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THE GORENSTEINNESS OF THE SYMBOLIC BLOW-UPS FOR CERTAIN SPACE MONOMIAL CURVES

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ABSTRACT. Let $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$ denote the prime ideal in the formal power series ring $A = k[[X, Y, Z]]$ over a field k defining the space monomial curve $X = T^{n_1}$, $Y = T^{n_2}$, and $Z = T^{n_3}$ with $\mathrm{GCD}(n_1, n_2, n_3) = 1$. Then the symbolic Rees algebras $R_s(\mathbf{p}) = \bigoplus_{n\geq 0} \mathbf{p}^{(n)}$ are Gorenstein rings for the prime ideals $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$ with $\min\{n_1, n_2, n_3\} = 4$ and $\mathbf{p} = \mathbf{p}(m, m+1, m+4)$ with $m \neq 9, 13$. The rings $R_s(\mathbf{p})$ for $\mathbf{p} = \mathbf{p}(9, 10, 13)$ and $\mathbf{p} = \mathbf{p}(13, 14, 17)$ are Noetherian but non-Cohen-Macaulay, if $\mathrm{ch}\, k = 3$.

1. INTRODUCTION

Let k be a field and let $A = k[[X, Y, Z]]$ and $S = k[[T]]$ be formal power series rings over k. Let $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$ denote, for positive integers n_1, n_2 and n_3 with $\mathrm{GCD}(n_1, n_2, n_3) = 1$, the kernel of the homomorphism $f\colon A \to S$ of k-algebras defined by $f(X) = T^{n_1}$, $f(Y) = T^{n_2}$, and $f(Z) = T^{n_3}$. We put $R_s(\mathbf{p}) = \sum_{n\geq 0} \mathbf{p}^{(n)} t^n$ (here t denotes an indeterminate over A) and call it the symbolic Rees algebra of $\mathbf{p}$.

In the previous paper [1] the authors studied the problem when $R_s(\mathbf{p})$ is a Gorenstein ring and gave a criterion for the case in terms of the elements f and g of $\mathbf{p}$ in Huneke's condition [6] for $R_s(\mathbf{p})$ to be Noetherian. With the criterion the authors proved that $R_s(\mathbf{p})$ are always Gorenstein for the prime ideals $\mathbf{p} = \mathbf{p}(m, m+1, m+3)$ with $m \geq 1$ and $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$ with $\min\{n_1, n_2, n_3\} = 3$.

To be the next targets we would like to choose the prime ideals $\mathbf{p} = \mathbf{p}(m, m+1, m+4)$ with $m \geq 1$ and $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$ with $\min\{n_1, n_2, n_3\} = 4$, and our conclusion for these ideals can be summarized into the following two theorems.

Theorem (1.1). *$R_s(\mathbf{p})$ is a Gorenstein ring for $\mathbf{p} = \mathbf{p}(m, m+1, m+4)$, if $m \neq 9, 13$.*

Theorem (1.2). *$R_s(\mathbf{p})$ is a Gorenstein ring for $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$, if*

$$\min\{n_1, n_2, n_3\} = 4.$$

In Theorem (1.2) the fact that $R_s(\mathbf{p})$ is Noetherian is due to [6]. Our contribution is its Gorensteinness. For $m = 9, 13$ in Theorem (1.1) the rings $R_s(\mathbf{p})$ are Noetherian but not Cohen-Macaulay, if $\mathrm{ch}\, k = 3$ (cf. [7] and (3.4)).

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Theorem (1.1) (resp. Theorem (1.2)) shall be proved in §3 (resp. §4). Section 2 is devoted to some preliminary steps. In his remarkable paper [6] Huneke gave a criterion for $R_s(\mathbf{p})$ to be Noetherian, by which he guaranteed the Noetherian property of $R_s(\mathbf{p})$ for $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$ with $\min\{n_1, n_2, n_3\} = 4$. To prove Theorem (1.2) we need his arguments as well as his results (that we will briefly summarize in §4). However the key is the criterion given by the authors [1] for $R_s(\mathbf{p})$ to be a Gorenstein ring, which we will recall in §2 for the sake of completeness.

Throughout this paper let $(A, \mathbf{m})$ be a regular local ring of $\dim A = 3$ and $\mathbf{p}$ a prime ideal in A with $\dim A/\mathbf{p} = 1$. For each finitely generated A-module M let $l_A(M)$ and $\mu_A(M)$ respectively denote the length of M and the number of elements in a minimal system of generators for M.

2. Preliminaries

First of all let us recall Huneke's criterion.

Proposition (2.1) [6]. *If there exist $f \in \mathbf{p}^{(k)}$ and $g \in \mathbf{p}^{(l)}$ with positive integers k, l such that $l_A(A/(f, g, x)A) = kl \cdot l_A(A/\mathbf{p} + xA)$ for some $x \in \mathbf{m}\backslash\mathbf{p}$, then $R_s(\mathbf{p})$ is Noetherian. When the field $A/\mathbf{m}$ is infinite, the converse is also true.*

The criterion given by the authors for $R_s(\mathbf{p})$ to be a Gorenstein ring is based on (2.1) and is stated as follows.

Theorem (2.2) [1]. *Let f and g be as in (2.1). Then the following two conditions are equivalent.*

(1) *$R_s(\mathbf{p})$ is a Gorenstein ring.*

(2) *$A/(f, g) + \mathbf{p}^{(n)}$ is a Cohen-Macaulay ring for any $1 \leq n \leq k + l - 2$.*

When this is the case, the A-algebra $R_s(\mathbf{p})$ is generated by $\{\mathbf{p}^{(n)}t^n\}_{1 \leq n \leq k+l-2}$, ft^k and gt^l, and the rings $A/(f) + \mathbf{p}^{(n)}$, $A/(g) + \mathbf{p}^{(n)}$ and $A/(f, g) + \mathbf{p}^{(n)}$ are Cohen-Macaulay for all $n \geq 1$.

Here let us note the following lemma that we will use to calculate the length of certain modules.

