

ON THE DISCRETE SERIES OF GENERALIZED STIEFEL MANIFOLDS

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ABSTRACT. A study of the discrete series of generalized Stiefel manifolds is made using the oscillator representation. New infinite families of such discrete series are constructed.

1. INTRODUCTION

Let D be one of the three division algebras \mathbf{R} , \mathbf{C} or \mathbf{H} (the quaternion algebra) over \mathbf{R} , with their usual involutions. Let V be a vector space over D , endowed with a nondegenerate hermitian form $(\ , \)$. Let $G = U(V)$ be the group of isometries of $(\ , \)$. Up to equivalence $(\ , \)$ is determined by its signature, say (p, q) . We write $G = U(p, q; D)$, or simply $G = U(p, q)$ when D is understood implicitly.

Fix a pair of integers (r, s) with $0 \leq r \leq p$, $0 \leq s \leq q$. We call the homogeneous space $U(p, q; D)/U(p-r, q-s; D)$ a generalized Stiefel manifold. This space carries a natural G -invariant measure, and hence there is a natural unitary representation of G on $L^2(U(p, q; D)/U(p-r, q-s; D))$ via left translations. We say that an irreducible unitary representation of G belongs to the discrete series of the (generalized) Stiefel manifold if it can be realized as a subrepresentation of $L^2(U(p, q; D)/U(p-r, q-s; D))$. The purpose of this paper is to introduce a new method for the study of this discrete series, and to construct new infinite families of discrete series for $U(p, q; D)/U(p-r, q-s; D)$. Let $n = r + s$. When $s = 0$, $U(p, q; D)/U(p-n, q; D)$ is what is called an indefinite Stiefel manifold [12], and its discrete series were studied by Schlichtkrull [18] and Kobayashi [12]. The results of the present paper will show that as soon as $n > 1$, the families of discrete series for $U(p, q; D)/U(p-n, q; D)$ constructed by Schlichtkrull and Kobayashi are not exhaustive. In fact there exist new infinite families of discrete series which are not in the lists of Schlichtkrull and Kobayashi.

Our understanding of the discrete series come from its relation with the oscillator representation. In §2 it will be shown that the study of discrete series of our generalized Stiefel manifolds is more or less equivalent to the study of the discrete spectrum of the oscillator representation associated to certain reductive dual pairs. This relationship was noticed before in special cases,

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notably by Gelbart [7] for $O(m)/O(n)$ and by Rallis and Schiffmann [17] for $O(p, q)/O(p - 1, q)$.

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2. THE OSCILLATOR REPRESENTATION AND STIEFEL MANIFOLDS

Let $G = U(p, q; D)$ be as in §1 and fix an integer $n \leq p + q$. We endow the D -vector space $V' = D^n \oplus (D^n)^*$ with a maximally split skew hermitian form $(,)'$, and let G' be its group of isometries. Thus $G' \simeq Sp_{2n}(\mathbf{R}), U(n, n)$ or $O^*(4n)$. We identify D^n and $(D^n)^*$ with the first and second summand of V' , and assume that they are both totally isotropic subspaces of V' . The form $(,)'$ identifies $D^n, (D^n)^*$ with duals of each other (hence the notation).

The pair of groups G, G' form a reductive dual pair [8] inside the symplectic group $Sp_{2m}(\mathbf{R})$, where

$$m = n(p + q) \cdot \dim_{\mathbf{R}}(D).$$

Let ω denote an oscillator representation of $Sp_{2m}(\mathbf{R})$. There are essentially two such representations depending on the choice of a nontrivial character of \mathbf{R} ; our choice here is specified by the formula (6) below. We also recall that ω is in fact a representation of the nontrivial two-fold cover of $Sp_{2m}(\mathbf{R})$. In the present setting, this two-fold cover is trivial when restricted to G . Hence the restriction of ω to G can and will be considered an ordinary representation of G .

To simplify the notations we write $V^n = \text{Hom}_D(D^n, V)$, the space of D -linear maps from D^n to V . The representation ω can be realized on $L^2(V^n)$, where the action of G is simply given by post-multiplication

$$(1) \quad \omega(g)f(x) = f(g^{-1}x)$$

for $g \in G, f \in L^2(V^n), x \in V^n$.

For each subgroup E of G' we let \tilde{E} be the pre-image of E in the two-fold cover of $Sp_{2m}(\mathbf{R})$. The action of \tilde{G}' is less straightforward. Let P be the stablizer of D^n in G' . This is a maximal parabolic subgroup of G' . If M denotes the subgroup of P consisting of those linear maps which preserve both D^n and $(D^n)^*$ then $M \simeq GL(n, D)$, and we have the Levi decomposition $P = MN$, where N is the unipotent radical of P . If $m \in M$ and \tilde{m} denotes a pre-image of m in \tilde{G}' then

$$(2) \quad \omega(\tilde{m})f(x) = \nu(\tilde{m})f(xm).$$

Here $\nu(\tilde{m})$ is a complex number whose absolute value $|\nu(\tilde{m})|$ is determined by the fact that the action defined by (2) is unitary. Also $\nu(\tilde{m})/|\nu(\tilde{m})|$ is at most a fourth root of unity.

