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A QUALITATIVE UNCERTAINTY PRINCIPLE FOR UNIMODULAR GROUPS OF TYPE I

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ABSTRACT. It has long been known that if $f \in L^2(\mathbf{R}^n)$ and the supports of f and its Fourier transform $\hat{f}$ are bounded then $f = 0$ almost everywhere. More recently it has been shown that the same conclusion can be reached under the weaker condition that the supports of f and $\hat{f}$ have finite measure. These results may be thought of as qualitative uncertainty principles since they limit the "concentration" of the Fourier transform pair $(f, \hat{f})$. Little is known, however, of analogous results for functions on locally compact groups. A qualitative uncertainty principle is proved here for unimodular groups of type I.

1. INTRODUCTION

Let G be a locally compact (LC) group equipped with left Haar measure m_G. $\widehat{G}$ will denote the dual of G (i.e., a maximal set of pairwise inequivalent continuous irreducible unitary representations of G). The Fourier transform $\hat{f}$ of $f \in L^1(G)$ is defined by

$$\langle \hat{f}(\pi)\zeta, \eta\rangle = \int_G f(x)\langle \pi(x)\zeta, \eta\rangle \, dm_G(x)$$

where $\pi \in \widehat{G}$, $\zeta, \eta \in \mathscr{H}_\pi$ (the representation space of π) and $\langle\ ,\ \rangle$ is the inner product on $\mathscr{H}_\pi$.

If f, g are measurable functions on G, we define their convolution $f * g$ by

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, dm_G(y)$$

whenever the integral exists.

For $f \in L^1(G)$, let $A_f = \{x \in G; f(x) \neq 0\}$ and $B_f = \{\pi \in \widehat{G}; \hat{f}(\pi) \neq 0\}$. In 1973, Matolcsi and Szücs [7] showed that if G is a locally compact abelian (LCA) group then, with notation as above, $m_G(A_f)m_{\widehat{G}}(B_f) < 1 \Rightarrow f = 0$ almost everywhere, where $m_{\widehat{G}}$ is the Haar measure on the dual group $\widehat{G}$, normalized so that the Plancherel identity is valid. In 1974, Benedicks [2] showed that if m_n is ordinary Lebesgue measure on $\mathbf{R}^n$ and $f \in L^1(\mathbf{R}^n)$, then $m_n(A_f)m_n(B_f) < \infty \Rightarrow f = 0$ almost everywhere, and in 1977 Amrein and Berthier [1] reached the same conclusion using Hilbert space techniques.

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Let $\mathscr{H}$ be a Hilbert space with inner product $\langle\ ,\ \rangle$ and $T \in \mathscr{B}(\mathscr{H})$ (the set of bounded linear operators on $\mathscr{H}$). Let $\{\phi_k\}$ be a complete orthonormal set in $\mathscr{H}$. We define the Hilbert-Schmidt norm $\|T\|_2$ of T by

$$\|T\|_2^2 = \sum_k \|T\phi_k\|^2 = \sum_k \sum_j |\langle T\phi_k\,,\,\phi_j\rangle|^2$$

and we say T is a Hilbert-Schmidt operator if $\|T\|_2 < \infty$. The class $\mathrm{End}_2(\mathscr{H})$ of Hilbert-Schmidt operators on $\mathscr{H}$ is itself a Hilbert space with inner product

$$\langle T\,,\,S\rangle = \sum_k \langle T\phi_k\,,\,S\phi_k\rangle$$

for $T\,,\,S \in \mathrm{End}_2(\mathscr{H})$.

Let G be a unimodular group of type I. Then there exists a measure $m_{\widehat{G}}$ on $\widehat{G}$ such that

$$L^2(G) \simeq \int_{\widehat{G}}^{\oplus} \mathrm{End}_2(\mathscr{H}_\pi)\,dm_{\widehat{G}}(\pi) \tag{1.1}$$

i.e., the Fourier transform $f \mapsto \hat{f}$ may be extended to $L^2(G)$ to furnish an isometric isomorphism between $L^2(G)$ and the measurable fields h on $\widehat{G}$ for which $\int_{\widehat{G}} \|h(\pi)\|_2^2 dm_{\widehat{G}}(\pi) < \infty$. (For a discussion, see [4 or 10].)