Lemma (2.3) [1]. *Let R be a two-dimensional Cohen-Macaulay local ring and let x, y be a system of parameters of R. For given sequences $p_0 = 0 < p_1 \leq p_2 \leq \cdots \leq p_n$ and $q_0 \geq q_1 \geq \cdots \geq q_{n-1} > q_n = 0$ of integers, let*

$$I = (x^{p_i} y^{q_i} | 0 \leq i \leq n)R.$$

Then

$$l_R(R/I) = l_R(R/(x, y)) \cdot \sum_{i=1}^{n} q_{i-1}(p_i - p_{i-1}).$$

Proof. We may assume that $n \geq 2$ and that our assertion is true for $n - 1$. Then considering the sequences $p'_i = p_i$ $(0 \leq i \leq n-1)$, $q'_i = q_i$ $(0 \leq i \leq n-2)$ and $q'_{n-1} = 0$, we get by the hypothesis on n that

$$l_R(R/I') = l_R(R/(x, y)) \cdot \sum_{i=1}^{n-1} q_{i-1}(p_i - p_{i-1}),$$

[1] The formulation of this lemma is due to the referee. The authors are grateful to the referee for his suggestion.

where $I' = (x^{p_i'}y^{q_i'} \mid 0 \le i \le n-1)R$. Since $I' = I + (x^{p_{n-1}})$ and $I : x^{p_{n-1}} = (x^{p_n - p_{n-1}}, y^{q_{n-1}})$, we have

$$\begin{aligned} l_R(R/I) &= l_R(R/I') + l_R(I + (x^{p_{n-1}})/I) \\ &= l_R(R/(x, y)) \cdot \sum_{i=1}^{n-1} q_{i-1}(p_i - p_{i-1}) + l_R(R/(x^{p_n - p_{n-1}}, y^{q_{n-1}})) \\ &= l_R(R/(x, y)) \cdot \sum_{i=1}^{n} q_{i-1}(p_i - p_{i-1}) \end{aligned}$$

as required.

Now let us assume that our ideal $\mathbf{p}$ is generated by the maximal minors of the matrix

$$M = \begin{bmatrix} X^\alpha & Y^{\beta'} & Z^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{bmatrix},$$

where X, Y, Z is a regular system of parameters for A and $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive integers. Then after suitable permutations of the rows and columns of M, we may assume that the matrix M is one of the following type.

(I) $\alpha \le \alpha'$, $\beta \le \beta'$ and $\gamma \le \gamma'$,
(II) $\alpha' < \alpha$, $\beta < \beta'$ and $\gamma < \gamma'$.

As was proved by Herzog and Ulrich [3], $\mathbf{p}$ is self-linked (resp. not self-linked) if and only if M has type (I) (resp. type (II)). And in any case it is already known that $\mu_A(\mathbf{p}^{(2)}/\mathbf{p}^2) = 1$ and $\mathbf{p}^{(n)} \ne \mathbf{p}^n$ for all $n \ge 2$ (cf. [5]). However, later we will need so frequently the assertions for the prime ideals $\mathbf{p}$ whose matrices M have type (I) that we would like to give a brief proof for the case. (See [7] for the case of type (II).)

So assume that $\alpha \le \alpha'$, $\beta \le \beta'$ and $\gamma \le \gamma'$. Let $a = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'}$, $b = X^{\alpha+\alpha'} - Y^\beta Z^{\gamma'}$ and $c = Y^{\beta+\beta'} - X^\alpha Z^\gamma$. Hence $\mathbf{p} = (a, b, c)$ and any pair of a, b and c forms a regular system of parameters for $A_{\mathbf{p}}$. We begin with the following

Lemma (2.4). *$\alpha < \alpha'$, $\beta < \beta'$ or $\gamma < \gamma'$.*

Proof. Suppose that $\alpha = \alpha'$, $\beta = \beta'$ and $\gamma = \gamma'$. Then since $a - b = (X^\alpha + Y^\beta + Z^\gamma)(Z^\gamma - X^\alpha)$, we have $X^\alpha + Y^\beta + Z^\gamma \in \mathbf{p}$ or $Z^\gamma - X^\alpha \in \mathbf{p}$, while $\mathbf{p} \subseteq (X^\alpha, Y^\beta, Z^\gamma)^2$. Hence $Z^\gamma \in (X^\alpha, Y^\beta, Z^{2\gamma})$, which is absurd.

Proposition (2.5). *There exists $d_2 \in \mathbf{p}^{(2)}$ such that*

$$X^\alpha d_2 = acZ^{\gamma'-\gamma} - b^2 Y^{\beta'-\beta},$$
$$Y^\beta d_2 = ab - c^2 X^{\alpha'-\alpha} Z^{\gamma'-\gamma} \quad \text{and} \quad Z^\gamma d_2 = -a^2 + bcX^{\alpha'-\alpha}Y^{\beta'-\beta}.$$

If $\alpha < \alpha'$, then $d_2 \equiv -Z^{\gamma+2\gamma'} \operatorname{mod}(X)$.

Proof. Because $X^\alpha a + Y^{\beta'} b + Z^{\gamma'} c = Y^\beta a + Z^\gamma b + X^{\alpha'} c = 0$, we see

$$(X^\alpha a + Y^{\beta'} b) \cdot b = -Z^{\gamma'} bc = (Y^\beta a + X^{\alpha'} c) \cdot cZ^{\gamma'-\gamma}$$

so that $X^\alpha(ab - c^2 X^{\alpha'-\alpha} Z^{\gamma'-\gamma}) = Y^\beta(acZ^{\gamma'-\gamma} - b^2 Y^{\beta'-\beta})$, whence $X^\alpha d_2 = acZ^{\gamma'-\gamma} - b^2 Y^{\beta'-\beta}$ and $Y^\beta d_2 = ab - c^2 X^{\alpha'-\alpha} Z^{\gamma'-\gamma}$ for some $d_2 \in \mathbf{p}^{(2)}$. Notice that

$$\begin{aligned}(Z^{\gamma} d_2)b &= (Z^{\gamma} b)\, d_2 = (-Y^{\beta} a - X^{\alpha'} c)\, d_2 \\ &= (Y^{\beta} d_2)(-a) + (X^{\alpha} d_2)(-cX^{\alpha'-\alpha}) \\ &= (-a^2 + bcX^{\alpha'-\alpha} Y^{\beta'-\beta})b\end{aligned}$$

and we get $Z^{\gamma} d_2 = -a^2 + bcX^{\alpha'-\alpha}Y^{\beta'-\beta}$, too. If $\alpha < \alpha'$, we have $Y^{\beta} d_2 \equiv ab \equiv -Y^{\beta} Z^{\gamma+2\gamma'} \operatorname{mod}(X)$ so that $d_2 \equiv -Z^{\gamma+2\gamma'} \operatorname{mod}(X)$.

Corollary (2.6) [5]. (1) $\mathbf{p}^{(2)} = (d_2) + \mathbf{p}^2$.
(2) $\mu_A(\mathbf{p}^{(2)}) \leq 5$.
(3) $\mathbf{p}^{(n)} \neq \mathbf{p}^n$ *if* $n \geq 2$.

