BASIC CONSTRUCTIONS IN THE $K$-THEORY OF HOMOTOPY RING SPACES

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Abstract. Using the language of category theory and universal algebra we formalize the passage from the permutative category of finitely generated free $R$-modules to the algebraic $K$-theory $KR$ of $R$ and thus make it applicable to homotopy ring spaces. As applications we construct a Waldhausen type of algebraic $K$-theory for arbitrary homotopy ring spaces, show its equivalence with constructions of May and Steiner, prove its Morita invariance and show that the algebraic $K$-theory $KX$ of an $E_{\infty}$ ring $X$ is itself an $E_{\infty}$ ring. Finally we investigate the monomial map $Q(\Omega X_+^*) \to KX$.

1. Introduction

It has been known for some time that the algebraic $K$-theory $KR$ of a commutative ring $R$ admits a homotopy commutative and associative product structure (e.g. see [L] or [W3]). May showed that it even has an $E_{\infty}$ ring structure [M6, Proposition 10.12], but he found little reason to believe that the $K$-theory $KX$ of an $A_{\infty}$ or $E_{\infty}$ ring $X$ has any structure beyond that of an $H$-space [M6, Remark 10.3]. Fortunately, Steiner could refute this pessimistic outlook [St2]. Using a construction for algebraic $K$-theory different from May's he proved that for an $A_{\infty}$ ring $X$ his $KX$ is an infinite loop space which agrees as infinite loop space with Waldhausen's algebraic $K$-theory $A(Y)$ of the space $Y$ for the $A_{\infty}$ ring $X = Q(\Omega Y_+, +)$. Here $Q$ is stable homotopy and $\Omega_+$ the addition of an extra base point. So it is natural to ask whether $KX$ has even more structure if $X$ is an $E_{\infty}$ ring. If one tries to attack this question using the methods of [St2] one runs into nasty bookkeeping problems making a general procedure desirable which takes care of those. Such a machine is suggested by May's passage from bipermutative categories to $E_{\infty}$ spectra [M8]. The purpose of this paper is twofold. We first develop methods which adapt this passage to the theory of $A_{\infty}$ and $E_{\infty}$ rings. They are of interest in their own right and have been applied successfully in [FSSV]. They will also be an essential tool in the forthcoming papers [FSV, FOV, and SV6]. We then use this approach to prove

1.1 Theorem. If $X$ is an $E_{\infty}$ ring so is $KX$.

Throughout this paper we use $A_{\infty}$ and $E_{\infty}$ rings in the sense of (2.5) below which can be considered as the homotopy invariant extensions of May's
definitions [M6, M4] based on operad pairs. Our machine produces a $K$-theory space $KX$ which a priori differs from the constructions of May and Steiner, but we have

1.2 Proposition. Let $X$ be an $A_\infty$ ring.

1. $KX$ can be realized as a plus construction. In particular, $KX$ agrees with May's construction [M6] for $A_\infty$ rings structured by operad pairs.

2. $KX$ is an infinite loop space which is equivalent as infinite loop space to Steiner's construction.

This result also provides the first explicit proof of the equivalence of the constructions of May and Steiner. In [SV4] we showed that the space $M_nX$ of $n$-squared matrices over an $A_\infty$ ring has a natural $A_\infty$-structure. As an immediate consequence of our machine and of (1.2) we obtain

1.3 Proposition. $KX$ is Morita invariant, i.e. there is an infinite loop equivalence $K(M_nX) \simeq KX$.

Finally we apply our methods to the monomial map:

1.4 Proposition. Let $X^*$ the space of homotopy units in $X$ and $BX^*$ its classifying space. Then there is a monomial infinite loop map $Q((BX^*)_+) \to K(X)$ which is a map of $E_\infty$ rings if $X$ is an $E_\infty$ ring.

Our translation mechanism of May's techniques to the $A_\infty$ and $E_\infty$ world forces us to express the well-known classical constructions in abstract terms. To help the reader to keep track of what is going on we recall the steps in May's set-up, explain the idea of the translation procedure, and indicate the necessary changes (for precise definitions see §2).

1.5 The steps in May's construction. 1. One starts with the permutative category $\mathscr{A}$ of finitely generated projective modules over the ring $R$.

2. $\mathscr{A}$ gives rise to a lax functor $A : \mathcal{F} \to \text{Cat}$ from the category $\mathcal{F}$ of based finite sets into the category of small categories [M7, T1].

3. One rectifies $A$ to a genuine functor $SA$ by Street's or Segal's rectification process [Str, Se].

4. Composition of $SA$ with the classifying space functor gives an $\mathcal{F}$-diagram of topological spaces. Using homotopy theory of categories one checks that this $\mathcal{F}$-space is special in the sense of [MT] (a $T$-space in the notation of [Se]).

5. Each special $\mathcal{F}$-space has an associated $\Omega$-spectrum. The $0$th space of the $\mathcal{F}$-space of (4) is the algebraic $K$-theory $KR$.

We want to make these five steps accessible to $A_\infty$ and $E_\infty$ ring theory. The translation is as follows:

1.6 An $A_\infty$ ring (and similarly an $E_\infty$ ring) is structured by a functor $X : \Theta \to \mathcal{T}_R$ from a theory $\Theta$ which is augmented over the theory $\Theta_r$ of semirings (no additive inverses are codified), and the augmentation $F_\Theta : \Theta \to \Theta_r$ is a homotopy equivalence over simple morphisms in $\Theta_r$.