If $E \subseteq \widehat{G}$ is a measurable subset, we define $\mu(E)$ by

$$\mu(E) = \int_E \dim(\mathscr{H}_\pi)\,dm_{\widehat{G}}(\pi)\,.$$

With this notation, the LC group G is said to satisfy the qualitative uncertainty principle (QUP) if, for each $f \in L^2(G)$,

$$m_G(A_f) < m_G(G)\,, \qquad \mu(B_f) < \mu(\widehat{G}) \Rightarrow f = 0 \quad m_G\text{-a.e.}$$

Of course, if the Plancherel measure is supported on the set of infinite-dimensional representations (e.g., if G is a nonabelian nilpotent Lie group or semisimple Lie group) the QUP is meaningless since the condition $\mu(B_f) < \mu(\widehat{G})$ cannot be complied with by a nonzero $f \in L^2(G)$. A theorem due to Moore [8] states that $\dim(\mathscr{H}_\pi) < M < \infty$ for all $\pi \in \widehat{G}$ if and only if G contains an open abelian subgroup of finite index so that in this case the condition $\mu(B_f) < \mu(\widehat{G})$ reduces to the more natural $m_{\widehat{G}}(B_f) < m_{\widehat{G}}(\widehat{G})$.

In [9], Price and Sitaram prove qualitative uncertainty principles of a similar nature to that described above for functions on a wide variety of locally compact groups including those of the form $\mathbf{R}^n \times H$ where H is a compact connected Lie group, the motion group, the affine group, the Heisenberg group or $SL(2\,,\,\mathbf{R})$. In [3], Cowling, Price, and Sitaram give a qualitative uncertainty principle valid for functions on all noncompact semisimple Lie groups with finite centres.

Suppose $t \in L^2(X \times X)$ where X is a measure space with measure m_X. Define an operator T on $L^2(X)$ by

$$(T\phi)(y) = \int_X t(x\,,\,y)\phi(x)\,dm_X(x) \qquad (\phi \in L^2(X))\,.$$

Then $T \in \mathscr{B}(L^2(X))$ and

$$\|T\| \le \|T\|_2 = \|t\|_2 < \infty \tag{1.2}$$

where $\|T\|$ denotes the usual operator norm of T and $\|t\|_2$ the $L^2(X \times X)$ norm of t.

Now let P and Q be orthogonal projections on a Hilbert space $\mathscr{H}$ and let $P \cap Q$ denote the unique orthogonal projection onto $\mathscr{H}_1$, the intersection of the ranges of P and Q. Clearly $PQ(\phi) = \phi$ for all $\phi \in \mathscr{H}_1$, hence, since the Hilbert-Schmidt norm of a projection is equal to the dimension of its range,

$$\|P \cap Q\|_2 \leq \|PQ\|_2 . \tag{1.3}$$

If E is a measurable subset of an LC group G, we denote its characteristic function by ξ_E and its complement by E' or $G \backslash E$. The annihilator $A(\widehat{G}, H)$ of a closed subset H of G is defined by

$$A(\widehat{G}, H) = \{\pi \in \widehat{G}; \pi(h) = \mathbf{1}_{\mathscr{H}_\pi} \text{ for all } h \in H\} .$$

The identity element of G will be denoted by e. The left regular representation L of G on $L^2(G)$ is strongly continuous and defined by

$$(L(a)\phi)(x) = \phi(a^{-1}x) \qquad (a \in G, \ \phi \in L^2(G)) .$$

2. The QUP

The satisfaction (or otherwise) of the QUP for a particular LC group G is largely determined by the topology of G.

Lemma 2.1. *Let G be an LC group with noncompact identity component G_0 and C be a measurable subset of G with $0 < m_G(C) < \infty$. Given $\varepsilon > 0$ and a measurable subset $C_0 \subseteq C$ with $m_G(C_0) > 0$, there exists $a \in G_0$ such that*

$$m_G(C) < m_G(C \cup aC_0) < m_G(C) + \varepsilon .$$

Proof. Define $h\colon G_0 \to \mathbf{R}^+ \ (= \{y \in \mathbf{R};\ y > 0\})$ by $h(a) = m_G(C \cup aC_0)$. Then h may also be written as

$$h(a) = \|L(a)\xi_{C_0} - \xi_C\|_2^2 + \langle L(a)\xi_{C_0}, \xi_C\rangle .$$

The strong continuity of L implies the continuity of h on G_0.

