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WHEN CANTOR SETS INTERSECT THICKLY

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ABSTRACT. The thickness of a Cantor set on the real line is a measurement of its "size". Thickness conditions have been used to guarantee that the intersection of two Cantor sets is nonempty. We present sharp conditions on the thicknesses of two Cantor sets which imply that their intersection contains a Cantor set of positive thickness.

1. INTRODUCTION

Newhouse defined [5] a nonnegative quantity called the "thickness" of a Cantor set in order to formulate conditions which will guarantee that two Cantor sets intersect. (All Cantor sets considered in this paper lie in $\mathbb{R}^1$.) These conditions have been used [5, 6, 7, 8, 9] in the study of two-dimensional dynamical systems to deduce the existence of tangencies between stable and unstable manifolds whose one-dimensional cross sections are Cantor sets.

Thickness may be thought of as a measure of how large a Cantor set is relative to the intervals in its complement. Henceforth, these intervals will be referred to as *gaps*; the two unbounded intervals in the complement are each included in our use of the term gap. Newhouse's result [5, 7, 8] is that two Cantor sets must intersect if the product of their thicknesses is at least one, and neither set lies in a gap of the other. When this latter condition is satisfied, the sets are said to be *interleaved*. In [10], Williams observed the surprising fact that two interleaved Cantor sets can have thicknesses well above one and still only intersect in a single point. One might hope that under sufficiently strong thickness conditions, the intersection would be a Cantor set. However, the intersection of two arbitrarily thick interleaved Cantor sets can contain isolated points, so Williams posed the question of what conditions on the thicknesses of two interleaved Cantor sets will guarantee that their intersection contains another Cantor set. Williams obtained such a condition, though it is not sharp. In this paper we obtain the sharp condition. More precisely, we find a curve in (τ_1, τ_2)-space such that if the ordered pair (τ_1, τ_2) of thicknesses of two interleaved Cantor sets lies above the curve, their intersection contains a Cantor set, but if the pair of thicknesses lies below the curve, there exist examples for which the intersection is a single point. Kraft [2] has independently arrived at this condition. We

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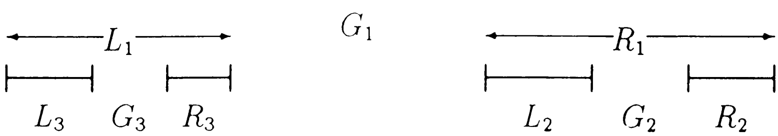

FIGURE 1. Constructing a Cantor set

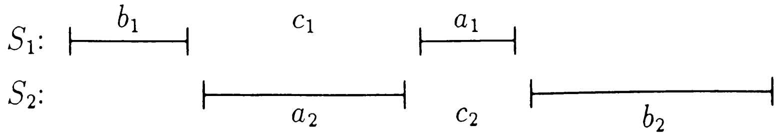

FIGURE 2. Nonintersecting interleaved sets

further show that if the thickness pair lies above the curve, the intersection must contain a Cantor set of positive thickness. This is the only result that addresses in terms of thickness how large the intersection of two Cantor sets must be. There are well-known probabilistic results concerning the Hausdorff dimensions of intersections of Cantor sets (cf. [1, 3, 4]).

One may think of a Cantor set as being constructed by starting with a closed interval and successively removing open gaps in order of decreasing length. Williams' formulation of the thickness of a Cantor set may then be thought of as follows. Each gap G_n is removed from a closed interval I_n, leaving behind closed intervals L_n, the left piece of $I_n - G_n$, and R_n on the right (see Figure 1). Let ρ_n be the ratio of the length of the smaller of L_n and R_n to the length of G_n. The thickness of the set is the infimum of ρ_n over all n.

We consider as an example the "middle-third" Cantor set, constructed as follows. Start with the closed interval $[0, 1]$, and remove the open interval $(1/3, 2/3)$, the middle third of the original interval. Then from each of the two remaining intervals, remove their middle thirds; repeat this process infinitely often. Each gap G_n is the same length as the adjacent intervals L_n and R_n, so $\rho_n = 1$ for all n. Thus the thickness of the middle-third Cantor set is one.

There is a connection between the thickness of a Cantor set and its fractal dimension, which depends in part on how the ratios ρ_n are distributed as $n \to \infty$. However, two large gaps close together make the thickness of a set very small, while its dimension can still be large. It was shown in [7] that the Hausdorff dimension of a Cantor set with thickness τ is bounded below by $\log 2/\log(2 + 1/\tau)$. This lower bound is sharp for the middle-third Cantor set (whose dimension is $\log 2/\log 3$).

We offer here a new formulation of the definition of thickness which we state for all compact sets, not just Cantor sets. (The results in this and previous papers are found to be valid for all compact sets.) We define nondegenerate intervals to have infinite thickness, while singletons are defined to have thickness zero. In fact, any set containing an isolated point will be seen to have thickness zero. To define the thickness of a compact set S which is not an interval, we consider a type of subset of S obtained by intersecting S with a closed interval. We call such an intersection P a *chunk* of S if P is a proper subset of S and has a positive distance from $S - P$, the complement of P in S. (Notice that for P to be a chunk both P and $S - P$ must be closed and nonempty.) We then define the thickness of S to be the infimum over all chunks P of the ratio between the diameter of P and the distance from P to $S - P$. In the case of the middle-third Cantor set, the given ratio can be shown to be smallest when the chunk P is obtained by intersecting S with an interval L_n or R_n, in which case the ratio is one. In §2 we will show that our new definition is equivalent to the old one for all Cantor sets.

The reason thickness is an appropriate quantity for determining when one can guarantee that two compact sets intersect is illustrated by considering an example where each of the two sets is an union of two disjoint intervals. For $i = 1, 2$ let S_i consist of closed intervals of lengths a_i and b_i with $a_i \leq b_i$, separated by a distance c_i. Then each S_i has only two chunks, and is found to have thicknesses a_i/c_i. If the product of the thickness $a_1 a_2/c_1 c_2$ is at least one, then either $a_1 \geq c_2$ or $a_2 \geq c_1$ (or both); assume $a_1 \geq c_2$. Then since $b_1 \geq a_1$, neither interval of S_1 can lie in the gap of S_2; hence if the two sets are interleaved, they must intersect. If on the other hand $a_1 a_2/c_1 c_2 < 1$, then with an affine map, we can scale the sets so that $a_1 < c_2$ and $a_2 < c_1$, and position them so that the component of S_1 with length a_1 lies inside the gap of S_2, and vice versa. The two sets are then interleaved, but they do not intersect (see Figure 2). This example could of course be made to involve Cantor sets by constructing very thick Cantor sets in each interval of each S_i.

An important point which is apparent in the above example is that the union of two sets can have a smaller thickness than either of the original sets. In other words, adding points to a set can decrease its thickness. By the same token, one may be able to increase the thickness of a set by removing appropriate subsets. This observation is useful in the following way. No matter how thick two interleaved compact sets are, their intersection may have thickness zero because it may contain isolated points, or arbitrarily small chunks which are relatively isolated from the rest of the intersection. Nonetheless we are able to show that if the original sets are thick enough, then by throwing out the relatively isolated parts of their intersection we can obtain a set of positive thickness in the intersection.

To define the set C of thickness pairs (τ_1, τ_2) for which a Cantor set of intersection can be guaranteed, we make use of the functions

$$f(\tau) = \frac{\tau^2 + 3\tau + 1}{\tau^2}, \qquad g(\tau) = \frac{(2\tau + 1)^2}{\tau^3}.$$

Let C be the set of pairs (τ_1, τ_2) for which one of the following sets of conditions holds:

$$\tau_1 \geq \tau_2, \quad \tau_1 > f(\tau_2), \quad \text{and} \quad \tau_2 > g(\tau_1) \tag{1.1}$$

or

$$\tau_2 \geq \tau_1, \quad \tau_2 > f(\tau_1), \quad \text{and} \quad \tau_1 > g(\tau_2) \tag{1.2}$$

(see Figure 3). Our main result is as follows.