Proof. By (2.4) we may assume that $\alpha < \alpha'$. Then as $d_2 \equiv -Z^{\gamma+2\gamma'} \operatorname{mod}(X)$ by (2.5) and as $(X) + \mathbf{p} = (X) + (Z^{\gamma+\gamma'}, Y^{\beta} Z^{\gamma'}, Y^{\beta+\beta'})$, we have

(#) $(X, d_2) + \mathbf{p}^2 = (X) + (Z^{\gamma+2\gamma'}, Y^{2\beta} Z^{2\gamma'}, Y^{\beta+\beta'} Z^{\gamma+\gamma'}, Y^{2\beta+\beta'} Z^{\gamma'}, Y^{2(\beta+\beta')})$

whence $l_A(A/(X, d_2) + \mathbf{p}^2) = 3(\beta\gamma + \beta\gamma' + \beta'\gamma')$ by (2.3). Let $e_{XA}(A/\mathbf{p}^{(2)})$ denote the multiplicity of $A/\mathbf{p}^{(2)}$ relative to the parameter X. Then

$$l_A(A/(X) + \mathbf{p}^{(2)}) = e_{XA}(A/\mathbf{p}^{(2)})$$

since $A/\mathbf{p}^{(2)}$ is a Cohen-Macaulay ring, while we get by the associative formula [8, p. 126] of multiplicity that

$$\begin{aligned}e_{XA}(A/\mathbf{p}^{(2)}) &= l_{A_{\mathbf{p}}}(A_{\mathbf{p}}/\mathbf{p}^2 A_{\mathbf{p}}) \cdot e_{XA}(A/\mathbf{p}) = 3 \cdot l_A(A/(X) + \mathbf{p}) \\ &= 3 \cdot l_A(A/(X) + (Z^{\gamma+\gamma'}, Y^{\beta} Z^{\gamma'}, Y^{\beta+\beta'})) \\ &= 3(\beta\gamma + \beta\gamma' + \beta'\gamma')\end{aligned}$$

(cf. (2.3)). Hence $l_A(A/(X, d_2) + \mathbf{p}^2) = l_A(A/(X) + \mathbf{p}^{(2)})$, which yields $(X) + \mathbf{p}^{(2)} = (X, d_2) + \mathbf{p}^2$ so that $\mathbf{p}^{(2)} = (d_2) + \mathbf{p}^2 + X\mathbf{p}^{(2)}$. Thus Nakayama's lemma proves the assertion (1). Notice that $\mu_A(\mathbf{p}^{(2)}) = \mu_A((X) + \mathbf{p}^{(2)}/(X)) \leq 5$ by the above equality (#) and we have the assertion (2). As $(X)+\mathbf{p}^2 \subseteq (X, Y, Z^{2(\gamma+\gamma')})$ and as $d_2 \equiv -Z^{\gamma+2\gamma'} \operatorname{mod}(X)$, we have $d_2 \notin (X) + \mathbf{p}^2$ so that $d_2 \notin \mathbf{p}^2$; hence $\mathbf{p}^{(2)} \neq \mathbf{p}^2$. Let $n \geq 3$ be an integer and assume that $\mathbf{p}^{(n)} = \mathbf{p}^n$. Hence $d_2 t \cdot (at)^{n-2} \in \mathbf{p}^n t^{n-1}$. We put $R = \sum_{i \geq 0} \mathbf{p}^i t^i$ and $G = R/\mathbf{p}R$ $(= \bigoplus_{i \geq 0} \mathbf{p}^i/\mathbf{p}^{i+1})$. Then because at is G-regular (cf., e.g., [4, 2.1]), we have $d_2 t \in \mathbf{p}R$, that is $d_2 \in \mathbf{p}^2$ which cannot happen as we have checked above. Thus $\mathbf{p}^{(n)} \neq \mathbf{p}^n$ for all $n \geq 2$.

3. Proof of Theorem (1.1)

We begin with the following

Theorem (3.1). *Suppose that* $\mathbf{p}$ *is generated by the maximal minors of the matrix*

$$\begin{bmatrix} X & Y^3 & Z^{n+1} \\ Y & Z^3 & X^n \end{bmatrix},$$

where X, Y, Z *is a regular system of parameters for* A *and* n *is a positive integer. Then* $R_s(\mathbf{p})$ *is a Gorenstein ring.*

Proof. If $n = 1$, then after renaming X, Y and Z, we may assume that $\mathbf{p}$ is generated by the maximal minors of the matrix

$$M = \begin{bmatrix} X & Y & Z^3 \\ Y & Z^2 & X^3 \end{bmatrix}.$$

Let us maintain the same notation as in §2. Then the matrix M is of type (I) and so we have by (2.5) that $d_2 \equiv -Z^8 \bmod(X)$. Hence $(c, d_2, X) = (X, Y^2, Z^8)$ and

$$l_A(A/(c, d_2, X)) = 16 = 1 \cdot 2 \cdot l_A(A/(X) + \mathbf{p}),$$

because $l_A(A/(X) + \mathbf{p}) = l_A(A/(X) + (Z^5, YZ^3, Y^2)) = 8$ (cf. (2.3)). Thus $R_s(\mathbf{p})$ is a Gorenstein ring by (2.2).

Suppose that $n \geq 2$ and recall that $Xd_2 = acZ^{n-2} - b^2Y^2$ and $Yd_2 = ab - c^2X^{n-1}Z^{n-2}$ (cf. (2.5)). Then as

$$(Xd_2 + b^2Y^2)b = Z^{n-2}abc = (Yd_2 + c^2X^{n-1}Z^{n-2})cZ^{n-2},$$

we have $X(bd_2 - c^3X^{n-2}Z^{2n-4}) = Y(cd_2Z^{n-2} - b^3Y)$ so that

(1) $Xd_3 = cd_2Z^{n-2} - b^3Y$ and

(2) $Yd_3 = bd_2 - c^3X^{n-2}Z^{2n-4}$,

for some $d_3 \in \mathbf{p}^{(3)}$. When $n = 2$, we have $d_2 \equiv -Z^9 \bmod(X)$ (cf. (2.5)). Hence as $Yd_3 \equiv (Z^{12} - Y^{11})Y \bmod(X)$ by the equation (2), we get $d_3 \equiv Z^{12} - Y^{11} \bmod(X)$. Therefore $(b, d_3, X) = (X, YZ^3, Z^{12} - Y^{11})$ so that

$$\begin{aligned} l_A(A/(b, d_3, X)) &= l_A(A/(X, Y, Z^{12} - Y^{11})) + l_A(A/(X, Z^3, Z^{12} - Y^{11})) \\ &= 45 = 1 \cdot 3 \cdot l_A(A/(X) + \mathbf{p}), \end{aligned}$$

since $l_A(A/(X)+\mathbf{p}) = l_A(A/(X)+(Z^6, YZ^3, Y^4)) = 15$. Thus $R_s(\mathbf{p})$ is Noetherian by (2.1). Because $\mathbf{p}^{(2)} = (d_2) + \mathbf{p}^2$ (cf. (2.6)(1)), we have $(X, b) + \mathbf{p}^{(2)} = (X, Z^9, YZ^3, Y^8)$ whence

$$l_A(A/(X, b) + \mathbf{p}^{(2)}) = 30 = e_{XA}(A/(b) + \mathbf{p}^{(2)}),$$

that is $A/(b) + \mathbf{p}^{(2)}$ is Cohen-Macaulay and so $R_s(\mathbf{p})$ is a Gorenstein ring by (2.2).

