V \otimes \Lambda^2(I_{-2})$.

Finally, since $E^\infty_{0,3}$ is a submodule of $E^2_{0,3} \cong \Lambda^3(I_{-2})$, we see that

$$H_3(L_-^{(3)}) \cong V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2}) \oplus (V \otimes \Lambda^2(I_{-2})) \oplus M ,$$

where M is a direct sum of level 6 irreducible representations of $A_1^{(1)}$. Therefore we have

$$H_3(L_-^{(3)})_{-4} \cong V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2}), \tag{6.4}$$

and

$$\begin{aligned} I_{-4} &\cong V \otimes I_{-3}/H_3(L_-^{(3)})_{-4} \\ &\cong V \otimes I_{-3}/(V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2})). \end{aligned} \tag{6.5}$$

To determine I_{-5}, we study the short exact sequence

$$0 \to N_-^{(3)} \to L_-^{(4)} \to L_-^{(3)} \to 0,$$

and the corresponding spectral sequence $\{E_{p,q}^r\}$ converging to $H_*(L_-^{(4)})$ such that

$$E_{p,q}^2 \cong H_p(L_-^{(3)}) \otimes \Lambda^q(I_{-3}).$$

We will compute $H_3(L_-^{(4)})_{-5}$ from this spectral sequence. It is easy to show that $d_2\colon E_{2,0}^2 \to E_{0,1}^2$ is an isomorphism and that $E_{2,0}^\infty = 0$.

Consider the sequence

$$0 \to E_{3,0}^2 \xrightarrow{d_2} E_{1,1}^2 \to 0.$$

We have

$$E_{3,0}^2 \cong V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2}) \oplus (V \otimes \Lambda^2(I_{-2})) \oplus M,$$

and

$$E_{1,1}^2 \cong H_1(L_-^{(3)}) \otimes I_{-3} \cong V \otimes I_{-3}.$$

By elementary linear algebra, we have $E_{3,0}^2 \cong \operatorname{Ker} d_2 \oplus \operatorname{Im} d_2$. Since d_2 is a homomorphism between $A_1^{(1)}$-modules, by comparing levels, we see that d_2 maps $(V \otimes \Lambda^2(I_{-2})) \oplus M$ to zero. Thus $\operatorname{Im} d_2$ is isomorphic to a submodule of $V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2})$. By Lemma 3.2 and (6.5), we have

$$H_2(L_-^{(4)}) \cong I_{-4} \cong V \otimes I_{-3}/(V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2})).$$

Since

$$E_{1,1}^\infty = E_{1,1}^3 = E_{1,1}^2/\operatorname{Im} d_2 \cong V \otimes I_{-3}/\operatorname{Im} d_2$$

is a direct summand of $H_2(L_-^{(4)})$, $\operatorname{Im} d_2$ contains a submodule isomorphic to $V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2})$. Hence we conclude that

$$\operatorname{Im} d_2 \cong V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2}).$$

Therefore

$$E_{3,0}^3 \cong (V \otimes \Lambda^2(I_{-2})) \oplus M$$

and $E_{1,1}^\infty = E_{1,1}^3 \cong I_{-4}$. This implies that $E_{0,2}^\infty = E_{0,2}^4 = 0$. Thus the homomorphism d_3 is surjective in the sequence

$$0 \to E_{3,0}^3 \xrightarrow{d_3} E_{0,2}^3 \to 0.$$

Since $E_{0,2}^3$ is a submodule of $E_{0,2}^2 = \Lambda^2(I_{-3})$, by comparing levels, we see that $\operatorname{Ker} d_3$ must contain $V \otimes \Lambda^2(I_{-2})$. It follows that

$$E_{3,0}^\infty = E_{3,0}^4 \cong (V \otimes \Lambda^2(I_{-2})) \oplus M',$$

where M' is a direct sum of level 6 irreducible highest weight representations of $A_1^{(1)}$. Therefore $(E_{3,0}^\infty)_{-5} \cong V \otimes \Lambda^2(I_{-2})$. It is easy to see that

$$(E_{2,1}^\infty)_{-5} = (E_{1,2}^\infty)_{-5} = (E_{0,3}^\infty)_{-5} = 0.$$

Therefore we have

$$H_3(L_-^{(4)})_{-5} \cong V \otimes \Lambda^2(I_{-2}) \tag{6.6}$$

and

$$(6.7) \qquad I_{-5} \cong V \otimes I_{-4}/H_3(L_-^{(4)})_{-5} \cong V \otimes I_{-4}/V \otimes \Lambda^2(I_{-2}).$$

Case 2. $HA_2^{(2)}$.