The group N can be lifted to a unique subgroup of \tilde{G}' ; and that subgroup will be identified with N itself. It is easy to see that N is isomorphic to the space of hermitian forms on $(D^n)^*$, or equivalently, the space of $n \times n$ hermitian matrices with entries in D . We let $B((D^n)^*)$ denote that space. For each $\beta \in B((D^n)^*)$ we let n_β be the corresponding element of N . Similarly

we let $B(D^n)$ be the space of hermitian forms on D^n . Consider the natural involution

$$*: \text{End}_D(V') \rightarrow \text{End}_D(V'), \quad T \rightarrow T^*,$$

defined by the identity

$$(Tu, v)' = (u, T^*v)' \quad (u, v \in V').$$

This involution preserves $\text{Hom}_D(D^n, (D^n)^*)$, which is to be identified with the subspace of those endomorphisms of V' that vanish on $(D^n)^*$ and have their images contained in $(D^n)^*$. Similarly it preserves $\text{Hom}_D((D^n)^*, D^n)$. We have natural isomorphisms

$$B((D^n)^*) \simeq \{\beta \in \text{Hom}_D((D^n)^*, D^n) \mid \beta^* = \beta\},$$

$$B(D^n) \simeq \{\gamma \in \text{Hom}_D(D^n, (D^n)^*) \mid \gamma^* = \gamma\}.$$

We identify $B(D^n)$ and $B((D^n)^*)$ with linear duals of each other as follows. According to the above, given $\beta \in B((D^n)^*)$ and $\gamma \in B(D^n)$, the composite $\beta\gamma$ is an endomorphism of D^n . Let tr denote the map which takes a matrix to the sum of its diagonal entries, followed by the trace map of D over \mathbf{R} . The bilinear form

$$(3) \quad \langle \beta, \gamma \rangle = \text{tr}(\beta\gamma)$$

exhibits the duality involved.

We now consider the orbit parameter map

$$(4) \quad \tau: V^n = \text{Hom}_D(D^n, V) \rightarrow B(D^n)$$

where for $x \in \text{Hom}_D(D^n, V)$, $\tau(x)$ is the hermitian form defined by

$$(5) \quad \tau(x)(u, v) = (x(u), x(v)) \quad (u, v \in D^n).$$

The action of N on $L^2(X)$ is now given by

$$(6) \quad \omega(n_\beta)f(x) = \psi(\text{tr}(\tau(x)\beta))f(x) \quad (\beta \in B((D^n)^*)).$$

Here ψ is the character defined by $\psi(x) = e^{2\pi ix}$.

The group G acts on V^n by post-multiplications. Its generic orbits are classified by the orbit parameter map (4). More precisely we have the following version of Witt's theorem:

Lemma 2.1. *Let $x, x' \in V^n$. If $\tau(x) = \tau(x')$ and this hermitian form is non-degenerate then there is a $g \in G$ such that $gx = x'$.*

On the other hand, $M = GL(n, D)$ acts on V^n via pre-multiplication; and there is also a natural action of M on $B(D^n)$ so that the orbit parameter map τ is M -equivariant. An M -orbit on $B(D^n)$ is nothing but an equivalence class of hermitian forms on D^n . Let r, s be integers with $0 \leq r \leq p, 0 \leq s \leq q$ and $r + s = n$. Set

$$(7) \quad \Omega_{r,s} = \{x \in V^n \mid \tau(x) \text{ is of signature } (r, s)\}.$$

The above discussion shows that under the action of $G \times M$, the open orbits in V^n are precisely the sets $\Omega_{r,s}$ defined above. It follows that we have the direct sum decomposition

$$(8) \quad L^2(V^n) = \bigoplus L^2(\Omega_{r,s}), \quad (0 \leq r \leq p, 0 \leq s \leq q, r + s = n).$$

From (1), (2) and (6) we see that each summand in the above decomposition is stable under G and under \tilde{P} .

Fix a pair of integers r, s as above and choose $X_0 \in \Omega_{r,s}$. We get a fibration

$$(9) \quad G \times M \rightarrow \Omega_{r,s}, \quad (g, m) \rightarrow gX_0m.$$

Let $V_0 \subseteq V$ be the image of X_0 and let H be the stabilizer of V_0 in G . One has $H = H_1 \times H_2$ with $H_1 = U(V_0) \simeq U(r, s; D)$, $H_2 = U(V_0^\perp) = U(p-r, q-s; D)$. Note that X_0 defines a linear isomorphism from D^n to V_0 ; hence $X_0^{-1}: V_0 \rightarrow D^n$ is defined.

Lemma 2.2. *Let $g \in G, m \in M$. If $gX_0m = X_0$ then $g = h_1h_2$ with $h_1 \in H_1, h_2 \in H_2$ and $m = X_0^{-1}h_1^{-1}X_0$.*

Proof. We have $\tau(X_0) = \tau(gX_0m) = \tau(X_0m)$, which means m is in the isometry group of $\tau(X_0)$. Hence $m = X_0^{-1}hX_0$ with $h \in H_1$. Now $X_0 = ghX_0$, hence $gh \in H_2$, and the lemma follows by taking $h_1 = h^{-1}, h_2 = gh$.

