GAUSS MAP OF MINIMAL SURFACES WITH RAMIFICATION

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Abstract. We prove that for any complete minimal surface $M$ immersed in $\mathbb{R}^n$, if in $\mathbb{CP}^{n-1}$ there are $q > n(n+1)/2$ hyperplanes $H_j$ in general position such that the Gauss map of $M$ is ramified over $H_j$ with multiplicity at least $e_j$ for each $j$ and

$$\sum_{j=1}^{q} \left(1 - \frac{n-1}{e_j}\right) > n(n+1)/2,$$

then $M$ must be flat.

1. Introduction

Let $x: M \to \mathbb{R}^n$ be a (smooth, oriented) minimal surface immersed in $\mathbb{R}^n$. Make $M$ into a Riemann surface by decreeing that the 1-form $d\xi_1 + id\xi_2$ is of type $(1,0)$, where $(\xi_1, \xi_2)$ are any local isothermal coordinates of $M$. The Gauss map of $x$ is defined to be

$$G: M \to Q_{n-2}(C) \subset \mathbb{CP}^{n-1}, \quad G(z) = [(\partial x/\partial z)]$$

where $[(\cdot)]$ denotes the complex line in $\mathbb{C}^n$ through the origin and $(\cdot)$, $z = \xi_1 + i\xi_2$ is the holomorphic coordinate of $M$, and

$$Q_{n-2}(C) = \{(w_0 : \cdots : w_{n-1}: w_0^2 + \cdots + w_{n-1}^2 = 0) \subset \mathbb{CP}^{n-1}.$$

By the assumption of minimality of $M$, $G$ is a holomorphic map of $M$ into $\mathbb{CP}^{n-1}$. It is a natural question to study the “value distribution” properties of the Gauss map $G$. Fujimoto (see [8]) has shown that the Gauss map of a nonflat minimal surfaces can omit at most $n(n+1)/2$ hyperplanes in general position in $\mathbb{CP}^{n-1}$ under the assumption that $G$ is nondegenerate. The “nondegenerate” assumption was removed by the author (see [13]). The purpose of this paper is to study more general “value distribution” properties of the Gauss map. In particular, we study the Gauss map with ramification.

One says that $G$ is ramified over a hyperplane $H = \{[w] \in \mathbb{CP}^{n-1} : a_0w_0 + \cdots + a_{n-1}w_{n-1} = 0\}$ with multiplicity at least $e$ if all the zeros of the function $g_H = (G, A)$ have orders at least $e$, where $A = (a_0, \ldots, a_{n-1})$. If the image of $G$ omits $H$, we shall say that $G$ is ramified over $H$ with multiplicity $\infty$.

Our main result is the following:

Received by the editors July 10, 1991.

1991 Mathematics Subject Classification. Primary 53A10, 32H30.
Theorem 1. Let \( M \) be a complete minimal surface immersed in \( \mathbb{R}^n \) and assume that the Gauss map \( G \) of \( M \) is \( k \)-nondegenerate (that is \( G(M) \) is contained in a \( k \)-dimensional linear subspace of \( \mathbb{C}P^{n-1} \), but none of lower dimension), \( 1 \leq k \leq n - 1 \). Let \( H_i \subset \mathbb{C}P^{n-1} \) be \( q \) hyperplanes in general position. If \( G \) is ramified over \( H_i \) with multiplicity at least \( e_i \) for each \( i \). Then

\[
\sum_{j=1}^{q} \left( 1 - \frac{k}{e_j} \right) \leq (k + 1) \left( n - \frac{k}{2} - 1 \right) + n.
\]

In particular, for any complete minimal surface \( M \) immersed in \( \mathbb{R}^n \), if in \( \mathbb{C}P^{n-1} \) there are \( q > n(n + 1)/2 \) hyperplanes in general position such that its Gauss map \( G \) is ramified over \( H_j \) with multiplicity at least \( e_j \) for each \( j \) and

\[
\sum_{j=1}^{q} \left( 1 - \frac{(n-1)}{e_j} \right) > n(n + 1)/2,
\]

then \( M \) must be flat.

In the case \( m = 3 \), \( Q_1(C) \) can be identified with \( \mathbb{C}P^1 \). We have a better result.

Theorem 2. Let \( M \) be a complete minimal surface \((\subset \mathbb{R}^3)\). If there are \( q(q > 4) \) distinct points \( a_1, \ldots, a_q \in \mathbb{C}P^1 \) such that the Gauss map of \( M \) is ramified over \( a_j \) with multiplicity at least \( e_j \) for each \( j \) and \( \sum_{j=1}^{q}(1 - 1/e_j) > 4 \), then \( M \) must be flat.

In particular, if the Gauss map omits five distinct points, then \( M \) must be flat.

2. FACTS ON HOLONOMIC CURVES INTO PROJECTIVE SPACES

We shall recall some known results in the theory of holomorphic curves.

(A) Associated curve. Let \( f \) be a nondegenerated holomorphic map of \( \Delta_R: \{z: |z| < R\} \) into \( \mathbb{C}P^k \), where \( 0 < R < \infty \). Take a reduced representation 

\[
f = [Z_0: \cdots: Z_k], \text{ where } Z = (Z_0, \ldots, Z_k): \Delta_R \to \mathbb{C}^k+1 - \{0\}.
\]

Denote by \( Z^{(j)} \) the \( j \)th derivative of \( Z \) and define

\[
\Lambda_j = Z^{(0)} \wedge \cdots \wedge Z^{(j)}: \Delta_R \to \wedge^{j+1} \mathbb{C}^{k+1}
\]

for \( 0 \leq j \leq k \). Evidently \( \Lambda_{k+1} \equiv 0 \).