For the structure of I_{-4}, we study the spectral sequence $\{E_{p,q}^r\}$ corresponding to the short exact sequence

$$0 \to N_-^{(2)} \to L_-^{(3)} \to L_-^{(2)} \to 0.$$

As in the case of $HA_1^{(1)}$, it is easy to deduce that the map $d_2\colon E_{2,0}^2 \to E_{0,1}^2$ is an isomorphism and that $E_{2,0}^\infty = 0$.

By comparing the levels of the terms in the sequence

$$0 \to E_{3,0}^2 \xrightarrow{d_2} E_{1,1}^2 \to 0,$$

we get $d_2 = 0$. So

$$E_{3,0}^3 = E_{3,0}^2 = V(-6\alpha_{-1} - 5\alpha_0 - \alpha_1),$$

and

$$E_{1,1}^\infty = E_{1,1}^3 = E_{1,1}^2 \cong V \otimes I_{-2} = I_{-3}.$$

Since $H_2(L_-^{(3)}) \cong I_{-3}$, we have $E_{0,2}^\infty = E_{0,2}^4 = 0$. Hence the homomorphism $d_3\colon E_{3,0}^3 \to E_{0,2}^3$ is surjective. But since $E_{0,2}^\infty$ is a submodule of $E_{0,2}^2 \cong \Lambda^2(I_{-2})$, we see that d_3 is trivial. This implies

$$E_{3,0}^\infty = E_{3,0}^4 = E_{3,0}^3 = V(-6\alpha_{-1} - 5\alpha_0 - \alpha_1)$$

and $E_{0,2}^3 = 0$. Thus the homomorphism $d_2\colon E_{2,1}^2 \to E_{0,2}^2$ is surjective in the following sequence

$$0 \to E_{4,0}^2 \to E_{2,1}^2 \to E_{0,2}^2 \to 0.$$

Again by comparing levels, we deduce that the homomorphism $d_2\colon E_{4,0}^2 \to E_{2,1}^2$ must be trivial. Therefore

$$E_{4,0}^3 = E_{4,0}^2 = V(-9\alpha_{-1} - 8\alpha_0 - 2\alpha_1)$$

and

$$\begin{aligned} E_{2,1}^\infty = E_{2,1}^3 &= \mathrm{Ker}(d_2\colon E_{2,1}^2 \to E_{0,2}^2)/\mathrm{Im}(d_2\colon E_{4,0}^2 \to E_{2,1}^2) \\ &\cong \mathrm{Ker}(d_2\colon E_{2,1}^2 \to E_{0,2}^2). \end{aligned}$$

Since

$$\mathrm{Im}\, d_2 \cong \Lambda^2(I_{-2}) \cong E_{2,1}^2/\mathrm{Ker}\, d_2 \cong I_{-2} \otimes I_{-2}/\mathrm{Ker}\, d_2,$$

we have $\mathrm{Ker}\, d_2 \cong S^2(I_{-2})$. Therefore $E_{2,1}^\infty \cong S^2(I_{-2})$.

Now consider the sequence

$$0 \to E_{5,0}^2 \to E_{3,1}^2 \to E_{1,2}^2 \to 0.$$

By comparing levels, we see that the homomorphism $d_2\colon E_{3,1}^2 \to E_{1,2}^2$ is trivial. Thus $E_{1,2}^3 = E_{1,2}^2 \cong V \otimes \Lambda^2(I_{-2})$. Again by comparing the levels of the terms in the sequence

$$0 \to E_{4,0}^3 \xrightarrow{d_3} E_{1,2}^3 \to 0,$$

we conclude $d_3 = 0$. Therefore

$$E^{\infty}_{1,2} = E^4_{1,2} = E^3_{1,2} \cong V \otimes \Lambda^2(I_{-2}).$$

Finally, since $E^{\infty}_{0,3}$ is a submodule of $E^2_{0,3} \cong \Lambda^3(I_{-2})$, we see that

$$H_3(L^{(3)}_-) \cong V(-6\alpha_{-1} - 5\alpha_0 - \alpha_1) \oplus S^2(I_{-2}) \oplus (V \otimes \Lambda^2(I_{-2})) \oplus M,$$

where M is a direct sum of level 6 irreducible highest weight representations of $A^{(2)}_2$. Therefore we obtain

$$H_3(L^{(3)}_-)_{-4} \cong S^2(I_{-2}), \tag{6.8}$$

and

$$I_{-4} \cong V \otimes I_{-3}/H_3(L^{(3)}_-)_{-4} \cong V \otimes I_{-3}/S^2(I_{-2}). \tag{6.9}$$