Theorem 1. *There is a function $\varphi(\tau_1, \tau_2)$ which is positive in region C such that for all interleaved compact sets $S_1, S_2 \subset \mathbb{R}$ with $\tau(S_1) \geq \tau_1$ and $\tau(S_2) \geq \tau_2$, there is a set $S \subset S_1 \cap S_2$ with thickness at least $\varphi(\tau_1, \tau_2)$.*

Notice that a compact set with positive thickness can have no isolated points, and thus must either be a Cantor set or contain an interval; either way it contains a Cantor set.

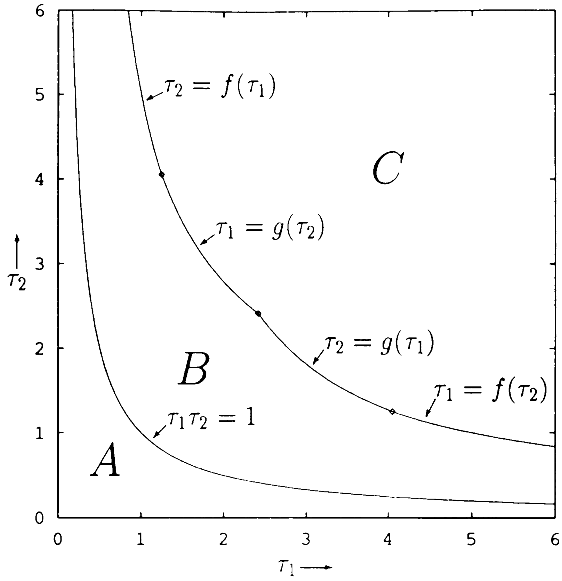

FIGURE 3. The intersection of two interleaved compact sets with thicknesses τ_1 and τ_2 can be empty for (τ_1, τ_2) in region A, must be nonempty but can be a single point in region B, and must contain a set of positive thickness in region C.

We remark that (τ_1, τ_2) is in C if both thicknesses are greater than $\sqrt{2}+1$. This is the critical value Williams found for the case of interleaved Cantor sets with the same thickness. Also, no matter how small one thickness is, the other thickness can be chosen large enough so that the pair lies in C. Our results and the results of Newhouse are summarized in Figure 3.

In §2 we give a proof of Newhouse's result, which will illustrate some of the methods to be used later. Then we present for all pairs (τ_1, τ_2) not in C an example of interleaved compact sets with thicknesses τ_1 and τ_2 whose intersection is a single point (except when (τ_1, τ_2) is on the boundary of C, in which case our example gives a countable intersection). This example shows that Theorem 1 is sharp in that its conclusion cannot hold for any larger set of thickness pairs (τ_1, τ_2). In §3 we prove Theorem 1, and in §4 we discuss some further properties of $S_1 \cap S_2$. The positive thickness set $S \in S_1 \cap S_2$ constructed in §3 need not be dense in $S_1 \cap S_2$; however we find that there are subsets with thickness at least $\varphi(\tau_1, \tau_2)$ arbitrarily near any accumulation point of $S_1 \cap S_2$. In addition, we find bounds on the diameter of S which allow us to obtain thickness conditions that imply that the intersection of three Cantor sets is nonempty.

2. PRELIMINARIES

Let us define precisely the concepts and notation we will use.

Definition 1. We say two sets S_1, $S_2 \subset \mathbb{R}$ are interleaved if each set intersects the interior of the convex hull of the other set (that is, neither set is contained in the closure of a gap of the other set).

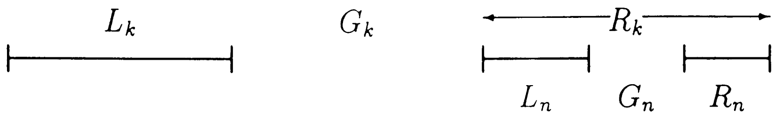

FIGURE 4. Chunks and gaps of a Cantor set $(k < n)$

We define the distance between two nonempty sets S_1, S_2 to be

$$d(S_1, S_2) = \inf\{|x-y| \,|\, x \in S_1, y \in S_2\},$$

and write $S_2 - S_1$ for the intersection of S_2 with the complement of S_1. We say that a set S_1 is a *chunk* of a set S_2, and write $S_1 \propto S_2$, if S_1 is the intersection of a closed interval with S_2, is a proper subset of S_2, and $d(S_1, S_2 - S_1) > 0$. Notice that a closed set S has a chunk if and only if it is not connected. We denote the diameter of a set S (the length of its convex hull) by $|S|$.

Definition 2. Given a compact set $S \subset \mathbb{R}$, we define the thickness of S to be

$$\tau(S) = \inf_{P \propto S} \frac{|P|}{d(P, S-P)} \tag{2.1}$$

provided S has a chunk. Otherwise, we let $\tau(S) = 0$ if S is empty or consists of a single point, and $\tau(S) = \infty$ if S is an interval with positive length.

The following simple proposition demonstrates that Definition 2 agrees with Williams' definition of thickness for Cantor sets [10].

Proposition 2. *Let S be a Cantor set, and define the ratios ρ_n as in the introduction. Then the quantity $\tau(S)$ given by* (2.1) *is equal to the infimum of ρ_n over all n.*

Proof. The intervals L_n and R_n defined in the introduction are the convex hulls of chunks $A_n = L_n \cap S$ and $B_n = R_n \cap S$ of S. Since the gap G_n is not larger than any previously removed gap G_k, $k < n$, it follows that

$$d(A_n, S - A_n) = d(B_n, S - B_n) = |G_n|$$

(see Figure 4). Thus for all n,

$$\rho_n = \min\left(\frac{|L_n|}{|G_n|}, \frac{|R_n|}{|G_n|}\right) = \min\left(\frac{|A_n|}{d(A_n, S-A_n)}, \frac{|B_n|}{d(B_n, S-B_n)}\right) \geq \tau(S).$$

Next, if P is a chunk of S, it must be bordered on each side by a gap of S; let G_n be the smaller of these two gaps. Then $|G_n| = d(P, S-P)$ and $|P| \geq \min(|L_n|, |R_n|)$. Therefore

$$\tau(S) = \inf_{P \propto S} \frac{|P|}{d(P, S-P)} \geq \inf_n \rho_n,$$

which completes the proof. □

We now prove Newhouse's result in a way that will motivate our later examples and methods.

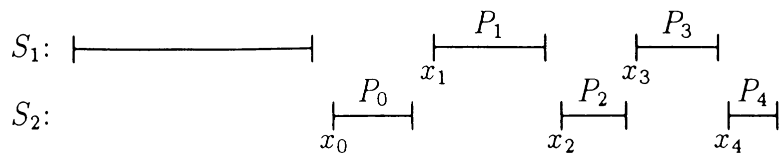

FIGURE 5. The points x_n and chunks P_n

Proposition 3. *If S_1 and S_2 are interleaved compact sets with $\tau(S_1)\cdot\tau(S_2) \geq 1$, then $S_1 \cap S_2$ is not empty.*

Proof. Let S_1 and S_2 be as above, and let

$$x_0 = \max\left(\inf_{x\in S_1} x,\ \inf_{x\in S_2} x\right),$$

the greater of the leftmost points of S_1 and S_2. Assume without loss of generality that $x_0 \in S_2$. We will show that $S_1 \cap S_2$ is nonempty by looking for the leftmost point of this set. Let x_1 be the leftmost point of S_1 which is at least as great as x_0. Since S_1 and S_2 are interleaved, x_1 must exist (otherwise S_1 would lie entirely to the left of S_2; see Figure 5). Next, let x_2 be the leftmost point of S_2 greater than or equal to x_1. Once again the interleaving assumption implies that x_2 exists, for otherwise S_2 would lie inside a gap of S_1. We similarly define $x_3, x_4, \ldots$; if each of these points can be shown to exist, we claim to be done. Then $\{x_n\}$ will be a nondecreasing sequence which is bounded above (since S_1 and S_2 are bounded), so it approaches a limit. This limit must belong to both S_1 and S_2 since these sets are closed and the odd numbered terms of $\{x_n\}$ belong to S_1, the even ones to S_2.