By the regularity of Haar measure, there is a compact set $K \subseteq C$ with $m_G(C \backslash K) < m_G(C_0)/2$. Let $M = KK^{-1}$, a compact subset of G. $M \cap G_0$ is then either compact in G_0 or empty and since G_0 is not compact, we may choose $a \in G_0 \backslash (M \cap G_0)$. With this choice of a, $aK \cap K$ is empty and so,

$$\begin{aligned} aC_0 \cap K &= a((C_0 \cap K) \cup (C_0 \cap K')) \cap K \\ &= (aC_0 \cap aK \cap K) \cup (aC_0 \cap aK' \cap K) \\ &= aC_0 \cap aK' \cap K \subseteq a(C_0 \cap K') . \end{aligned}$$

Therefore,

$$m_G(aC_0 \cap K) \leq m_G(C \cap K') < m_G(C_0)/2 \tag{2.1}$$

and

$$\begin{aligned} h(a) &= m_G(C \cup aC_0) \\ &= m_G((C \cap K) \cup (C \cap K') \cup (aC_0 \cap K) \cup (aC_0 \cap K')) \\ &\geq m_G((C \cap K) \cup (aC_0 \cap K')) \\ &= m_G(C) - m_G(C \cap K') + m_G(aC_0) - m_G(aC_0 \cap K) \\ &> m_G(C) \quad \text{(by the choice of } K \text{ and (2.1))} \\ &= h(e) . \end{aligned}$$

So h is a nonconstant continuous function on the connected set G_0. We may then choose $a \in G_0$ with

$$m_G(C) = h(e) < h(a) = m_G(C \cup aC_0) < h(e) + \varepsilon = m_G(C) + \varepsilon. \quad \square$$

Now let G be a unimodular group of type I equipped with Haar measure m_G and $\widehat{G}$ be its dual object equipped with Plancherel measure $m_{\widehat{G}}$. For $h \in \int_{\widehat{G}}^{\oplus} \mathrm{End}_2(\mathscr{H}_\pi) dm_{\widehat{G}}(\pi)$, denote by $\check{h}$ that element of $L^2(G)$ for which $(\check{h})^\wedge(\pi) = h(\pi)$ for almost every $\pi \in \widehat{G}$. Let $A \subseteq G$, $B \subseteq \widehat{G}$ be measurable subsets with $m_G(A) < m_G(G)$, $\mu(B) < \mu(\widehat{G})$. With slightly abusive notation, we denote by $\xi_B^\vee$ that element of $L^2(G)$ for which $(\xi_B^\vee)^\wedge(\pi) = \xi_B(\pi)\mathbf{1}_{\mathscr{H}_\pi}$ for almost every $\pi \in \widehat{G}$. Define projections E_A and F_B on $L^2(G)$ by

$$E_A f(x) = \xi_A(x) f(x), \qquad F_B f(x) = (\xi_B \hat{f})^\vee(x). \tag{2.2}$$

G then satisfies the QUP if, for all such subsets A and B, $(E_A \cap F_B)L^2(G) = \{0\}$. It is easily shown that if $f \in L^1(G) \cap L^2(G)$ and $g \in L^2(G)$, then $g * f \in L^2(G)$ and $(g * f)^\wedge = \hat{g}\hat{f}$ so that $g * f = (g * f)^{\wedge\vee} = (\hat{g}\hat{f})^\vee$. Therefore, if $f \in L^1(G) \cap L^2(G)$, $F_B f$ may be written as the convolution

$$F_B f(x) = \xi_B^\vee * f(x) = \int_G \xi_B^\vee(y) f(y^{-1}x) dm_G(x). \tag{2.3}$$

Theorem 2.2. *If G is a unimodular group of type* I *with noncompact identity component, then G satisfies the QUP.*