If $h \in H_2$ then $hX_0 = X_0$, and hence (9) induces a fibration

$$(10) \quad \phi: G/H_2 \times M \rightarrow \Omega_{r,s}$$

where each fiber is isomorphic to H_1 . In other words we have

$$(11) \quad \Omega_{r,s} \simeq G/H_2 \times_{H_1} M.$$

Set $K = X_0^{-1}H_1X_0$. This is a closed subgroup of M . Let $\{M'_i\}$ be a countable family of left K -invariant open subsets of M whose union is all of M , such that for each i there is a subset R'_i of M with $M'_i = KR'_i$, and that the map $K \times R'_i \rightarrow M'_i$ is an isomorphism (of topological spaces). Now set

$$M_1 = M'_1, \quad M_{i+1} = M'_{i+1} - \bigcup_{j \leq i} M'_j.$$

Then M is the disjoint union of the M_i 's and for each i we have $M_i = KR_i$ for some $R_i \subseteq R'_i$. By Lemma 2.2 we have

$$G/H_2 \times_{H_1} M_i \simeq G/H_2 \times R_i.$$

Hence (11) implies

$$(12) \quad L^2(\Omega_{r,s}) = \bigoplus_i L^2(G/H_2 \times R_i)$$

where on the right-hand side G/H_2 is endowed with the unique (up to constants) G -invariant measure, while the measure on R_i comes from an M -invariant measure on $K \backslash M$.

There is one case where the above decomposition is particularly simple: suppose $r = 0$ or $s = 0$. Then K is a maximal compact subgroup of M . Let $M = KAN$ be an Iwasawa decomposition. Then (11) gives $\Omega_{r,s} \simeq G/H_2 \times AN$.

Theorem 2.3. *The unitary representation of G on $L^2(\Omega_{r,s})$ is a multiple of $L^2(G/H_2)$.*

Theorem 2.4. *The restriction of the oscillator representation ω to $G = U(p, q; D)$ is the direct sum of $L^2(U(p, q; D)/U(p-r, q-s; D))$, $r \leq p, s \leq q, r+s = n$, with multiplicities.*

These are obvious consequences of (8) and (12).

3. L^p -PROPERTIES

Let ρ be a unitary representation of G realized on the Hilbert space H_ρ .

Definition 3.1. Let $p > 0$ be a real number. We say that ρ is *strongly L^p* if there is a dense subset $H \subseteq H_\rho$ such that for any $\xi, \eta \in H$, the associated matrix coefficient $x \rightarrow (\rho(x)\xi, \eta)$ belongs to L^p . We say that ρ is *strongly $L^{p+\epsilon}$* if it is strongly $L^{p'}$ for any $p' > p$.

Definition 3.2. The *spectrum* of ρ is the set of irreducible unitary representations of G weakly contained in ρ .

Let $\omega_n = \omega|_G$ be the restriction to G of the oscillator representation (cf. §2).

Lemma 3.3. (a) *The representation ω_n is strongly $L^{p_n+\epsilon}$, where*

$$(13) \quad p_n = \frac{1}{n} \left(p + q - \frac{2}{\dim_{\mathbf{R}} D} \right).$$

(b) *The restriction of ω to G' is strongly L^r for any r satisfying*

$$(14) \quad r > \frac{4}{p+q} \left(n - 1 + \frac{1}{\dim_{\mathbf{R}} D} \right).$$

Proof. See [13, Theorem 3.2].

Theorem 3.4. *Let $2k$ be the smallest even integer satisfying*

$$(15) \quad 2k \geq \frac{1}{n} \left(p + q - \frac{2}{\dim_{\mathbf{R}} D} \right).$$

Let

$$\omega_n = \int_X^\oplus \pi_x d\mu(x)$$

be the direct integral decomposition of ω_n into irreducible unitary representations of G . Then for almost all x , the representation π_x is strongly $L^{2k+\epsilon}$.

Proof. Let K and P_1 be a maximal compact subgroup and a minimal parabolic subgroup of G , such that the Iwasawa decomposition $G = KP_1$ holds. Let δ be the left modular function of P_1 . Extend δ to a function on G by the formula $\delta(kp) = \delta(p)$ for $k \in K, p \in P_1$. Let Ξ be the function defined by Harish-Chandra

$$\Xi(x) = \int_K \delta^{-1/2}(xk) dk \quad (x \in G).$$

Let μ be an irreducible representation of K and ξ, η vectors in the space of ω_n which transforms under K according to a multiple of μ . By the corollary on [5, p. 108], we have for any $x \in G$,

$$(16) \quad |(\omega_n(x)\xi, \eta)| \leq (\dim \mu)^2 \|\xi\| \|\eta\| \Xi^{1/k}(x).$$

If ω_n is decomposed into a direct integral as stated in the theorem then the same estimate (16) is valid for almost all π_x . But, as the Harish-Chandra function $\Xi \in L^{2+\epsilon}(G)$, this implies that for almost all x, π_x is strongly $L^{2k+\epsilon}$. Q.E.D.

Remark. The appearance of the even integer $2k$ is an artifact resulting from the technical difficulties involved in the proof of [5]. Let p_n be as defined by (13). It is natural to wonder whether in the above theorem the representations π_x (for almost all x) are in fact strongly $L^{p_n+\varepsilon}$, as long as $p_n \geq 2$. Such an estimate would then be sharp, as can be seen from the next example.