Given a small category $\mathcal{D}$, a functor $D : \mathcal{D} \to \Theta_r$, and an $A_\infty$ ring $X$ we
can form the diagram

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{HD} & \Theta \\
\downarrow \nu & & \downarrow F_0 \\
\mathcal{D} & \xrightarrow{D} & \Theta_r
\end{array}
\]

If \( D \) takes simple morphisms as values only, \( \nu \) is an equivalence so that the \( \mathcal{P} \)-diagram \( X \circ HD \) can be considered a \( \mathcal{D} \)-diagram up to coherent homotopy. We can rectify it to an equivalent genuine \( \mathcal{D} \)-diagram using Segal's homotopy pushdown construction [Se, Appendix B] or a related process.

The simple principle is that constructions for rings which are universal, i.e. which do not depend on the particular ring and hence can be described in terms of the morphisms of \( \Theta_r \), can be performed up to coherent homotopy in the \( A_\infty \)-world, expressed by the inverse images of these morphisms in \( \Theta \), and can be replaced by strict data constructions by the process (1.6), provided only simple morphisms are involved.

The \( \mathcal{I} \)-space associated with the algebraic \( K \)-theory of a genuine ring \( R \) has as underlying space the disjoint union of the classifying spaces \( BG\ell_n(R) \) of the general linear groups. To allow constructions of this kind we have to enlarge \( \Theta_r \) to a category \( \Sigma \Theta_r \) by formally adjoining all categorical sums. By taking disjoint unions in \( \mathcal{I} \)-spaces an \( A_\infty \) ring \( X: \Theta \to \mathcal{I} \)-extends to a functor \( \Sigma \Theta \to \mathcal{I} \)-, and we are in precisely the same situation as in (1.6) with \( \Theta \) and \( \Theta_r \) substituted by \( \Sigma \Theta \) and \( \Sigma \Theta_r \).

After the recollection in §2 of the basic definitions of \( A_\infty \) and \( E_\infty \) monoids and rings and their more combinatorial equivalent analogues, the special \( \mathcal{I} \)-spaces and \( \mathcal{I} \)-\( \mathcal{I} \)-spaces, we start the development of our machinery in §3 defining the substitute for \( \text{Cat} \), a category \( \text{Cat}(\Sigma \Theta_r) \) of category objects in \( \Sigma \Theta_r \). The existence of certain iterated pullbacks in \( \Sigma \Theta_r \) makes this possible. The next step is to check whether Street's rectification extends to a lax functor into \( \text{Cat}(\Sigma \Theta_r) \). This is done in §4 by translating it into category theoretical constructions and checking that those can be executed in \( \text{Cat}(\Sigma \Theta_r) \). In Step (1.5.4) the homotopy theory in \( \text{Cat}(\Sigma \Theta_r) \) requires some attention because Segal's pushdown construction is only natural up to homotopy with respect to subdiagrams of \( \mathcal{D} \). Since we are only interested in homotopy types this suffices (see §5). The applications start with §6: We give a formal description of the permutative category of finitely generated free \( R \)-modules as an object in \( \text{Cat}(\Sigma \Theta_r) \) (the category of finitely generated projective \( R \)-modules cannot be described in universal terms because projectivity depends on the particular ring). We then put this object through the machinery. In the comparison with Steiner's and May's constructions, §§7 and 8, we partly extend our methods: Steiner's construction, considered from our viewpoint, uses Segal's rectification of a lax functor, and the comparison of both rectifications described in [M7] can be transited into our set-up. The proofs of (1.1), ..., (1.4) are now simple consequences.

We are indebted to R. Steiner for several illuminating conversations, in particular in connection with §8. We are grateful to P. May for publishing [M8] which clarified the connection of \( E_\infty \) rings and \( \mathcal{I} \)-\( \mathcal{I} \)-spaces and described the passage from bipermutative categories to \( E_\infty \) ring spectra. Finally, we want to
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2. Homotopy monoids and rings, $\mathcal{T}$- and $\mathcal{T} \setminus \mathcal{T}$-spaces

For the reader’s convenience we recall the definitions of $A_{\infty}$ and $E_{\infty}$ monoids and rings from [SV4], where a more detailed discussion is given and the relation to other definitions in the literature is explained. The modifications needed for a module approach to the theory will be given in [SV6]. We work in the category $\mathcal{T}_{fg}$ of compactly generated topological spaces in the sense of [V].

Let $\Theta_m$, $\Theta_{cm}$, $\Theta_r$, and $\Theta_{cr}$ be the theories of monoids, commutative monoids, semirings, and commutative semirings respectively. These are categories with objects 0, 1, 2, ... . The morphisms from $n$ to $k$ in $\Theta_m$ are $k$-tuples of monomials

$$x_1^{r_1} \cdots x_n^{r_n}$$

in $n$ noncommuting variables $x_1, \ldots, x_n$, in $\Theta_{cm}$ they are $k$-tuples of monomials

$$x_1^{r_1} \cdots x_n^{r_n}$$

in $n$ commuting variables, in $\Theta_r$ they are $k$-tuples of finite sums of monomials

$$c \cdot x_1^{r_1} \cdots x_n^{r_n}$$

of type (2.1) with coefficients $c \in \mathbb{N}$, the set of nonnegative integers, and in $\Theta_{cr}$ they are $k$-tuples of finite sums of monomials

$$c \cdot x_1^{r_1} \cdots x_n^{r_n}$$

of type (2.2) with coefficients $c \in \mathbb{N}$. Composition is given by substitution. A morphism from $n$ to 1 is called simple if all its coefficients $c$ are $\leq 1$; in the commutative cases we assume in addition that all exponents $r_i$ are $\leq 1$. A morphism from $n$ to $k$ is simple if all its $k$ components are. In particular, every morphism in $\Theta_m$ is simple. In all other cases the simple morphisms do not form a subcategory. In the commutative monoid case there is a remedy (see (2.9)).