If at any step x_n exists and equals x_{n-1}, then $x_{n+1}, x_{n+2}, \ldots$ will also equal x_{n-1}, and we will have found a point in $S_1 \cap S_2$. Henceforth we assume $x_0 < x_1 < \cdots$ as long as they are defined. We know at least that x_0, x_1, and x_2 exist, so there is a chunk P_0 of S_2 which lies in $[x_0, x_1)$, whose diameter is thus less than $x_1 - x_0$, and whose distance from the rest of S_2 is greater than $x_2 - x_1$ (see again Figure 5). Then

$$\frac{x_1 - x_0}{x_2 - x_1} > \frac{|P_0|}{d(P_0, S_2 - P_0)} \geq \tau(S_2). \tag{2.2}$$

Let P_1 be the largest chunk of S_1 which lies in $[x_1, x_2)$. If x_3 did not exist, in other words if all points in S_1 were less than x_2, then $S_1 - P_1$ would lie to the left of P_1, and the distance between these sets would be greater than $x_1 - x_0$. But then using (2.2) and $\tau(S_1)\cdot\tau(S_2) \geq 1$ we would have

$$\frac{|P_1|}{d(P_1, S_1 - P_1)} < \frac{x_2 - x_1}{x_1 - x_0} < \frac{1}{\tau(S_2)} \leq \tau(S_1),$$

contradicting the definition of the thickness of S_1. Thus x_3 exists, and similarly to (2.2) we obtain

$$\frac{x_2 - x_1}{x_3 - x_2} > \frac{|P_1|}{d(P_1, S_1 - P_1)} \geq \tau(S_1). \tag{2.3}$$

Likewise (2.3) can be used to show the existence of x_4, and so forth. The proof is completed by induction. □

One could similarly find the rightmost point in $S_1 \cap S_2$, but as Williams observed it may coincide with the leftmost point, even if both thicknesses are significantly greater than 1. We next present an example which will give a single point of intersection for thickness pairs (τ_1, τ_2) not in the closure of region C, and a countable intersection for (τ_1, τ_2) on the boundary of C. In our example both sets are countable unions of closed intervals, but they could be replaced by Cantor sets with the same thicknesses by constructing a very thick Cantor set in each of the closed intervals.

Let τ be a positive constant, and define the intervals

$$A_0 = [\tau^2 + 3\tau + 1, (2\tau + 1)^2], \qquad B_0 = [\tau^2, \tau^2 + 3\tau + 1],$$
$$A_n = \left(-\frac{\tau}{2\tau + 1}\right)^n A_0, \qquad B_n = \left(-\frac{\tau}{2\tau + 1}\right)^n B_0,$$

where multiplication of a set by a scalar means the set obtained by multiplying each element of the original set by the given scalar. Let

$$S_1 = \left(\bigcup_{n=0}^{\infty} A_n\right) \cup \{0\}, \quad S_2 = \left(\bigcup_{n=0}^{\infty} B_n\right) \cup \{0\}.$$

Notice that B_n is the closure of the interval between A_n and A_{n+2} for all n, and A_n is the closure of the interval between B_{n-2} and B_n for $n \geq 2$. Thus $S_1 \cap S_2$ is countable, containing only the point 0 and endpoints of the intervals A_n and B_n. Furthermore, the intersection could by reduced to only the point 0 by shrinking the intervals which make up one of the sets by a factor arbitrarily close to one.

Let us compute the thicknesses of the sets S_1 and S_2. Observe that

$$|A_n| = d(B_{n-2}, B_n) = \left(\frac{\tau}{2\tau + 1}\right)^n \tau(3\tau + 1),$$
$$|B_n| = d(A_n, A_{n+2}) = \left(\frac{\tau}{2\tau + 1}\right)^n (3\tau + 1).$$

The intervals A_n are ordered from left to right $A_1, A_3, A_5, \ldots, A_4, A_2, A_0$, so any chunk P of S_1 which does not contain 0 must be a finite union of consecutive even or odd numbered A_n. Let A_n be the interval in P with the largest index; then

$$\frac{|P|}{d(P, S_1 - P)} \geq \frac{|A_n|}{d(A_n, A_{n+2})} = \tau,$$

with equality holding when $P = A_n$. On the other hand, if a chunk P of S_1 contains zero, let n be the larger index of the leftmost and rightmost A_k in P. Then P must contain A_{n-1}, and since P is not all of S_1, $n \geq 2$, so

$$\frac{|P|}{d(P, S_1 - P)} \geq \frac{|A_n \cup A_{n-1}|}{d(A_n, A_{n-2})} = \frac{(\tau/(2\tau + 1))^{n-1}(3\tau + 1)(2\tau + 1)}{(\tau/(2\tau + 1))^{n-2}(3\tau + 1)} = \tau.$$

Therefore the thickness of S_1 is τ.

Similarly, if P is a chunk of S_2, then for an appropriately chosen B_n, either

$$\frac{|P|}{d(P, S_2 - P)} \geq \frac{|B_n|}{d(B_n, B_{n+2})} = \frac{(2\tau + 1)^2}{\tau^3} = g(\tau)$$

or

$$\frac{|P|}{d(P, S_2 - P)} \geq \frac{|B_n \cup B_{n-1}|}{d(B_n, B_{n-2})}$$
$$= \frac{(\tau/(2\tau+1))^{n-1}((3\tau+1)/(2\tau+1))(\tau^2+3\tau+1)}{(\tau/(2\tau+1))^{n-2}\tau(3\tau+1)}$$
$$= \frac{\tau^2+3\tau+1}{\tau^2} = f(\tau).$$

Thus

$$\tau(S_2) = \min(f(\tau), g(\tau)).$$

As we pointed out before, by reducing the thickness of S_2 by an arbitrarily small amount we can shrink the intersection of S_1 and S_2 to a single point. Let τ_1 denote the thickness of the set S_1, and let τ_2 be the thickness of S_2. Then up to a change of indices, the above construction demonstrates that a single point of intersection can be obtained when either

$$\tau_1 < \min(f(\tau_2), g(\tau_2)) \tag{2.4}$$

or

$$\tau_2 < \min(f(\tau_1), g(\tau_1)). \tag{2.5}$$

Also, if either (2.4) or (2.5) is an equality instead, the intersection can be countable. (Kraft [2] has analyzed this borderline case and determined when the intersection can be finite.) Therefore we can only hope to guarantee an uncountable intersection if

$$\tau_1 > \min(f(\tau_2), g(\tau_2)) \tag{2.6}$$

and

$$\tau_2 > \min(f(\tau_1), g(\tau_1)). \tag{2.7}$$

One may check that $g(\tau) > f(\tau) > \sqrt{2}+1$ for $\tau < \sqrt{2}+1$ and $g(\tau) < f(\tau) < \sqrt{2}+1$ for $\tau > \sqrt{2}+1$. Therefore (2.6) and (2.7) are equivalent to (1.1) in the case $\tau_1 \geq \tau_2$, and to (1.2) when $\tau_2 \geq \tau_1$.

3. Proof of main result

We now prove Theorem 1 by constructing a set S with positive thickness in $S_1 \cap S_2$.