Example. Consider the case $D = \mathbf{R}$, $q = 1$. Then $G = O(p, 1)$, $G' = Sp_{2n}(\mathbf{R})$. Let G_0 be the connected component of the identity in G . The spherical complementary series of G_0 may be parametrized by the interval $0 \leq \nu \leq \rho$ with $\rho = \frac{p-1}{2}$. Let π_ν be the representation corresponding to ν . Then π_0 is a tempered representation while π_ρ is the trivial one. For $0 < \nu < \rho$, π_ν is strongly $L^{(p-1)/(\rho-\nu)+\varepsilon}$ but not strongly $L^{(p-1)/(\rho-\nu)}$. Suppose $n < (p-1)/2$ then it can be shown that $\pi_{\rho-n}$ is a subrepresentation of (the restriction of) ω_n . Note that $p_n = (p-1)/(\rho-\nu)$ here!

Lemma 3.5. *Let τ be a strongly L^2 representation on the Hilbert space H . Let π be an irreducible subrepresentation of τ . Then π belongs to the discrete series of G .*

Proof. Let $H_\pi \subseteq H$ be the closed subspace affording the representation π . There is a dense subspace $H_0 \subseteq H$ such that all matrix coefficients associated to vectors in H_0 are in $L^2(G)$. We fix a vector $v_0 \in H_0$ which is not orthogonal to H_π . The map

$$\phi: H \rightarrow L^2(G)$$

which takes $v \in H$ to the matrix coefficient $(\tau(x)v, v_0)$ is a densely defined G -intertwining operator with a closable graph. By the general Schur's lemma [2], we get a unitary embedding of $(\ker \phi)^\perp$ into $L^2(G)$. Since H_π is not orthogonal to v_0 and π is irreducible, we see that $H_\pi \subseteq (\ker \phi)^\perp$. Thus π is an irreducible subrepresentation of $L^2(G)$. Q.E.D.

Theorem 3.6. *There is a one-to-one correspondence $\pi \rightarrow \sigma$ between irreducible subrepresentations of ω_n and $\omega|_{\tilde{G}'}$. The representations π and σ are Howe duals of each other in the sense of [8]. Furthermore, if $p + q \leq 2n$ then π belongs to the discrete series of G , i.e., it is square integrable. If $p + q \geq 2n$ then σ is square integrable.*

Proof. The one-to-one correspondence is established in [9]. Let H be the largest closed subspace of $L^2(V^n)$ on which G acts by a multiple of π . Then H is stable under G and \tilde{G}' . By [9, §7], the joint action of $G \times \tilde{G}'$ on H is of the form $\pi \otimes \sigma$, where σ is an irreducible unitary representation of \tilde{G}' . In particular, σ is an irreducible subrepresentation of $\omega|_{\tilde{G}'}$. Assume $p + q \leq 2n$. We have $p_n < 2$ in this case. According to Lemma 3.3, ω_n is strongly L^2 . So π is square integrable by Lemma 3.5. By [14, §2], we see that π and σ are the Howe duals of each other in the sense defined in [8].

The proof in the case $p + q \geq 2n$ is similar by reversing the role of G and G' . Note that in this case $\omega|_{\tilde{G}'}$ is strongly L^2 unless $p + q = 2n$ and $D = \mathbf{R}$. However in this special case our result is valid by [16] or [14]. Q.E.D.

Combined with results obtained in §2, we immediately deduce the following results about Stiefel manifolds.

Theorem 3.7. *Let $2k$ be the smallest even integer satisfying (15). Let*

$$L^2(U(p, q; D)/U(p - r, q - s; D)) = \int_X^{\oplus} \pi_x d\mu(x)$$

be the direct integral decomposition into irreducible unitary representations of G . Then for almost all x , the representation π_x is strongly $L^{2k+\epsilon}$.

Corollary 3.8. *Suppose $p + q \leq 2n$ then the spectrum of*

$$L^2(U(p, q; D)/U(p - r, q - s; D))$$

consists of tempered representations.

Theorem 3.9. *Let π be an irreducible unitary representation of G belonging to the discrete series of $U(p, q; D)/U(p - r, q - s; D)$. There is a unique irreducible unitary representation σ of \tilde{G}' which corresponds to π via Howe duality. Suppose $p + q \leq 2n$. Then π belongs to the discrete series of G itself; while if $p + q \geq 2n$ then σ is square integrable.*

Corollary 3.10. *Let $s = 0$. Suppose $p + q \leq 2n$ and $n \leq p$. Then the discrete series of the symmetric space $U(p, q; D)/U(n) \times U(p - n, q; D)$ must be contained in the discrete series of $U(p, q; D)$.*

Corollary 3.11. *Take $D = \mathbf{R}$ so that $U(p, q; D) = O(p, q)$. Suppose p, q are both odd and $p + q \leq 2n$. Then $O(p, q)/O(p - r, q - s)$ contains no discrete series. If also $n \leq p$ then the symmetric space $O(p, q)/O(n) \times O(p - n, q)$ contains no discrete series.*

Of course Corollary 3.10 and the last sentence of Corollary 3.11 would also follow from the known description of discrete series for symmetric spaces [15].