To determine I_{-5}, we study the spectral sequence $\{E^r_{p,q}\}$ corresponding to the short exact sequence

$$0 \to N^{(3)}_- \to L^{(4)}_- \to L^{(3)}_- \to 0.$$

As in the case of $HA^{(1)}_1$, we know that

$$(E^{\infty}_{3,0})_{-5} \cong V \otimes \Lambda^2(I_{-2})$$

and that

$$(E^{\infty}_{2,1})_{-5} = (E^{\infty}_{1,2})_{-5} = (E^{\infty}_{0,3})_{-5} = 0.$$

Therefore we have

$$H_3(L^{(4)}_-)_{-5} \cong V \otimes \Lambda^2(I_{-2}) \tag{6.10}$$

and

$$I_{-5} \cong V \otimes I_{-4}/H_3(L^{(4)}_-)_{-5} \cong V \otimes I_{-4}/V \otimes \Lambda^2(I_{-2}). \tag{6.11}$$

Case 3. $HA^{(1)}_l$, $l \geq 2$, $HC^{(1)}_2$, $HD^{(3)}_4$.

We first study the case of $HA^{(1)}_l$. Consider the short exact sequence

$$0 \to N^{(2)}_- \to L^{(3)}_- \to L^{(2)}_- \to 0,$$

and the corresponding spectral sequence $\{E^r_{p,q}\}$. As we have seen before, $d_2\colon E^2_{2,0} \to E^2_{0,1}$ is an isomorphism and hence $E^{\infty}_{2,0} = 0$.

In the sequence

$$0 \to E^2_{3,0} \xrightarrow{d_2} E^2_{1,1} \to 0,$$

we have

$$E^2_{3,0} \cong H_3(L^{(2)}_-) \cong V(-3\alpha_{-1} - 2\alpha_0 - \alpha_1) \oplus V(-3\alpha_{-1} - 2\alpha_0 - \alpha_l),$$

and $E^2_{1,1} \cong V \otimes I_{-2}$. As we have seen before, we have

$$\operatorname{Im} d_2 \cong V(-3\alpha_{-1} - 2\alpha_0 - \alpha_1) \oplus V(-3\alpha_{-1} - 2\alpha_0 - \alpha_l).$$

Hence $E^3_{3,0} = 0$, which implies $E^{\infty}_{3,0} = 0$.

Moreover, since $E^{\infty}_{1,1} = E^3_{1,1} \cong I_{-3}$, we have $E^{\infty}_{0,2} = E^4_{0,2} = 0$. Thus the homomorphism d_3 is surjective in the sequence

$$0 \to E^3_{3,0} \xrightarrow{d_3} E^3_{0,2} \to 0.$$

Since $E^3_{3,0} = 0$, we also have $E^3_{0,2} = 0$. That is, the homomorphism $d_2\colon E^2_{2,1} \to E^2_{0,2}$ is surjective in the sequence

$$0 \to E^2_{4,0} \to E^2_{2,1} \to E^2_{0,2} \to 0.$$

Note that

$$E^2_{4,0} \cong V(-4\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - \alpha_2) \oplus V(-4\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - \alpha_{l-1})$$

and that

$$E^2_{2,1} \cong I_{-2} \otimes I_{-2} \cong V(-2\alpha_{-1} - \alpha_0) \otimes V(-2\alpha_{-1} - \alpha_0).$$

In the decomposition of $V(-2\alpha_{-1} - \alpha_0) \otimes V(-2\alpha_{-1} - \alpha_0)$, every irreducible component must have the form $V(\lambda - m\delta)$, with $m \geq 0$ and

$$-4\alpha_{-1} - 2\alpha_0 - \delta < \lambda \leq -4\alpha_{-1} - 2\alpha_0.$$

Since $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_l$, λ must satisfy

$$-4\alpha_{-1} - 2\alpha_0 - \lambda < \alpha_0 + \alpha_1 + \cdots + \alpha_l.$$

But we have

$$-4\alpha_{-1} - 2\alpha_0 - (-4\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - \alpha_2) = \alpha_0 + 2\alpha_1 + \alpha_2 \not< \delta,$$

and

$$-4\alpha_{-1} - 2\alpha_0 - (-4\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - \alpha_{l-1}) = \alpha_0 + 2\alpha_1 + \alpha_{l-1} \not< \delta.$$