Proof. Suppose $f_0 \in (E_A \cap F_B)L^2(G)$, $f_0 \not\equiv 0$. Let $A_0 = \{x \in G;\, f_0(x) \neq 0\}$ $(m_G(A_0) > 0)$. Choose $N \in \mathbf{Z}^+$ $(= \mathbf{Z} \cap \mathbf{R}^+)$ such that $m_G(A_0)\mu(B) < N/2$. We define a sequence $\{a_i\}_{i=1}^N \subseteq G$ and an increasing sequence of measurable sets $\{A_i\}_{i=0}^N$ satisfying

$$m_G(A_i) < m_G(A_i \cup a_{i+1}A_0) < m_G(A_i) + 1/(2\mu(B)).$$

If we set $C = C_0 = A_0$, $\varepsilon = 1/(2\mu(B))$ in Lemma 2.1, then we are assured of the existence of $a_1 \in G$ with $m_G(A_0) < m_G(A_0 \cup a_1 A_0) < m_G(A_0) + 1/(2\mu(B))$. In general, with $C = A_i$, $C_0 = A_0$, and $\varepsilon = 1/(2\mu(B))$, continued application of Lemma 2.1 yields the required sequences. If $f \in L^2(G)$ then $E_{A_i} f \in L^1(G) \cap L^2(G)$ and by (2.3)

$$\begin{aligned} F_B E_{A_i} f(x) &= \int_G \xi_B^\vee(y) \xi_{A_i}(y^{-1}x) f(y^{-1}x) dm_G(y) \\ &= \int_G \xi_B^\vee(xy^{-1}) \xi_{A_i}(y) f(y) dm_G(y). \end{aligned} \tag{2.4}$$

Then, by (1.2) and (2.4),

$$\|F_B E_{A_i}\|_2^2 = \int_G \int_G |\xi_B^\vee(xy^{-1}) \xi_{A_i}(y)|^2 dm_G(y) dm_G(x) = m_G(A_i)\mu(B) \tag{2.5}$$

and by (1.3) and (2.5),

$$\begin{aligned} \dim(E_{A_N} \cap F_B)L^2(G) &\leq m_G(A_N)\mu(B) \\ &< [m_G(A_0) + N/(2\mu(B))]\mu(B) \\ &< N. \end{aligned} \tag{2.6}$$

Let $f_i = L(a_i)f_0$ so that $(f_i)^\wedge(\pi) = \pi(a_i)(f_0)^\wedge(\pi)$ for all $\pi \in \widehat{G}$. Then $F_B f_i = f_i$ $(0 \le i \le N)$. Since $A_m = A_0 \cup a_1 A_0 \cup \cdots \cup a_m A_0$ and $f_i = 0$ m_G-a.e. on $(a_i A_0)'$, we see that $E_{A_m} f_i = f_i$ for $0 \le i \le m$. Also $E_{A_m \setminus A_{m-1}} f_i = 0$ for $0 \le i \le m-1$ and $E_{A_m \setminus A_{m-1}} f_m \ne 0$. Therefore f_m is not a linear combination of $f_0, \dots, f_{m-1}$ and so $\{f_0, \dots, f_N\}$ is a set of $N+1$ linearly independent functions in $(E_{A_N} \cap F_B)L^2(G)$, thus contradicting (2.6). We conclude that $(E_A \cap F_B)L^2(G) = \{0\}$. □

A simple argument extends this result to functions in $L^P(G)$, $1 \le p \le \infty$.

Corollary 2.3. *Let G be as in the statement of Theorem 2.2, $\widehat{G}$ be the dual object of G, $A \subseteq G$, $B \subseteq \widehat{G}$ be measurable subsets with $m_G(A) < m_G(G)$, $\mu(B) < \mu(\widehat{G})$, and $1 \le p \le \infty$. If $f \in L^p(G)$, $f(x) = 0$ m_G-a.e. on A' and $\hat{f}(\pi) = 0$ $m_{\widehat{G}}$-a.e. on B', then $f = 0$ m_G-a.e.*

Proof. For $f \in L^1(G)$, $\sup_{\pi \in \widehat{G}} \|\hat{f}(\pi)\| \le \|f\|_1$. Therefore if $\pi \in \widehat{G}$ and $\{\phi_k\}$ is a complete orthonormal set in $\mathscr{H}_\pi$,

$$\|\hat{f}(\pi)\|_2^2 = \sum_k \|\hat{f}(\pi)\phi_k\|^2 \le \sum_k \sup_{\sigma \in \widehat{G}} \|\hat{f}(\sigma)\|^2 \|\phi_k\|^2 \le \|f\|_1^2 \dim(\mathscr{H}_\pi).$$