In the following definition $\Theta_\ast$ stands for $\Theta_m$, $\Theta_{cm}$, $\Theta_r$, $\Theta_{cr}$, whatever is appropriate, and $s\Theta_\ast$ its subset of simple morphisms.

2.5 Definition. An $A_{\infty}$ or (in the commutative case) $E_{\infty}$ monoid or ring theory is a topological theory $\Theta$ together with a theory functor $F = F_\Theta : \Theta \rightarrow \Theta_\ast$ such that

1. $\text{ob}\, \Theta \subset \text{mor}\, \Theta$ is a closed cofibration.
2. $F : \text{mor}\, \Theta \rightarrow \text{mor}\, \Theta_\ast$ is bijective on path components and a homotopy equivalence over $s\Theta_\ast$.

A $\Theta$-space, i.e. a continuous functor $X : \Theta \rightarrow \mathcal{T}_{fg}$ such that the maps $X(n) \rightarrow (X(1))^n$ induced by the projection set operations are homotopy equivalences, is called an $A_{\infty}$ or $E_{\infty}$ monoid or ring.
Remark. By changing $X$ within its homotopy type if necessary, we can arrange $X$ to be a strictly product preserving functor [M8].

We will often find it convenient to refer to $X$ when we mean its underlying space $X(1)$, and vice versa.

As maps we use $h$-morphisms (homotopy homomorphisms), which arise naturally when one transfers classical homomorphisms to the $A_\infty$ or $E_\infty$ world, or hammocks introduced in [DK], which describe the functoriality of our constructions in simple terms. For a short summary see [SV4], a more detailed study of their relationship is given in [SV5]. After restriction to a suitable universe the $\Theta$-spaces and hammocks form a simplicial category $\mathcal{F}_p$. $\Theta^*$.

A technical aspect of our constructions is the interplay of the above definition of homotopy monoids and rings with the more combinatorial descriptions of Segal [Se] and Woolfson [Wo]. We shall use the detailed description of May [M8].

Let $\mathcal{F}$ be the category of based sets $n = \{0, 1, 2, \ldots, n\}$ with basepoint 0 and based maps. We single out the morphisms

$$\pi_i = \pi_{i,n} : n \to 1, \quad 1 \leq i \leq n,$$

and $\mu_n : n \to 1$ where $\pi_i$ sends $i$ to 1 and the rest to 0 while $\mu_n$ sends all $i > 0$ to 1.

2.6 Definition. An $\mathcal{F}$-space (G-space in Segal's terminology) is a functor $X : \mathcal{F} \to \mathcal{F}_p$. We call $X(1)$ its underlying space. An $\mathcal{F}$-space is called special, if

1. $(\pi_1, \ldots, \pi_n) : X(n) \to (X(1))^n$ is homotopy equivalence and $X(0)$ is contractible,

2. for each injection $\phi : m \to n$ the map $\phi : X(m) \to X(n)$ is a $\Sigma_\phi$-equivariant cofibration, where $\Sigma_\phi \subset \Sigma_n$ is the subgroup of all permutations $\sigma$ satisfying $\sigma(\text{Im}\, \phi) = \text{Im}\, \phi$.

An $\mathcal{F}$-space, such that $(\pi_1, \ldots, \pi_n) : X(n) \to (X(1))^n$ is a homeomorphism for all $n$, is a commutative monoid with the multiplication

$$X(1)^2 \cong X(2) \xrightarrow{\mu_2} X(1).$$

This is a first indication that special $\mathcal{F}$-spaces are combinatorial descriptions of $E_\infty$ monoids. The precise passage from $E_\infty$-monoids to $\mathcal{F}$-spaces is given by the Segal push-down [Se, Appendix B].

Segal's push-down construction.

2.7 Definition. A category of operators in the sense of [Se] is a small category $\mathcal{C}$ with topologized morphism sets and continuous composition such that $\text{ob} C \subset \text{mor} C$ is a closed cofibration. A $\mathcal{C}$-space is a continuous functor $X : \mathcal{C} \to \mathcal{F}_p$. A homomorphism of $\mathcal{C}$-spaces is a natural transformation $\tau : X \to Y$ of such functors. $\tau$ is called a weak equivalence if $\tau(C) : X(C) \to Y(C)$ is a homotopy equivalence for all $C \in \text{ob} \mathcal{C}$. An equivalence $F : \mathcal{C} \to \mathcal{D}$ of categories of operators is a continuous functor which is bijective on objects and a homotopy equivalence of morphism spaces.

Let $\mathcal{F}_p \mathcal{C}$ denote the category of $\mathcal{C}$-spaces. A continuous functor $F : \mathcal{C} \to \mathcal{D}$ induces a pull-back functor

$$F^* : \mathcal{F}_p \mathcal{D} \to \mathcal{F}_p \mathcal{C}.$$
Segal constructed a functor

\[ F_* : \mathcal{T}_\mathcal{P} \rightarrow \mathcal{T}_\mathcal{P} \]

by defining \( F_*(X) = B(\mathcal{D}, \mathcal{C}, X) \), where the right side is the two-sided bar construction. In detail, \( F_*(X)(D) \) for \( D \in \text{ob} \mathcal{D} \) is the topological realization of the simplicial space \( B_*(\mathcal{D}(-, D), \mathcal{C}, X) \) given by

\[
[n] \mapsto \coprod_{C_j \in \text{ob} \mathcal{C}} \mathcal{D}(F(C_n), D) \times \mathcal{C}(C_{n-1}, C_n) \times \cdots \times \mathcal{C}(C_0, C_1) \times X(C_0)
\]

with the obvious simplicial structure. The \( \mathcal{D} \)-structure is given by composition from the left.