Proof of Theorem 1. Let S_1 and S_2 be interleaved compact sets with $\tau(S_1) \geq \tau_1$ and $\tau(S_2) \geq \tau_2$ for some (τ_1, τ_2) in region C of Figure 3. Let the gaps of S_1 be $I_0, I_1, I_2, \ldots$, with I_0 and I_1 unbounded, I_0 to the left of I_1, and $|I_2| \geq |I_3| \geq \cdots$. For S_2 we define $J_0, J_1, J_2, \ldots$ similarly. We refer to the intervals I_n and J_n collectively as the "original gaps". Our goal is to construct the complement of S as a union of disjoint open intervals $K_0, K_1, K_2, \ldots$ with K_0 and K_1 unbounded, and with every original gap contained in some K_m (whence $S \subset S_1 \cap S_2$). To get a lower bound on the thickness of S, observe that every chunk P of S is bordered on each side by a gap of S, with at least one of the bordering gaps being bounded. Pick a chunk P, and say P is bordered by K_m and K_n with $m > n$ and $m \geq 2$. Then

$$\frac{|P|}{d(P, S-P)} = \frac{d(K_m, K_n)}{\min(|K_m|, |K_n|)} \geq \frac{d(K_m, K_n)}{|K_m|}.$$

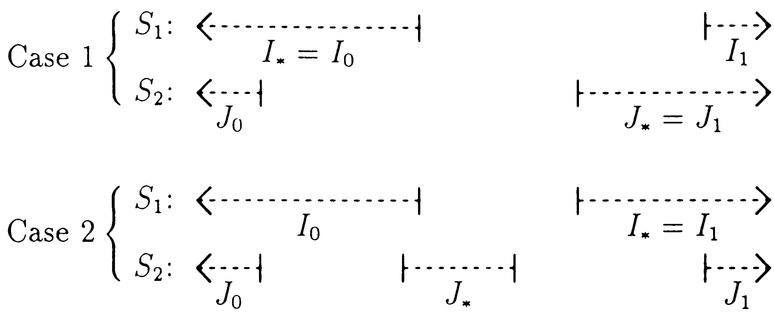

FIGURE 6. Cases in the construction of I_* and J_*

The theorem will therefore be proved when we show for some $\varphi(\tau_1, \tau_2) > 0$ that whenever $m > n$ and $m \geq 2$,

$$\frac{d(K_m, K_n)}{|K_m|} \geq \varphi(\tau_1, \tau_2). \tag{3.1}$$

We begin by finding a pair of original gaps I_* and J_* between which S will lie; that is, I_* and J_* will be contained in K_0 and K_1. The properties we desire of I_* and J_* are that they are a positive distance apart, that all gaps of S_1 with an endpoint between the closures of I_* and J_* are bounded and no larger than I_*, and likewise (in comparison to J_*) for gaps of S_2 between I_* and J_*. We will show later that once I_* and J_* have been determined, the diameter of S can be bounded below by a constant depending on τ_1 and τ_2 times the distance between I_* and J_*.

Assume without loss of generality that $J_0 \subset I_0$. If $I_1 \subset J_1$ (Case 1 of Figure 6), then $I_* = I_0$ and $J_* = J_1$ have the above properties; they must be separated by a positive distance since S_1 and S_2 are interleaved. If $J_1 \subset I_1$ (Case 2 of Figure 6), let J_* be the largest gap of S_2 with an endpoint between I_0 and I_1, and let I_* be whichever of I_0 and I_1 is farthest from J_*. At least one of I_0 and I_1 must be a positive distance from J_* since S_1 and S_2 are interleaved.

Next, let t be a positive constant whose precise value will be chosen later; for now we assume that $t < (\tau_1\tau_2 - 1)/(\tau_1 + \tau_2 + 2) < \min(\tau_1, \tau_2)$. Assume without loss of generality that I_* lies to the left of J_*. We begin constructing K_0 by requiring that it contain I_*. We then require that K_0 contain the rightmost bounded J_n with $d(I_*, J_n) \leq t|J_n|$ (we will verify that there is a rightmost gap satisfying this condition when we later examine our construction in more detail). If there does not exist such a J_n that is not already contained in I_*, we stop the construction and let $K_0 = I_*$. Otherwise, we further require that K_0 contain the rightmost bounded I_m that is within t times its length of the previously added J_n. Again, if this requirement does not extend K_0 any farther rightward, we stop the construction. If not, we then add to K_0 the rightmost J_l which is within t times its length of I_m and is at most as large as J_n (see Figure 7). If a next step is necessary, we consider gaps of S_1 which are no larger than I_m, and so forth. We may have to continue this process infinitely often, but if so we must converge to a right endpoint for K_0, since there is no way this construction can extend past the rightmost point in $S_1 \cup S_2$.

We define K_1 similarly, starting with the requirement that K_1 contain J_*

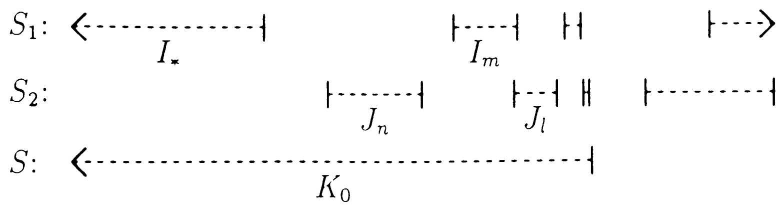

FIGURE 7. The construction of K_0

and extending K_1 to the left if necessary in the same way we constructed K_0. Next, to construct K_2 we first require that it contain the largest original gap (choose any one in the case of a tie) not contained in $K_0 \cup K_1$ (if no such gap exists, we leave K_2 undefined and let S be the complement of $K_0 \cup K_1$). Then we extend it on both the left and right in the same manner as before, but considering only gaps that are at most as large as the one we started with, to obtain the endpoints of K_2. We next start with the largest original gap not contained in $K_0 \cup K_1 \cup K_2$, proceeding similarly to define K_3, and so forth. Any given original gap must eventually be contained in some K_n because there can be only finitely many original gaps that are as large or larger than the given one. We do not yet know that the K_n are disjoint from each other; this will follow when we prove (3.1), though.

Let us now examine our construction more closely. Define $l(I)$ and $r(I)$ to be respectively the left and right endpoints of an interval I. For a given K_n, let G_0 be the gap we started with in its construction, which for $n \geq 2$ must be the largest original gap it contains (or at least tied for the largest). For simplicity we assume here that G_0 is a gap of S_1. Consider the collection E of all J_n with $|J_n| \leq |G_0|$, $r(J_n) > r(G_0)$, and $d(G_0, J_n) \leq t|J_n|$. We claim that the members of E (if any) are increasing in size from left to right. If J_m, $J_n \in E$ with J_m to the left of J_n, then since $r(J_m) < r(J_n)$, it follows that $d(J_m, J_n) < d(G_0, J_n) \leq t|J_n|$. Since $t < \tau_2$ and $d(J_m, J_n) \geq \tau_2 \min(|J_m|, |J_n|)$, it must then be the case that $|J_n| > |J_m|$. Thus if E is not empty, it must have a rightmost member, which we call G_1 (notice that G_1 is also the largest member of E). If E is empty, we let G_1 be empty, but in order to facilitate future formalism, we define $|G_1| = 0$ and $r(G_1) = r(G_0)$. One must keep in mind this degenerate case in verifying the assertions and formulas that follow.

We likewise define G_2 to be the rightmost gap of S_1 which is at most as large as G_0 and lies within t times its length of G_1; again if no such gap exists with $r(G_2) > r(G_1)$ we say that $|G_2| = 0$ and $r(G_2) = r(G_1)$. Next, to define G_3 we consider only gaps of S_2 which are at most as large as G_1, for G_4 we look only at gaps of S_1 no larger than G_2, and so forth. Define $G_{-1}, G_{-2}, \ldots$ similarly to be the leftmost (and largest) gaps added to K_n at each stage of the process of extending K_n leftward. Then we may think of the open interval K_n as being defined by

$$l(K_n) = \lim_{m \to -\infty} l(G_m), \quad r(K_n) = \lim_{m \to \infty} r(G_m).$$

Each limit exists because it is the limit of a bounded monotonic sequence.

In the above construction, the even-numbered G_m are gaps of S_1 and the odd-numbered ones are gaps of S_2, but if G_0 had been a gap of S_2 it would be the other way around. In any case, G_0 is the largest even-numbered G_m and either G_1 or G_{-1} is the largest odd-numbered one. Also, the even-numbered

G_m decrease monotonically in size as one moves either rightward or leftward from the largest, and the same statement holds for the odd-numbered G_m. We call a given G_m either a "1-gap" or "2-gap" of K_n according to whether it is a gap of S_1 or S_2. Notice that not all original gaps contained in K_n are 1-gaps or 2-gaps, only those that have been given a label G_m in the construction of K_n. When we refer henceforth to left-to-right ordering or adjacency among the 1-gaps and 2-gaps of a given K_n, it is with respect to the ordering $\dots, G_{-2}, G_{-1}, G_0, G_1, G_2, \dots$. (Thus, for instance, 1-gaps can only be adjacent to 2-gaps and vice versa.)