Now, if $f \in L^p(G)$ for some $p \in [1, \infty]$ and $f(x) = 0$ m_G-a.e. on A' then $f \in L^1(G)$ since

$$\|f\|_1 = \int_G |f(x)| \xi_A(x) dm_G(x) \le \|f\|_p m_G(A)^{1/p'}$$

(by Hölder's inequality with $p' = p/(p-1)$ for $1 < p < \infty$, $1' = \infty$ and $\infty' = 1$). So $\hat{f}(\pi)$ is well defined and $f \in L^2(G)$ since

$$\begin{aligned} \|f\|_2^2 &= \int_B \|\hat{f}(\pi)\|_2^2 dm_{\widehat{G}}(\pi) \\ &\le \|f\|_1^2 \int_B \dim(\mathscr{H}_\pi) dm_{\widehat{G}}(\pi) \\ &= \|f\|_1^2 \mu(B) < \infty. \end{aligned}$$

Applying Theorem 2.2 we see that $f = 0$ m_G-a.e. □

We say that the LC group G is a Plancherel group if the dual object $\widehat{G}$ can be equipped with a measure $m_{\widehat{G}}$ (the Plancherel measure) such that

$$\int_G |f(x)|^2 dm_G(x) = \int_{\widehat{G}} \|\hat{f}(\pi)\|_2^2 dm_{\widehat{G}}(\pi)$$

for all $f \in L^1(G) \cap L^2(G)$. (For a discussion of these groups, see [12].) The QUP, originally stated for unimodular groups of type I only, remains meaningful for Plancherel groups.

It is natural to ask whether the conditions given in Theorem 2.2 for the QUP to be satisfied are necessary. For example, does there exist a Plancherel group G and $f \in L^1(G) \cap L^2(G)$ with $m_G(A_f) < m_G(G)$, $\mu(B_f) < \mu(\widehat{G})$? Theorem 2.4 provides a partial answer to this question.

Theorem 2.4. *Let G be a noncompact, nondiscrete Plancherel group with a compact open normal subgroup G_1. Then the QUP is violated by G.*

Proof. Let $\alpha = m_G(G_1) > 0$ and ν_{G_1} be the restriction of m_G to G_1. Then $m_{G_1} = \alpha^{-1}\nu_{G_1}$ is a left Haar measure on G_1 for which $m_{G_1}(G_1) = 1$. Let $f = \xi_{G_1}$. Then $f \in L^1(G) \cap L^2(G)$ and $\|f\|_2^2 = \alpha$. Since G_1 is nondiscrete, it is certainly nontrivial so we have the strict inclusion $A(\widehat{G}, G_1) \subset \widehat{G}$. If $\pi \in A(\widehat{G}, G_1)$, then $\hat{f}(\pi) = \alpha \mathbf{1}_{\mathscr{H}_\pi}$. If $\pi \notin A(\widehat{G}, G_1)$, $\zeta, \eta \in \mathscr{H}_\pi$ and $a \in G_1$, then

$$\begin{aligned}
\langle \hat{f}(\pi)\zeta, \eta\rangle &= \int_G \xi_{G_1}(x)\langle \pi(x)\zeta, \eta\rangle dm_G(x) \\
&= \int_{G_1} \langle \pi(x)\zeta, \eta\rangle \alpha dm_{G_1}(x) \\
&= \int_{G_1} \langle \pi(xa)\zeta, \eta\rangle \alpha dm_{G_1}(x) \\
&= \int_G \xi_{G_1}(x)\langle \pi(x)\pi(a)\zeta, \eta\rangle dm_G(x) \\
&= \langle \hat{f}(\pi)\pi(a)\zeta, \eta\rangle .
\end{aligned}$$

Therefore, $\hat{f}(\pi)(\mathbf{1}_{\mathscr{H}_\pi} - \pi(a))\zeta = 0$ for each $\zeta \in \mathscr{H}_\pi$ and $a \in G_1$, i.e., $\hat{f}(\pi)\eta = 0$ for each $\eta \in V$ where V is the smallest closed subspace of $\mathscr{H}_\pi$ containing $\bigcup_{a\in G_1} R(\mathbf{1}_{\mathscr{H}_\pi} - \pi(a))$. (If T is a linear operator on a Hilbert space $\mathscr{H}$ then $R(T)$ denotes the range of T.) But for all $g \in G$, $a \in G_1$, and $\zeta \in \mathscr{H}_\pi$,

$$\begin{aligned}
\pi(g)(\mathbf{1}_{\mathscr{H}_\pi} - \pi(a))\zeta &= (\pi(g) - \pi(ga))\zeta = (\pi(g) - \pi(gag^{-1}g))\zeta \\
&= (\mathbf{1}_{\mathscr{H}_\pi} - \pi(gag^{-1}))\pi(g)\zeta
\end{aligned}$$