Let \( P: \wedge^{j+1} \mathbb{C}^{k+1} - \{0\} \to \mathbb{C}P^N_j \) denote the canonical projection, where \( N_j = \binom{k+1}{j+1} - 1 \). The \( j \)th associated curve of \( f \) is the map \( f_j = P(\Lambda_j) \).

It is well known [4] (also see [16]) that the pull-back \( \Omega_j \) of the Fubini-study metric on \( \mathbb{C}P^N_j \) by \( f_j \) is given by

\[
\Omega_j = dd^c \log |\Lambda_j|^2 = \frac{i}{2\pi} \frac{|\Lambda_{j-1}|^2 |\Lambda_{j+1}|^2}{|\Lambda_j|^4} dz \wedge d\bar{z},
\]

for \( 0 \leq j \leq k \) and by convention \( \Lambda_{-1} \equiv 1 \). Note that \( \Omega_k \equiv 0 \). It follows that

\[
\text{Ric } \Omega_j = \Omega_{j-1} + \Omega_{j+1} - 2\Omega_j.
\]
Take a hyperplane $H: (W, A) = 0$, where $A = (a_0, \ldots, a_k)$ is a unit vector. Define

$$\varphi_j(H) = \frac{|A_j \lor A|^2}{|A_j|^2 |A|^2}.$$ 

Note that $0 \leq \varphi_j(H) \leq \varphi_{j+1}(H) \leq 1$ for $0 \leq j \leq k$ and $\varphi_k(H) = 1$.

We need the following well-known lemma (see [4, 16 and 17]).

**Lemma 2.1.** Let $H$ be a hyperplane in $CP^k$, then for any constant $N > 1$, for $0 < p < k - 1$, 

$$(2.3) \quad \frac{1}{N - \log \varphi_p(H)} \geq \frac{\varphi_{p+1}(H)}{\varphi_p(H)(N - \log \varphi_p(H))} \left( \frac{1}{N} \right) \Omega_p,$$

on $\Delta_R - \{ 0 \} = \{ \varphi_p = 0 \}$.

(B) **Nochka weights and product to sum estimate.** We consider $q$ hyperplanes $H_j$ $(1 \leq j \leq q)$ in $CP^k$ which are given by $H_j: (W, A_j) = 0$. According to Chen [2], we give the following definition.

**Definition 2.2.** We say that hyperplanes $H_1, \ldots, H_q$ are in $n$-subgeneral position if, for every $1 \leq j_0 < \cdots < j_n \leq q$, $A_{j_0}, A_{j_1}, \ldots, A_{j_n}$ generate $C^{k+1}$.

In [11] (see also [2]), Nochka has given the following lemma to prove the Cartan conjecture.

**Lemma 2.3.** Let $H_1, \ldots, H_q$ be hyperplanes in $CP^k$ located in the $n$-subgeneral position, where $q > 2n - k + 1$. Then there are some constants $(\omega(1), \ldots, \omega(q)$ and $\theta$ satisfying the following condition:

(i) $0 < \omega(j) \theta \leq 1$ $(1 \leq j \leq q)$,

(ii) $\theta (\sum_{j=1}^{q} \omega(j) - k - 1) = q - 2n + k - 1$,

(iii) $1 \leq (n + 1)/(k + 1) \leq \theta \leq (2n - k + 1)/(k + 1)$,

(iv) if $R \subset Q$ and $0 < \# R \leq n + 1$, then $\sum_{j \in R} \omega(j) \leq d(R)$.

For the proof, see [2] or [11].

**Definition 2.4.** We call constants $\omega(j)$ $(1 \leq j \leq q)$ and $\theta$ above Nochka weights and a Nochka constant for $H_1, \ldots, H_q$ respectively.

Nochka weights are useful because of the following lemma.

**Lemma 2.5.** Under the above assumptions. Let $E_1, \ldots, E_q$ be a sequence of real numbers with $E_j \geq 1$ for all $j$. Then for any subset $B$ of the set $\{ 1, 2, \ldots, q \}$ with $0 < \# B \leq n + 1$, there exists a subset $C$ of $B$ such that $\{ A_j | j \in C \}$ is a base of the linear space spanned by $\{ A_j | j \in B \}$ and

$$\prod_{j \in B} E_j^{\omega(j)} \leq \prod_{j \in C} E_j,$$

where $\omega(j)$ are the Nochka weights associated to hyperplanes $H_j: (A_j, W) = 0$, $j = 1, 2, \ldots, q$.

For the proof, see [2] or [11].

We also have the following product to sum estimate.
Lemma 2.6 (see Chen [2]). Under the above assumptions. For \(0 < p \leq k - 1\), any constant \(N > 1\), \(1/q \leq \lambda_p \leq 1/(k - p)\), there exists a positive constant \(c_p > 0\) only depends on \(p\) and the given hyperplanes such that

\[
c_p \prod_{j=1}^{q} \left( \frac{\varphi_{p+1}(H_j)\omega(j)}{\varphi_p(H_j)} \frac{1}{(N - \log \varphi_p(H_j))^2} \right)^{\lambda_p} \leq \sum_{j=1}^{q} \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)(N - \log \varphi_p(H_j))^2},
\]

on \(\Delta_R - \{\varphi_p = 0\}\).

3. Metrics with negative curvature

We retain the notation of the last section. Let \(f: \Delta_R \to \mathbb{C}P^k\) be a nondegenerate holomorphic map. Take a reduced representation \(f = [Z_0 : \cdots : Z_k]\) where \(Z = (Z_0, \ldots, Z_k): \Delta_R \to \mathbb{C}^{k+1} - \{0\}\) is a holomorphic map. Let \(H_1, \ldots, H_q\) be hyperplanes in \(\mathbb{C}P^k\) located in \(n\)-subgeneral position. Let \(\omega(j)\) be their Nochka weights.