The following lemma will be used in bounding both the numerator and denominator of the left side of (3.1). It establishes for all $m \geq 0$ a bound on how far K_n can extend to the right of G_m in terms of how far G_{m+1} extends past G_m, and similarly for $m \leq 0$ on the left.

Lemma 4. *Assume* $t < (\tau_1\tau_2 - 1)/(\tau_1 + \tau_2 + 2)$. *Let*

$$\sigma_1 = \frac{(\tau_1 - t)(\tau_2 + 1)}{(\tau_1 - t)(\tau_2 - t) - (1+t)^2}$$

and

$$\sigma_2 = \frac{(\tau_2 - t)(\tau_1 + 1)}{(\tau_1 - t)(\tau_2 - t) - (1+t)^2}.$$

Let G *be a* 1*-gap of* K_n *which is at least as large as all* 1*-gaps of* K_n *to its right. Let* H *be the next* 2*-gap of* K_n *to the right of* G. *Then*

$$r(K_n) - r(G) \leq \sigma_2(r(H) - r(G)).$$

The same statement with "1" *and* "2" *interchanged holds, as do the corresponding results for left endpoints.*

Proof. Let I be the next 1-gap of K_n to the right of H. Then since $|I| \leq |G|$,

$$\tau_1|I| \leq d(G, I) \leq d(H, I) + r(H) - r(G) \leq t|I| + r(H) - r(G),$$

which, because $t < \tau_1$, implies that

$$|I| \leq \frac{r(H) - r(G)}{\tau_1 - t}.$$

Hence

$$r(I) - r(H) \leq |I| + d(H, I) \leq (1+t)|I| \leq \frac{1+t}{\tau_1 - t}(r(H) - r(G)). \tag{3.2}$$

Likewise the next rightward 2-gap of K_n extends at most

$$\frac{1+t}{\tau_2 - t}(r(I) - r(H))$$

beyond I, and by induction

$$\begin{aligned} r(K_n) - r(G) &= r(H) - r(G) + r(I) - r(H) + \cdots \\ &\leq \left(1 + \frac{1+t}{\tau_1 - t} + \frac{1+t}{\tau_1 - t}\frac{1+t}{\tau_2 - t} + \cdots\right)(r(H) - r(G)) \\ &= \sigma_2(r(H) - r(G)). \end{aligned}$$

The geometric series converges, and the denominator of σ_2 is positive, because of our assumption that $t < (\tau_1\tau_2 - 1)/(\tau_1 + \tau_2 + 2)$. □

The next lemma builds on Lemma 4 to obtain a positive lower bound on the distance between a given K_m and K_n, provided we can find a 1-gap of K_m and a 2-gap of K_n which are respectively larger than all 1-gaps and 2-gaps between them. The proof is difficult and will be handled later.

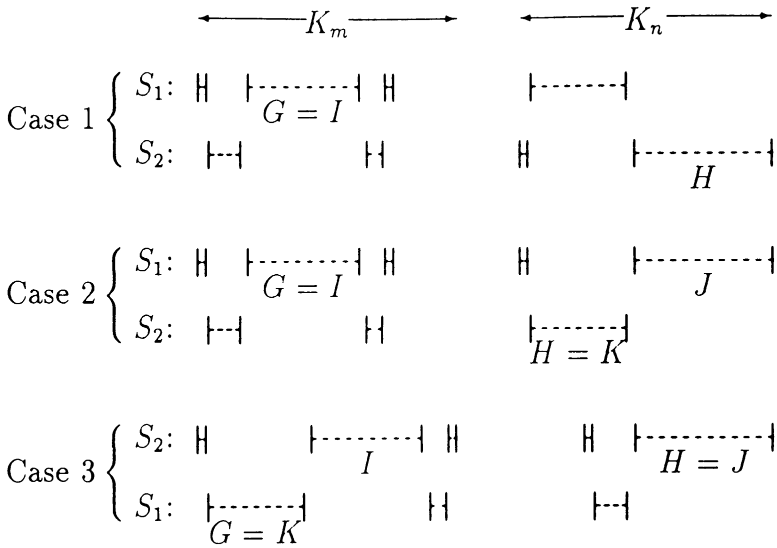

FIGURE 8. Cases in the proof of (3.1)

Lemma 5. *There exists a function* $\psi_t(\tau_1, \tau_2)$ *that is positive whenever* (τ_1, τ_2) *is in region* C *and* t *is sufficiently small, and for which the following statement holds. For* $m \neq n$, *let* G *be a* 1*-gap of* K_m *and* H *be a* 2*-gap of* K_n. *If all* 1*-gaps of* K_m *or* K_n *with at least one endpoint between the closures of* G *and* H *are bounded and at most as large as* G, *and all similarly situated* 2*-gaps are bounded and at most as large as* H, *then*

$$d(K_n, K_m) \geq \psi_t(\tau_1, \tau_2) d(G, H).$$

Recall that to construct K_0 and K_1, we chose I_* and J_* to satisfy the above hypotheses. Thus we now know that K_0 and K_1 are disjoint and separated by a positive distance (which is at least $\psi_t(\tau_1, \tau_2)$ times the distance between I_* and J_*).

Now suppose $0 \leq n < m$ and $m \geq 2$; we will prove (3.1) by finding a G and H which satisfy the hypotheses of Lemma 5. Assume without loss of generality that K_m lies to the left of K_n. Let I be the largest original gap in K_m; say I is a 1-gap. If all 1-gaps of K_n are smaller than I (Case 1 of Figure 8), let H be the largest original gap in K_n. Since $m > n$, K_n was constructed before K_n, so H must be at least as large as I, and thus is a 2-gap. Let $G = I$; then G and H satisfy the hypotheses of Lemma 5. Also, $d(G, H) > t|G|$, since otherwise G would have been included in the construction of K_n. If on the other hand there are 1-gaps of K_n which are at least as large as I (Cases 2 and 3 of Figure 8), let J be the closest such gap to I. Consider all 2-gaps of K_m or K_n to the left of J; let K be the largest such 2-gap (any one will do in case of a tie). Notice that K must be adjacent to I or J. If K is in K_n (Case 2), let $G = I$ and $H = K$; then G and H satisfy the hypotheses of Lemma 5, and $d(G, H) > t|G|$ because G was not included in K_n. Otherwise (Case 3), let $G = K$ and reverse the indices "1" and "2" (so that G is a 1-gap). If the 1-gap in K_n next to J on its right is at least as large as G, then let $H = J$; otherwise let H be the 2-gap in K_n on the left of the leftmost 1-gap in K_n

which is at least as large as G, or let H be the largest 2-gap in K_n if there is no 1-gap in K_n as large as G. Then once again, G and H satisfy the hypotheses of Lemma 5 and $d(G, H) > t|G|$. Notice also that in all cases, G is the largest 1-gap of K_m, and H is at least as large as all 2-gaps of K_m.