4. THE N -SPECTRUM AND DISCRETE SERIES

Definition 4.1. If π and σ are as in Theorem 3.7 we write $\pi = \theta(\sigma)$ and $\sigma = \theta(\pi)$. We let $\theta(\sigma) = 0$ if σ does not occur in Howe's correspondence.

In this section we assume $p + q \geq 2n$. Let σ be an irreducible square integrable representation of \tilde{G}' . By §3 we know that $\theta(\sigma)$ is in the discrete series of $U(p, q)/U(p - r, q - s)$ for some r, s with $r + s = n$. Our task here is to determine exactly which r, s . The answer to this lies in the notion of N -spectrum introduced in [10].

Consider the parabolic $P = MN$ of G' introduced in §2. Let \hat{N} be the Pontrjagin dual of N . We have $N \simeq B((D^n)^*)$ (cf. §2) and we can identify $B(D^n)$ with \hat{N} , by associating to $\gamma \in B(D^n)$ the character ψ_γ defined by

$$(17) \quad \psi_\gamma(x) = \psi(\text{tr}(x\gamma)).$$

Here ψ and other things are as in (6). Let σ be a unitary representation of P . By the direct integral theory [6], the restriction $\sigma|_N$ is determined up to unitary equivalence by a projection valued measure π_σ on \hat{N} . This projection valued measure, and the unitary equivalence class it defines, is called the N -spectrum of σ (see [10, §2]). Let \mathcal{O} be an Ad^*P -orbit in \hat{N} . As explained in [10], the restriction of π_σ to \mathcal{O} is of uniform multiplicity, and the N -spectrum of σ is determined by these multiplicities for all the orbits. Following [10], we say that

an orbit \mathcal{O} occurs in σ if its multiplicity is nonzero. We say σ is supported on the orbits which occur in σ . Via the identification of \widehat{N} with $B(D^n)$, an Ad^*P -orbit is the same thing as an equivalence class of hermitian forms on D^n , which is determined by the signature, say (r, s) , of any form in the class. We denote by $\mathcal{O}_{r,s}$ the orbit corresponding to forms of signature (r, s) . We have $r + s \leq n$ and $\mathcal{O}_{r,s}$ is open in \widehat{N} if and only if $r + s = n$. Finally, if σ is a representation of \widetilde{G}' we can restrict it first to \widetilde{P} and then to N . Hence the notion of N -spectrum applies to σ . We let $NS(\sigma)$ be the union of the Ad^*P -orbits which occur in σ .

Note that the decomposition (8) is preserved by \widetilde{P} . By (6) we have

Lemma 4.2. *In (8) the N -spectrum of the representation of \widetilde{P} on $L^2(\Omega_{r,s})$ is supported on the open orbit $\mathcal{O}_{r,s}$.*

Let σ be an irreducible square integrable representation for \widetilde{G}' . By the above lemma and (8), (12), we see that $\theta(\sigma)$ occurs in $L^2(U(p, q)/U(p - r, q - s))$ if and only if $\mathcal{O}_{r,s} \subseteq NS(\sigma)$. Write \widetilde{G}'_d for the set of (equivalence classes of) irreducible square integrable representations of \widetilde{G}' . We have

Theorem 4.3. *The discrete series of $U(p, q)/U(p - r, q - s)$ is precisely the set*

$$(18) \quad \{\theta(\sigma) \mid \sigma \in \widetilde{G}'_d, \mathcal{O}_{r,s} \subseteq NS(\sigma)\}.$$

To complete the description of the discrete series of $U(p, q)/U(p - r, q - s)$, it remains to

- (a) calculate the N -spectrum of $\sigma \in \widetilde{G}'_d$;
- (b) determine the parameter of $\theta(\sigma)$, here parameter means say the Langlands-Vogan parameter.

To a large degree, the answer to the first question is essentially known, at least conjecturally. The rest of this section will be devoted to a discussion of problem (a).

Let σ be a representation of \widetilde{G}' on a Hilbert space H . We recall from [11] the notion of the wave front set of σ . Let $J_1(H)$ be the trace class operators on H . Given $T \in J_1(H)$, we let $\text{tr}_\sigma(T)$ be the distribution defined by

$$\text{tr}_\sigma(T)(f) = \int_{\widetilde{G}'} f(g) \text{tr}(\sigma(g)T) dg.$$

We define $WF(\sigma)$ to be the closure of the union of the wave front sets at 0 for all $T \in J_1(H)$. (In the notation of [11], $WF(\sigma)$ is called WF_σ^0 .) Let \mathfrak{g} be the Lie algebra of G' . It is known that $WF(\sigma)$ is a union of nilpotent co-adjoint orbits in \mathfrak{g}^* . Now suppose σ has a distribution character Θ_σ and let $AS(\Theta_\sigma)$ be its asymptotic support, as defined in [3]. By [11 and 4, p. 159], we have the relation

$$(19) \quad AS(\Theta_\sigma) \subseteq WF(\sigma).$$

(Presumably this should be an equality; however no written proof is known to the author.)