Let \(f\) be ramified over \(H_j\) with multiplicity at least \(e_j\) for each \(j\). Assume that

\[
\sum_{j=1}^{q} \left(1 - \frac{k}{e_j}\right) > 2n - k + 1,
\]

we shall construct a continuous pseudo-metric on \(\Delta_R\) such that its Gauss curvature is less than or equal to \(-1\). So that we can use Schwarz lemma to obtain our main inequality.

Let \(\Omega_p = \frac{i}{2\pi} h_p(z) dz \wedge d\bar{z}\). Let

\[
\sigma_p = c_p \prod_{j=1}^{q} \left( \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)} \right)^{\omega(j)(1-1/e_j)} \frac{1}{(N - \log \varphi_p(H_j))^2} h_p.
\]

Where \(c_p\) is the constant in the product to sum estimate,

\[
\lambda_p = 1/ \left( (k - p) + (k - p)^2 \frac{2q}{N} \right),
\]

and \(N > 1\).

We take the geometric mean of the \(\sigma_p\) and define

\[
\Gamma = i \frac{1}{2\pi} c \prod_{p=0}^{k-1} \sigma_p^{\beta_p/\lambda_p} dz \wedge d\bar{z}.
\]

where \(\beta_k = 1/(\sum_{p=0}^{k-1} \lambda_p^{-1})\), and \(c = 2(\prod_{p=0}^{k-1} \lambda_p^{\beta_p})\).

Let

\[
\Gamma = i \frac{1}{2\pi} h(z) dz \wedge d\bar{z}.
\]

We now compute \(h(z)\). By (3.1) and (3.2), we have

\[
h(z) = c \prod_{j=1}^{q} \frac{\varphi_0(H_j)^{\omega(j)(1-1/e_j)} \beta_k}{\varphi_0(H_j)} \prod_{p=0}^{k-1} \frac{h_p^{\beta_p/\lambda_p}}{(N - \log \varphi_p(H_j))}.\]
By (2.1),
\[ h_p^{1/\beta_p} = \left( \frac{|A_{p-1}|^2 |A_{p+1}|^2}{|A_p|^4} \right)^{(k-p)+(k-p)^2q/N}, \]
so
\[ \prod_{p=0}^{k-1} h_p^{1/\beta_p} = |A_0|^{-2(k+1)-(k^2+2k-1)4q/N} |A_1|^{8q/N} \cdots |A_{k-1}|^{8q/N} |A_k|^{2+4q/N}. \]

Notice that \( |A_0| = |Z| \), and \( \varphi_0(H_j) = |(Z, A_j)|^2/|Z|^2 \), therefore
\[ (3.5) \quad h(z) = c \left[ \frac{|Z|^{\sum_j \omega(j)(1-k/e_j)-(k+1)-(k^2+2k-1)q/N} |(Z, A_1)\cdots (Z, A_{k-1})|^{4q/N} |A_k|^{1+2q/N}}{\prod_{j=1}^q |(Z, A_j)|^{\omega(j)(1-k/e_j)} \prod_{p=0}^{k-1} (N - \log \varphi_p(H_j))} \right]^{2\beta_k}. \]

**Lemma 3.1.** The function
\[ \frac{|A_k|}{\prod_{j=1}^q |(Z, A_j)|^{\omega(j)(1-k/e_j)}} \]

is continuous on \( \Delta_R \).

**Proof.** We shall prove that the function
\[ P = \left[ \frac{|A_k|^2}{\prod_{j=1}^q \varphi_0(H_j)^{\omega(j)(1-k/e_j)}} \right]^e \]
is continuous where \( e = e_1 \cdots e_q \). Lemma 3.1 follows from this. According to the expression of \( P(z) \), we only need to consider the points at which \((Z, A_j)\) vanishes. For zero point \( z_0 \) of \((Z, A_j)\), since \( f \) is ramified over \( H_j \) with multiplicity at least \( e_j \) for each \( j \), we have
\[ (Z, A_j) = (z-z_0)^{\nu_j} Q_j(z) \]
where \( Q_j(z_0) \neq 0 \), and \( \nu_j \geq e_j \) or \( \nu_j = 0 \). The \( n \)-subgeneral position implies that, at each point \( z \), there are at most \( n \) of hyperplanes \( H_j \), such that \((Z(z), A_j) = 0 \). Thus there exists a constant \( c_0 \) (depending only on the given hyperplanes) such that
\[ \#B = \# \{ j \mid |(Z(z), A_j)|/|A_j||Z(z)| \leq c_0 \} \leq n. \]

Let \( E_j = 1/\varphi_0(H_j)^{\omega(j)(1-k/e_j)} \), then \( E_j \leq 1 \). If \( j \notin B \), then \( \varphi_0(H_j) > c_0 \), so \( E_j \leq c_1 \) (depending only on the given hyperplanes).