2.8 Properties. (1) \( F_* \) is a functor which preserves weak equivalences.

(2) There are homomorphisms of \( \mathcal{C} \)-spaces

\[
F^*F_*X \xrightarrow{\alpha(X)} \text{Id}^* \text{Id}_*X \xrightarrow{\epsilon(X)} X
\]

natural in \( X \), and \( \epsilon(X) \) is always a weak equivalence while \( \alpha(X) \) is a weak equivalence if \( F \) is an equivalence.

(3) There is a natural homomorphism \( \beta(Y) : F_*F_*Y \rightarrow Y \) for \( Y \in \mathcal{T}_\mathcal{P} \), which is a weak equivalence if \( F \) is an equivalence.

(4) Given a commutative diagram of categories of operators

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{U} & \mathcal{C} & \xrightarrow{X} & \mathcal{T}_\mathcal{P} \\
& \Downarrow G & & \Downarrow F \\
\mathcal{B} & \xrightarrow{V} & \mathcal{D}
\end{array}
\]

there is a homomorphism of \( \mathcal{B} \)-spaces

\[
\omega(U, V) : G_*(X \circ U) \rightarrow F_*(X) \circ V
\]

functorial in \( (U, V) \) in the obvious sense. If \( F \) and \( G \) are equivalences, \( \omega(U, V) \) is a weak equivalence.

(5) If \( \text{mor} \mathcal{D} \) is discrete, then any monomorphism \( g : D_1 \rightarrow D_2 \) in \( \mathcal{D} \) induces an \( \text{End}_g D_2 \)-equivariant closed cofibration

\[
F_*X(g) : F_*(X(D_1)) \rightarrow F_*(X(D_2))
\]

where \( \text{End}_g D_2 \) is the monoid of all \( h \in \text{End} D_2 \) such that \( F_*(X(h)) \) is an endomorphism of \( \text{Im} F_*(X(g)) \).

(1), (2) and (3) are proved in [Se], (4) is a consequence of (1) and (2), while (5) holds trivially on the simplicial level, because \( F_*(X(g)([n])) \) is the inclusion of disjoint summands. Topological realization preserves the cofibration property.

Passage from \( E_\infty \) monoids to \( \mathcal{T} \)-spaces. We construct a functor

\[ M_\circ : \mathcal{T} \rightarrow \Theta_{cm} \]

In \( \mathcal{T} \) we have a wedge sum \( m \vee n \), which is to be identified with \( m + n \) in blocks, and a smash product \( m \wedge n \) to be identified with \( mn \) via lexicographical ordering. A morphism \( \phi : m \rightarrow n \) then decomposes into

\[
m^\sigma \xrightarrow{\tau_0 \vee (\tau_1 \vee \cdots \vee \tau_u)} \tau_1 \vee \cdots \vee \tau_u \quad \xrightarrow{\mu_1 \vee \cdots \vee \mu_n} \tau = n
\]
where $r_j$ is the number of elements in $\phi^{-1}(j)$ and $\sigma^*$ is a suitable permutation. The functor $M_0$ sends $\phi$ to $(\mu_1 \times \cdots \times \mu_k) \circ \hat{\sigma}^*$ in $\Theta_{cm}$, where $\hat{\sigma}^*: X^m \to X^k$; $k = \sum_{i=1}^n r_i$; is the composition of the permutation of coordinates corresponding to $\sigma^*$ and the projection corresponding to $O_{\theta}$. The morphisms $\mu_i$ are given by $\mu_n = x_1 \cdots x_n$, for $n \geq 1$, $\mu_0$ is the unit. It is easily seen that

(2.9) $M_0$ maps $\mathcal{F}$ isomorphically onto a subcategory $\mathcal{F}'$ of simple morphisms. In addition, $F_0: \Theta \to \Theta_{cm}$ is an $E_{\infty}$ monoid theory iff (2.5) holds with $\Omega_{cm}$ replaced by $\mathcal{F}'$.

Given an $E_{\infty}$ monoid $X: \Theta \to \Theta_{op}$ with $\Theta \to \Theta_{cm}$ an $E_{\infty}$ monoid theory, we form the pullback

\[ \begin{array}{ccc}
\mathcal{F} & \xrightarrow{M_0} & \Theta \\
G \downarrow & & \downarrow F_0 \\
\mathcal{F}_0 & \xrightarrow{M_0} & \Theta_{cm}
\end{array} \]

(2.10)

By (2.9), $G$ is an equivalence, and the properties 2.8 imply that $G_* (X \circ M_0)$ is a special $\mathcal{F}$-space such that $G_* (X \circ M_0)(1) \simeq X(1)$.

The passage from special $\mathcal{F}$-spaces to $E_{\infty}$ monoids is more involved. We refer the reader to [M7] and [SV3].