Thus $V(-4\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - \alpha_2)$ and $V(-4\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - \alpha_{l-1})$ do not occur as irreducible components in the decomposition of

$$V(-2\alpha_{-1} - \alpha_0) \otimes V(-2\alpha_{-1} - \alpha_0).$$

Therefore the homomorphism $d_2\colon E^2_{4,0} \to E^2_{2,1}$ is trivial and we obtain

$$E^\infty_{2,1} = E^3_{2,1} \cong I_{-2} \otimes I_{-2} / \operatorname{Im}(d_2\colon E^2_{2,1} \to E^2_{0,2}) \cong S^2(I_{-2}).$$

It is easy to show that $(E^\infty_{1,2})_{-4} = (E^\infty_{0,3})_{-4} = 0$. Hence we get

(6.12) $$H_3(L_-^{(3)})_{-4} \cong S^2(I_{-2}),$$

and

(6.13) $$I_{-4} \cong V \otimes I_{-3} / S^2(I_{-2}).$$

The proofs for the other cases are similar.

Case 4. $HA^{(2)}_{2l}$, $HD^{(2)}_{l+1}$, $l \geq 2$.

By the same argument of Case 1, we have

(6.14) $$H_3(L_-^{(3)})_{-4} \cong S^2(I_{-2}) \oplus V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1),$$

and therefore

(6.15) $$I_{-4} \cong V \otimes I_{-3} / (V(-4\alpha_{-1} - 3\alpha_0 - \alpha_1) \oplus S^2(I_{-2})). \quad \square$$

7. The principally specialized affine characters and the root multiplicities

In this section, we compute the principally specialized affine characters and the root multiplicities of the Lorentzian Kac-Moody Lie algebras $\mathfrak{g}(\widetilde{A})$ for certain higher levels. We define the *signum function* as follows. If α is an integer, then

$$\begin{aligned}\varepsilon(\alpha, t) &= 1 \quad \text{if } \alpha \text{ is divisible by } t, \\ &= 0 \quad \text{otherwise},\end{aligned}$$

and if α is a root of a Kac-Moody Lie algebra $\mathfrak{g}(A)$, then

$$\begin{aligned}\varepsilon(\alpha, t) &= 1 \quad \text{if } \alpha \text{ is divisible by a root of level } t, \\ &= 0 \quad \text{otherwise}.\end{aligned} \tag{7.1}$$

Let $F(q) = \sum_{n=0}^{\infty} f(n)q^n$, $X(q)$, $Y(q)$, and $Z(q)$ be the principally specialized characters of the integrable irreducible representations of the affine Kac-Moody Lie algebras $\mathfrak{g}(A)$ of level 1, 2, 3, and 4 given in [Kang].

In §2, we have seen that the Witt formula gives the principally specialized affine characters of the homogeneous subspaces of the maximal graded Lie algebra G:

$$\begin{aligned}\mathrm{ch}_q G_{-l} &= \sum_{m\geq 0} (\dim G_{(l,m)}) q^m \\ &= \sum_{m\geq 0} \left(\sum_{(r,s)|(l,m+l)} \mu\left(\frac{(l, m+l)}{(r, s)}\right) \frac{(r, s)}{(l, m+l)} \widehat{B}(r, s) \right) q^m,\end{aligned}$$

where the function $\widehat{B}(r, s)$ is defined by the formula (2.7). We introduce the following functions:

$$\begin{aligned}W_2(q) &= \mathrm{ch}_q G_{-2} \\ &= \sum_{m\geq 0} \left(\widehat{B}(2, m+2) - \frac{1}{2}\varepsilon(m, 2) f\left(\frac{m}{2}\right) \right) q^m,\end{aligned} \tag{7.2}$$

$$\begin{aligned}W_3(q) &= \mathrm{ch}_q G_{-3} \\ &= \sum_{m\geq 0} \left(\widehat{B}(3, m+3) - \frac{1}{3}\varepsilon(m, 3) f\left(\frac{m}{3}\right) \right) q^m,\end{aligned} \tag{7.3}$$

$$\begin{aligned}W_4(q) &= \mathrm{ch}_q G_{-4} \\ &= \sum_{m\geq 0} \left(\widehat{B}(4, m+4) - \frac{1}{2}\varepsilon(m, 2) \widehat{B}\left(2, \frac{m}{2}+2\right) \right) q^m,\end{aligned} \tag{7.4}$$

$$\begin{aligned}W_5(q) &= \mathrm{ch}_q G_{-5} \\ &= \sum_{m\geq 0} \left(\widehat{B}(5, m+5) - \frac{1}{5}\varepsilon(m, 5) f\left(\frac{m}{5}\right) \right) q^m.\end{aligned} \tag{7.5}$$