Let \mathfrak{n} be the Lie algebra of N . Then $\mathfrak{n} \simeq B((D^n)^*)$ and via the formula (17) we have $\widehat{N} \simeq \mathfrak{n}^* \simeq B(D^n)$. We will use the following relation without proof,

since it is implicit in [11]: Let $\overline{NS(\sigma)}$ be the closure of $NS(\sigma)$ in \mathfrak{n}^* . Then

$$(20) \quad WF(\sigma|_N) = \overline{NS(\sigma)}.$$

Let

$$q: \mathfrak{g}^* \rightarrow \mathfrak{n}^*$$

be the restriction map. By [11, Proposition 1.5] we have

$$(21) \quad q(WF(\sigma)) \subseteq WF(\sigma|_N).$$

This together with (19)–(20) gives

$$(22) \quad q(AS(\Theta_\sigma)) \subseteq \overline{NS(\sigma)}.$$

It is this formula that we will use to get some information about $NS(\sigma)$, at least when σ is square-integrable.

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra. Choose compatible orderings for the root systems $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ (here the subscript \mathbb{C} indicates complexifications). Let \mathfrak{t}^+ be the set of regular elements in a positive Weyl chamber for \mathfrak{k} and assume $\mathfrak{t}^+ = \bigcup_{j=1}^s \mathfrak{t}^j$, where \mathfrak{t}^j are positive chambers for \mathfrak{g} .

Let now σ be a discrete series representation of \tilde{G}' with Harish-Chandra parameter $\lambda \in \mathfrak{t}^*$. We write $\sigma = \sigma_\lambda$ and let Θ_λ be its distribution character. We choose a nondegenerate symmetric bilinear invariant form $B(\cdot, \cdot)$ on \mathfrak{g} , by means of which we identify \mathfrak{g} with \mathfrak{g}^* . Then λ can be identified with a $Z_\lambda \in \mathfrak{t}^j$. By a result of Barbasch and Vogan [3, Proposition 3.7], we have

$$(23) \quad AS(\Theta_\lambda) = \{X \in \mathfrak{g} \mid \text{there exist } x_i \in G' \text{ and } t_i \in \mathbf{R} \text{ with } t_i \rightarrow 0, \\ \text{such that } X = \lim t_i \text{Ad } x_i Z_\lambda\}.$$

Further, $AS(\Theta_\lambda)$ depends only on the chamber \mathfrak{t}^j which contains Z_λ .

Let N_- be the unipotent radical of the parabolic opposite to P and let \mathfrak{n}_- be its Lie algebra. Then under the identification of \mathfrak{g} with \mathfrak{g}^* we have $\mathfrak{n}_- = \mathfrak{n}^* \simeq B(D^n)$ and the projection q is identified with a map $q: \mathfrak{g} \rightarrow \mathfrak{n}_-$. We now realize elements of \mathfrak{g} as $2n \times 2n$ matrices. Write $M_n(D)$ for the set of $n \times n$ matrices with entries in D . We have

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^t\bar{A} \end{pmatrix} \mid A, B, C \in M_n(D), {}^t\bar{B} = B, {}^t\bar{C} = C \right\}, \\ \mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid {}^t\bar{C} = C \right\}.$$

The projection $q: \mathfrak{g} \rightarrow B(D^n) = \mathfrak{n}_-$ is then given by the formula

$$(24) \quad q \begin{pmatrix} A & B \\ C & -{}^t\bar{A} \end{pmatrix} = B.$$

We take

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid {}^t\bar{A} = -A, {}^t\bar{B} = B \right\}.$$

Let i, j, k be the standard basis for the quaternion algebra so that $\mathbf{H} = \mathbf{R} \oplus \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k$. We consider $\mathbf{R} \subseteq \mathbf{C} = \mathbf{R} \oplus \mathbf{R}i \subseteq \mathbf{H}$. For the Cartan subalgebra \mathfrak{t} we take the set of matrices in \mathfrak{k} where A, B are both diagonal, and that the entries of A and B are in $\mathbf{R}i$ and \mathbf{R} respectively.

We write

$$(25) \quad Z_\lambda = \begin{pmatrix} R & S \\ -S & R \end{pmatrix}$$

where R, S are diagonal matrices as just described. In order to calculate the set $AS(\Theta_\lambda)$ given by (23), it is clearly sufficient to take the x_i there in a fixed open dense subset of G' . Now N_-P is open dense in G' , and if $x = n_-p$ with $n_- \in N_-$ and

$$p = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \in P$$

we have

$$\text{Ad}(x)(Z_\lambda) = \begin{pmatrix} * & A(S + BR - RB + BSB){}^tA \\ & * \end{pmatrix}$$

and hence by (24) we have

$$(26) \quad q(\text{Ad}(x)(Z_\lambda)) = A(S + BR - RB + BSB){}^tA.$$

Let \mathcal{O}_β denote the orbit, or equivalently the equivalence class, of $\beta \in B(D^n)$. Suppose \mathcal{O}_β is open in $B(D^n)$. We see that in order for $\mathcal{O}_\beta \subseteq q(AS(\Theta_\lambda))$, it is necessary that β be equivalent to a hermitian form of the form given by the right-hand side of (26). Conversely, it is trivially seen that this is also sufficient. We obtain

Proposition 4.4. *The open orbit \mathcal{O}_β is contained in $q(AS(\Theta_\lambda))$ if and only if there is a hermitian matrix B such that*

$$(27) \quad \beta \simeq S + BR - RB + BSB.$$

Here \simeq denotes equivalence of hermitian forms.