The $E_{\infty}$ ring case: The replacement for $\mathcal{F}$ in this case is Woolfson's category $\mathcal{F} \triangleright \mathcal{F}$. We use May's version of it. The category $\mathcal{F} \triangleright \mathcal{F}$ is a clever way of combining simple additive and simple multiplicative morphisms via the distributive law to a new category of simple morphisms. This is achieved by enlarging the set of objects in order to separate additive and multiplicative simple morphisms as much as the distributive law allows. $\text{ob} \mathcal{F} \triangleright \mathcal{F} = \bigcup_{n \geq 0} \mathbb{N}^n$, with elements denoted $(m; R)$ where $R = (r_1, \ldots, r_m)$. A morphism $(\phi, \varphi): (m; R) \to (n; S)$ consists of a morphism $\phi: m \to n$ in $\mathcal{F}$ and a collection $\varphi = (\varphi_1, \ldots, \varphi_n)$ of morphisms

\[ \varphi_j: \bigwedge_{\phi(i) = j} r_i \to s_j \]

in $\mathcal{F}$ (the empty smash product is 1). Composition is the obvious one. The morphism $(\phi, \varphi)$ is called an injection and each $\varphi_j: r_i \to s_j$ is injective, $\phi(i) = j$; if $j \notin \text{Im} \phi$, then $\varphi_j: 1 \to s_j$ can be any morphism in $\mathcal{F}$. For an injection $(\phi, \varphi)$ let $\Sigma(\phi, \varphi)$ be the group of automorphisms $(\sigma; \tau): (n; S) \to (n; S)$ such that $(\sigma; \tau) \text{Im}(\phi; \varphi) = \text{Im}(\phi, \varphi)$. Here we interpret $(\phi, \varphi)$ as a map of based sets

\[ \varphi_1 \wedge \cdots \wedge \varphi_n: E_{\phi^{-1}(1)} \wedge \cdots \wedge E_{\phi^{-1}(n)} \to \xi_1 \wedge \cdots \wedge \xi_n \]

and similarly $(\sigma; \tau)$.

2.11 Definition. An $\mathcal{F} \triangleright \mathcal{F}$-space (hyper-$\Gamma$-space in Woolfson's terminology) is a functor $X: \mathcal{F} \triangleright \mathcal{F} \to \Theta_{op}$. We call $X(1; 1)$ its underlying space. An $\mathcal{F} \triangleright \mathcal{F}$-space is called special, if

1. $X(0; *)$ contracts to a nondegenerate base point.
2. $X(1; 0) \simeq *$.
3. $X(1; n) \to (X(1; 1))^n$ induced by $(id; \pi_i)$, $i = 1, \ldots, n$, is a homotopy equivalence.
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\( X(n; S) \to \prod_{j=1}^n X(1; s_j) \) induced by \((\pi_i; id), i = 1, \ldots, n,\) is a homotopy equivalence.

(5) If \((\phi; \varphi): (m; R) \to (n; S)\) is an injection, then \((\phi; \varphi): X(m; R) \to X(n; S)\) is a \(\Sigma(\phi; \varphi)\)-equivariant cofibration.

A special \(\mathcal{T} \times \mathcal{T}\)-space such that the equivalences (1), \ldots, (4) of (2.11) are homeomorphisms determines and is determined by a topological commutative semiring [M8, 2.4]. Addition is given by \((id_1, \tilde{\mu}_2)\) and multiplication by \((\tilde{\mu}_2; id)\).

**Passage from** \(E_\infty\) **rings to** \(\mathcal{T} \times \mathcal{T}\)-**spaces.** As in the \(E_\infty\) monoid case we construct a functor

\[
R_i: \mathcal{T} \times \mathcal{T} \to \Theta_{cr}.
\]

It sends the object \((m; R)\) to \(r_1 + \cdots + r_m\) and the morphism \((\phi, \varphi): (m; R) \to (n; S)\) to the following \((s_1 + \cdots + s_n)\)-tuple of polynomials \(z_{11}, \ldots, z_{1s_1}, \ldots, z_{n1}, \ldots, z_{ns_n}\) in \(r_1 + \cdots + r_m\) variables \(x_{11}, \ldots, x_{1r_1}, \ldots, x_{mr_1}, \ldots, x_{mr_m}\): Recall that

\[
\varphi_j: \bigwedge_{\phi(i) = j} r_i \to s_j.
\]

Then

\[
z_{jq} = \sum_{\varphi_j(U) = q} \prod_{\phi(i) = j} x_{iu_i}
\]

where \(U\) runs over the lexicographically ordered set of sequences with \(i\)th term \(u_i\) is satisfying \(1 \leq u_i \leq r_i\) for \(i \in \phi^{-1}(j)\). Again the empty sum is 0 and the empty product is 1. Each such polynomial is simple. The equivalent of (2.9) does not hold: the image of \(R_i\) is not a subcategory and \(R_i\) does not pick up all simple morphisms in \(\Theta_{cr}(m, 1)\); e.g. \(x_1x_2 + x_1\) is not in the image.

We can proceed as in (2.10) to obtain a special \(\mathcal{T} \times \mathcal{T}\)-space from an \(E_\infty\) ring. For the passage from \(\mathcal{T} \times \mathcal{T}\)-spaces to \(E_\infty\) rings we use May’s machinery of [M8].

**2.12 Remark.** Since the standard CW-approximation of an \(E_\infty\) theory is again an \(E_\infty\) theory, we may assume that each \(E_\infty\) structure is codified by a CW-category, and up to weak equivalence we may assume that all our \(E_\infty\) rings and \(\mathcal{T} \times \mathcal{T}\)-spaces are CW-complexes.