Applying Lemma 2.5 with \( E_j \) above, we obtain
\[ \frac{|A_k|^2}{\prod_{j=1}^q \varphi_0(H_j)^{\omega(j)(1-k/e_j)}} \leq c_2 \frac{|A_k|^2}{\prod_{j \in B} \varphi_0(H_j)^{\omega(j)(1-k/e_j)}} \leq c_2 \frac{|A_k|^2}{\prod_{j \in C} \varphi_0(H_j)^{(1-k/e_j)}}. \]

We may assume the index set \( C = \{ 1, 2, \ldots, l \} \) and \( l \leq k + 1 \), therefore
\[ \left[ \prod_{j \in C} (Z(z), A_j)^{(1-k/e_j)} \right]^e = (z-z_0)^b R(z) \]
where \( b = \sum_{j=1}^{l} e v_j (1 - k/e_j) \) and \( R \) is a holomorphic function such that \( R(z_0) \neq 0 \). Since

\[
|\Lambda_k| = \det \begin{vmatrix}
Z_0 & Z_1 & Z_2 & \cdots & Z_k \\
Z'_0 & Z'_1 & Z'_2 & \cdots & Z'_k \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Z^{(k)}_0 & Z^{(k)}_1 & Z^{(k)}_2 & \cdots & Z^{(k)}_k \\
(Z, A_1) & (Z, A_2) & (Z, A_3) & \cdots & \\
(Z, A_1)' & (Z, A_2)' & (Z, A_3)' & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(Z, A_1)^{(k)} & (Z, A_2)^{(k)} & (Z, A_3)^{(k)} & \cdots & \\
\end{vmatrix},
\]

we have \( \Lambda_k = (z - z_0)^{\nu} S(z) \), where \( \nu = \nu_1 + \nu_2 - 1 + \cdots + \nu_l - k \) and \( S \) is a holomorphic function. Hence we obtain

\[
P(z) \leq |(z - z_0)^{2p} T(z)|,
\]

where

\[
p = \frac{e_k}{e_1} + \frac{e}{e_2} (k \nu_2 - e_2) + \frac{e}{e_3} (k \nu_3 - 2e_3) + \cdots + \frac{e}{e_l} (k \nu_l - (l - 1)e_l) \geq 0,
\]

and \( T(z) \) is continuous at \( z_0 \). Therefore \( P(z) \) is bounded around \( z_0 \). Therefore \( P(z) \) is continuous. Q.E.D.

Lemma 3.2. If \( \sum_{j=1}^{q} (1 - k/e_j) \geq 2n - k + 2 \), and

\[
2q/N < \left( \sum_{j=1}^{q} \omega(j)(1 - k/e_j) - (k + 1) \right)/(k^2 + 2k),
\]

we have

(i) Ric \( \Gamma \geq \Gamma \) on \( \Delta_R - \bigcup \{ \varphi_0(H_j) = 0 \} \).

(ii) \( \Gamma \) is a continuous pseudo-metric on \( \Delta_R \).

Proof. From (3.3) and (3.4) it follows that

\[
\text{Ric } \Gamma = - \beta_k \sum_{j=1}^{k} \omega(j) \left( 1 - \frac{k}{e_j} \right) d^c \log \varphi_0(H_j)
+ \beta_k \sum_{j=1}^{k} \sum_{p=0}^{k-1} d^c \log(1/(N - \log \varphi_p(H_j))^2)
+ \beta_k \sum_{p=0}^{k-1} (1/\lambda_p) \text{Ric } \Omega_p .
\]

By Lemma 2.1, (2.2), and that \( d^c \log \varphi_0(H_j) = -\Omega_0 \), we have

\[
\text{Ric } \Gamma \geq \beta_k \left( \sum_{j=1}^{q} \omega(j) \left( 1 - \frac{k}{e_j} \right) \right) \Omega_0 + 2 \sum_{j=1}^{q} \sum_{p=0}^{k-1} \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)(N - \log \varphi_p(H_j))^2} \Omega_p
- \frac{2q}{N} \sum_{p=0}^{k-1} \Omega_p + \sum_{p=0}^{k-1} \left[ (k - p) + (k - p) \right] \frac{2q}{N} \left\{ \Omega_{p+1} - 2\Omega_p + \Omega_{p-1} \right\} .
\]
Using Lemma 2.6, we obtain
\[
\sum_{j=1}^{q} \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)(N - \log \varphi_p(H_j))^2} \Omega_p
\leq c_p \left[ \prod_{j=1}^{q} \left( \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)} \right)^{\omega(j)} \frac{1}{(N - \log \varphi_p(H_j))^2} \right]^{\lambda_p} \Omega_p
\geq \frac{i}{2\pi} \sigma_p dz \wedge d\bar{z}.
\]

We also notice that \( \Omega_k = 0 \) so that
\[
\sum_{p=0}^{k-1} (k-p)(\Omega_{p+1} - 2\Omega_p + \Omega_{p-1}) = -(k+1)\Omega_0
\]
and therefore
\[
\text{Ric} \Gamma \geq \beta_k \left( \sum_{j=1}^{q} \omega(j) \left( 1 - \frac{k}{e_j} \right) \Omega_0 + \frac{2i}{2\pi} \sum_{p=0}^{k-1} \sigma_p dz \wedge d\bar{z} - (k+1)\Omega_0 - (k^2 + 2k) \frac{2q}{N} \Omega_0 \right.
\]
\[
+ \sum_{p=1}^{k-2} \left( (k-p+1)^2 - 2(k-p)^2 + (k-p-1)^2 - 1 \right) \frac{2q}{N} \Omega_p + \frac{2q}{N} \Omega_{k-1} \bigg). \]

The following is an elementary inequality:
For all the positive numbers \( x_1, \ldots, x_n \) and \( a_1, \ldots, a_n \),
\[
(3.6) \quad a_1x_1 + \cdots + a_nx_n \geq (a_1 + \cdots + a_n)(x_1^{a_1} \cdots x_n^{a_n})^{1/(a_1 + \cdots + a_n)}.
\]