which is in V by the normality of G_1. So V is a closed invariant subspace of $\mathscr{H}_\pi$ which is proper since G_1 is nontrivial and $\pi \notin A(\widehat{G}, G_1)$. By the irreducibility of π, $V = \mathscr{H}_\pi$ and consequently $\hat{f}(\pi) = 0$. Therefore $\hat{f}(\pi) = m_G(G_1)\xi_{A(\widehat{G}, G_1)}(\pi)\mathbf{1}_{\mathscr{H}_\pi}$ $(\pi \in \widehat{G})$. But

$$\begin{aligned}
m_G(G_1) = \|f\|_2^2 &= \int_{\widehat{G}} \|m_G(G_1)\xi_{A(\widehat{G}, G_1)}(\pi)\mathbf{1}_{\mathscr{H}_\pi}\|_2^2 dm_{\widehat{G}}(\pi) \\
&= m_G(G_1)^2\mu(A(\widehat{G}, G_1)).
\end{aligned}$$

So, if $f = \xi_{G_1}$, we have $A_f = G_1$, $B_f = A(\widehat{G}, G_1)$, and $m_G(A_f)\mu(B_f) = 1$, and since $A(\widehat{G}, G_1)$ is a closed subset of $\widehat{G}$, the QUP is violated. □

If G is an LCA group, the dual object $\widehat{G}$ is also an LCA group. The Plancherel identity

$$\int_{\widehat{G}} |\hat{f}(\pi)|^2 dm_{\widehat{G}}(\pi) = \int_G |f(x)|^2 dm_G(x)$$

holds and the Plancherel measure $m_{\widehat{G}}$ is a suitable multiple of the Haar measure on $\widehat{G}$. All subgroups of G are normal so the identity component is compact if and only if G contains a compact open (normal) subgroup. Further, if $B \subseteq \widehat{G}$ is measurable, $\mu(B) = m_{\widehat{G}}(B)$. Applying these simplifications to Theorems 2.2 and 2.4 we get the following result which first appeared in [5].

Corollary 2.5. *If G is a noncompact nondiscrete LCA group with identity component G_0, the QUP is satisfied by G if and only if G_0 is noncompact.*

If G is an LCA group, Corollary 2.5 leaves only the cases where G is either compact or discrete to be dealt with. However, the above discussion and Theorem 8.4.1 of Rudin [11] may be adapted to show that an infinite compact abelian group satisfies the QUP if and only if it is connected and (by the duality between compact and discrete LCA groups and the symmetry between G and $\widehat{G}$ in the statement of the QUP) an infinite discrete abelian group satisfies the QUP if and only if it is torsion-free.

Our last result relates to the QUP for compact groups and the author is indebted to Michael Cowling for helpful suggestions regarding the proof.

Theorem 2.6. *Let G be an infinite compact group. Then G satisfies the QUP if and only if it is connected.*

Proof. Let G be an infinite compact disconnected group with identity component G_0. The quotient group G/G_0 is nontrivial and totally disconnected. Let $\phi\colon G \to G/G_0$ be the canonical homomorphism and $U \subset G/G_0$ be a proper open neighbourhood of the identity. Since G/G_0 is compact and totally disconnected, there exists a compact open normal subgroup K of G/G_0 with $K \subseteq U$. Let $G_1 = \phi^{-1}(K)$, a proper compact open normal subgroup of G. Then if $f = \xi_{G_1}$, we have $\hat{f} = m_G(G_1)\xi_{A(\widehat{G}, G_1)}\mathbf{1}_{\mathscr{H}_\pi}$ so $A_f = G_1$, $B_f = A(\widehat{G}, G_1)$, $m_G(A_f)\mu(B_f) = 1$ and, as before, the closedness of $A(\widehat{G}, G_1)$ and the nonemptiness of $A(\widehat{G}, G_1)'$ imply that the QUP is violated.

