TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 341, Number 1, January 1994

ON THE RANK AND THE CRANK MODULO 4 AND 8

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ABSTRACT. In this paper we prove some identities, conjectured by Lewis, for the rank and crank of partitions concerning the modulo 4 and 8. These identities are similar to Dyson's identities for the rank modulo 5 and 7 which give a combinatorial interpretation to Ramanujan's partition congruences. For this, we use multisection of series and some of the results that Watson established for the third order mock theta functions.

1. INTRODUCTION

Let π denote an integer partition with the parts arranged in nonincreasing order. Dyson [D1] defined the "rank" of a partition as the difference between the largest part of the number of parts. Denoting by $N(m, t, n)$ the number of partitions of n with rank congruent to m modulo t, he conjectured several relations between the numbers $N(m, t, kn+s)$ when $t = k = 5$ and when $t = k = 7$, which were proved by Atkin and Swinnerton-Dyer [A-S]. Of special importance are the following relations

$$N(0, 5, 5n+4) = N(1, 5, 5n+4) = \cdots = N(4, 5, 5n+4), \tag{1}$$

$$N(0, 7, 7n+5) = N(1, 7, 7n+5) = \cdots = N(6, 7, 7n+5) \tag{2}$$

since they provide a combinatorial interpretation to Ramanujan congruences [Ra], viz. $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$.

Dyson also conjectured the existence of a partition statistic, similar to the rank, which could be used to divide the partitions of $11n+6$ into eleven equal classes, and he even gave a name to this partition statistic: "crank." Denoting by $M(m, t, n)$ the number of partitions of n whose crank is congruent to m $\pmod t$, he conjectured

$$M(m, t, n) = M(t-m, t, n), \tag{3}$$

$$M(0, 11, 11n+6) = M(1, 11, 11n+6) = \cdots = M(6, 11, 11n+6) \tag{4}$$

as well as other relations for the crank modulo 11.

Garvan [G1, G2] discovered a protocrank. He defined certain "vector partitions" and assigned to each such partition a "rank." Denoting by $N_v(m, t, n)$ the weighted count of the vector partitions of n with rank congruent to m

Received by the editors October 15, 1990 and, in revised form, November 15, 1991.
1991 *Mathematics Subject Classification*. Primary 11P76.
Key words and phrases. Partitions, rank, crank, mock theta functions.

(mod t), he showed

$$N_v(m, t, n) = N_v(t-m, t, n), \tag{5}$$

$$N_v(0, 11, 11n+6) = N_v(1, 11, 11n+6) = \cdots = N_v(6, 11, 11n+6). \tag{6}$$

Garvan's protocrank not only satisfies (6) but also statements equivalent to (1) and (2) which provide another combinatorial interpretation to Ramanujan's congruences.

Recently, Andrews and Garvan [A-G], discovered the crank whose existence has been conjectured by Dyson. For a partition π, $\lambda(\pi)$ denotes the largest part, $\nu(\pi)$ denotes the number of ones, and $\mu(\pi)$ the number of parts larger than $\nu(\pi)$. The crank of π, denoted by $c(\pi)$, is defined as follows

$$c(\pi) = \begin{cases} \lambda(\pi), & \text{when } \nu(\pi) = 0, \\ \mu(\pi) - \nu(\pi), & \text{otherwise}. \end{cases} \tag{7}$$

Lately, Garvan, Kim, and Stanton [G-K-S] have found a new partition statistic (which they also call crank) which not only explains combinatorially all the previously mentioned Ramanujan's congruences, but also some generalizations of these congruences which were stated by Ramanujan.

The purpose of this paper is to prove some conjectures stated by Lewis [L1, L2], concerning the rank and the crank of a partition with respect to the modulo 4 and 8. Similar relations were proved by Garvan [G2] and Lewis [L1], but those relations concern the modulo 5 and 7. Lewis's statements are the following:

$$N(2, 4, 2n) = M(1, 4, 2n), \quad \text{for all } n \geq 1, \tag{8}$$

$$N(0, 4, 2n+1) = M(1, 4, 2n+1), \quad \text{for all } n \geq 1, \tag{9}$$

$$M(1, 8, 4n) = M(3, 8, 4n) = N(2, 8, 4n) = N(4, 8, 4n), \quad \text{for all } n \geq 1, \tag{10}$$

$$\begin{aligned} &M(0, 8, 4n+1) + M(1, 8, 4n+1) \\ &\quad = M(3, 8, 4n+1) + M(4, 8, 4n+1) \\ &\quad = N(1, 8, 4n+1) + N(2, 8, 4n+1) \\ &\quad = N(3, 9, 4n+1) + N(4, 8, 4N+1), \quad \text{for all } n \geq 0, \end{aligned} \tag{11}$$

$$\begin{aligned} M(1, 8, 4n+2) &= M(3, 8, 4n+2) = N(2, 8, 4n+2) \\ &= N(0, 8, 4n+2), \quad \text{for all } n \geq 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &M(0, 8, 4n+3) + M(1, 8, 4n+3) \\ &\quad = M(3, 8, 4n+3) + M(4, 8, 4n+3) \\ &\quad = N(0, 8, 4n+3) + N(1, 8, 4n+3) \\ &\quad = N(2, 8, 4n+3) + N(3, 8, 4n+3), \quad \text{for all } n \geq 0, \end{aligned} \tag{13}$$

$$N(3, 8, 4n) = M(2, 8, 4n), \quad \text{for all } n \geq 1, \tag{14}$$

$$N(3, 8, 4n+1) = M(2, 8, 4n+1), \quad \text{for all } n \geq 0, \tag{15}$$

$$N(1, 8, 4n+2) = M(2, 8, 4n+2), \quad \text{for all } n \geq 0, \tag{16}$$

$$N(1, 8, 4n+3) = M(2, 8, 4n+3), \quad \text{for all } n \geq 0. \tag{17}$$

Equations (8) and (9) have been proved by Lewis [L2]. To do this Lewis uses similar methods to the ones devised by Atkin and Swinnerton-Dyer [A-S] to prove Dyson's conjectures, concerning the rank modulo 5 and 7. In our proof we only use a convenient partial fraction decomposition followed by a multisection of series, and all this leads to one of the relations for the third order mock theta functions established by Watson in [W]. Equations (14) to (17) are proved in a similar way, and for the equalities involving only the rank in equations (10) and (13), we use the same method but at the end we have to prove a theta identity.