Letting \( a_p = \lambda_p^{-1} \) in (3.6), we have
\[
\sum_{p=0}^{k-1} \sigma_p \geq \frac{c}{2\beta_k} \sum_{p=0}^{k-1} \sigma_p^{\beta_p/\lambda_p}
\]
and therefore
\[
\text{Ric} \Gamma \geq \beta_k \left( \left( \sum_{j=1}^{q} \omega(j) \left( 1 - \frac{k}{e_j} \right) - (k+1) - (k^2 + 2k) \frac{2q}{N} \right) \right. \Omega_0
\]
\[
+ \sum_{p=0}^{k-2} \frac{2q}{N} \Omega_p + \frac{2q}{N} \Omega_{k-1} \bigg) + \Gamma.
\]

By Lemma 2.2, we find
\[
\theta \left( \sum_{j=1}^{q} \omega(j) \left( 1 - \frac{k}{e_j} \right) - k - 1 \right) = \theta \left( \sum_{j=1}^{q} \omega(j) - k - 1 \right) - \frac{\sum_{j=1}^{q} \omega(j) \theta k}{e_j}
\]
\[
= q - 2n + k - 1 - \frac{\sum_{j=1}^{q} \omega(j) \theta k}{e_j} \geq q - 2n + k - 1 - \frac{k}{e_j}
\]
\[
= \sum_{j=1}^{q} \left( 1 - \frac{k}{e_j} \right) - 2n + k - 1 > 0
\]
and $\theta > 0$, so
\[ \sum_{j=1}^{q} \omega(j) \left( 1 - \frac{k}{e_j} \right) - (k + 1) > 0. \]

This implies $\text{Ric} \Gamma \geq \Gamma$. Thus (i) is satisfied.

(ii) follows from Lemma 3.1, (3.3) and (3.5). Q.E.D.

We recall the following generalization of the Schwarz lemma.

**Lemma 3.3.** Let $\Gamma = \frac{1}{2\pi} \int h(z) \, dz \wedge dz$ be a continuous pseudo-metric on $\Delta_R$ whose curvature is bounded above by a negative constant. Then, for some positive $c_0$, $h(z) \leq c_0 \left( \frac{2R}{(R^2 - |z|^2)} \right)^2$. For the proof, see [1, pp. 12–14].

The purpose of this section is to obtain the following lemma.

**Main Lemma.** Let $f = [Z_0 : \cdots : Z_k] : \Delta_R \to \mathbb{C} P^k$ be a nondegenerate holomorphic map, $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{C} P^k$ in n-subgeneral position, $\omega(j)$ be their Nochka weights. Let $H_j : (W, A_j) = 0$ and $Z = (Z_0, \ldots, Z_k)$. If $f$ is ramified over $H_j$ with multiplicity at least $e_j$ for each $j$, $\sum_{j=1}^{q} (1 - k/e_j) > 2n - k + 1$ and $N > 2q(k^2 + 2k)/\left( \sum_{j=1}^{q} \omega(j)(1 - k/e_j) - (k + 1) \right)$, then there exists a positive constant $c$ such that
\[ |Z| \sum_{j=1}^{q} \omega(j)(1-k/e_j) - (k+1) - (2k^2+2k-1)2q/N \prod_{p=0}^{k-1} \prod_{j=1}^{q} |A_p \vee A_j|^{4/N} |A_k|^{1+2q+N} \prod_{j=1}^{q} |(Z, A_j)^{\omega(j)(1-k/e_j)}| \leq c \left( \frac{2R}{(R^2 - |z|^2)} \right)^{k(k+1)/2 + \sum_{p=0}^{k-1} (k-p)^2 2q/N}. \]

**Proof.** Using the above Schwarz lemma for $\Gamma$, we obtain
\[ h(z) \leq c_0 \left( \frac{2R}{(R^2 - |z|^2)} \right)^2. \]

So by (3.5) we have
\[ |Z| \sum_{j=1}^{q} \omega(j)(1-k/e_j) - (k+1) - (2k^2+2k-1)2q/N \prod_{p=0}^{k-1} \prod_{j=1}^{q} |A_p \vee A_j|^{4/N} |A_k|^{1+2q+N} \prod_{j=1}^{q} |(Z, A_j)^{\omega(j)(1-k/e_j)}| \leq c_0 \left( \frac{2R}{R^2 - |z|^2} \right)^{1/\beta_k}. \]

Set $K := \sup_{0 < x \leq 1} x^{2/N} (N - \log x)$. Since $\varphi_p(H_j) < 1$ for all $p$ and $j$ we have
\[ \frac{1}{(N - \log \varphi_p(H_j))} \geq \frac{1}{K} \varphi_p(H_j)^{2/N} = \frac{1}{K} \frac{|A_p \vee A_j|^{4/N}}{|A_p|^{4/N}}. \]

Substituting these into (3.7), we obtain the desired conclusion.

**4. Proof of Theorem 1**

The proof of Theorem 1 basically follows the argument in [13] using the main lemma (see also the arguments in [6, 7 and 8]). We include our proof here for the convenience of the reader.

We may assume $M$ is simply connected, otherwise we consider its universal covering. By Koebe’s uniformization theorem, $M$ is biholomorphic to $C$ or to the unit disc. For the case $M = C$, Nochka (see [10], also see [16]) proved
that if a $k$-nondegenerate holomorphic map from $C$ to $CP^{n-1}$ is ramified over hyperplanes $H_j$ ($1 \leq j \leq q$) with multiplicity at least $e_j$, where $H_j$ are in general position, then

$$\sum_{j=1}^{q} \left(1 - \frac{k}{e_j}\right) \leq 2(n - 1) - k + 1;$$

in this case our Theorem 1 is true. For our purpose it suffices to consider the case $M = \Delta$.