Theorem 7.1. *Let $L = \bigoplus_{n\in Z} L_n$ be the realization of the Lorentzian Kac-Moody Lie algebra $\mathfrak{g}(\widetilde{A})$. Then the principally specialized affine characters of L_{-n}, $n = 1, 2, \ldots, 5$, are given by the following formulas.*

$HA_1^{(1)}$:
$$\begin{aligned}
\operatorname{ch}_q L_{-1} &= F(q),\\
\operatorname{ch}_q L_{-2} &= W_2(q) - X(q),\\
\operatorname{ch}_q L_{-3} &= W_3(q) - F(q)X(q),\\
\operatorname{ch}_q L_{-4} &= W_4(q) - F(q)^2X(q) + Z(q) + \tfrac{1}{2}(X(q)^2 + X(q^2)),\\
\operatorname{ch}_q L_{-5} &= W_5(q) - F(q)^3X(q) + F(q)Z(q) + F(q)X(q)^2.
\end{aligned}$$

$HA_2^{(2)}$:
$$\begin{aligned}
\operatorname{ch}_q L_{-1} &= F(q),\\
\operatorname{ch}_q L_{-2} &= W_2(q) - X(q),\\
\operatorname{ch}_q L_{-3} &= W_3(q) - F(q)X(q),\\
\operatorname{ch}_q L_{-4} &= W_4(q) - F(q)^2X(q) + \tfrac{1}{2}(X(q)^2 + X(q^2)),\\
\operatorname{ch}_q L_{-5} &= W_5(q) - F(q)^3X(q) + F(q)X(q)^2.
\end{aligned}$$

$HA_l^{(1)}$, $l \geq 2$:
$$\begin{aligned}
\operatorname{ch}_q L_{-1} &= F(q),\\
\operatorname{ch}_q L_{-2} &= W_2(q) - X(q),\\
\operatorname{ch}_q L_{-3} &= W_3(q) - F(q)X(q) + 2Y(q),\\
\operatorname{ch}_q L_{-4} &= W_4(q) - F(q)^2X(q) + 2F(q)Y(q) + \tfrac{1}{2}(X(q)^2 + X(q^2)).
\end{aligned}$$

$HC_2^{(1)}$, $HD_4^{(3)}$:
$$\begin{aligned}
\operatorname{ch}_q L_1- &= F(q),\\
\operatorname{ch}_q L_{-2} &= W_2(q) - X(q),\\
\operatorname{ch}_q L_{-3} &= W_3(q) - F(q)X(q) + Y(q),\\
\operatorname{ch}_q L_{-4} &= W_4(q) - F(q)^2X(q) + F(q)Y(q) + \tfrac{1}{2}(X(q)^2 + X(q^2)).
\end{aligned}$$

$HA_{2l}^{(2)}$, $HD_{l+1}^{(2)}$, $l \geq 2$:
$$\begin{aligned}
\operatorname{ch}_q L_{-1} &= F(q),\\
\operatorname{ch}_q L_{-2} &= W_2(q) - X(q),\\
\operatorname{ch}_q L_{-3} &= W_3(q) - F(q)X(q),\\
\operatorname{ch}_q L_{-4} &= W_4(q) - F(q)^2X(q) + Z(q) + \tfrac{1}{2}(X(q)^2 + X(q^2)).
\end{aligned}$$

$HA_{2l-1}^{(2)}$, $HB_l^{(1)}$, $HC_l^{(1)}$, $l \geq 3$, $HD_l^{(1)}$, $l \geq 4$, $HF_4^{(1)}$, $HG_2^{(1)}$, $HE_6^{(1)}$, $HE_7^{(1)}$, $HE_8^{(1)}$, $HE_6^{(2)}$:
$$\begin{aligned}
\operatorname{ch}_q L_{-1} &= F(q),\\
\operatorname{ch}_q L_{-2} &= W_2(q) - X(q),\\
\operatorname{ch}_q L_{-3} &= W_3(q) - F(q)X(q) + Y(q).
\end{aligned}$$