Now assume that $n \geq 3$. Then since

$$(Xd_3 + b^3Y) = bcd_2Z^{n-2} = (Yd_3 + c^3X^{n-2}Z^{2n-4})cZ^{n-2}$$

by the equations (1) and (2), we have $X(bd_3 - c^4X^{n-3}Z^{3n-6}) = Y(cd_3Z^{n-2} - b^4)$ so that

(3) $Yd_4 = bd_3 - c^4X^{n-3}Z^{3n-6}$

for some $d_4 \in \mathbf{p}^{(4)}$. Notice that $d_3 \equiv Z^{3n+6} \bmod(X)$ by the equation (2) and we get $d_4 \equiv -Z^{4n+7} - X^{n-3}Y^{15}Z^{3n-6} \bmod(X)$ by the equation (3). Hence $(c, d_4, X) = (X, Y^4, Z^{4n+7})$ so that

$$l_A(A/(c, d_4, X)) = 4 \cdot (4n + 7) = 1 \cdot 4 \cdot l_A(A/(X) + \mathbf{p}).$$

Thus $R_s(\mathbf{p})$ is Noetherian by (2.1). To check that $R_s(\mathbf{p})$ is Gorenstein, it is enough by (2.2) to see that $A/(c) + \mathbf{p}^{(2)}$ and $A/(c) + \mathbf{p}^{(3)}$ are Cohen-Macaulay. As $(X, c) + \mathbf{p}^{(2)} = (X) + (Z^{2n+5}, Y^2Z^{2n+2}, Y^4)$ (cf. (2.6)(1)), we have

$$l_A(A/(X, c) + \mathbf{p}^{(2)}) = 2 \cdot (4n + 7) = e_{XA}(A/(c) + \mathbf{p}^{(2)})$$

whence $A/(c)+\mathbf{p}^{(2)}$ is Cohen-Macaulay. Because $d_3 \equiv Z^{3n+6} \bmod(X)$, we have

$$(X, d_3)+\mathbf{pp}^{(2)} = (X)+(Z^{3n+6}, Y^3Z^{3n+3}, Y^4Z^{2n+5}, Y^6Z^{2n+2}, Y^8Z^{n+4}, Y^9Z^{n+1}, Y^{12})$$

by (2.6)(1). Therefore

$$l_A(A/(X, d_3)+\mathbf{pp}^{(2)}) = 6\cdot(4n+7) = l_A(A/(X)+\mathbf{p}^{(3)})$$

so that $(X)+\mathbf{p}^{(3)} = (X, d_3)+\mathbf{pp}^{(2)}$. Hence

$$(X, c)+\mathbf{p}^{(3)} = (X)+(Z^{3n+6}, Y^3Z^{3n+3}, Y^4)$$

and so we get

$$l_A(A/(X, c)+\mathbf{p}^{(3)}) = 3\cdot(4n+7) = e_{XA}(A/(c)+\mathbf{p}^{(3)}).$$

Thus $A/(c)+\mathbf{p}^{(3)}$ is Cohen-Macaulay.

To prove Theorem (1.1) we need one more result.

Proposition (3.2). *Suppose that* $\mathbf{p}$ *is generated by the maximal minors of the matrix*

$$\begin{bmatrix} X^2 & Y^2 & Z^3 \\ Y & Z^2 & X^2 \end{bmatrix}$$

where X, Y, Z *is a regular system of parameters for* A. *Then* $R_s(\mathbf{p})$ *is a Gorenstein ring.*

Proof. The matrix has type (I) and so by (2.5), $Yd_2 = ab - c^2Z$ and $Z^2d_2 = -a^2+bcY$. Therefore as

$$(Yd_2+c^2Z)a = a^2b = (bcY - Z^2d_2)b,$$

we get $Y(ad_2 - b^2c) = Z(-ac^2 - bd_2Z)$ so that $Yd_3 = -ac^2 - bd_2Z$ and $Zd_3 = ad_2 - b^2c$ for some $d_3 \in \mathbf{p}^{(3)}$. Notice that

$$\begin{aligned} &d_2 \equiv -Z^8 \bmod(Y), \qquad d_2 \equiv -X^6Y \bmod(Z), \\ &d_3 \equiv -Z^{12}+X^{10}Z \bmod(Y) \quad\text{and}\quad d_3 \equiv X^2Y^7 \bmod(Z). \end{aligned}$$

Then we have $c^2d_2 + bd_3 \equiv 0 \bmod(Z)$, whence $Zd_4 = c^2d_2+bd_3$ for some $d_4 \in \mathbf{p}^{(4)}$. Because $d_4 \equiv X^{14} - 2X^4Z^{11} \bmod(Y)$, we see

$$l_A(A/(d_2, d_4, Y)) = l_A(A/(X^{14}, Y, Z^8)) = 112 = 2\cdot 4\cdot l_A(A/(Y)+\mathbf{p}).$$

Thus $R_s(\mathbf{p})$ is Noetherian by (2.1). To check that $R_s(\mathbf{p})$ is Gorenstein, let $I = (d_2, d_3)+\mathbf{p}^3$ $(\subseteq (d_2)+\mathbf{p}^{(3)})$. Then

$$(Y)+I = (Y)+(Z^8, X^6Z^6, X^8Z^4, X^{10}Z, X^{12})$$

so that $l_A(A/(Y)+I) = 70$ by (2.3), while

$$\begin{aligned} e_{YA}(A/(d_2)+\mathbf{p}^{(3)}) &= l_{A_\mathbf{p}}(A_\mathbf{p}/d_2A_\mathbf{p}+\mathbf{p}^3A_\mathbf{p})\cdot e_{YA}(A/\mathbf{p}) \\ &= 5\cdot 14 = 70 \end{aligned}$$

by the associative formula of multiplicity (cf. [1, (3.1)(3)], too). Hence by the inequalities

$$l_A(A/(Y)+I) \geq l_A(A/(Y, d_2)+\mathbf{p}^{(3)}) \geq e_{YA}(A/(d_2)+\mathbf{p}^{(3)}),$$

we get that $A/(d_2)+\mathbf{p}^{(3)}$ is Cohen-Macaulay. Let $J=(d_2, d_4)+d_3\mathbf{p}+\mathbf{p}^4$ ($\subseteq (d_2)+\mathbf{p}^{(4)}$). Then

$$(Y)+J=(Y)+(Z^8, X^{10}Z^6, X^{12}Z^3, X^{14})$$

so that $l_A(A/(Y)+J)=98=e_{YA}(A/(d_2)+\mathbf{p}^{(4)})$, whence by the inequalities

$$l_A(A/(Y)+J)\geq l_A(A/(Y, d_2)+\mathbf{p}^{(4)})\geq e_{YA}(A/(d_2)+\mathbf{p}^{(4)}),$$

we find that $A/(d_2)+\mathbf{p}^{(4)}$ is Cohen-Macaulay. Thus $R_s(\mathbf{p})$ is a Gorenstein ring by (2.2).