We now estimate how large K_m can be. Let I and J be the 2-gaps of K_m adjacent to G on its left and right, respectively. Since I is at most as large as H,

$$\tau_2|I| \le d(I, H) \le d(I, G) + |G| + d(G, H) \le t|I| + |G| + d(G, H),$$

or in other words

$$|I| \le \frac{1}{\tau_2 - t}(|G| + d(G, H)). \tag{3.3}$$

The same bound holds also for J, so by Lemma 4,

$$\begin{aligned} |K_m| &= |G| + l(G) - l(K_m) + r(K_m) - r(G) \\ &\le |G| + \sigma_2(l(G) - l(I)) + \sigma_2(r(J) - r(G)) \\ &\le |G| + \sigma_2(1+t)(|I| + |J|) \\ &\le |G| + 2\sigma_2\frac{1+t}{(\tau_2 - t)}(|G| + d(G, H)) \\ &\le \left(\frac{1}{t} + 2\sigma_2\frac{1+t}{(\tau_2 - t)}\left(\frac{1}{t} + 1\right)\right) d(G, H) \\ &= \frac{(\tau_1 - t)(\tau_2 - t) + (1+t)^2(2\tau_1 + 1)}{t((\tau_1 - t)(\tau_2 - t) - (1+t)^2)} d(G, H). \end{aligned} \tag{3.4}$$

If on the other hand G is a 2-gap and H is a 1-gap, we obtain the same bound as (3.4), but with the indices "1" and "2" interchanged. Then in either case,

$$|K_m| \le \frac{(\tau_1 - t)(\tau_2 - t) + (1+t)^2(2\max(\tau_1, \tau_2) + 1)}{t((\tau_1 - t)(\tau_2 - t) - (1+t)^2)} d(G, H).$$

Finally, by Lemma 5,

$$\frac{d(K_m, K_n)}{|K_m|} \ge \frac{t((\tau_1 - t)(\tau_2 - t) - (1+t)^2)\psi_t(\tau_1, \tau_2)}{(\tau_1 - t)(\tau_2 - t) + (1+t)^2(2\max(\tau_1, \tau_2) + 1)}. \tag{3.5}$$

The right side of (3.5) is positive as long as $\psi_t(\tau_1, \tau_2) > 0$ and t is between 0 and $(\tau_1\tau_2 - 1)/(\tau_1 + \tau_2 + 2)$, and it goes to zero when t approaches any of the borderline values. Therefore the right side of (3.5) attains a maximum value, call it $\varphi(\tau_1, \tau_2)$, at some allowable value of t, say t_*. We thus carry out the construction of S with $t = t_*$; then (3.1) holds, and the proof is complete. □

Let $\psi(\tau_1, \tau_2) = \psi_{t_*}(\tau_1, \tau_2)$; then

$$\varphi(\tau_1, \tau_2) = \frac{t_*((\tau_1 - t_*)(\tau_2 - t_*) - (1+t_*)^2)\psi(\tau_1, \tau_2)}{(\tau_1 - t_*)(\tau_2 - t_*) + (1+t_*)^2(2\max(\tau_1, \tau_2) + 1)}.$$

Remark. We will see in the proof of Lemma 5 that $\psi(\tau_1, \tau_2)$, and hence $\varphi(\tau_1, \tau_2)$, must be very small when (τ_1, τ_2) is near the boundary of region C. However, if both τ_1 and τ_2 are large and t is small compared with the two thicknesses, it is not hard to check that $\psi_t(\tau_1, \tau_2)$ is close to one. Then

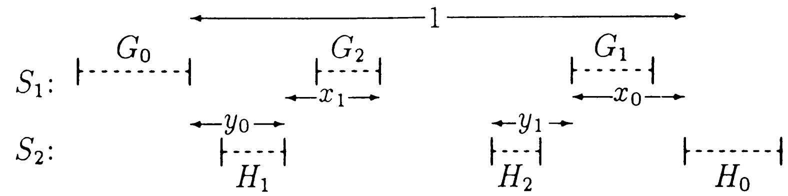

FIGURE 9. The gaps G_i and H_i

if $\tau_1, \tau_2 \gg 1$, one finds that t_* is of order $\sqrt{\min(\tau_1, \tau_2)}$, whence $\varphi(\tau_1, \tau_2)$ is of order $\sqrt{\min(\tau_1, \tau_2)}$ also. Thus when the thicknesses of S_1 and S_2 are large, the lower bound we obtain on the thickness of S is reasonably large.

We now prove our main technical lemma.

Proof of Lemma 5. Let G be a 1-gap of K_m and H be a 2-gap of K_n satisfying the hypotheses. We assume without loss of generality that $\tau_1 \geq \tau_2$; then by (1.1) the condition $(\tau_1, \tau_2) \in C$ implies

$$\tau_1 > f(\tau_2) = \frac{\tau_2^2 + 3\tau_2 + 1}{\tau_2^2} \tag{3.6}$$

and

$$\tau_2 > g(\tau_1) = \frac{(2\tau_1 + 1)^2}{\tau_1^3}. \tag{3.7}$$

If $d(G, H) = 0$, the inequality to be proven is trivial. Otherwise, let us normalize $d(G, H)$ to be one, and assume G lies to the left of H. Let $G_0 = G$ and $H_0 = H$. Let G_1 be the 1-gap of K_n adjacent to H_0 on its left, and let H_1 be the 2-gap of K_m adjacent to G_0 on its right. Let G_2 be the adjacent 1-gap of K_m rightward from H_1, and likewise define $H_2, G_3, H_3, \ldots$ (see Figure 9). For $i \geq 0$ let

$$x_i = \begin{cases} l(H_i) - l(G_{i+1}), & i \text{ even}, \\ r(G_{i+1}) - r(H_i), & i \text{ odd}, \end{cases}$$

and

$$y_i = \begin{cases} r(H_{i+1}) - r(G_i), & i \text{ even}, \\ l(G_i) - l(H_{i+1}), & i \text{ odd}. \end{cases}$$

Let $R_i = d(G_i, H_i)$; then $R_0 = 1$ and $R_{i+1} = \max(R_i - x_i - y_i, 0)$ for $i \geq 0$. Let R_∞ be the limit as i goes to infinity of R_i. Then $d(K_m, K_n) = R_\infty$, so we wish to show that there is a positive lower bound on R_∞ which depends only on τ_1, τ_2, and t.

In the same way as we obtained (3.2) it follows that for all i,

$$x_{i+1} \leq \frac{1+t}{\tau_1 - t} y_i \tag{3.8}$$

and

$$y_{i+1} \leq \frac{1+t}{\tau_2 - t} x_i. \tag{3.9}$$

Furthermore, by Lemma 4 we have that

$$y_i + x_{i+1} + y_{i+2} + \cdots \leq \sigma_2 y_i$$

and

$$x_i + y_{i+1} + x_{i+2} + \cdots \leq \sigma_1 x_i .$$

Thus, for each i,

$$R_\infty \geq R_i - x_i - y_i - x_{i+1} - y_{i+1} - \cdots \geq R_i - \sigma_1 x_i - \sigma_2 y_i . \tag{3.10}$$

We will show that for some i, the right side of (3.10) is positive.

Next let us obtain upper bounds on x_0 and y_0. We know that

$$x_0 = l(H_0) - l(G_1) \leq |G_1| + d(G_1, H_0) \leq (1+t)|G_1| , \tag{3.11}$$

and by hypothesis $|G_1| \leq |G_0|$, so

$$x_0 = l(H_0) - r(G_0) - (l(G_1) - r(G_0)) = 1 - d(G_0, G_1) \leq 1 - \tau_1 |G_1| . \tag{3.12}$$

Eliminating $|G_1|$ from these inequalities yields

$$x_0 \leq \frac{1+t}{\tau_1 + 1 + t} . \tag{3.13}$$

Similarly,

$$y_0 \leq \frac{1+t}{\tau_2 + 1 + t} . \tag{3.14}$$

We can obtain similar bounds on x_i and y_i for $i \geq 1$, but the bounds are complicated by the fact that we do not know in general that $|G_{i+1}| \leq |G_i|$ (or $|H_{i+1}| \leq |H_i|$). The analogues of (3.11) and (3.12) are thus

$$x_i \leq (1+t)|G_{i+1}|$$

and

$$x_i \leq R_i - \tau_1 \min(|G_i|, |G_{i+1}|) . \tag{3.15}$$

If $|G_{i+1}| \leq |G_i|$, then as in (3.13) it follows that

$$x_i \leq \frac{1+t}{\tau_1 + 1 + t} R_i . \tag{3.16}$$

If $|G_{i+1}| > |G_i|$, then by (3.15),

$$x_i \leq R_i - \tau_1 |G_i| \leq R_i - \frac{\tau_1}{1+t} x_{i-1} . \tag{3.17}$$