The equalities involving only the crank in (10) to (13) were proved by Garvan [G3]. The equalities in (10) and (12) that relate the rank and the crank follow from (8), (9), and some simple observations about the rank and the crank, as noted by Lewis [L2]. Using an observation from Lewis [L1] we will prove the equalities relating the rank and the crank in (11) and (13).

Our methods can be extended to prove all of Lewis' conjectures [L1]; these results will appear in a forthcoming paper by one of us. Some of the results of this paper are contained in one of the authors' doctoral thesis [S].

Note. For the summation symbols we will use the following conventions

$$\sum_n = \sum_{n=-\infty}^{\infty}, \qquad \sum_{n\neq 0} = \sum_{n=-\infty\,(n\neq 0)}^{\infty} .$$

2. Preliminary results

In this section we present most of the results that will be needed to prove equations (8) to (17). The following theorem appears in Riordan's book [Ri, p. 131],

Theorem 1. *If $f(q)$ is the generating function for the coefficients A_n, so*

$$f(q) = \sum_{n=0}^{\infty} A_n q^n ,$$

and ξ is a primitive t-root of 1, then

$$\sum_{m=0}^{\infty} A_{tm+r} q^{tm+r} = \frac{1}{t}\sum_{k=0}^{t-1} \xi^{(t-r)k} f(\xi^k q) .$$

Next, we generalize equation (6.2) from Atkin and Swinnerton-Dyer [A-S]. For a, b positive integers, define

$$S_b(a, \alpha) = \sum_{n\neq 0} \frac{(-1)^n q^{n(an+1)/2+\alpha n}}{1-q^{bn}} .$$

Writing $-n$ for n we have

$$S_b(a, \alpha) = -S_b(a, b-1-\alpha) . \tag{18}$$

Also we need the generating function for $N(m, t, n)$, given by

$$\sum_{n=0}^{\infty} N(m, t, n) q^n = \frac{1}{(q;q)_\infty} \sum_{n\neq 0} \frac{(-1)^n q^{n(3n+1)/2}[q^{mn} + q^{(t-m)n}]}{1-q^{tn}} . \tag{19}$$

From (19) it is clear that

$$N(m, t, n) = N(t-m, m, n). \tag{20}$$

We also need the following results, established by Andrews and Garvan [A-G]

$$M(m, n) = N_v(m, n), \quad \text{for all } n > 1. \tag{21}$$

$$M(m, t, n) = N_v(m, t, n), \quad \text{for all } n > 1. \tag{22}$$

A combinatorial proof of (21) has been found by Dyson [D2].

Note. Examining (8) to (17) we notice that $M(m, t, n)$ can be replaced by $N_v(m, t, n)$ in all the equations, since the only problem that could arise would be in (11) or (15), with $n = 0$. In both cases this can be overcome since, for instance, $N_v(2, 8, 1) = N(3, 8, 1) = M(2, 8, 1) = 0$ and the new version of (15) holds. We also notice that Lewis used Dyson's crank (defined in [D2]), which is slightly different from the Andrews-Garvan crank. With Dyson's crank equations (8) to (17) hold for all $n \geq 0$.

Next, we need the generating function for $N_v(m, n)$ which is equation (7.20) of [G1]

$$\sum_{n=0}^{\infty} N_v(m, n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2+n|m|}(1-q^n). \tag{23}$$

Using equality (23) Santa-Gadea [S] proved that the generating function for $N_v(m, t, n)$ is given by

$$\sum_{n=0}^{\infty} N_v(m, t, n) q^n = \frac{1}{(q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(n+1)/2}[q^{mn} + q^{(t-m)n}]}{1-q^{tn}}. \tag{24}$$

From [W] we need the following definitions and results, concerning the third order mock theta functions:

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = \frac{2}{(q;q)_\infty} \sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}, \tag{25}$$

$$\phi_3(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} = \frac{2}{(q;q)_\infty} \sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}}, \tag{26}$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} = \frac{1}{(q^4;q^4)_\infty} \sum_n \frac{(-1)^n q^{6n(n+1)+1}}{1-q^{4n+1}}, \tag{27}$$

$$\begin{aligned} 2\phi_3(-q) f(q) = f(q) + 4\psi(-q) &= \phi_3(-q) + 2\psi(-q) \\ &= \frac{(q;q)_\infty}{(-q;q)_\infty^2} = \Pi_2(q). \end{aligned} \tag{28}$$

Equation (28) is only one of the many relations that Watson [W] proved using the following theorem

Theorem 2. *Let r be any integer and let α, β and γ, be any constants such that*

$$\vartheta_1(\beta-\gamma)\vartheta_1(\gamma-\alpha)\vartheta_1(\alpha-\beta) \neq 0.$$

Then

(29)
$$\frac{e^{(2r-1)iz}\vartheta_i(0)}{\vartheta_2(z-\alpha)\vartheta_2(z-\beta)\vartheta_2(z-\gamma)} = 2\sum_{\alpha,\beta,\gamma}\sum_m (-1)^{m+r}q^{m(3m+1)+2mr}\frac{e^{2mi(2\alpha-\beta-\gamma)+(2r+1)i\alpha}}{\vartheta_1(\alpha-\beta)\vartheta_1(\alpha-\gamma)(e^{2iz}+q^{2m}e^{2i\alpha})}$$

where $\vartheta_1(z)$, $\vartheta_2(z)$ *denote* $\vartheta_1(z, q)$ *and* $\vartheta_2(z, q)$, *defined by*

$$\vartheta_1(z, q) = -iq^{1/4}e^{iz}(q^2; q^2)_\infty(e^{2iz}q^2; q^2)_\infty(e^{-2iz}; q^2)_\infty,$$
$$\vartheta_2(z, q) = q^{1/4}e^{iz}(q^2; q^2)_\infty(-e^{2iz}q^2; q^2)_\infty(-e^{-2iz}; q^2)_\infty$$

with $q = \exp(i\tau)$.