Remark (3.3). The prime ideal $\mathbf{p}=\mathbf{p}(11, 14, 10)$ corresponds to the ideal considered in (3.2).

Proof of Theorem (1.1). We write $m=4n+r$ with $0\leq r<4$. If $r=0$, then $\mathbf{p}=(X^{n+1}-Z^n, Y^4-X^3Z)$ which is a complete intersection in $A=k[[X, Y, Z]]$. Hence $\mathbf{p}^{(n)}=\mathbf{p}^n$ for any $n\geq 1$ and we have an isomorphism $R_s(\mathbf{p})\cong A[T_1, T_2]/(f)$ of A-algebras, where $A[T_1, T_2]$ is a polynomial ring and $0\neq f\in A[T_1, T_2]$. Thus $R_s(\mathbf{p})$ is certainly Gorenstein.

(1) ($r=1$). If $n=0$, then $Y-X^2\in\mathbf{p}$ and $\mathbf{p}$ is a complete intersection in A. If $n=1$, then $\mathbf{p}=(Y^3-Z^2, X^3-YZ)$, which is a complete intersection in A. Thus we may assume $n\geq 2$. Then $\mathbf{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^3 & Y^3 & Z^n \\ Y & Z^2 & X^{n-1} \end{bmatrix}$$

(cf. [2]), whence the assertion follows from [1, (4.1)] if $n\geq 4$. The cases $n=2, 3$ are the exceptional ones, that is $m=9, 13$.

(2) ($r=2$). We may assume $n\geq 1$, because $Z-Y^2\in\mathbf{p}$ if $n=0$. Hence $\mathbf{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^3 & Y^2 & Z^n \\ Y^2 & Z & X^n \end{bmatrix}$$

so that the assertion follows from [1, (4.1)] if $n\geq 3$. When $n=1$, notice that $\mathbf{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} Y^2 & Z & X^3 \\ Z & X & Y^2 \end{bmatrix}$$

and we have $R_s(\mathbf{p})$ to be a Gorenstein ring again by [1, (4.1)]. If $n=2$, $\mathbf{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} Y^2 & Z^2 & X^3 \\ Z & X^2 & Y^2 \end{bmatrix}$$

so that $R_s(\mathbf{p})$ is Gorenstein by (3.2).

(3) ($r=3$). We may assume $n\geq 1$, as $Z-XY\in\mathbf{p}$ if $n=0$. Hence $\mathbf{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} Z & Y^3 & X^{n+1} \\ Y & X^3 & Z^n \end{bmatrix}$$

so that the assertion follows from (3.1). This completes the proof of Theorem (1.1).

The symbolic Rees algebras $R_s(\mathbf{p})$ for $\mathbf{p}=\mathbf{p}(9, 10, 13)$ is Noetherian but not Cohen-Macaulay, if $\operatorname{ch} k=3$ (cf. [7]). The same is true for $\mathbf{p}=\mathbf{p}(13, 14, 17)$ too, if $\operatorname{ch} k=3$. We shall prove it in the following

Example (3.4). Let $\mathfrak{p} = \mathfrak{p}(13, 14, 17)$ and let $\mathbf{M}$ denote the unique graded maximal ideal of $R_s(\mathfrak{p})$. Then $R_s(\mathfrak{p})$ is a Noetherian ring with $\dim R_s(\mathfrak{p})_{\mathbf{M}} = 4$ and depth $R_s(\mathfrak{p})_{\mathbf{M}} = 3$, if $\operatorname{ch} k = 3$.

Proof. The ideal $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$M = \begin{bmatrix} X^3 & Y^3 & Z^3 \\ Y & Z & X^2 \end{bmatrix}$$

of type (II). Let $a = Z^4 - X^2Y^3$, $b = X^5 - YZ^3$ and $c = Y^4 - X^3Z$ (hence $\mathfrak{p} = (a, b, c)$). Then as $X^3a + Y^3b + Z^3c = Ya + Zb + X^2c = 0$, we have $Y^3a^3 + Z^3b^3 + X^6c^3 = 0$. Therefore because

$$(Z^3b^3 + X^6c^3)b = -Y^3a^3b = (X^3a + Z^3c)a^3$$

we see $X^3(a^4 - bc^3X^3) = Z^3(b^4 - a^3c)$ so that $Z^3d_4 = a^4 - bc^3X^3$ for some $d_4 \in \mathfrak{p}^{(4)}$. Notice that $c \equiv Y^4$ and $d_1 \equiv Z^{13} \bmod(X)$ and we find

$$l_A(A/(c, d_4, X)) = 52 = 1 \cdot 4 \cdot l_A(A/(X) + \mathfrak{p}),$$

whence $R_s(\mathfrak{p})$ is Noetherian by (2.1) but non-Cohen-Macaulay by (2.2) and [7, (2.4)]. Because depth $R_s(\mathfrak{p})_{\mathbf{M}} \geq 3$ by [1, (2.1) and (3.7)(3)] and $\dim R_s(\mathfrak{p})_{\mathbf{M}} = 4$, we get depth $R_s(\mathfrak{p})_{\mathbf{M}} = 3$ as required.

4. Proof of Theorem (1.2)

Let $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$ with $n_1 = 4$ and assume that $\mathfrak{p}$ is not a complete intersection in $A = A[[X, Y, Z]]$. Hence by [2] the ideal $\mathfrak{p}$ is generated by maximal minors of a matrix of the following form

$$\begin{bmatrix} X^\alpha & Y^{\beta'} & Z^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{bmatrix}$$

with positive integers $\alpha, \beta, \gamma, \alpha', \beta'$ and γ'. Then as $(X) + \mathfrak{p} = (X) + (Z^{\gamma+\gamma'}, Y^\beta Z^{\gamma'}, Y^{\beta+\beta'})$, we have $l_A(A/(X) + \mathfrak{p}) = \beta\gamma + \beta\gamma' + \beta'\gamma'$ (cf. (2.3)), while $e_{XA}(A/\mathfrak{p}) = 4\ (= n_1)$. Hence $\beta = \gamma' = 1$ and $\gamma + \beta' = 3$, as $\beta\gamma + \beta\gamma' + \beta'\gamma' = 4$. We may assume $\gamma = 1$ and $\beta' = 2$ so that solving the equations

$$4(\alpha + \alpha') = n_2 + n_3, \quad 3n_2 = 4\alpha + n_3, \quad 2n_3 = 4\alpha' + 2n_2,$$

we get $n_2 = 2\alpha + \alpha'$ and $n_3 = 2\alpha + 3\alpha'$; hence α' is odd, as $\mathrm{GCD}(4, n_2, n_3) = 1$. Thus Theorem (1.2) follows from the next more general

Theorem (4.1). *Let $\mathfrak{p}$ be a prime ideal in a 3-dimensional regular local ring A and assume that $\mathfrak{p}$ is generated by the maximal minors of a matrix of the form*

$$\begin{bmatrix} X^q & Y^2 & Z \\ Y & Z & X^p \end{bmatrix}$$

where X, Y, Z is a regular system of parameters for A and p, q are positive integers with p odd. Then $R_s(\mathfrak{p})$ is a Gorenstein ring.