Proof. The results follow directly from Theorem 6.2 and formula (2.8). □

Now we compute the root multiplicities. We first consider the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$. Let V be the basic representation of $A_1^{(1)}$ and let $S = \{\tau_i | i = 1, 2, 3, \dots\}$ be an enumeration of all the weights of V. Define the function $B^{(1)}(\tau)$ by

$$B^{(1)}(\tau) = \sum_{(n)\in T(\tau)} \frac{((\sum n_i) - 1)!}{\prod(n_i!)} \prod \left(p\left(1 - \frac{(\tau_i|\tau_i)}{2}\right)\right)^{n_i}, \tag{7.6}$$

where $(\ |\ )$ denotes the standard invariant symmetric bilinear form on $\mathfrak{h}^*$ [Kac2]. Then by the Witt formula, we have

$$\dim(G_-)_\alpha = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B^{(1)}(\tau). \tag{7.7}$$

Thus for the roots of level 2, the Witt formula gives

$$\dim(G_{-2})_\alpha = B^{(1)}(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2)p(1 - \tfrac{1}{8}(\alpha|\alpha)). \tag{7.8}$$

Since $I_{-2} \cong V(-2\alpha_{-1} - \alpha_0)$, by [F-L], we have

$$\dim(I_{-2})_\alpha = E\left(3 - \frac{(\alpha|\alpha)}{2}\right), \tag{7.9}$$

where the coefficients $E(n)$ are given by the equation

$$\sum_{n=0}^{\infty} E(n)q^n = \frac{1}{\prod_{n\geq 1}(1 - q^{4n})(1 - q^{4n-1})(1 - q^{4n-3})} = \frac{\phi(q^2)}{\phi(q)\phi(q^4)}.$$

Therefore we obtain

$$\dim(L_{-2})_\alpha = B^{(1)}(\alpha) - \frac{1}{2}\varepsilon(\alpha, 2)p\left(1 - \frac{1}{8}(\alpha|\alpha)\right) - E\left(3 - \frac{(\alpha|\alpha)}{2}\right). \tag{7.10}$$

For the roots of level 3, by the Witt formula, we have

$$\dim(G_{-3})_\alpha = B^{(1)}(\alpha) - \tfrac{1}{3}\varepsilon(\alpha, 3)p(1 - \tfrac{1}{18}(\alpha|\alpha)). \tag{7.11}$$

By Theorem 5.2,

$$I_{-3} \cong V \otimes I_{-2} \cong V(-\alpha_{-1}) \otimes V(-2\alpha_{-1} - \alpha_0).$$

In [Fe1], Feingold showed that I_{-3} has a decomposition into a direct sum of level 3 irreducible representations of the affine Kac-Moody Lie algebra $A_1^{(1)}$:

$$I_{-3} = \sum_{m\geq 0} (a_m V(-3\alpha_{-1} - \alpha_0 - m\delta) + b_m V(-3\alpha_{-1} - (m+1)\delta)),$$

where the coefficients a_m and b_m are given by

$$a_m = \sum_{j\in\mathbf{Z}} (p(m - j(20j+3)) - p(m - (4j+3)(5j+3)))$$

and

$$b_m = \sum_{j\in\mathbf{Z}} (p(m - (20j^2 + 11j + 1)) - p(m - (20j^2 + 19j + 4))).$$

Therefore, by [F-L], we have

$$\begin{aligned} N^{(1)}(\alpha) &= \dim(I_{-3})_\alpha \\ &= \sum_{m\geq 0} \left(a_m H\left(1 - 3m - \frac{(\alpha|\alpha)}{2}\right) + b_m G\left(6 - 3m - \frac{(\alpha|\alpha)}{2}\right)\right), \end{aligned} \tag{7.12}$$

where the functions G and H are defined recursively as follows. For $m > 0$,

$$0 = \sum_{n\in\mathbf{Z}} (-1)^n G(3m - \tfrac{1}{2}n(5n+3)),$$

$$0 = \sum_{n\in\mathbf{Z}} (-1)^n G(3m - 1 - \tfrac{1}{2}n(5n+1)),$$

with the initial conditions $G(0) = 1$, $G(-3m) = G(-3m+2) = 0$, and for $m > 0$,

$$0 = \sum_{n\in\mathbf{Z}} (-1)^n H(3m - \tfrac{1}{2}n(5n+3)),$$

$$0 = \sum_{n\in\mathbf{Z}} (-1)^n H(3m + 2 - \tfrac{1}{2}n(5n+1)),$$

with the initial conditions $H(2) = 1$, $H(-3m+3) = H(-3m+2) = 0$. Therefore we have

$$\dim(L_{-3})_\alpha = B^{(1)}(\alpha) - \tfrac{1}{3}\varepsilon(\alpha, 3)p(1 - \tfrac{1}{18}(\alpha|\alpha)) - N^{(1)}(\alpha). \tag{7.13}$$

We summarize these results in the following theorem.