We also need equation (7.15) of Garvan [G1] which can be restated in terms of bilateral series as follows

$$(1-z)(1-z^{-1})\sum_n \frac{(-1)^n q^{n(n+1)/2}}{1-(z+z^{-1})q^n+q^{2n}} = \frac{(q;q)_\infty^2}{(zq;q)_\infty(z^{-1}q;q)_\infty}. \tag{30}$$

From Hickerson [H] we need the following notation and results. He defines $j(x, q)$ (for $x \neq 0$ and $|q| < 1$) as follows:

$$\begin{aligned} j(x, q) &= (x; q)_\infty(q/x; q)_\infty(q; q)_\infty \\ &= \sum_\infty (-1)^n q^{n(n-1)/2}x^n \end{aligned} \tag{31}$$

and uses the following notation

$$J_{a,m} = j(q^a, q^m), \quad \overline{J}_{a,m} = j(-q^a, q^m), \quad J_m = J_{m,3m} = (q^m; q^m)_\infty.$$

In the same paper Hickerson proves several relations for $j(x, q)$. We need the following result which follows letting $y = x$ in Theorem 1.1 of [H]

$$j^2(x, q) = j(-x^2, q^2)j(-q, q^2) - xj(-x^2q, q^2)j(-1, q^2). \tag{32}$$

Next, we quote Theorem (1.1) of [A-H]:

Theorem 2. *If* n *is a positive integer,* $0 < |q| < 1$, *and* $x \neq 0$, *then*

$$j(x, q) = \sum_{k=0}^{n-1}(-1)^k q^{k(k-1)/2}x^k j[(-1)^{n+1}q^{n(n-1)/2+kn}x^n, q^{n^2}]. \tag{33}$$

In particular if $n = 2$ we obtain

$$j(x, q) = j(-x^2q, q^4) - xj(-x^2q^3, q^4). \tag{34}$$

Finally we need the following theorem, which is Theorem (4.1) of Santa-Gadea [S]

Theorem 3. *If* $|q| < 1$, $z \neq 0, 1, -1, i, -i$, *then*

$$\begin{aligned} &\frac{1}{(-q;-q)_\infty}\sum_n \frac{(-1)^n(-q)^{3n(n+1)/2}}{1+z^2q^{2n}} + \frac{1}{(q;q)_\infty}\sum_n \frac{(-1)^n q^{3n(n+1)/2}}{1-z^2q^{2n}} \\ &= \frac{2(q^4;q^4)_\infty^2(-q^2;q^2)_\infty(-z^2q;q^4)_\infty(-z^{-2}q^3;q^4)_\infty}{(1-z^4)(q^2;q^2)_\infty(z^4q^4;q^4)_\infty(z^{-4}q^4;q^4)_\infty}. \end{aligned} \tag{35}$$

Proof. Equation (1) of [J] can be rewritten as follows.

$$\begin{aligned}(36)\quad &\sum_n (-1)^n q^{3n(n+1)/2}\left[\frac{(\xi^{-3n}+\xi^{3n+3})-zq^n(\xi^{-3n+1}+\xi^{3n+2})}{1-zq^n(\xi+\xi^{-1})+z^2q^{2n}}\right]\\ &= \frac{\xi P(\xi^2, q)}{P(\xi, q)}\sum_n \frac{(-1)^n q^{3n(n+1)/2}}{1-zq^n} + \frac{P(\xi, q)P(\xi^2, q)(q;q)_\infty^2}{P(z\xi^{-1}, q)P(z, q)P(z\xi, q)}\end{aligned}$$

where

$$P(z, q) = (1-z)(zq;q)_\infty(z^{-1}q;q)_\infty = \frac{j(z, q)}{J_1}.$$

Letting $\xi = i$ in (36) we first notice that

$$\begin{aligned}(37)\quad &(\xi^{-3n}+\xi^{3n+3})-zq^n(\xi^{-3n+1}+\xi^{3n+2})\\ &= i^{3n}[(-1)^n - i - zq^n(-1)^n i + zq^n]\\ &= i^{3n}[(-1)^n - i][1+(-1)^n zq^n].\end{aligned}$$

But,

$$i^{3n}[(-1)^n - i] = (1-i)(-1)^{3n(n+1)/2}, \qquad \text{for all } n.$$

Thus the left-hand side of (36) becomes

$$(38)\qquad L = (1-i)\sum_n \frac{(-1)^n(-q)^{3n(n+1)/2}[1+(-1)^n zq^n]}{1+z^2q^{2n}}.$$

Meanwhile, the right-hand side of (36) equals

$$(39)\qquad R = \frac{iP(-1, q)}{P(i, q)}\sum_n \frac{(-1)^n q^{3n(n+1)/2}}{1-zq^n} + \frac{P(i, q)P(-1, q)(q;q)_\infty^2}{P(-zi, q)P(z, q)P(zi, q)}.$$

Replacing z by $-z$ in (38) and (39) and adding the resulting equations to (38) and (39), respectively, we obtain

$$\begin{aligned}(40)\quad 2(1-i)\sum_n \frac{(-1)^n(-q)^{3n(n+1)/2}}{1+z^2q^{2n}} &= \frac{2iP(-1, q)}{P(i, q)}\sum_n \frac{(-1)^n q^{3n(n+1)/2}}{1-z^2q^{2n}}\\ &+ \frac{P(i, q)P(-1, q)(q;q)_\infty^2}{P(-zi, q)P(zi, q)}\left[\frac{1}{P(z, q)} + \frac{1}{P(-z, q)}\right].\end{aligned}$$