**2.13 Remark.** The analogue of \(\mathcal{T} \times \mathcal{T}\) for \(A_\infty\) rings is \(\Delta^{op} \times \mathcal{T}\) where \(\Delta^{op}\) is the image of the canonical object preserving functor \(\Delta^{op} \to \mathcal{T}\) [MT, 3.5]. For an analysis of \(\Delta^{op}\) see [T2, p. 224]. The above definition of \(R_i\) adapts to this case and May’s machinery [M8] can easily be extended to this situation.

**3. Category objects in theories**

In this section we introduce the framework in which we define and study the algebraic \(K\)-theory of \(A_\infty\) and \(E_\infty\) rings.

Let \(\Theta\) be an \(A_\infty\) ring theory. We have shown in [SV4] that the structure of the \(A_\infty\) ring of \(n\)-squared matrices \(M_n X\) over a \(\Theta\)-space \(X\) can be expressed in terms of \(\Theta\). So we have a good grip on the “linear maps” between free \(X\)-modules in terms of matrices. The idea is to give a formal description of the category of finitely generated free \(R\)-modules for a genuine ring \(R\) in terms of operations in \(\Theta_r\), which then can be lifted to \(\Theta\) using the augmentation
Let \( \Theta \to \Theta_r \) (thus systematizing [St2]). The appropriate term is the notion of a suitable category object.

3.1 Definition. Let \( \mathcal{C} \) be an arbitrary category. A category object in \( \mathcal{C} \) is a simplicial object \( \mathcal{A} : \Delta^{op} \to \mathcal{C} \) in \( \mathcal{C} \) such that \( \mathcal{A}[n] \) is an iterated pullback of

\[
\begin{array}{cccccc}
\mathcal{A}[1] & \mathcal{A}[1] & \ldots & \mathcal{A}[1] & \mathcal{A}[1] \\
\downarrow & \downarrow & \ldots & \downarrow & \downarrow \\
\mathcal{A}[0] & \mathcal{A}[0] & \ldots & \mathcal{A}[0] & \mathcal{A}[0] \\
\end{array}
\]

for \( n > 1 \).

3.2 Remark. Usually a category object in \( \mathcal{C} \) consists of the object \( \mathcal{A}_0 \) of objects, the object \( \mathcal{A}_1 \) of morphisms, a source map \( s = d^1 : \mathcal{A}_1 \to \mathcal{A}_0 \), a target map \( t = d^0 : \mathcal{A}_1 \to \mathcal{A}_0 \), the inclusion of identities \( u = s^0 : \mathcal{A}_0 \to \mathcal{A}_1 \) and a composition \( c = d^1 : \mathcal{A}_2 \to \mathcal{A}_1 \) where \( \mathcal{A}_2 \) is a pullback of

\[
\begin{array}{ccc}
\mathcal{A}_1 & \mathcal{A}_1 \\
\downarrow & \downarrow \\
\mathcal{A}_0 & \mathcal{A}_0 \\
\end{array}
\]

subject to the usual axioms. For the homotopy theory of categories we need its nerve, which is given in Definition 3.1, while the nerve functor in the less restricted definition is only defined up to natural isomorphism. If \( \mathcal{C} \) has canonical iterated pullbacks the usual definition suffices for our purposes.

We intend to study category objects in \( \Theta \). Since we also have to codify direct sum operations, \( \Theta \) is too small: it does not have categorical sums. We adopt Steiner's extension who adjoined arbitrary direct sums to \( \Theta \) to obtain a category \( \Sigma \Theta \).

Let \( \mathcal{C} \) be any category. Then \( \Sigma \mathcal{C} \) is the category of pairs \((S; (A_s))_{s \in S}\) consisting of a set \( S \) and an \( S \)-indexed family of objects \( A_s \) in \( \mathcal{C} \). A morphism \((f, \phi) : (S; (A_s)) \to (T; (B_t))\) consists of a set map \( f : S \to T \) and an \( S \)-indexed family \( \phi = (\phi_s) \) of morphisms \( \phi_s : A_s \to B_{f(s)} \) in \( \mathcal{C} \). Composition is the obvious one.

\((S; (A_s))\) is the categorical sum \( \coprod_{s \in S} (\ast ; A_s) \) where \( \ast \) is a single element set. Hence we have

3.3 Lemma. (1) \( \Sigma \mathcal{C} \) admits arbitrary canonical sums.

(2) If \( \mathcal{D} \) is a category with canonical sums and \( F : \mathcal{C} \to \mathcal{D} \) is a functor, there is a canonical extension functor

\[
\coprod F : \Sigma \mathcal{C} \to \mathcal{D}, \quad (S; (A_s)) \mapsto \coprod_{s \in S} F(A_s).
\]

Now let \( \Theta \) be an arbitrary theory. Let \((0)\) stand for families consisting of copies of the object \( 0 \) of \( \Theta \). The following results are quite obvious.

3.4 Lemma. The inclusion functor

\[
\text{Sets} \to \Sigma \Theta, \quad S \mapsto (S; (0))
\]
is right adjoint to the forgetful functor

\[ \Sigma \mathcal I \Theta \rightarrow \mathcal Sets, \quad (S; (n_s)) \mapsto S. \]

In particular, it preserves (inverse) limits.

3.5 Lemma. The inclusion functor \( \mathcal Sets \rightarrow \Sigma \mathcal I \Theta \) preserves colimits.

3.6 Lemma. \( (S \times T; (m_s + n_t)_{(s,t)}) \) is the product of \( (S; (m_s)) \) with \( (T; (n_t)) \) in \( \Sigma \mathcal I \Theta \).