Note that an open orbit in $\overline{NS(\sigma_\lambda)}$ is necessarily contained in $NS(\sigma_\lambda)$. From (22) we have

Corollary 4.5. *Let Z_λ be given by (25). Suppose there is a hermitian matrix B such that $S + BR - RB + BSB$ has signature (r, s) , where $r + s = n$. Then the orbit $\mathcal{O}_{r,s} \subseteq NS(\sigma_\lambda)$.*

5. CONSTRUCTION OF SOME DISCRETE SERIES

We now write $S = \text{diag}(s_1, \dots, s_n)$, $R = i \cdot \text{diag}(r_1, \dots, r_n)$ and assume

$$(28) \quad s_1, \dots, s_k > 0 > s_{k+1}, \dots, s_n.$$

By taking $B = 0$ in (27) we see $\mathcal{O}_{k,l} \subseteq NS(\sigma_\lambda)$, where $l = n - k$.

Proposition 5.1. *Let $m_+(S)$ (resp. $m_-(S)$) be the largest integer t for which there are sequences of integers $0 < i_1, \dots, i_t \leq k$ and $k < j_1, \dots, j_t \leq n$, such that $s_{i_\mu} + s_{j_\mu} > |r_{i_\mu} - r_{j_\mu}|$ (resp. $s_{i_\mu} + s_{j_\mu} < -|r_{i_\mu} - r_{j_\mu}|$) for $\mu = 1, \dots, t$. Then*

$$(29) \quad NS(\sigma_\lambda) \supseteq \mathcal{O}_{k+\mu, l-\mu} \quad \text{for } -m_-(S) \leq \mu \leq m_+(S).$$

Proof. Consider first the case $n = 2$. We assume $s_1 > 0 > s_2$ (this being the only nontrivial case). We already know $\mathcal{O}_{1,1} \subseteq NS(\sigma_\lambda)$. Taking $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in (27) we obtain

$$S + BSB + BR - RB = \begin{pmatrix} s_1 + s_2 & -i(r_1 - r_2) \\ i(r_1 - r_2) & s_1 + s_2 \end{pmatrix}$$

which is positive definite or negative definite according as $s_1 + s_2 > |r_1 - r_2|$ or $s_1 + s_2 < -|r_1 - r_2|$. Hence by Corollary 4.5 we have $\mathcal{O}_{2,0} \subseteq NS(\sigma_\lambda)$ or $\mathcal{O}_{0,2} \subseteq NS(\sigma_\lambda)$ according as $s_1 + s_2 > |r_1 - r_2|$ or $s_1 + s_2 < -|r_1 - r_2|$. This proves the proposition when $n = 2$. For general n we divide the matrix B in (27) into suitable blocks; then it is easily seen that the result follows from the case $n = 2$. Q.E.D.

The theta lifting $\theta(\sigma_\lambda)$ is known in a significant number of cases, in fact it is known in most cases [1, 16, 17]. We can use this together with Proposition 5.1 to exhibit new infinite families of discrete series for the Stiefel manifolds $U(p, q; D)/U(p - r, q - s; D)$ for any pair of integers (r, s) with $r \leq p, s \leq q, r + s = n \leq \frac{1}{2}(p + q)$, except when $D = \mathbf{R}, p + q = 2n$ and p, q are both odd (cf. Corollary 3.11). To avoid carrying an extra baggage of notations along, we shall restrict ourselves to the real case for the rest of this section.

So assume $D = \mathbf{R}$. Then $R = 0$ and S alone determines λ . Suppose that $s_1 > \dots > s_k > 0 > s_{k+1} > \dots > s_n$ and set $l = n - k$. We assume $k \leq p/2$ and $l \leq q/2$. We may assume that s_1, \dots, s_n are in fact integers. To the parameter λ we may associate a parabolic subalgebra $\mathfrak{q} = \mathfrak{q}(\lambda)$, of the complexified Lie algebra of G . The precise definition of $\mathfrak{q}(\lambda)$ is given as in [14]. Let $L(\lambda)$ be the normalizer of \mathfrak{q} in G . We have

$$(30) \quad L(\lambda) \simeq T^n \times O(p - 2k, q - 2l)$$

where T is the circle group. Since the Pontrjagin dual of T^n is \mathbf{Z}^n we may identify the n -tuple of integers (s_1, \dots, s_n) with a character of T^n . This character has a unique extension to $L(\lambda)$, and we will abuse notation to call this λ . Consider the derived functor module $\mathcal{R}_{\mathfrak{q}}^S(\lambda)$, as defined in [19] and with normalization also as in [19]. The main result of [14] in the real case is then (see also [1] when $2n < \min(p, q)$ and [16] when p, q are both even).

Proposition 5.2. *Suppose $|s_j| \geq \frac{1}{2}(p + q) - n$ for $j = 1, \dots, n$. Then $\theta(\sigma_\lambda) = \mathcal{R}_{\mathfrak{q}}^S(\lambda)$.*

Fix (r, s) with $r \leq p, s \leq q, r + s = n \leq \frac{1}{2}(p + q)$. We want to show that $\mathcal{R}_{\mathfrak{q}}^S(\lambda)$ is in the discrete series for $O(p, q)/O(p - r, q - s)$ for appropriate λ . By Proposition 5.1 it suffices to have $k + m_+(S) \geq r$ and $l + m_-(S) \geq s$. This is easily arranged since $k + l = r + s = n$. We have thus constructed infinite families of discrete series for $O(p, q)/O(p - r, q - s)$.