We divide the proof of Theorem (4.1) into a few parts. First we put $a = Z^2 - X^pY^2$, $b = X^{p+q} - YZ$ and $c = Y^3 - X^qZ$. Hence $\mathfrak{p} = (a, b, c)$ and any pair of a, b and c forms a regular system of parameters for $A_\mathfrak{p}$. Choose $0 \leq k \in Z$ so that $kp < q \leq (k+1)p$. Then we get by [6, Proof of 3.14] the following

Lemma 4.2. *There exist elements $e_n \in \mathbf{p}^{(n)}$ $(1 \le n \le k+2)$ and $f \in \mathbf{p}^{(2k+3)}$ such that*

$$\begin{aligned} e_n &\equiv Y^{2n+1} \bmod(X) && (1 \le n \le k+1), \\ e_{k+2} &\equiv (-1)^k Z^{2k+3} \bmod(X) && \text{if } q < (k+1)p, \\ &\equiv Y^{2k+5} + (-1)^k Z^{2k+3} \bmod(X) && \text{if } q = (k+1)p, \\ f &\equiv -Z^{4k+4} \bmod(X) && \text{if } q - kp < (k+1)p - q, \\ &\equiv Y^{4k+8} \bmod(X) && \text{if } q - kp > (k+1)p - q > 0. \end{aligned}$$

The Noetherian property of $R_s(\mathbf{p})$ now directly follows from (2.1) *and* (4.2), *because*

(4.3)
$$\begin{aligned} l_A(A/(b, e_{k+2}, X)) &= l_A(A/(X, YZ, Y^{2k+5} + (-1)^k Z^{2k+3})) \\ &= 1 \cdot (k+2) \cdot 4 \quad \text{if } q = (k+1)p, \end{aligned}$$

(4.4)
$$\begin{aligned} l_A(A/(e_{k+1}, f, X)) &= l_A(A/(X, Y^{2k+3}, Z^{4k+4})) \\ &= (k+1) \cdot (2k+3) \cdot 4 \quad \text{if } q - kp < (k+1)p - q \end{aligned}$$

and

(4.5)
$$\begin{aligned} l_A(A/(e_{k+2}, f, X)) &= l_A(A/(X, Y^{4k+8}, Z^{2k+3})) \\ &= (k+2) \cdot (2k+3) \cdot 4 \quad \text{if } q - kp > (k+1)p - q > 0 \end{aligned}$$

(*notice that $q - kp \ne (k+1)p - q$, as p is odd*).

To see that $R_s(\mathbf{p})$ is a Gorenstein ring we need further informations about the ideals $\mathbf{p}^{(n)}$. We begin with the following

Proposition (4.6). $\mathbf{p}^{(n)} = \mathbf{p}^n + \sum_{j=1}^n e_j \mathbf{p}^{n-j}$ *for* $1 \le n \le k+1$.

Proof. Let $I = \mathbf{p}^n + \sum_{j=1}^n e_j \mathbf{p}^{n-j}$ and

$$\begin{aligned} J = (X) &+ (Z^{2n}, YZ^{2n-1}, \dots, Y^{n-1}Z^{n+1}, Y^n Z^n) \\ &+ (Y^{n+2}Z^{n-1}, Y^{n+3}Z^{n-2}, \dots, Y^{2n}Z, Y^{2n+1}). \end{aligned}$$

Then $(X) + I \supseteq J$, because

$$\begin{aligned} a^{n-j} b^j &\equiv Y^j Z^{2n-j} \bmod(X) \quad \text{for } 0 \le j \le n, \\ b^{n-1-j} e_{j+1} &\equiv Y^{n+2+j} Z^{n-1-j} \bmod(X) \quad \text{for } 0 \le j \le n-1. \end{aligned}$$

As $l_A(A/J) = 4\binom{n+1}{2} = e_{XA}(A/\mathbf{p}^{(n)})$ (cf. (2.3)), by the canonical inequalities

$$l_A(A/J) \ge l_A(A/(X) + I) \ge l_A(A/(X) + \mathbf{p}^{(n)}) \ge e_{XA}(A/\mathbf{p}^{(n)})$$

we get $J = (X) + I = (X) + \mathbf{p}^{(n)}$. Hence $\mathbf{p}^{(n)} = I + X\mathbf{p}^{(n)}$ so that $\mathbf{p}^{(n)} = I$ by Nakayama's lemma.

Corollary (4.7). *$R_s(\mathbf{p})$ is a Gorenstein ring, if $q = (k+1)p$.*

Proof. By (4.6) and its proof we see $(X, b) + \mathbf{p}^{(n)} = (X) + (Z^{2n}, YZ, Y^{2n+1})$ so that

$$l_A(A/(X, b) + \mathbf{p}^{(n)}) = 4n = e_{XA}(A/(b) + \mathbf{p}^{(n)})$$

for $1 \leq n \leq k+1$. Hence $A/(b)+\mathbf{p}^{(n)}$ is a Cohen-Macaulay ring, which proves by (2.2) and (4.3) the assertion.