We also need the existence of particular pullbacks: Let \( \mathcal S \) be the category of finite sets \( (n) = \{1, 2, \ldots, n\} \) and set maps. Let \( \mathcal S^{\text{op}} \subset \Theta \) be the subcategory of set operations \( \sigma^*: m \rightarrow n \) for \( \sigma: (n) \rightarrow (m) \) in \( \mathcal S \). Given a diagram

\[ (S, (m_s)) \xrightarrow{(f_1; (g_{s1}))} (T, (n_t)) \xleftarrow{(g_2; (a_{s2}))} (R, (k_r)) \]

where \( \sigma_r: (n_{g(r)}) \rightarrow (k_r) \) is an injection in \( \mathcal S \) for all \( r \in R \), we can construct a canonical pullback diagram

\[ (S \times_T R; (l_{(s,r)})) \xrightarrow{(q_1; (\rho_{s1,r}))} (R; (k_r)) \]

\[ \downarrow (g_2; (a_{s2})) \]

\[ (S; (m_s)) \xrightarrow{(f_1; \varphi)} (T; (n_t)) \]

where \( \rho_{(s,r)}: (m_s) \rightarrow (l_{(s,r)}) \) again is an injection; \( q_1 \) and \( q_2 \) are the projections, \( (l_{(s,r)}) = (m_s) \bigsqcup (n_t) \) is the disjoint union in \( \mathcal S \) (identified with \( (m_s + n_t) \) in blocks), where \( (n_t) \) is the "complement" of \( \text{Im}(\sigma_r: (n_{g(r)}) \rightarrow (k_r)) \), so that \( (k_r) \cong (n_{g(r)}) \bigsqcup (n_t) \). With these identifications \( \rho_{(s,r)}: (m_s) \rightarrow (l_{(s,r)}) = (m_s) \bigsqcup (n_t) \) is the inclusion and

\[ \pi_{(r,s)}: m_s \times v_r \xrightarrow{\varphi_{s \times id}} n_{f(s)} \times v_r \cong k_r \]

with the isomorphism induced from \( (k_r) \cong (n_{g(r)}) \bigsqcup (n_t) \). Here observe that \( g(r) = f(s) \). Since the \( \rho^* \) again come from injections we can iterate this process to obtain

3.8 Lemma. \( \Sigma \mathcal I \Theta \) has canonical iterated pullbacks of diagrams

\[ (S_0; (n_{0,s_0})) \xrightarrow{(f_1; \varphi_1)} (S_1; (n_{1,s_1})) \xrightarrow{(g_1; \sigma^*_1)} \ldots \xrightarrow{(g_i; \sigma^*_i)} (S_i; (n_{i,s_i})) \]

where the \( \sigma^*_i \) are \( S_i \)-indexed families of set operations induced by injections.

This result enables us to construct category objects in \( \Sigma \mathcal I \Theta \). If the source or the target map \( \mathcal A_i \rightarrow \mathcal A_0 \) is of the form \( (f; (\sigma^*)) \) where \( (\sigma^*) \) is a family of set operations from injections \( \sigma \), the required canonical iterated pullbacks exist. To get concise statements for our machinery, let \( \mathcal Cat(\Sigma \mathcal I \Theta) \) denote the category of all those category objects in \( \Sigma \mathcal I \Theta \) for which the source map is of that form.

The inclusion \( \mathcal Sets \subset \Sigma \mathcal I \Theta \) induces an embedding \( \mathcal Cat \subset \mathcal Cat(\Sigma \mathcal I \Theta) \) of the category of small categories as a full subcategory. \( \mathcal Cat(\Sigma \mathcal I \Theta) \) has products
because $\Sigma \Theta$ has. In particular, we can define a natural transformation $\alpha: F \to G$ of functors $F, G: \mathcal{C} \to \mathcal{D}$ in $\text{Cat}(\Sigma \Theta)$ as an appropriate functor $\mathcal{L}_n \times \mathcal{C} \to \mathcal{D}$, where $\mathcal{L}_n$ in the linear category

$$0 \to 1 \to 2 \to \cdots \to n.$$ 

(3.9) If $X: \Theta \to \mathcal{Top}$ is a $\Theta$-space, the functor $\prod X: \Sigma \Theta \to \mathcal{Top}$ maps the canonical pullbacks (3.7) to the canonical pullbacks in $\mathcal{Top}$. Hence $X$ induces a functor $\text{Cat}(X): \text{Cat}(\Sigma \Theta) \to \text{Cat}(\mathcal{Top})$ into the category of topological categories. Similarly, a theory functor $F: \Theta_1 \to \Theta_2$ induces a functor $\text{Cat}(F): \text{Cat}(\Sigma \Theta_1) \to \text{Cat}(\Sigma \Theta_2)$.

### 4. Rectification of lax functors into $\text{Cat}(\Sigma \Theta)$

In this section we extend Street's first rectification construction to arbitrary functors $A: \mathcal{C} \to \text{Cat}(\Sigma \Theta)$. To fix notation let us recall the basic definitions.