The result follows after some simplifications if we use

$$(41)\qquad j(z, q) + j(-z, q) = 2j(-z^2q, q^4).$$

3. The rank and the crank modulus 4

To prove (8) and (9) we first place $M(1, 4, 2n)$ by $N_v(1, 4, 2n)$ and $M(1, 4, 2n+1)$ by $N_v(1, 4, 2n+1)$. Moreover, as $N(2, 4, 0) = 0 = N_v(1, 4, 0)$ and $N(0, 4, 1) = 1 = N_v(1, 4, 1)$, we can rewrite (8) and (9) as follows

$$(42)\qquad N(2, 4, 2n) = N_v(1, 4, 2n), \qquad \text{for all } n \geq 0.$$

$$(43)\qquad N(0, 4, 2n+1) = N_v(1, 4, 2n+1), \qquad \text{for all } n \geq 0.$$

The generating function for $N(2, 4, n)$ is equal to

$$\begin{aligned}\sum_{n=0}^{\infty} N(2, 4, n)q^n &= \frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}\cdot 2q^{2n}}{1-q^{4n}} \\ &= \frac{1}{(q)_\infty}\left[\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}}{1-q^{2n}} - \sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}}\right] \\ &= A'(q) - B'(q)\end{aligned} \tag{44}$$

where

$$A'(q) = \frac{1}{(q)_\infty}\left[\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}}{1-q^{2n}}\right] = \frac{1}{2(q)_\infty}\left[\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{n}}\right]$$

since

$$\frac{1}{1-q^{2n}} = \frac{1}{2}\left[\frac{1}{1+q^n} + \frac{1}{1-q^n}\right]$$

and from (18) $S_1(3, 0) = 0$. Here we define

$$B'(q) = \frac{1}{(q)_\infty}\left[\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}}\right].$$

Similarly, we find that the generating function for $N(0, 4, n)$ is equal to

$$\begin{aligned}\sum_{n=0}^{\infty} N(0, 4, n)q^n &= \frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}\cdot (1+q^{4n})}{1-q^{4n}} \\ &= -1 + \frac{1}{(q)_\infty} + A'(q) + B'(q)\end{aligned} \tag{45}$$

where the last line follows by partial fraction decomposition and the fact that $S_1(1, 0) = 0$.

On the other hand the generating function for $N_v(1, 4, n)$ is equal to

$$\begin{aligned}\sum_{n=0}^{\infty} N_v(1, 4, n)q^n &= \frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(n+1)/2}\cdot (q^n+q^{3n})}{1-q^{4n}} \\ &= \frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(n+1)/2} q^n}{1-q^{2n}} \\ &= -\frac{1}{2(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(n+1)/2}}{1+q^n} \\ &= -C'(q).\end{aligned} \tag{46}$$

We define $A(q)$, $B(q)$ and $C(q)$ respectively to be the same as $A'(q)$, $B'(q)$ and $C'(q)$ respectively except the term corresponding to $n = 0$ in the summation part has been added; i.e. $A = A' + 1/4(q)^\infty$, $B = B' + 1/2(q)^\infty$ and $C = C' + 1/4(q)^\infty$. Now we can express the generating function for $N(2, 4, n) - N_v(1, 4, n)$ as follows

$$D_1(q) = \sum_{n=0}^{\infty}[N(2, 4, n) - N_v(1, 4, n)]q^n = A(q) - B(q) + C(q). \tag{47}$$

Also, the generating function for $N(0, 4, n) - N_v(1, 4, n)$ can be written as follows

$$D_2(q) = \sum_{n=0}^{\infty}[N(0, 4, n) - N_v(1, 4, n)]q^n = A(q) + B(q) + C(q) - 1. \tag{48}$$

In terms of the corresponding generating functions (42) is equivalent to the fact that $D_1(q)$ is odd, while (43) is equivalent to saying that $D_2(q)$ is an even function. Therefore to prove (42) and (43) we only need to show, respectively,

$$A(q) - B(q) + C(q) = -A(-q) + B(-q) - C(-q), \tag{49}$$

$$A(q) + B(q) + C(q) = A(-q) + B(-q) + C(-q). \tag{50}$$

Adding (49) and (50) side by side we obtain,

$$A(q) + C(q) = B(-q) \tag{51}$$

and (51) is equivalent to (49) and (50) together.

Now, by (25), (26) and the definitions for $A(q)$ and $B(q)$

$$A(q) = \frac{1}{2(q)_\infty}\left[\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}\right] = \frac{f(q)}{4}, \tag{52}$$

$$B(q) = \frac{1}{(q)_\infty}\left[\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}}\right] = \frac{\phi_3(q)}{2}. \tag{53}$$

On the other hand, by equation (3.35) of [A]

$$C(q) = \frac{1}{2(q)_\infty}\sum_n \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} = \frac{(q\,;q)_\infty}{4(-q\,;q)_\infty^2}. \tag{54}$$

Replacing (52) to (54) in (51) we obtain (28). Therefore, (42) and (43) are true, and so are (8) and (9).

4. The rank modulus 8

Now, we prove the equalities involving only the rank in equations (10) to (13). We start studying the generating functions for the corresponding differences.

The generating function for $N(2, 8, n) - N(4, 8, n)$ equals

$$\alpha_0(q) = \frac{1}{(q)_\infty}\sum_{n\neq 0} \frac{(-1)^n q^{n(3n+1)/2}(q^{2n} + q^{6n} - 2q^{4n})}{1-q^{8n}} \tag{55}$$

but,

$$\frac{q^{2n} + q^{6n} - 2q^{4n}}{1-q^{8n}} = \frac{q^{2n}(1-q^{2n})^2}{(1-q^{2n})(1+q^{2n})(1+q^{4n})} = \frac{1}{1+q^{4n}} - \frac{1}{1+q^{2n}}$$

and in the summation we can include the term which corresponds to $n = 0$ since this equals 0. Therefore (55) becomes

$$\begin{aligned}\alpha_0(q) &= \frac{1}{(q)_\infty}\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{4n}} - \frac{1}{2}\left[\frac{2}{(q)_\infty}\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}}\right] \\ &= L(q) - \frac{\phi_3(q)}{2}\end{aligned} \tag{56}$$

where

$$L(q)=\frac{1}{(q)_\infty}\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{4n}}.$$

Similarly, using partial fractions decomposition, the generating function for $N(0,8,n)-N(2,8,n)$ can be written as follows

$$\alpha_2(q)=L(q)+\frac{\alpha_3(q)}{2}-1. \tag{57}$$

Now, the generating function for $N(1,8,n)+N(2,8,n)-N(3,8,n)-N(4,8,n)$ is given by

(58)

$$\alpha_1(q)=\frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}(q^n+q^{7n}+q^{2n}+q^{6n}-q^{3n}-q^{5n}-2q^{4n})}{1-q^{8n}}.$$