Proposition (4.8). *Suppose $q < (k+1)p$. Then $\mathbf{p}^{(n)} = \mathbf{p}^n + \sum_{j=1}^{k+2} e_j \mathbf{p}^{(n-j)}$ for $k+2 \leq n \leq 2k+2$.*

Proof. Let $I = \mathbf{p}^n + \sum_{j=1}^{k+2} e_j \mathbf{p}^{(n-j)}$ and

$$\begin{aligned} J = (X) &+ (Z^{2n-1}, YZ^{2n-2}, \dots, Y^{n-k-2}Z^{n+k+1}) \\ &+ (Y^{n-k}Z^{n+k}, Y^{n-k+1}Z^{n+k-1}, \dots, Y^nZ^n) \\ &+ (Y^{n+2}Z^{n-1}, Y^{n+3}Z^{n-2}, \dots, Y^{n+k+2}Z^{n-k-1}) \\ &+ (Y^{n+k+4}Z^{n-k-2}, Y^{n+k+5}Z^{n-k-3}, \dots, Y^{2n+2}). \end{aligned}$$

Then $(X) + I \supseteq J$, because

$$\begin{aligned} &a^{n-k-2-j}b^je_{k+2} \equiv (-1)^{k+j}Y^jZ^{2n-j-1} \bmod(X) \quad \text{for } 0 \leq j \leq n-k-2, \\ &a^{k-j}b^{n-k+j} \equiv (-1)^{n-k+j}Y^{n-j+k}Z^{n+k-j} \bmod(X) \quad \text{for } 0 \leq j \leq k, \\ &b^{n-j-1}e_{j+1} \equiv (-1)^{n-j-1}Y^{n+j+2}Z^{n-j-1} \bmod(X) \quad \text{for } 0 \leq j \leq k, \\ &b^{n-k-2-j}e_{k+1}e_{j+1} \equiv (-1)^{n-k+j}Y^{n+k+4+j}Z^{n-k-2-j} \bmod(X) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \leq j \leq n-k-2. \end{aligned}$$

Therefore as $l_A(A/J) = e_{XA}(A/\mathbf{p}^{(n)})$, we get similarly as in the proof of (4.6) that $J = (X) + I = (X) + \mathbf{p}^{(n)}$. Hence $\mathbf{p}^{(n)} = I$.

Proposition (4.9). *Suppose that $q - kp < (k+1)p - q$. Then*

$$\mathbf{p}^{(n)} = \mathbf{p}^n + f\mathbf{p}^{(n-2k-3)} + \sum_{j=1}^{k+2} e_j \mathbf{p}^{(n-j)}$$

for $2k+3 \leq n \leq 3k+3$.

Proof. We put $I = \mathbf{p}^n + f\mathbf{p}^{(n-2k-3)} + \sum_{j=1}^{k+2} e_j\mathbf{p}^{(n-j)}$ and

$$\begin{aligned} J = (X) &+ (Z^{2n-2}, YZ^{2n-3}, \dots, Y^{n-2k-3}Z^{n+2k+1}) \\ &+ (Y^{n-2k-1}Z^{n+2k}, Y^{n-2k}Z^{n+2k-1}, \dots, Y^{n-k-2}Z^{n+k+1}) \\ &+ (Y^{n-k}Z^{n+k}, Y^{n-k+1}Z^{n+k-1}, \dots, Y^nZ^n) \\ &+ (Y^{n+2}Z^{n-1}, Y^{n+3}Z^{n-2}, \dots, Y^{n+k+2}Z^{n-k-1}) \\ &+ (Y^{n+k+4}Z^{n-k-2}, Y^{n+k+5}Z^{n-k-3}, \dots, Y^{n+2k+4}Z^{n-2k-2}) \\ &+ (Y^{n+2k+6}Z^{n-2k-3}, Y^{n+2k+7}Z^{n-2k-4}, \dots, Y^{2n+3}). \end{aligned}$$

Then $(X) + I \supseteq J$, because

$$\begin{aligned} &a^{n-2k-3-j}b^jf \equiv (-1)^{j+1}Y^jZ^{2n-2-j} \bmod(X) \quad \text{for } 0 \leq j \leq n-2k-3, \\ &a^{k-j-1}b^{n-2k-1+j}f \equiv (-1)^{n+j}Y^{n-2k+1+j}Z^{n+2k-j} \bmod(X) \quad \text{for } 0 \leq j \leq k-1, \\ &a^{k+j}b^{n-k-j} \equiv (-1)^{n-k-j}Y^{n-k-j}Z^{n+k+j} \bmod(X) \quad \text{for } 0 \leq j \leq k, \\ &b^{n-1-j}e_{j+1} \equiv (-1)^{n-1-j}Y^{n+2+j}Z^{n-1-j} \bmod(X) \quad \text{for } 0 \leq j \leq k, \\ &b^{n-k-2-j}e_{k+1}e_{j+1} \equiv (-1)^{n-k-j}Y^{n+k+4+j}Z^{n-k-2-j} \bmod(X) \quad \text{for } 0 \leq j \leq k \text{ and} \\ &b^{n-2k-3-j}(e_{k+1})^2e_{j+1} \equiv (-1)^{n-1-j}Y^{n+2k+6+j}Z^{n-2k-3-j} \bmod(X) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \leq j \leq n-2k-3. \end{aligned}$$

Hence we have $J = (X) + I = (X) + \mathbf{p}^{(n)}$ for $2k+3 \le n \le 3k+3$ by the same reason as in the proof of (4.6). Thus $\mathbf{p}^{(n)} = I$.

Corollary (4.10). *$R_s(\mathbf{p})$ is a Gorenstein ring, if $q - kp < (k+1)p - q$.*

Proof. It suffices to see that $A/(e_{k+1}, f) + \mathbf{p}^{(n)}$ is a Cohen-Macaulay ring for each $k+2 \le n \le 3k+2$ (cf. (2.2) and (4.4)); that is enough to check $l_A(A/(X, e_{k+1}, f) + \mathbf{p}^{(n)}) \le e_{XA}(A/(e_{k+1}, f) + \mathbf{p}^{(n)})$. However, because

$$e_{XA}(A/(e_{k+1}, f) + \mathbf{p}^{(n)}) = 4 \cdot l_{A_\mathbf{p}}(A_\mathbf{p}/(e_{k+1}, f)A_\mathbf{p} + \mathbf{p}^n A_\mathbf{p})$$

by the associative formula of multiplicity (cf. [8]) and because e_{k+1}, f forms a super regular sequence in $A_\mathbf{p}$ (cf. [1, (3.1)(3)]), we can easily compute the exact value of $e_{XA}(A/(e_{k+1}, f) + \mathbf{p}^{(n)})$ in terms of n and k, that is

$$\begin{aligned} &e_{XA}(A/(e_{k+1}, f) + \mathbf{p}^{(n)}) \\ &\quad = 2(2n-k)(k+1) \qquad (k+2 \le n \le 2k+2) \\ &\quad = 2(6kn - 5k^2 - 11k - n^2 + 7n - 6) \qquad (2k+3 \le n \le 3k+2), \end{aligned}$$

while we now explicitly have the ideal $(X, e_{k+1}, f) + \mathbf{p}^{(n)}$ by (4.6), (4.8) and (4.9) (cf. their proofs, too). Therefore the required inequality $l_A(A/(X, e_{k+1}, f) + \mathbf{p}^{(n)}) \le e_{XA}(A/(e_{k+1}, f) + \mathbf{p}^{(n)})$ can be directly checked, which we would like to leave to the readers.