We first prove the first part of Theorem 1.

Assume the first part of Theorem 1 is not true, namely $G$ is ramified over hyperplanes $H_1, \ldots, H_q$ in $CP^{n-1}$ in general position with multiplicity $e_j$ and

$$\sum_{j=1}^{q} (1 - k/e_j) > (k + 1)(n - k/2 - 1) + n.$$

Let $\omega(j)$ be Nochka weights of $\{H_j\}$. Because $G$ is $k$-nondegenerate, we may assume $G(\Delta) \subset CP^k$, so that $G = [g_0 : \cdots : g_k]: \Delta \to CP^k$ is nondegenerate. We consider hyperplanes $H_j \cap CP^k$, obviously these hyperplanes are in $(n - 1)$-subgeneral position in $CP^k$. For the convenience, we still denote these hyperplanes by $\{H_j\}$.

Let $\tilde{G} = (g_0, \ldots, g_k): \Delta \to CP^{k+1} - \{0\}$; then the metric $ds^2$ on $M$ induced from the standard metric on $R^n$ is given by

$$ds^2 = 2|\tilde{G}|^2|dz|^2.$$

By Lemma 2.2,

$$q - 2(n - 1) + k - 1 = \theta \left(\sum_{j=1}^{q} \omega(j) - k - 1\right), \quad 0 < \omega(j) \theta \leq 1,$$

and

$$\theta \leq \frac{2(n - 1) - k + 1}{k + 1} = \frac{2n - k - 1}{k + 1},$$

so

$$2 \left(\sum_{j=1}^{q} \omega(j) \left(1 - \frac{k}{e_j}\right) - k - 1\right) = \frac{2\theta \left(\sum_{j=1}^{q} \omega(j) - k - 1\right)}{\theta} - 2 \sum_{j=1}^{q} \frac{k \omega(j) \theta}{\theta e_j}$$

$$= \frac{2(q - 2n + k + 1)}{\theta} - 2 \sum_{j=1}^{q} \frac{k \omega(j) \theta}{\theta e_j}$$

$$\geq \frac{2(q - 2n + k + 1)}{\theta} - 2 \sum_{j=1}^{q} \frac{k}{\theta e_j}$$

$$= \frac{2 \left(\sum_{j=1}^{q} (1 - k/e_j) - 2n + k + 1\right)}{\theta}$$

$$\geq \frac{2 \left(\sum_{j=1}^{q} (1 - k/e_j) - 2n + k + 1\right) (k + 1)}{(2n - k - 1)}$$

$$> k(k + 1) \quad (by \ (4.1)).$$
Consider numbers

\[
\rho = \frac{k(k+1)/2 + \sum_{p=0}^{k-1} (k-p)^2 2q/N}{\sum_{j=1}^{q} \omega(j)(1-k/e_j) - (k+1) - (k^2 + 2k - 1)2q/N},
\]

(4.3)

\[
\gamma = \frac{k(k+1)/2 + qk(k+1)/N + 2q/N \sum_{p=0}^{k-1} p(p+1)}{\sum_{j=1}^{q} \omega(j)(1-k/e_j) - (k+1) - (k^2 + 2k - 1)2q/N},
\]

(4.4)

\[
\delta = \frac{1}{(1-\gamma) \left( \sum_{j=1}^{q} \omega(j)(1-k/e_j) - (k+1) - (k^2 + 2k - 1)2q/N \right)}.
\]

(4.5)

Choose some \( N \) with

\[
\sum_{j=1}^{q} \omega(j)(1-k/e_j) - (k+1) - k(k+1)/2 > 2q/N > \frac{k^2 + 2k - 1 + \sum_{p=0}^{k} (k-p)^2}{1/q + (k^2 + 2k - 1) + k(k+1)/2 + \sum_{p=0}^{k-1} p(p+1)}
\]

so that

\[
0 < \rho < 1, \quad 2\delta/N > 1.
\]

(4.6)

Consider the open subset

\[
M' = M - \left( \{ \tilde{G}_k = 0 \} \cup \bigcup_{1 \leq j \leq q, \; 0 \leq p \leq k-1} \{ \tilde{G}_p \lor A_j = 0 \} \right)
\]

of \( M \) and define the function

\[
v = \left( \frac{\prod_{j=1}^{q} \lvert (G, A_j)^{[\omega(j)(1-k/e_j)]} \rvert}{\prod_{p=0}^{k-1} \prod_{j=1}^{q} \lvert \tilde{G}_p \lor A_j \rvert^{4/N} \lvert \tilde{G}_k \rvert^{1+2q/N}} \right)^\delta
\]

on \( M' \), where \( \tilde{G}_p = \tilde{G}^{(0)} \land \cdots \land \tilde{G}^{(p)} \). By Lemma 3.1, \( \nu(z) \) is strictly positive and continuous on \( M' \).