Using (18) we notice that $S_8(3,1)+S_8(3,6)=0$, $S_8(3,3)+S_8(3,4)=0$, $S_8(3,7)=-S_8(3,0)$, $S_8(3,2)=-S_8(3,5)$; so we can rewrite (58) as follows

$$\alpha_1(q)=\frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}(-1+2q^{2n}-q^{4n})}{1-q^{8n}} \tag{59}$$

but,

$$\frac{-1+2q^{2n}-q^{4n}}{1-q^{8n}}=-\frac{(1-q^{2n})^2}{(1-q^{2n})(1+q^{2n})(1+q^{4n})}=\frac{q^{2n}}{1+q^{4n}}-\frac{1}{1+q^{2n}}.$$

Noticing that in the summation we can include the term which corresponds to $n=0$ since this equals 0, we obtain

$$\alpha_1(q)=H(q)-\frac{\phi_3(q)}{2} \tag{60}$$

where

$$H(q)=\frac{1}{(q)_\infty}\sum_n \frac{(-1)^n q^{n(3n+1)/2}q^{2n}}{1+q^{4n}}.$$

Similarly, the generating function for $N(0,8,n)+N(1,8,n)-N(2,8,n)-N(3,8,n)$ is equal to

$$\alpha_3(q)=H(q)+\frac{\phi_3(q)}{2}-1. \tag{61}$$

By Theorem (1) if $f(q)=\sum A_n q^n$, then

$$\sum_{n=0}^{\infty} A_{4n}q^{4n}=\frac{1}{4}[f(q)+f(-q)+f(iq)+f(-iq)], \tag{62}$$

$$\sum_{n=0}^{\infty} A_{4n+1}q^{4n+1}=\frac{1}{4}[f(q)-f(-q)-if(iq)+if(-iq)], \tag{63}$$

$$\sum_{n=0}^{\infty} A_{4n+2}q^{4n+2}=\frac{1}{4}[f(q)+f(-q)-f(iq)-f(-iq)], \tag{64}$$

$$\sum_{n=0}^{\infty} A_{4n+3}q^{4n+3}=\frac{1}{4}[f(q)-f(-q)+if(iq)-if(-iq)]. \tag{65}$$

Using (62) to (65) we can state the equalities for the rank in equations (10) to (13) in terms of the generating functions for the corresponding differences, which are given by $\alpha_0(q)$, $\alpha_1(q)$, $\alpha_2(q)$ and $\alpha_j(q)$. Thus, we obtain

$$\alpha_0(q)+\alpha_0(-q)+\alpha_0(iq)+\alpha_0(-iq)=0, \tag{66}$$

$$\alpha_1(q)-\alpha_1(-q)-i\alpha_1(iq)+i\alpha_1(-iq)=0, \tag{67}$$

$$\alpha_2(q)+\alpha_2(-q)-\alpha_2(iq)-\alpha_2(-iq)=0, \tag{68}$$

$$\alpha_3(q)-\alpha_3(-q)+i\alpha_3(iq)-i\alpha_3(-iq)=0. \tag{69}$$

Now $\alpha_0(q)$, $\alpha_1(q)$, $\alpha_2(q)$ and $\alpha_3(q)$ can be written in terms of $\phi_3(q)$, $L(q)$ and $H(q)$. If we do so then (68) to (71) become

$$\begin{aligned}&\phi_3(q)+\phi_3(-q)+\phi_3(iq)+\phi_3(-iq)\\&\qquad=2[L(q)+L(-q)+L(iq)+L(-iq)],\end{aligned} \tag{70}$$

$$\begin{aligned}&\phi_3(q)-\phi_3(-q)-i\phi_3(iq)+i\phi_3(-iq)\\&\qquad=2[H(q)-H(-q)-iH(iq)+iH(-iq)],\end{aligned} \tag{71}$$

$$\begin{aligned}&\phi_3(q)+\phi_3(-q)-\phi_3(iq)-\phi_3(-iq)\\&\qquad=2[-L(q)-L(-q)+L(iq)+L(-iq)],\end{aligned} \tag{72}$$

$$\begin{aligned}&\phi_3(q)-\phi_3(-q)+i\phi_3(iq)-i\phi_3(-iq)\\&\qquad=2[-H(q)+H(-q)-iH(iq)+iH(-iq)].\end{aligned} \tag{73}$$

Equations (70) to (73) represent a system of 4 equations with 4 unknowns ($\phi_3(q)$, $\phi_3(-q)$, $\phi_3(iq)$ and $\phi_3(-iq)$). But these unknowns are related to each other in such a way that if we determine one of them all the others can be found by a change of variable. If we add (70) to (73) side by side we obtain

$$\phi_3(q)=L(iq)+L(-iq)-iH(iq)+iH(-iq). \tag{74}$$

Therefore, if we show the truth of (74) then the equalities for the rank in (10) to (13) are true. To facilitate our work we replace q by iq in (74) to obtain

$$\begin{aligned}\phi_3(iq)&=[L(q)+iH(q)]+L(-q)-iH(-q)]\\&=\frac{1}{(q;q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}}{(1-iq^{2n})}\\&=\frac{1}{(-q;-q)_\infty}\sum_n\frac{(-1)^{3n(n+1)/2}q^{n(3n+1)/2}}{(1+iq^{2n})}\\&=\frac{i}{(q;q)_\infty}\sum_n\frac{(-1)^n q^{3n(3n+1)/2}}{(1+iq^{2n})}\\&\quad-\frac{i}{(-q;-q)_\infty}\sum_n\frac{(-1)^{3n(n-1)/2}q^{3n(3n+1)/2}}{(1-iq^{2n})}\end{aligned} \tag{75}$$

where the last line is obtained writing $-n$ for n.