Now let G be an infinite compact connected group. Given an open neighbourhood U of the identity in G, there is a family $\mathscr{L}$ of compact normal subgroups of G such that $H \in \mathscr{L} \Rightarrow H \subseteq U$ and the quotient group G/H is a compact connected Lie group. $\mathscr{L}$ is in fact a directed set and we write

$$G = \varprojlim_{H \in \mathscr{L}} G/H$$

i.e., G is the projective limit of compact connected Lie groups G/H $(H \in \mathscr{L})$. Lipsman shows in [6] that

$$\widehat{G} = \varinjlim_{H \in \mathscr{L}} A(\widehat{G}, H)$$

i.e., $\widehat{G}$ is the corresponding injective limit of the annihilators $A(\widehat{G}, H)$ $(H \in \mathscr{L})$. In particular, given $\pi \in \widehat{G}$, there exists $H \in \mathscr{L}$ such that $\pi \in A(\widehat{G}, H)$. Now suppose $f \in L^2(G)$, $m_G(A_f) < 1$ $(= m_G(G))$ and $\mu(B_f) < \mu(\widehat{G})$. Then f is a trigonometric polynomial and we write

$$f(x) = \sum_{i=1}^{n} \dim(\mathscr{H}_{\pi_i}) \operatorname{tr}(A_i \pi_i(x))$$

where $\hat{f}(\pi_i) = A_i$ $(1 \le i \le n)$. There exist compact normal subgroups $H_i \subseteq G$ with $\pi_i \in A(\widehat{G}, H_i)$ $(1 \le i \le n)$. Let $H = \bigcap_{i=1}^n H_i$. Then $H \in \mathscr{L}$ and $\pi_i \in A(\widehat{G}, H)$ $(1 \le i \le n)$. If $h \in H$, $f(xh) = f(x)$ since $\pi_i(h) = \mathbf{1}_{\mathscr{H}_{\pi_i}}$. Let $\phi\colon G \to G/H$ be the canonical homomorphism and define $f_1 \in L^1(G/H)$

by $f_1(\phi(x)) = f(x)$ $(x \in G)$, a trigonometric polynomial on the compact connected Lie group G/H. Since f is right H-invariant, $A'_f = A'_f H$, so $f_1(\phi(x)) = 0$ for all $x \in A'_f$ and $m_{G/H}(\phi(A'_f)) = m_G(A_f) > 0$. Lemma 0.3 of [9] (which, in the language of this paper, states that a compact Lie group satisfies the QUP if and only if it is connected) now applies, giving $f_1 \equiv 0$. Hence $f \equiv 0$. □

It is perhaps also worthy of note that if G is a noncompact LC group with $\dim(\mathscr{H}_\pi) < \infty$ for all $\pi \in \widehat{G}$, the QUP is satisfied by G if and only if the identity component of G is noncompact since in this case, compactness of the identity component is equivalent to the existence of a compact open normal subgroup.

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References

1. W. O. Amrein and A. M. Berthier, *On support properties of L^p-functions and their Fourier transforms*, J. Funct. Anal. **24** (1977), 258–267.
2. M. Benedicks, *On Fourier transforms of functions supported on sets of finite Lebesgue measure*, J. Math. Anal. Appl. **106** (1985), 180–183.
3. M. Cowling, J. F. Price, and A. Sitaram, *A qualitative uncertainty principle for semisimple Lie groups*, J. Austral. Math. Soc. (Ser. A) **45** (1988), 127–132.
4. J. Dixmier, *C^*-algebras*, North-Holland, Amsterdam, 1977.
5. J. A. Hogan, *A qualitative uncertainty principle for locally compact abelian groups*, Proc. Centre Math. Anal. Austral. Nat. Univ. **16** (1988), 133–142.
6. R. Lipsman, *Representation theory of almost connected groups*, Pacific J. Math. **42** (1972), 453–467.
7. T. Matolcsi and J. Szücs, *Intersection des mesures spectrales conjugées*, C. R. Acad. Sci. Paris Ser. A. **277** (1973), 841–843.
8. C. Moore, *Groups with finite-dimensional irreducible representations*, Trans. Amer. Math. Soc. **166** (1972), 401–410.
9. J. F. Price and A. Sitaram, *Functions and their Fourier transforms with supports of finite measure for certain locally compact groups*, J. Funct. Anal. **79** (1988), 166–182.
10. A. Robert, *Introduction to the representation theory of compact and locally compact groups*, Cambridge Univ. Press, Cambridge, 1983.
11. W. Rudin, *Fourier analysis on groups*, Interscience, New York and London, 1967.
12. G. Warner, *Harmonic analysis on semisimple Lie groups*. I and II, Springer, Berlin, 1972.

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