Proposition (4.11). *Suppose that $q - kp > (k+1)p - q > 0$. Then we have*

$$\text{(1)} \qquad \mathbf{p}^{(2k+3)} = \mathbf{p}^{2k+3} + (f) + \sum_{j=1}^{k+2} e_j \mathbf{p}^{(2k+3-j)}.$$

$$\text{(2)} \qquad \mathbf{p}^{(n)} = \mathbf{p}^n + f\mathbf{p}^{(n-2k-3)} + \sum_{j=1}^{k+2} e_j \mathbf{p}^{(n-j)} \quad \textit{for } 2k+4 \le n \le 3k+4.$$

Proof. (1) Let $I = \mathbf{p}^{2k+3} + (f) + \sum_{j=1}^{k+2} e_j \mathbf{p}^{(2k+3-j)}$ and

$$\begin{aligned} J = (X) &+ (Z^{4k+5}, YZ^{4k+4}, \dots, Y^{k+1}Z^{3k+4}) \\ &+ (Y^{k+3}Z^{3k+3}, Y^{k+4}Z^{3k+2}, \dots, Y^{2k+3}Z^{2k+3}) \\ &+ (Y^{2k+5}Z^{2k+2}, Y^{2k+6}Z^{2k+1}, \dots, Y^{3k+5}Z^{k+2}) \\ &+ (Y^{3k+7}Z^{k+1}, Y^{3k+8}Z^{k}, \dots, Y^{4k+8}). \end{aligned}$$

Then $(X) + I \supseteq J$,because

$$\begin{aligned} &a^{k+1-j}b^j e_{k+2} \equiv (-1)^{j+k} Y^j Z^{4k+5-j} \bmod(X) \quad \text{for } 0 \le j \le k+1, \\ &a^{k-j}b^{k+3+j} \equiv (-1)^{k+1+j} Y^{k+3+j} Z^{3k+3-j} \bmod(X) \quad \text{for } 0 \le j \le k, \\ &b^{2k+2-j} e_{j+1} \equiv (-1)^j Y^{2k+5+j} Z^{2k+2-j} \bmod(X) \quad \text{for } 0 \le j \le k, \\ &b^{k+1-j} e_{k+1} e_{j+1} \equiv (-1)^{k+1-j} Y^{3k+7+j} Z^{k+1-j} \bmod(X) \quad \text{for } 0 \le j \le k \end{aligned}$$

and

$$f \equiv Y^{4k+8} \bmod(X).$$

As $l_A(A/J) = e_{XA}(A/\mathbf{p}^{(2k+3)})$, we get $J = (X) + I = (X) + \mathbf{p}^{(2k+3)}$ whence $\mathbf{p}^{(2k+3)} = I$.

(2) Let $I = \mathbf{p}^n + f\mathbf{p}^{(n-2k-3)} + \sum_{j=1}^{k+2} e_j\mathbf{p}^{(n-j)}$ and

$$\begin{aligned} J = (X) &+ (Z^{2n-2}, YZ^{2n-3}, \dots, Y^{n-2k-4}Z^{n+2k+2}) \\ &+ (Y^{n-2k-2}Z^{n+2k+1}, Y^{n-2k-1}Z^{n+2k}, \dots, Y^{n-k-2}Z^{n+k+1}) \\ &+ (Y^{n-k}Z^{n+k}, Y^{n-k+1}Z^{n+k-1}, \dots, Y^nZ^n) \\ &+ (Y^{n+2}Z^{n-1}, Y^{n+3}Z^{n-2}, \dots, Y^{n+k+2}Z^{n-k-1}) \\ &+ (Y^{n+k+4}Z^{n-k-2}, Y^{n+k+5}Z^{n-k-3}, \dots, Y^{n+2k+5}Z^{n-2k-3}) \\ &+ (Y^{n+2k+7}Z^{n-2k-4}, Y^{n+2k+8}Z^{n-2k-5}, \dots, Y^{2n+3}). \end{aligned}$$

Then $(X) + I \supseteq J$, as

$$a^{n-2k-4-j}b^j(e_{k+2})^2 \equiv (-1)^jY^jZ^{2n-2-j} \bmod(X) \quad \text{for } 0 \le j \le n-2k-4,$$
$$a^{k-j}b^{n-2k-2+j}e_{k+2} \equiv (-1)^{n-k+j}Y^{n-2k-2+j}Z^{n+2k+1-j} \bmod(X) \quad \text{for } 0 \le j \le k,$$
$$a^{k-j}b^{n-k+j} \equiv (-1)^{n-k+j}Y^{n-k+j}Z^{n+k-j} \bmod(X) \quad \text{for } 0 \le j \le k,$$
$$b^{n-1-j}e_{j+1} \equiv (-1)^{n-1-j}Y^{n+2+j}Z^{n-1-j} \bmod(X) \quad \text{for } 0 \le j \le k,$$
$$b^{n-k-2-j}e_{k+1}e_{j+1} \equiv (-1)^{n-k-j}Y^{n+k+4+j}Z^{n-k-2-j} \bmod(X) \quad \text{for } 0 \le j \le k,$$
$$b^{n-2k-3}f \equiv (-1)^{n-1}Y^{n+2k+5}Z^{n-2k-3} \bmod(X) \quad \text{and}$$
$$b^{n-2k-4-j}fe_{j+1} \equiv (-1)^{n-j}Y^{n+2k+7+j}Z^{n-2k-4-j} \bmod(X) \quad \text{for } 0 \le j \le n-2k-4.$$

Because $l_A(A/J) = e_{XA}(A/\mathbf{p}^{(n)})$, we have $J = (X) + I = (X) + \mathbf{p}^{(n)}$, whence $\mathbf{p}^{(n)} = I$.

Corollary (4.12). $R_s(\mathbf{p})$ *is a Gorenstein ring, if* $q - kp > (k+1)p - q > 0$.

Proof. By (2.2) and (4.5) we have only to check that $l_A(A/(X, e_{k+2}, f) + \mathbf{p}^{(n)}) \le e_{XA}(A/(e_{k+2}, f) + \mathbf{p}^{(n)})$ for $k + 3 \le n \le 3k + 3$. Because we explicitly know the ideals $(X, e_{k+2}, f) + \mathbf{p}^{(n)}$ by (4.6), (4.8) and (4.11) and because

$$\begin{aligned} &e_{XA}(A/(e_{k+2}, f) + \mathbf{p}^{(n)}) \\ &\quad = 4kn - 2k^2 + 8n - 6k - 4 \qquad (k+3 \le n \le 2k+3), \\ &\quad = 12kn - 2n^2 + 18n - 10k^2 - 26k - 16 \qquad (2k+4 \le n \le 3k+3) \end{aligned}$$

we are able to directly check the required inequality. This completes the proof of Theorem (4.1) as well as that of (4.12).

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