Therefore, if we define

$$M(q)=\frac{1}{(q;q)_\infty}\sum_n\frac{(-1)^n q^{3n(n+1)/2}}{1-iq^{2n}}, \tag{76}$$

$$N(q)=\frac{1}{(q;q)_\infty}\sum_n\frac{(-1)^n q^{3n(n+1)/2}}{1+iq^{2n}} \tag{77}$$

and multiply (75) by i, we obtain that (74) is equivalent to

$$i\phi_3(iq)=M(-q)-N(q). \tag{78}$$

To prove (78) we first establish some other relations among $M(q)$, $N(q)$ and some of the third order mock theta functions. Now, if in Theorem 3 we let $z^2=-i$, we obtain

$$\begin{aligned} M(-q)+N(q)&=\frac{(q^4;q^4)^2_\infty(-q^2;q^2)_\infty(iq;q^4)_\infty(-iq^3;q^4)_\infty}{(-q^4;q^4)^2_\infty(q^2;q^2)_\infty}\\ &=(-q^2;q^4)^2_\infty j(iq,q^4)=\Pi_1(q). \end{aligned} \tag{79}$$

Replacing q by $-iq$ in (28) and multiplying the resulting relation by i, we obtain after transposing some terms

$$i\phi_3(iq)=-2i\psi(iq)+i\Pi_2(-iq). \tag{80}$$

Replacing (79) and (80) in (78) we find that proving (78) is equivalent to proving

$$-2i\psi(iq)+i\Pi_2(-iq)=\Pi_1(q)-2N(q). \tag{81}$$

Or equivalently (replacing q by $-q$ and then q by $q^{1/2}$)

$$2[i\psi(-iq^{1/2})-N(-q^{1/2})]=i\Pi_2(iq^{1/2})-\Pi_1(-q^{1/2}). \tag{82}$$

If in Theorem (2) we let $r=1$, $z=0$, $\alpha=\pi/4$, $\beta=\pi/4+\pi\tau/2$, $\gamma=x/4+\pi\tau/4$ (where $q=\exp(i\pi\tau)$). Then the left-hand side of (29) becomes

$$L=\frac{\vartheta_1'(0)}{\vartheta_2(\pi/4)\vartheta_2(\pi/4+\pi\tau/2)\vartheta_2(\pi/4+\pi\tau/4)} \tag{83}$$

since ϑ_2 is an even function. On the other hand, since ϑ_1 is an odd function, we find that the right-hand side becomes

(84)

$$\begin{aligned} R=-2\frac{i^{3/2}}{\vartheta_1(\pi\tau/2)\vartheta_1(\pi\tau/4)}&\left[\sum_m\frac{(-1)^m q^{3m(m+1/2)}}{1+iq^{2m}}+\sum_m\frac{(-1)^m q^{3(m+1/2)(m+1)}}{1+iq^{2m+1}}\right]\\ &+\frac{2i^{3/2}q^{1/4}}{[\vartheta_1(\pi\tau/4)]^2}\left[\sum_m\frac{(-1)^m q^{3m(m+1)+1/2}}{1+iq^{2m+1/2}}\right]. \end{aligned}$$

If in (77) we replace q by $-q$ and then q by $q^{1/2}$ we have

$$\begin{aligned} N(-q^{1/2})(-q^{1/2};-q^{1/2})_\infty&=\sum_n\frac{(-1)^{n+3n(n+1)/2}q^{3n(n+1)/4}}{1+iq^n}\\ &=\sum_m\frac{(-1)^m q^{3(2m(2m+1))/4}}{1+iq^{2m}}+\sum_m\frac{(-1)^m q^{3(2m+1)(2m+2)}}{1+iq^{2m+1}} \end{aligned} \tag{85}$$

where in the last line we are using the fact that if $n = 2m + r$, $r = 0$ or 1, then $n + 3n(n+1)/2 \equiv m \pmod 2$.

Replacing q by $-iq$ in (27) and using the facts that $i^{4t} = 1$ and $6n(n+1) \equiv 0 \pmod{12}$ we obtain

$$\psi(-iq)(q^4;q^4)_\infty = -i\sum_n \frac{(-1)^n q^{6n(n+1)+1}}{1+iq^{4n+1}}. \tag{86}$$

Replacing $q^{1/2}$ for q in (86) and multiplying by i,

$$i\psi(-iq^{1/2}(q^2;q^2)_\infty = \sum_n \frac{(-1)^n q^{3n(n+1)+1/2}}{1+iq^{2n+1/2}}. \tag{87}$$

If we replace all these results in (29) and multiply the whole equation by $i^{-3/2}q^{-1/4}[\vartheta_1(\pi\tau/4)]^2$ we obtain

$$\begin{aligned}&\frac{i^{-3/2}\vartheta_1'(0)\vartheta_1^2(\pi\tau/4)q^{-1/4}}{\vartheta_2(\pi/4)\vartheta_2(\pi/4+\pi\tau/2)\vartheta_2(\pi/4+\pi\tau/4)}\\ &\quad= -\frac{2N(-q^{1/2})q^{-1/4}\vartheta_1(\pi\tau/4)(-q^{1/2};-q^{1/2})_\infty}{\vartheta_1(\pi\tau/2)}\\ &\qquad+2i\psi(-iq^{1/2})(q^2;q^2)_\infty.\end{aligned} \tag{88}$$

But, as a result on p. 70 of [W],

$$\frac{q^{-1/4}\vartheta_1(\pi\tau/4)(-q^{1/2};-q^{1/2})_\infty}{\vartheta_1(\pi\tau/2)} = (q^2;q^2)_\infty. \tag{89}$$

Multiplying (88) by J_2^{-1} and using (89) we find after some reductions that

$$\begin{aligned}&2[i\psi(-iq^{1/2}) - N(-q^{1/2})]\\ &\quad= \frac{i^{-3/2}\vartheta_1'(0)\vartheta_1^2(\pi\tau/4)q^{-1/4}}{\vartheta_2(\pi/4)\vartheta_2(\pi/4+\pi\tau/2)\vartheta_2(\pi/4+\pi\tau/4)J_2}\\ &\quad= \frac{(-1+i)(q^{1/2};q^{1/2})_\infty^2 j(iq^{1/2},q^2)}{(q^2;q^2)_\infty(q;q)_\infty}.\end{aligned} \tag{90}$$

Comparing (90) with (82) we realize that we only need to prove that the right-hand sides of these equations are equal. Replacing q by q^2 in both expressions we obtain

$$i\Pi_2(iq) - \Pi_1(-q) = \frac{(-1+i)J_1^2 j(iq,q^4)}{J_2J_4}. \tag{91}$$

From (28) we know

$$\begin{aligned}\Pi_2(q) &= \frac{(q;q)_\infty}{(-q;q)_\infty^2} = \frac{(q;q)_\infty^3(-q^2;q^4)_\infty^2(-q^4;q^4)_\infty^2}{(q^4;q^4)_\infty^2}\\ &= \frac{j^3(q,q^4)(q^2;q^4)_\infty^3(-q^2;q^4)_\infty^2(-q^4;q^4)_\infty^2}{J_4^2}\\ &= \frac{j^3(q,q^4)(q^2;q^4)_\infty}{J_4^2}\end{aligned} \tag{92}$$

where the last line follows from

$$(-q^2;q^4)_\infty^2(q^2;q^4)_\infty^2(-q^4;q^4)_\infty^2 = (q^4;q^8)_\infty^2(-q^4;q^4)_\infty^2 = 1.$$