Let \( \pi: \tilde{M}' \to M' \) be the universal covering of \( M' \). Since \( \log v \circ \pi \) is harmonic on \( \tilde{M}' \) by the assumption, we can take a holomorphic function \( \beta \) on \( \tilde{M}' \) such that \( \lvert \beta \rvert = \nu \circ \pi \). Without loss of generality, we may assume that \( M' \) contains the origin 0 of \( C \). As in Fujimoto’s paper [6, 7, 8], for each point \( \tilde{p} \) of \( \tilde{M}' \) we take a continuous curve \( \gamma_{\tilde{p}}: [0, 1] \to M' \) with \( \gamma_{\tilde{p}}(0) = 0 \) and \( \gamma_{\tilde{p}}(1) = \pi(\tilde{p}) \), which corresponds to the homotopy class of \( \tilde{p} \). Let \( \tilde{0} \) denote the point corresponding to the constant curve 0. Set

\[
w = F(\tilde{p}) = \int_{\gamma_{\tilde{p}}} \beta(z) \, dz.
\]

Then \( F \) is a single-valued holomorphic function on \( M' \) satisfying the condition \( F(\tilde{0}) = 0 \) and \( dF(\tilde{p}) \neq 0 \) for every \( \tilde{p} \in \tilde{M}' \). Choose the largest \( R \) (\( \leq \infty \)) such that \( F \) maps an open neighborhood \( U \) of \( \tilde{0} \) biholomorphically onto an open disc \( \Delta_R \) in \( C \), and consider the map \( B = \pi \circ (F|U)^{-1}: \Delta_R \to M' \). By the Liouville theorem, \( R = \infty \) is impossible.
By the definition of $w = F(z)$ we have

\[(4.7) \quad |dw/dz| = v(z).\]

For each point $a \in \partial \Delta$ consider the curve

$$L_a : w = ta, \quad 0 \leq t < 1,$$

and the image $\Gamma_a$ of $L_a$ by $B$. We shall show that there exists a point $a_0$ in $\partial \Delta_R$ such that $\Gamma_{a_0}$ tends to the boundary of $M$. To this end, we assume the contrary. Then, for each $a \in \partial \Delta_R$, there is a sequence \( \{t_\nu : \nu = 1, 2, \ldots\} \) such that $\lim_{\nu \to \infty} t_\nu = 1$ and $z_0 = \lim_{\nu \to \infty} B(t_\nu a)$ exist in $M$. Suppose that $z_0 \notin M'$. Let $\delta_0 = 4\delta/N > 1$. Then by Lemma 3.1, we have

$$\liminf_{z \to z_0} |\tilde{G}_k|^{\delta_0} \prod_{1 \leq j \leq q, \ 1 \leq p \leq k-1} |\tilde{G}_p \lor A_j|^{2\delta_0} \cdot v > 0.$$

If $\tilde{G}_k(z_0) = 0$ or $|\tilde{G}_p \lor A_j|(z_0) = 0$ for some $p$ and $j$, we can find a positive constant $c$ such that $v \geq c/|z - z_0|^{\delta_0}$ in a neighborhood of $z_0$, so that we obtain

$$R = \int_{L_a} |dw| = \int_{L_a} \left|\frac{dw}{dz}\right| dz = \int v(z) |dz| \geq c \int_\Gamma \frac{1}{|z - z_0|^{\delta_0}} |dz| = \infty.$$

This is a contradiction. Therefore, we have $z_0 \in M'$.

Take a simply connected neighborhood $V$ of $z_0$ which is relatively compact in $M'$. Set $C' = \min_{z \in V} v(z) > 0$. Then $B(ta) \in V$ ($t_0 < t < 1$) for some $t_0$. In fact, if not, $\Gamma_a$ goes and returns infinitely often from $\partial V$ to a sufficiently small neighborhood of $z_0$ and so we get the absurd conclusion

$$R = \int_{L_a} |dw| \geq C' \int_{\Gamma_a} |dz| = \infty.$$

By the same argument, we can easily see that $\lim_{t \to 1} B(ta) = z_0$. Since $\pi$ maps each connected component of $\pi^{-1}(V)$ biholomorphically onto $V$, there exists the limit

$$\tilde{p}_0 = \lim_{t \to 1} (F|U)^{-1}(ta) \in \tilde{M}'.$$

Thus $(F|U)^{-1}$ has a biholomorphic extension to a neighborhood of $a$. Since $a$ is arbitrarily chosen, $F$ maps an open neighborhood of $\tilde{U}$ biholomorphically onto an open neighborhood of $\tilde{\Delta}_R$. This contradicts the property of $R$. In conclusion, there exists a point $a_0 \in \partial \Delta_R$ such that $\Gamma_{a_0}$ tends to the boundary of $M$. 

Our goal is to show that $\Gamma_{a_0}$ has finite length, contradicting the completeness of the given minimal surface $M$.

By (4.7) we obtain $|dw/dz| = v(z)$. So

\[(4.8) \quad \left|\frac{dw}{dz}\right| = |v(z)|^{1-\gamma} \left|\frac{dw}{dz}\right|^{\gamma} = \left(\frac{\prod_{j=1}^{q} |(\tilde{G}_j, A_j)|^{\omega(j)(1-k/j)}}{\prod_{p=0}^{k-1} \prod_{j=1}^{q} |\tilde{G}_p \lor A_j|^{4/N} |\tilde{G}_k|^{1+2q/N}}\right)^{1/\left(\sum \omega(j)(1-k/j) -(k+1) -(k^2+2k-1)2q/N\right)} \left|\frac{dw}{dz}\right|^{\gamma}.

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Let \( Z(w) = \tilde{G} \circ B(w) \), \( Z_0(w) = g_0 \circ B(w) \), \ldots , \( Z_k(w) = g_k \circ B(w) \). Then because

\[
Z \land Z' \land \cdots \land Z^{(p)} = (\tilde{G} \land \cdots \land \tilde{G}^{(p)}) \left( \frac{dz}{dw} \right)^{p+1/2},
\]

it is easy to see that

\[
\left( \frac{d}{dw} \right)^{1/2} \left( \sum_{j=1}^{k+1} \omega_j \right) = (\tilde{G} \land \cdots \land \tilde{G}^{(p)}) \left( \frac{dz}{dw} \right)^{p+1/2},
\]

where \( \Lambda_p = Z^{(0)} \land \cdots \land Z^{(p)} \).