If (3.16) fails, then using (3.17) together with the negation of (3.16), one finds that x_{i-1} is bounded above by the right side of (3.16). Thus regardless of the relative lengths of G_i and G_{i+1},

$$\min(x_i x_{i-1}) \leq \frac{1+t}{\tau_1 + 1 + t} R_i , \tag{3.18}$$

for $i \geq 1$. Likewise, regardless of the relative lengths of H_i and H_{i+1}, we have for all $i \geq 1$ that

$$\min(y_i, y_{i-1}) \leq \frac{1+t}{\tau_2 + 1 + t} R_i . \tag{3.19}$$

Let $a_i = x_i / R_i$ and $b_i = y_i / R_i$ provided $R_i > 0$; then

$$R_{i+1} = \max(1 - a_i - b_i, 0) R_i .$$

Thus a_{i+1} and b_{i+1} are defined as long as $1-a_i-b_i>0$. For $j=1,2$ let

$$\lambda_j = \frac{1+t}{\tau_j+1+t}, \qquad \mu_j = \frac{1+t}{\tau_j - t}. \tag{3.20}$$

The conditions (3.13), (3.14), (3.18), and (3.19) can then be written

$$\begin{gathered} a_0 \le \lambda_1, \qquad b_0 \le \lambda_2, \\ \min\left(a_{i+1}, \frac{a_i}{1-a_i-b_i}\right) \le \lambda_1, \end{gathered} \tag{3.21}$$

and

$$\min\left(b_{i+1}, \frac{b_i}{1-a_i-b_i}\right) \le \lambda_2.$$

Also, conditions (3.8) and (3.9) become

$$a_{i+1} \le \mu_1 \frac{b_i}{1-a_i-b_i}$$

and

$$b_{i+1} \le \mu_2 \frac{a_i}{1-a_i-b_i}. \tag{3.22}$$

Finally, our objective is to show that for some i,

$$1-\sigma_1 a_i - \sigma_2 b_i > 0, \tag{3.23}$$

which implies that the right side of (3.10) is positive.

We observe that a_{i+1} and b_{i+1} are defined at least as long as $a_i \le \lambda_1$ and $b_i \le \lambda_2$, because then

$$\begin{aligned} 1-a_i-b_i \ge 1-\lambda_1-\lambda_2 &= \frac{\tau_1\tau_2-(1+t)^2}{(\tau_1+t+1)(\tau_2+t+1)} \\ &> \frac{(\tau_1-t)(\tau_2-t)-(1+t)^2}{(\tau_1+t+1)(\tau_2+t+1)} > 0 \end{aligned}$$

(since $t<(\tau_1\tau_2-1)/(\tau_1+\tau_2+2)$). Also, as long as $a_i\le\lambda_1$, by (3.22) we have

$$b_{i+1} \le \frac{\mu_2\lambda_1}{1-\lambda_1-b_i}.$$

Let

$$h(b) = \frac{\mu_2\lambda_1}{1-\lambda_1-b}.$$

The equation $h(b)=b$ has two solutions,

$$b_\pm = \frac{1-\lambda_1 \pm \sqrt{(1-\lambda_1)^2-4\mu_2\lambda_1}}{2},$$

and if the roots are real, then $h(b)<b$ for $b_- < b < b_+$ (this can be verified by checking the value $b=(1-\lambda_1)/2$). We claim that for t sufficiently small, $b_\pm$ are real, with

$$b_+ > \lambda_2 \tag{3.24}$$

and

$$1-\sigma_1\lambda_1 - \sigma_2 b_- > 0. \tag{3.25}$$

Let us delay the verification of this claim until the end of the proof. Choose $b_* > b_-$ with $1 - \sigma_1\lambda_1 - \sigma_2 b_* > 0$. Now $b_0 \le \lambda_2 < b_+$, and as long as $a_i \le \lambda_1$ continues to hold, $b_{i+1} \le h(b_i) < b_i$ for $b_i \in (b_-, b_+)$. Then eventually $b_i \le b_*$, and furthermore since $b - h(b)$ must have a positive minimum value on $[b_*, \lambda_2]$ (if $b_* > \lambda_2$ then $b_0 < b_*$ already) there is a maximum number N (depending only on τ_1, τ_2, and t) of iterations it can take before $b_i \le b_*$. We therefore have shown that if $a_i \le \lambda_1$ for $i \le N$, then $b_i \le b_*$ for some $i \le N$, and hence

$$1 - \sigma_1 a_i - \sigma_2 b_i \ge 1 - \sigma_1\lambda_1 - \sigma_2 b_* > 0. \tag{3.26}$$

If on the other hand $a_{i+1} > \lambda_1$ for some $i \le N$, then let i be the smallest index for which this occurs. We claim that then (3.23) holds for i. By the results of the previous paragraph, $b_i < b_{i-1} < \cdots < b_0 \le \lambda_2$. Also, by (3.21), $a_i \le \lambda_1(1 - a_i - b_i)$, or in other words

$$a_i \le \frac{\lambda_1}{1+\lambda_1}(1 - b_i).$$

Then

$$1 - \sigma_1 a_i - \sigma_2 b_i \ge 1 - \frac{\sigma_1\lambda_1}{1+\lambda_1} - \left(\sigma_2 - \frac{\sigma_1\lambda_1}{1+\lambda_1}\right) b_i.$$

Now when $t = 0$,

$$\begin{aligned}\sigma_2 - \frac{\sigma_1\lambda_1}{1+\lambda_1} &= \frac{(\tau_1+1)\tau_2}{\tau_1\tau_2 - 1} - \frac{\tau_1(\tau_2+1)}{(\tau_1+2)(\tau_1\tau_2-1)} \\ &= \frac{\tau_1(\tau_1\tau_2 - 1) + 2\tau_1\tau_2 + 2\tau_2}{(\tau_1+2)(\tau_1\tau_2-1)} > 0,\end{aligned}$$

and thus for t sufficiently small it remains positive. Then since $b_i \le \lambda_2$,

$$\begin{aligned}1 - \sigma_1 a_i - \sigma_2 b_i &\ge 1 - \frac{\sigma_1\lambda_1}{1+\lambda_1} - \left(\sigma_2 - \frac{\sigma_1\lambda_1}{1+\lambda_1}\right)\lambda_2 \\ &= 1 - \sigma_2\lambda_2 - (1-\lambda_2)\frac{\sigma_1\lambda_1}{1+\lambda_1}.\end{aligned} \tag{3.27}$$

When $t = 0$, by (3.6)

$$\begin{aligned}1 - \sigma_2\lambda_2 = 1 - \frac{(\tau_1+1)\tau_2}{(\tau_2+1)(\tau_1\tau_2-1)} &= \frac{\tau_1\tau_2^2 - 2\tau_2 - 1}{(\tau_2+1)(\tau_1\tau_2-1)} \\ > \frac{\tau_2^2 + 3\tau_2 + 1 - 2\tau_2 - 1}{(\tau_2+1)(\tau_1\tau_2-1)} &= \frac{\tau_2}{\tau_1\tau_2 - 1},\end{aligned}$$

while

$$(1-\lambda_2)\frac{\sigma_1\lambda_1}{1+\lambda_1} = \frac{\tau_2}{\tau_2+1}\cdot\frac{\tau_1(\tau_2+1)}{(\tau_1+2)(\tau_1\tau_2-1)} = \frac{\tau_1}{\tau_1+2}\cdot\frac{\tau_2}{\tau_1\tau_2-1},$$

so the right side of (3.27) is positive for $t = 0$. It therefore remains positive for t sufficiently small.

To summarize, we have shown that if t is sufficiently small, then for some $i \le N$, either (3.26) or (3.27) holds. The right side of each of these equations is positive and depends only on τ_1, τ_2, and t. Furthermore, $a_j \le \lambda_1$ and $b_j \le \lambda_2$

for $j \le i$, so by (3.20), $R_{j+1} \ge (1-\lambda_1-\lambda_2)R_j$, and hence $R_i \ge (1-\lambda_1-\lambda_2)^i$. Then by (3.10),

$$R_\infty \ge R_i(1-\sigma_1 a_i - \sigma_2 b_i) \ge (1-\lambda_1-\lambda_2)^N(1-\sigma_1 a_i - \sigma_2 b_i),$$

where $1-\sigma_1 a_i - \sigma_2 b_i$ is in turn bounded below by the lesser of the right sides of (3.26) and (3.27). We have therefore shown for t sufficiently small how to obtain a positive lower bound on R_∞ which depends only on τ_1, τ_2, and t; we let $\psi_t(\tau_1, \tau_2)$ be this lower bound.