Replacing (92) and (79) in (91) we obtain

$$\begin{aligned}\frac{ij^3(iq, q^4)(-q^2; q^4)_\infty}{J_4^2} - j(-iq, q^4)(-q^2; q^4)_\infty^2 \\ = \frac{(-1+i)J_1^2 j(iq, q^4)}{J_2 J_4}.\end{aligned} \tag{93}$$

Multiplying (93) by $-ij(-iq, q^4)_\infty[(-q^2; q^4)_\infty]^{-2}$ and using

$$\begin{aligned} j(iq, q^4)j(-iq, q^4) \\ = (iq; q^4)_\infty(-iq; q^4)_\infty(-iq^3; q^4)_\infty(iq^3; q^4)_\infty J_4^2 \\ = (-q^2; q^8)_\infty(-q^6; q^8)_\infty J_4^2 = (-q^2; q^4)_\infty J_4^2 \end{aligned} \tag{94}$$

we can rewrite (93) as follows

$$j^2(iq, q^4) + ij^2(-iq, q^4) = (1+i)\frac{J_1^2 J_4}{(-q^2; q^4)_\infty J_2}. \tag{95}$$

If in (32) we replace q by q^4 and let $x = iq$ and $x = -iq$, respectively, we obtain

$$\begin{aligned} j^2(iq, q^4) + ij^2(-iq, q^4) \\ = (1+i)[j(q^2, q^8)j(-q^4, q^8) - qj(q^6, q^8)j(-1, q^8)] \\ = (1+i)J_{2,8}[\overline{J}_{4,8} - q\overline{J}_{0,8}] = (1+i)J_{2,8}J_{1,2} \end{aligned} \tag{96}$$

where the last line follows from (34) when q is replaced by q^2 and we set $x = q$.

To complete our proof we only need to show that the right-hand sides of (95) and (96) are equal, but from definition

$$\begin{aligned} J_{2,8}J_{1,2} = (q^2; q^8)_\infty(q^6; q^8)_\infty J_8(q, q^2)_\infty^2 J_2 \\ = \frac{(q^2; q^4)_\infty J_8 J_1(q; q^2)_\infty J_2(-q^2; q^4)}{J_2(-q^2; q^4)_\infty} = \frac{J_4 J_1^2}{(-q^2; q^4)_\infty J_2}. \end{aligned} \tag{97}$$

From all these we conclude that the equalities for the rank in equations (10) to (13) are true.

5. The rank and the crank modulo 8

To prove (14) to (17) we first need the generating function of $N_v(2, 8, n)$ which is equal to

$$\begin{aligned} \sum_{n=0}^{\infty} N_v(2, 8, n)q^n = \frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(n+1)/2}\cdot(q^{2n}+q^{6n})}{1-q^{8n}} \\ = \frac{1}{2(q)_\infty}\left[\sum_{n\neq 0}\frac{(-1)^n q^{n(n+1)/2}}{1-q^{2n}} - \sum_{n\neq 0}\frac{(-1)^n q^{n(n+1)/2}}{1+q^{2n}}\right] \\ = \frac{C'(q)}{2} - \frac{D'(q)}{2} \end{aligned} \tag{98}$$

where $C'(q)$ was defined in (46) and

$$D'(q) = \frac{1}{(q)_\infty}\left[\sum_{n\neq 0}\frac{(-1)^n q^{n(n+1)/2}}{1+q^{2n}}\right].$$

The generating function for $N(3, 8, n)$ is

$$\begin{aligned}\sum_{n=0}^{\infty} N(3, 8, n)q^n &= \frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}\cdot(q^{3n}+q^{5n})}{1-q^{8n}} \\ &= -\frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}\cdot(q^{2n}+q^{4n})}{1-q^{8n}}\end{aligned} \tag{99}$$

where the last line follows from (18). But,

$$\frac{-q^{4n}-q^{2n}}{1-q^{8n}} = \frac{1-q^{2n}}{2(1+q^{4n})} - \frac{1}{2(1-q^{2n})}.$$

Thus we can rewrite (99) as follows:

$$\sum_{n=0}^{\infty} N(3, 8, n)q^n = \frac{E(q)}{2} - \frac{A'(q)}{2} \tag{100}$$

where $A'(q)$ was defined in (44) and

$$E(q) = \frac{1}{(q)_\infty}\sum_{n}\frac{(-1)^n q^{n(3n+1)/2}(1-q^{2n})}{1+q^{4n}}.$$

Similarly, the generating function for $N(1, 8, n)$ equals

$$\begin{aligned}\sum_{n=0}^{\infty} N(1, 8, n)q^n &= \frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}\cdot(q^{n}+q^{7n})}{1-q^{8n}} \\ &= -\frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}\cdot(1+q^{6n})}{1-q^{8n}} = -\frac{E(q)}{2} - \frac{A'(q)}{2}\end{aligned} \tag{101}$$

where the last line is a consequence of

$$\frac{1+q^{6n}}{1-q^{8n}} = \frac{1-q^{2n}}{2(1+q^{4n})} + \frac{1}{2(1-q^{2n})}.$$

Let $\beta_0(q)$ denote the generating function for the difference $N_v(2, 8, n) - N(3, 8, n)$. Then,

$$\beta_0(q) = \tfrac{1}{2}[C'(q) - D'(q) - E(q) + A'(q)]. \tag{102}$$

Using (62) and (63) we obtain the following equations which are equivalent, respectively, to (14) and (15)

$$\beta_0(q) + \beta_0(-q) + \beta_0(iq) + \beta_0(-iq) = 0, \tag{103}$$

$$\beta_0(q) - \beta_0(-q) - i\beta_0(iq) + i\beta_0(-iq) = 0. \tag{104}$$

Adding (103) and (104) we obtain

$$\beta_0(q) = -\frac{(1-i)}{2}\beta_0(iq) - \frac{(1+i)}{2}\beta_0(-iq) \tag{105}$$

which is equivalent to (103) and (104) together.