Theorem 7.2. *Let $L = \bigoplus_{n\in Z} L_n$ be the realization of the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$. Then we have the following root multiplicity formulas:*

$$\dim(L_{-1})_\alpha = p\left(1 - \frac{(\alpha|\alpha)}{2}\right),$$

$$\dim(L_{-2})_\alpha = B^{(1)}(\alpha) - \frac{1}{2}\varepsilon(\alpha, 2)p\left(1 - \frac{1}{8}(\alpha|\alpha)\right) - E\left(3 - \frac{(\alpha|\alpha)}{2}\right),$$

$$\dim(L_{-3})_\alpha = B^{(1)}(\alpha) - \frac{1}{3}\varepsilon(\alpha, 3)p\left(1 - \frac{1}{18}(\alpha|\alpha)\right) - N^{(1)}(\alpha).$$

For the hyperbolic Kac-Moody Lie algebra $HA_2^{(2)}$, let $S = \{\tau_i | i = 1, 2, 3, \ldots\}$ be an enumeration of all the weights of the basic representation V of $A_2^{(2)}$. Define the function $B^{(2)}(\tau)$ by

$$B^{(2)}(\tau) = \sum_{(n)\in T(\tau)} \frac{\sum((n_i) - 1)!}{\prod(n_i!)} \prod \left(p\left(\frac{1}{2} - \frac{(\tau_i|\tau_i)}{2}\right)\right)^{n_i}. \tag{7.14}$$

Then by the Witt formula, we have

$$\dim(G_-)_\alpha = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B^{(2)}(\tau). \tag{7.15}$$

Thus for the roots of level 2, the Witt formula gives

$$\dim(G_{-2})_\alpha = B^{(2)}(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2)p(\tfrac{1}{2} - \tfrac{1}{8}(\alpha|\alpha)). \tag{7.16}$$

Since $I_{-2} \cong V(-2\alpha_{-1} - \alpha_0)$, by [F-L], we have

$$\dim(I_{-2})_\alpha = D(4 - (\alpha|\alpha)), \tag{7.17}$$

where the function D is defined recursively as follows. For $k > 0$,

$$\sum_{n\in\mathbf{Z}} D(4k - n(15n+4)) = \sum_{n\in\mathbf{Z}} D(4k - (3n+1)(5n+3)),$$

and for $m > 1$,

$$\sum_{n\in\mathbf{Z}} D(4m-3-n(15n+2)) = \sum_{n\in\mathbf{Z}} D(4m-3-(3n+1)(5n+1)),$$

with the initial conditions $D(1)=1$, $D(-4m+8)=D(-4m+5)=0$. Therefore we obtain

$$\dim(L_{-2})_\alpha = B^{(2)}(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2)p(\tfrac{1}{2} - \tfrac{1}{8}(\alpha|\alpha)) - D(4-(\alpha|\alpha)). \tag{7.18}$$

For the roots of level 3, by the Witt formula, we have

$$\dim(G_{-3})_\alpha = B^{(2)}(\alpha) - \tfrac{1}{3}\varepsilon(\alpha, 3)p(\tfrac{1}{2} - \tfrac{1}{18}(\alpha|\alpha)). \tag{7.19}$$

By Theorem 6.2,

$$I_{-3} \cong V \otimes I_{-2} \cong V(-\alpha_{-1}) \otimes V(-2\alpha_{-1} - \alpha_0)$$

and I_{-3} has a decomposition into a direct sum of level 3 irreducible representations of the affine Kac-Moody Lie algebra $A_2^{(2)}$ [Fel]:

$$I_{-3} = \sum_{m\geq 0} (a_m V(-3\alpha_{-1} - \alpha_0 - m\delta) + b_m V(-3\alpha_{-1} - (m+1)\delta)),$$

where the coefficients a_m and b_m are given by

$$a_m + \sum_{j\in\mathbf{Z}} (p(m - j(15j-2)) - p(m-(5j+1)(3j+1))),$$

and

$$b_m = \sum_{j\in\mathbf{Z}} (p(m - j(15j-7)) - p(m-(5j+1)(3j+2))).$$

Therefore by [F-L] we have

$$\begin{aligned} N^{(2)}(\alpha) &= \dim(I_{-3})_\alpha \\ &= \sum_{m\geq 0} \left(a_m K\left(\frac{9}{2} - 3m - \frac{(\alpha|\alpha)}{2}\right) + b_m J\left(\frac{3}{2} - 3m - \frac{(\alpha|\alpha)}{2}\right)\right), \end{aligned} \tag{7.20}$$