It remains for us to verify (3.24) and (3.25). We again show they are true for $t=0$, whence they hold for t sufficiently small by continuity. When $t=0$,

$$\begin{aligned} b_\pm &= \frac{\tau_1/(\tau_1+1) \pm \sqrt{\tau_1^2/(\tau_1+1)^2 - 4/((\tau_1+1)\tau_2)}}{2} \\ &= \frac{\tau_1\tau_2 \pm \sqrt{\tau_1^2\tau_2^2 - 4(\tau_1+1)\tau_2}}{2(\tau_1+1)\tau_2}. \end{aligned} \tag{3.28}$$

Now by (3.6),

$$\begin{aligned} \tau_1^2\tau_2^2 - 4(\tau_1+1)\tau_2 &= \tau_1(\tau_1\tau_2^2 - 4\tau_2) - 4\tau_2 \\ &> \tau_1(\tau_2^2 - \tau_2 + 1) - 4\tau_2 \\ &> \frac{(\tau_2^2+3\tau_2+1)(\tau_2^2-\tau_2+1) - 4\tau_2^3}{\tau_2^2} \\ &= \frac{(\tau_2^2-\tau_2-1)^2}{\tau_2^2}. \end{aligned} \tag{3.29}$$

Thus $b_\pm$ are real and distinct (and the same must then hold for t sufficiently small). Next by (3.28),

$$b_+ = \frac{\tau_2 + \sqrt{\tau_2^2 - 4\tau_2(1/\tau_1 + 1/\tau_1^2)}}{2\tau_2(1+1/\tau_1)}$$

from which we see that b_+ is increasing as a function of τ_1. Thus b_+ is greater than the value it would take on if (3.6) were an equality, which owing to (3.29) means

$$\begin{aligned} b_+ &> \frac{(\tau_2^2+3\tau_2+1)/\tau_2 + |\tau_2^2-\tau_2-1|/\tau_2}{2(2\tau_2^2+3\tau_2+1)/\tau_2} \\ &\ge \frac{\tau_2^2+3\tau_2+1-(\tau_2^2-\tau_2-1)}{2(\tau_2+1)(2\tau_2+1)} \\ &= \frac{1}{\tau_2+1} = \lambda_2. \end{aligned}$$

Hence (3.24) holds for $t=0$, and consequently for t sufficiently small.

When $t=0$, (3.25) can be written

$$b_- < \frac{1-\sigma_1\lambda_1}{\sigma_2} = \frac{\tau_1^2\tau_2 - 2\tau_1 - 1}{(\tau_1+1)^2\tau_2}. \tag{3.30}$$

The right side of (3.30) is an increasing function of τ_2, and since

$$b_- = \frac{\tau_1 - \sqrt{\tau_1^2 - 4(\tau_1+1)/\tau_2}}{2(\tau_1+1)},$$

b_- is a decreasing function of τ_2. Then by (3.7),

$$\begin{aligned} b_- &< \frac{\tau_1 - \sqrt{\tau_1^2 - 4\tau_1^3(\tau_1+1)/(2\tau_1+1)^2}}{2(\tau_1+1)} \\ &= \frac{\tau_1((2\tau_1+1) - \sqrt{(2\tau_1+1)^2 - 4(\tau_1^2+\tau_1)}}{2(\tau_1+1)(2\tau_1+1)} \\ &= \frac{\tau_1^2}{(\tau_1+1)(2\tau_1+1)}, \end{aligned}$$

while

$$\begin{aligned} \frac{\tau_1^2\tau_2 - 2\tau_1 - 1}{(\tau_1+1)^2\tau_2} &> \frac{(2\tau_1+1)^2/\tau_1 - (2\tau_1+1)}{(\tau_1+1)^2(2\tau_1+1)^2/\tau_1^3} \\ &= \frac{\tau_1^2(2\tau_1+1-\tau_1)}{(\tau_1+1)^2(2\tau_1+1)} = \frac{\tau_1^2}{(\tau_1+1)(2\tau_1+1)}. \end{aligned}$$

Thus (3.25) holds for $t = 0$, and for t sufficiently small. The proof of Lemma 5 is now complete. □

4. Intersecting three or more Cantor sets

In proving Theorem 1, we chose a subset S of $S_1 \cap S_2$ in order to guarantee positive thickness. In this section we demonstrate that positive thickness sets are in some sense generic in $S_1 \cap S_2$. We also explain how Theorem 1 is useful in finding conditions under which three or more Cantor sets must have a nonempty intersection.

The set S we constructed in §3 need not be dense in $S_1 \cap S_2$ nor even in the nonisolated points of $S_1 \cap S_2$. However, there are subsets of $S_1 \cap S_2$ with thickness at least $\varphi(\tau_1, \tau_2)$ near any accumulation point. To see this, let $\{q_n\}$ be a sequence of distinct points in $S_1 \cap S_2$ which converge to a point q. It is not hard to show that within any neighborhood N of q there are compact subsets $T_1 \subset S_1$ and $T_2 \subset S_2$, each of which contains all but finitely many q_n, with $\tau(T_1) \geq \tau(S_1)$ and $\tau(T_2) \geq \tau(S_2)$. Notice that any two compact sets which intersect in three or more points must be interleaved. Thus T_1 and T_2 are interleaved, and by Theorem 1 their intersection contains a set with thickness at least $\varphi(\tau_1, \tau_2)$. We conclude that arbitrarily near any nonisolated point of $S_1 \cap S_2$ there are subsets of $S_1 \cap S_2$ which have thickness at least $\varphi(\tau_1, \tau_2)$.

In addition to showing that there are many subsets of $S_1 \cap S_2$ with positive thickness, it is possible to obtain a lower bound on the diameter of the positive thickness subset S of $S_1 \cap S_2$. If the two sets S_1 and S_2 are interleaved in such a way that neither is contained in the convex hull of the other, then by the discussion following the statement of Lemma 5, the diameter of S is at least $\psi(\tau_1, \tau_2)$ times the length of overlap between the convex hulls of S_1 and S_2. Since the thickness of S is at least $\varphi(\tau_1, \tau_2)$, we immediately have the following result.

Corollary 6. *Let S_1 and S_2 be two interleaved compact sets whose thicknesses (τ_1, τ_2) lie in region C and for which the intersection Q of their convex hulls contains neither S_1 nor S_2. If S_3 is a compact set with largest bounded gap G such that*

(i) *the hull of* S_3 *contains* Q,
(ii) $|G| < \psi(\tau_1, \tau_2)|Q|$,
(iii) $\tau(S_3)\varphi(\tau_1, \tau_2) \geq 1$,
then $S_1 \cap S_2 \cap S_3$ *is nonempty.*

We note that if instead of condition (iii) we required the pair $\tau(S_3)$ and $\varphi(\tau_1, \tau_2)$ to lie in C, then $S_1 \cap S_2 \cap S_3$ would contain a set of thickness at least $\varphi(\tau(S_3), \varphi(\tau_1, \tau_2))$. Thus one can inductively find thickness conditions guaranteeing the nonempty intersection of any finite (or even countably infinite) number of compact sets, although the analogue of the interleaving condition gets more complicated.

If (τ_1, τ_2) is sufficiently far from the boundary of region C, then as discussed in the remark preceding the proof of Lemma 5 it is not hard to obtain explicit lower bounds on $\varphi(\tau_1, \tau_2)$ and $\psi(\tau_1, \tau_2)$. In particular, for τ_1 and τ_2 large we found that $\varphi(\tau_1, \tau_2)$ is at least of order $\sqrt{\min(\tau_1, \tau_2)}$, and $\psi(\tau_1, \tau_2)$ is approximately one.

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