Similarly, if $\beta_1(q)$ denotes the generating function of $N_v(2, 8, n) - N(1, 8, n)$ we obtain

$$\beta_1(q) = \tfrac{1}{2}[C'(q) - D'(q) + E(q) + A'(q)]. \tag{106}$$

Using (64) and (65) we obtain the following equations which are equivalent, respectively, to (16) and (17)

$$\beta_1(q) + \beta_1(-q) - \beta_1(iq) - \beta_1(-iq) = 0, \tag{107}$$

$$\beta_1(q) - \beta_1(-q) + i\beta_1(iq) - i\beta_1(-iq) = 0. \tag{108}$$

Adding (107) and (108) we obtain the following equation, which is equivalent to (107) and (108) together

$$\beta_1(q) = \frac{(1-i)}{2}\beta_1(iq) + \frac{(1+i)}{2}\beta_1(-iq). \tag{109}$$

Adding (105) and (109) we obtain

$$\beta_0(q) + \beta_1(q) = \frac{(1-i)}{2}[\beta_1(iq) - \beta_0(iq)] + \frac{(1+i)}{2}[\beta_1(-iq) - \beta_0(-iq)] \tag{110}$$

which is equivalent to (105) and (109) together. Therefore, if we show the truth of (110) then (14) to (17) are all true.

Using (102) and (106) we can rewrite (110) as follows

$$C'(q) + A'(q) - D'(q) = \frac{(1-i)}{2}E(iq) + \frac{(1+i)}{2}E(-iq). \tag{111}$$

To prove (111) we first notice that $E(q)$ is equal to

$$\begin{aligned} E(q) &= \frac{1}{(q)_\infty}\left[\frac{1-i}{2}\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+iq^{2n}} + \frac{1+i}{2}\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1-iq^{2n}}\right] \\ &= -\frac{(1+i)M(q)}{2} - \frac{(1-i)N(q)}{2} \end{aligned} \tag{112}$$

where the last line follows from equations (75) to (77).

Using (112) the right-hand side of (111) can be rewritten as follows:

$$\begin{aligned} R &= \tfrac{1}{4}[-(1-i^2)M(iq) - (1-i)^2N(iq) - (1+i)^2M(-iq) - (1-i^2)N(-iq)] \\ &= \tfrac{1}{2}[-(M(iq)+N(-iq)) + i(N(iq) - M(-iq))] \\ &= \frac{\phi_3(-q)}{2} - \frac{(q^2;q^4)_\infty J_1}{2} \end{aligned} \tag{113}$$

where the last line follows from (78) and (79) replacing q by iq in these equations.

Meanwhile, in the left-hand side of (111) we can replace $A'(q)$, $C'(q)$ and $D'(q)$ by $A(q)$, $C(q)$ and $D(q)$ (where these new functions are defined including the term for $n = 0$ in the summation), since the terms corresponding to $n = 0$ cancel out. But, $A(q)$, $C(q)$ and $D(q)$ can be written in terms of third order mock theta functions and certain infinite products using, respectively, (52), (54) and (30). Thus, the left-hand side of (111) is equal to

$$L = \frac{(q;q)_\infty}{4(-q;q)_\infty^2} + \frac{f(q)}{4} - \frac{(q^2;q^4)J_1}{2}. \tag{114}$$

Comparing (113) and (114) we realize that (111) is equivalent to (28), and so (14) to (17) are true.

Finally, we establish some simple observations that we need to prove the equalities between the rank and the crank in equations (10) to (13). These observations are the following:

$$N(m, t, n) = N(m, 2t, n) + N(m+t, 2t, n), \tag{115}$$

$$M(m, t, n) = M(m, 2t, n) + M(m+t, 2t, n), \tag{116}$$

$$\sum_{m=0}^{t-1} N(m, t, n) = \sum_{m=0}^{t-1} M(m, t, n) = p(n). \tag{117}$$

As noted by Lewis [L2] the equalities for the rank and the crank in (10) and (12) follow directly from (8), (115) and (116) using the fact that $M(1, 8, 2n) = M(3, 8, 2n)$ as proved by Garvan [G3].

To prove the equality between the rank and the crank in (11) we start with (117) with n replaced by $4n+1$ and $t = 8$. Thus, we obtain

$$\sum_{m=0}^{7} N(m, 8, 4n+1) = \sum_{m=0}^{7} M(m, 8, 4n+1). \tag{118}$$

Now, let X be equal to

$$\begin{aligned} X &= M(0, 8, 4n+1) + M(1, 8, 4n+1) \\ &= M(3, 8, 4n+1) + M(4, 8, 4n+1) \end{aligned} \tag{119}$$

and let Y be equal to

$$\begin{aligned} Y &= N(1, 8, 4n+1) + N(2, 8, 4n+1) \\ &= N(3, 8, 4n+1) + N(4, 8, 4n+1). \end{aligned} \tag{120}$$

With this notation what we want to prove is $X = Y$. Replacing (119) and (120) in (118), and using (3) and (20), we can rewrite (118) as follows

$$\begin{aligned} &3Y + N(0, 8, 4n+1) + N(3, 8, 4n+1) \\ &\quad = 2X + 2M(2, 8, 4n+1) + M(1, 8, 4n+1) + M(5, 8, 4n+1). \end{aligned} \tag{121}$$

Using (15) and (116) we can rewrite (121) as follows

$$\begin{aligned} &3Y + N(0, 8, 4n+1) + N(3, 8, 4n+1) \\ &\quad = 2X + 2N(3, 8, 4n+1) + M(1, 4, 4n+1). \end{aligned} \tag{122}$$

But, by (9) and (115) we have

$$M(1, 4, 4n+1) = N(0, 4, 4n+1) = N(0, 8, 4n+1) + N(4, 8, 4n+1). \tag{123}$$

Replacing (123) in (122) and transposing terms we obtain

$$3Y - N(4, 8, 4n+1) - N(3, 8, 4n+1) = 2X \tag{124}$$

and the equality follows from the definition of Y. The proof of the remaining equality in (13) is exactly the same.

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