where the functions J and K are defined recursively as follows. For $k > 0$,

$$\sum_{n\in\mathbf{Z}} J(3k - 3n(3n+1)) = \sum_{n\in\mathbf{Z}} J(3k-(3n+1)(3n+2)),$$

and for $m > 0$,

$$\sum_{n\in\mathbf{Z}} J(3m-2-(3n)^2) = \sum_{n\in\mathbf{Z}} J(3m-2-(3n+1)^2),$$

with the initial conditions $J(0)=1$, $J(-3m)=J(-3m+1)=0$. For $k>0$,

$$\sum_{n\in\mathbf{Z}} K(3k - 3n(3n+1)) = \sum_{n\in\mathbf{Z}} K(3k-(3n+1)(3n+2)),$$

and for $m > 1$,

$$\sum_{n\in\mathbf{Z}} K(3m-2-(3n)^2) = \sum_{n\in\mathbf{Z}} K(3m-2-(3n+1)^2),$$

with the initial conditions $K(1)=1$, $K(-3m+6)=K(-3m+4)=0$. Therefore we have

$$\dim(L_{-3})_\alpha = B^{(2)}(\alpha) - \tfrac{1}{3}\varepsilon(\alpha, 3)p(\tfrac{1}{2} - \tfrac{1}{18}(\alpha|\alpha)) - N^{(2)}(\alpha). \tag{7.21}$$

We summarize the above results in the following theorem.

Theorem 7.3. *Let $L = \bigoplus_{n \in \mathbf{Z}} L_n$ be the realization of the hyperbolic Kac-Moody Lie algebra $HA_2^{(2)}$. Then we have the following root multiplicity formulas:*

$$\begin{aligned}
\dim(L_{-1})_\alpha &= p\left(\frac{1}{2} - \frac{(\alpha|\alpha)}{2}\right), \\
\dim(L_{-2})_\alpha &= B^{(2)}(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2)p(\tfrac{1}{2} - \tfrac{1}{8}(\alpha|\alpha)) - D(4 - (\alpha|\alpha)), \\
\dim(L_{-3})_\alpha &= B^{(2)}(\alpha) - \tfrac{1}{3}\varepsilon(\alpha, 3)p(\tfrac{1}{2} - \tfrac{1}{18}(\alpha|\alpha)) - N^{(2)}(\alpha).
\end{aligned}$$

Remark 7.4. Recently, Kac has informed the author that he also discovered a level 3 root multiplicity formula for the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$. We introduce his formula in the following. Unfortunately, the proof has not been available to the author yet.

Define a function $\omega(k)$ by

$$\begin{aligned}
\omega(k) &= 1 \quad \text{if } k \equiv 0, 1 \pmod 3, \\
&= 0 \quad \text{if } k \equiv -1 \pmod 3.
\end{aligned} \tag{7.22}$$

Then we have

$$\dim(L_{-3})_\alpha = \nu\left(1 - \frac{(\alpha|\alpha)}{2}\right), \tag{7.23}$$

where the coefficients $\nu(n)$ are defined by

$$\begin{aligned}
\sum_{n \geq 0} \nu(n) q^{n+8} = \frac{\phi(q^{24})}{3\phi(q^3)^2\phi(q^6)} &\left(2\frac{\phi(q^{24})}{\phi(q^6)} \prod_{j \equiv \pm 3 \pmod{24}} (1+q^j) - 3\right) \\
&\times \left(q^2 \prod_{j \equiv \pm 9 \pmod{24}} (1+q^j) \sum_{k \in \mathbf{Z}} \omega(k) q^{2k^2+2k}\right. \\
&\qquad \left. + q^3 \prod_{j \equiv \pm 3 \pmod{24}} (1+q^j) \sum_{k \in \mathbf{Z}} \omega(k) q^{2k^2}\right) \\
&+ \frac{1}{3}\left(\frac{1}{\phi(q^3)\phi(q^6)} \sum_{k \in \mathbf{Z}} \omega(k) q^{2k^2} - \frac{1}{\phi(q^9)}\right).
\end{aligned} \tag{7.24}$$

Combining (7.23) with (7.13) yields the following combinatorial identity:

$$B^{(1)}(\alpha) - \frac{1}{3}\varepsilon(\alpha, 3)p\left(1 - \frac{1}{18}(\alpha|\alpha)\right) - N^{(1)}(\alpha) = \nu\left(1 - \frac{(\alpha|\alpha)}{2}\right). \tag{7.25}$$

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