On the other hand, the metric on \( \Delta_R \) induced from \( ds^2 = 2|\tilde{G}|^2 |dz|^2 \) through \( B \) is given by

\[
B^*ds^2 = 2|\tilde{G}(B(w))|^2 \left| \frac{dz}{dw} \right|^2 |dw|^2.
\]

Combining (4.7) and (4.8) gives

\[
B^*ds = 2|Z| \left( \frac{\Pi^{k-1}_{p=0} |\Lambda_p \land A_j|^{4/n} |\Lambda_k|^{1+2q/N}}{\Pi^{q}_{j=1} |(Z, A_j)|^{\omega(j)(1-k/e_j)}} \right)^{1/2} \left| \sum_{j=1}^{k+1} \omega(j)(1-k/e_j)-(k+1)-(k+2k-1)2q/N \right| |dw|.
\]

Using the main lemma, we have

\[
B^*ds \leq c \left( \frac{2R}{R^2 - |w|^2} \right)^\rho |dw|,
\]

where \( c \) is a positive constant. Since \( \rho < 1 \), it then follows that

\[
d(0) \leq \int_{\Gamma_{00}} B^*ds = \int_{L_{00}} B^*ds \leq c \int_{0}^R \left( \frac{2R}{R^2 - |w|^2} \right)^\rho |dw| < \infty,
\]

where \( d(0) \) denotes the distance from the origin 0 to the boundary of \( M \). This contradicts the assumption of completeness of \( M \). Hence the proof of the first part of Theorem 1 is complete.

We now prove the second part.

For any complete minimal surface \( M \) immersed in \( R^n \), if there are \( q > n(n + 1)/2 \) hyperplanes in general position in \( CP^{n-1} \) such that its Gauss map \( G \) is ramified over \( H_j \) with multiplicity at least \( \varepsilon_j \) for each \( j \) and

\[
\sum_{j=1}^{q} (1 - n/e_j) > n(n + 1)/2,
\]

we are going to prove that \( M \) is flat. Since \( M \) is flat if and only if its Gauss map is a constant map (see [12]), we only need to prove that \( G \) is a constant map.

If \( G \) is not a constant map, then we may assume that \( G \) is \( k \)-nondegenerate and \( 1 \leq k \leq n-1 \). By the first part of the theorem, we have

\[
\sum_{j=1}^{q} (1 - k/e_j) \leq (k+1)(n-k/2-1) + n.
\]

Since

\[
(k+1)(n-k/2-1) + n \leq n(n+1)/2,
\]
and
\[ \sum_{j=1}^{q} \left( 1 - \frac{n-1}{e_j} \right) \leq \sum_{j=1}^{q} \left( 1 - \frac{k}{e_j} \right), \]
we obtain
\[ \sum_{j=1}^{q} \left( 1 - \frac{n-1}{e_j} \right) \leq n(n+1)/2. \]
This contradicts the assumption. Therefore \( M \) is flat. Q.E.D.

5. Proof of Theorem 2

Let \( x = (x_1, x_2, x_3) : M \to \mathbb{R}^3 \) be a nonflat minimal surface and \( g : M \to CP^1 \) the Gauss map. Assume \( M = \Delta \) (as the argument above). Set \( \varphi_i = \frac{\partial x_i}{\partial z} \) \( (i = 1, 2, 3) \) and \( f = \varphi_1 - \sqrt{-1} \varphi_2 \). Then according to [12] or [7], the metric on \( M \) induced from \( \mathbb{R}^3 \) is given by
\[ ds^2 = |f|^2 (1 + |g|^2)^2 |dz|^2. \]
Take a reduced representation \( \tilde{g} = (g_0, g_1) \) of \( g \) on \( M \). Then we can rewrite
\[ ds^2 = |h|^2 |	ilde{g}|^4 |dz|^2, \]
where \( h = f/g_0^2 \), and moreover \( h \neq 0 \). The rest of the steps are the same as the proof of Theorem 1. If \( M \) is not flat, then \( g \) is not a constant map. Assume that \( g \) is ramified over \( a_j \) with multiplicity of \( e_j \) and \( \sum_{j=1}^{q} (1 - 1/e_j) > 4 \), we shall derive a contradiction. Let \( P(a_j) = a_j, \alpha_j \in C^2 \). Consider numbers
\[ \rho = \gamma = \frac{1 + 2q/N}{\sum_{j=1}^{q} (1 - 1/e_j) - 2 - 2q/N}, \]
\[ \delta = \frac{1}{(1 - \rho) \left( \sum_{j=1}^{q} (1 - 1/e_j) - 2 - 2q/N \right)}. \]
Choose some \( N \) with
\[ \frac{\sum_{j=1}^{q} (1 - 1/e_j) - 3}{3} > 2q/N > \frac{\sum_{j=1}^{q} (1 - 1/e_j) - 3}{3 + 1/q} \]
so that \( 0 < 2\rho < 1, \frac{2\delta}{N} > 1 \). Consider the open subset \( M' = M - \{ \tilde{g}_1 = 0 \} \) of \( M \) and define the function
\[ v = h^{1/(1-\gamma)} \left( \prod_{j=1}^{q} |(\tilde{g}, \alpha_j)|^{(1-1/e_j-4/N)} \right)^{\delta} \]
on \( M' \) where \( \tilde{g}_1 = \tilde{g} \wedge \tilde{g}' \).
By exactly the same argument as in the proof of Theorem 1, we can find a curve \( \Gamma_{a_0} \) tends to the boundary of \( M \), and we can estimate the pull-back metric, eventually we obtain that \( \Gamma_{a_0} \) has finite length, contradicting the completeness of the given minimal surface \( M \). Q.E.D.
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