Consider for example the simplest case $s = 0$. We have

Proposition 5.3. *Assume that $k \geq l$ and $s_j > |s_{k+j}|$ for $j = 1, \dots, l$. Then the representation $\mathcal{R}_{\mathfrak{q}}^S(\lambda)$, with $L(\lambda)$ given by (30), occurs in the discrete series of $O(p, q)/O(p - n, q)$.*

The family of discrete series for $O(p, q)/O(p - n, q)$ constructed by Kobayashi [12] are all of the form $\mathcal{R}_{\mathfrak{q}}^S(\lambda)$ where the normalizer of \mathfrak{q} in G is of the form $T^n \times O(p - 2n, q)$. Thus we have exhibited new infinite families of discrete series even in the case of indefinite Stiefel manifolds $O(p, q)/O(p - n, q)$. Of course similar results are valid in the complex and quaternionic cases.

6. THE SKEW-HERMITIAN CASE

We now state, and prove when necessary, some analogous results in the case when G is the isometry group of a skew-hermitian form. The results run parallel to the hermitian case. We will therefore omit some obvious details.

Suppose the form $(\ , \)$ on V is skew-hermitian (and nondegenerate). Let $G = U(V)$ be its isometry group. Let $V_1 \subseteq V$ be a nondegenerate subspace, and consider the generalized Stiefel manifold $G/G(V_1^\perp)$. It is only necessary to consider the case where $D = \mathbf{R}$ or \mathbf{H} . Then up to equivalence there is only one nondegenerate skew-hermitian form on V . Set $n = \dim_D V_1$ and let $(\ , \)'$ be a nondegenerate hermitian form on $D^n \oplus (D^n)^*$, so that D^n and $(D^n)^*$ are both totally isotropic subspaces with respect to $(\ , \)'$. Let G' be the group of isometries of $(\ , \)'$. Analogous to §2, we have the dual pair (G, G') and the oscillator representation ω acting on $L^2(V^n)$, where $V^n = \text{Hom}(D^n, V)$. Let P be the stabilizer of D^n in G' , with $M \simeq GL(n, D)$ the Levi component. Then G and P act via formulas analogous to (1), (2) and (6). Let $B'(D^n)$ denote the space of skew-hermitian forms on D^n , and define the orbit parameter map $\tau: V^n \rightarrow B'(D^n)$ as in (5). Set

$$(31) \quad \Omega = \{x \in V^n \mid \tau(x) \text{ is nondegenerate}\}.$$

Then Ω is open in V , and its complement in V^n is a subvariety of positive codimension, hence $L^2(V^n) = L^2(\Omega)$. Since there is only one equivalence class of nondegenerate skew-hermitian forms on D^n , Ω is a single orbit under $G \times M$. Now one has the obvious analogue of (12), which implies

Theorem 6.1. *The restriction of ω to G is a multiple of $L^2(G/H_2)$, where $H_2 = U(V_1^\perp)$.*

Corollary 6.2. *Let $2k$ be the smallest even integer satisfying*

$$(32) \quad 2k \geq \frac{1}{2} \left(\dim_D V - 2 + \frac{2}{\dim_{\mathbf{R}} D} \right).$$

Let

$$L^2(G/H_2) = \int_X^\oplus \pi_x d\mu(x)$$

be the direct integral decomposition of $L^2(G/H_2)$ into irreducible unitary representations of G . Then for almost all x , the representation π_x is strongly $L^{2k+\epsilon}$.

Proof. This follows from [13, Theorem 3.2] and Theorem 6.1 in the same way as Theorem 3.7 follows from Lemma 3.3 and Theorem 3.4.

Theorem 6.3. *The irreducible subrepresentations $L(G/H_2)$ are in one-to-one correspondence with the irreducible subrepresentations of $\omega|_{G'}$ by Howe's duality correspondence. If π and σ corresponds to each other we write $\pi = \theta(\sigma)$. Then*

- (a) *If $\dim_D V \leq 2n$ then π belongs to the discrete series of G .*
- (b) *Suppose $\dim_D V \geq 2n$ then the discrete series of G/H_2 is precisely the set*

$$(33) \quad \{\theta(\sigma) \mid \sigma \in \widehat{G}'_d\}.$$

Here \widehat{G}'_d denotes the discrete series representations of G' .

The proofs are similar to those of §§2–3, so we omit them. Note that here the restriction of the oscillator representation to G' can also be made into an ordinary representation, hence in the above we spoke of representations of G' rather than its two-fold cover.

We remark that by results in [1, 16, and 14], most of the representations $\theta(\sigma)$ in Theorem 6.3 are realized as Zuckerman functor modules $\mathcal{R}_{\mathfrak{q}}^S(\lambda)$, where the normalizer of \mathfrak{q} in G is of the form $T^n \times Sp_{m-2n}(\mathbf{R})$ in the real case, and $T^{2n} \times O^*(2(m-2n))$ in the quaternionic case. Here m is the dimension of V over D .

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