(4.1) Let $\mathcal{C} \in \text{Cat}$. A lax functor $A: \mathcal{C} \to \text{Cat}(\Sigma \Theta)$ is a pair of functions assigning a category object $A(n)$ to each $n \in \text{ob} \mathcal{C}$ and a functor $A(f): A(m) \to A(n)$ to each morphism $f: m \to n$ of $\mathcal{C}$ together with natural transformations

$\rho(n): A(id_n) \to id_{A(n)}$, $\sigma(f, g): A(f \circ g) \to A(f) \circ A(g)$

such that the following diagrams commute:

\[
\begin{array}{ccc}
A(f) \circ A(id_m) & \xrightarrow{\sigma(f, id_m)} & A(f) \\
\downarrow A(f) & & \downarrow A(f) \\
A(f) & & A(id_n) \circ A(f)
\end{array}
\]

\[
\begin{array}{ccc}
A(f \circ g \circ h) & \xrightarrow{\sigma(f, g, h)} & A(f \circ g) \circ A(h) \\
\downarrow A(f \circ g) & & \downarrow A(id_n) \\
A(f \circ A(g \circ h) & \xrightarrow{A(f) \circ \sigma(g, h)} & A(f) \circ A(g) \circ A(h)
\end{array}
\]

(4.2) Let $A, B: \mathcal{C} \to \text{Cat}(\Sigma \Theta)$ be lax functors. A (left) lax natural transformation $d: A \to B$ is a pair of functions assigning a functor $d(n): A(n) \to B(n)$ to each $n \in \text{ob} \mathcal{C}$ and a natural transformation $d(f): B(f) \circ d(m) \to d(n) \circ A(f)$ to each morphism $f: m \to n$ of $\mathcal{C}$ such that the following diagrams commute
and for \( g \circ f : m \rightarrow n \rightarrow p \)

\[
\begin{align*}
B(g \circ f) \circ d(m) & \xrightarrow{d(g \circ f)} d(p) \circ A(g \circ f) \\
& \xrightarrow{d(p)\sigma(g,f)} d(p) \circ A(g) \circ A(f) \\
\sigma(g,f)d(m) & \\
& \xrightarrow{d(g)A(f)} d(p) \circ A(g) \circ A(f) \\
B(g) \circ B(f) \circ d(m) & \xrightarrow{B(g)d(f)} B(g) \circ d(n) \circ A(f) \\
& \xrightarrow{B(g)d(f)} B(g) \circ d(n) \circ A(f)
\end{align*}
\]

One can compose lax natural transformations \( d : A \rightarrow B \) and \( e : B \rightarrow C \) by setting \( (e \circ d)(n) = e(n) \circ d(n) \) on objects and

\[
(e \circ d)(f) : C(f) \circ e(m) \circ d(m) \xrightarrow{e(f)d(m)} e(n) \circ B(f) \circ d(m)
\]

on morphisms. We obtain a category of lax functors and lax natural transformations.

(4.3) Let \( d, d' : A \rightarrow B \) be lax natural transformations of lax functors \( \mathcal{C} \rightarrow \mathcal{C}(\Sigma, \Theta) \). A natural homotopy \( \alpha : d \rightarrow d' \) consist of natural transformations \( \alpha(n) : d(n) \rightarrow d'(n) \) such that

\[
\begin{array}{ccc}
B(f)d(m) & \xrightarrow{B(f)\alpha(m)} & B(f)d'(m) \\
& \downarrow{d(f)} & \downarrow{d'(f)} \\
& B(f)A(f) & \xrightarrow{\alpha(n)A(f)} & d'(n)A(f)
\end{array}
\]

commutes. If \( \beta : d' \rightarrow d'' \) is another such natural homotopy then \( \beta \circ \alpha : d \rightarrow d'' \) is given by \( (\beta \circ \alpha)(n) = \beta(n)\alpha(n) \). If \( \gamma : e \rightarrow e' \) is a natural homotopy of lax natural transformations \( e, e' : B \rightarrow C \), then \( \gamma \circ e : B \rightarrow C \rightarrow \) is given by

\[
(\gamma \circ e)(n) = \gamma(n)d'(n) \circ e(n)\alpha(n) : e(n)d(n) \rightarrow e(n)d'(n) \rightarrow e'(n)d'(n)
\]

\[
= e'(n)\alpha(n) \circ \gamma(n)d(n) : e(n)d(n) \rightarrow e'(n)d(n) \rightarrow e'(n)d'(n).
\]

4.4 Proposition. Let \( \mathcal{C} \) be a small category and \( \Theta \) a theory. There is a functor \( \mathcal{C} \rightarrow SA \) from the category of lax functors \( \mathcal{C} \rightarrow \mathcal{C}(\Sigma, \Theta) \) and lax natural transformations to the category of genuine functors \( \mathcal{C} \rightarrow \mathcal{C}(\Sigma, \Theta) \) and genuine natural transformations with the following properties.

(1) For each \( n \in \text{ob} \mathcal{C} \), there is a pair of adjoint functors (i.e. front and back adjunction exist)

\[
\eta(n) : SA(n) \Rightarrow A(n) : \eta(n).
\]

The \( \eta(n) \) combine to a lax natural transformation \( \eta : A \rightarrow SA \) such that for a lax natural transformation \( d : A \rightarrow B \) the diagram

\[
\begin{array}{ccc}
A(n) & \xrightarrow{d(n)} & B(n) \\
\eta(n) & \downarrow & \eta(n) \\
SA(n) & \xrightarrow{Sd(n)} & SB(n)
\end{array}
\]

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commutes. If \( A \) is a genuine functor then the \( \varepsilon(n) \) combine to a genuine natural transformation \( \varepsilon: SA \to A \).

(2) If \( F: \mathcal{D} \to \mathcal{C} \) is a functor, then \( A \circ F: \mathcal{D} \to \text{Cat}(\Sigma \downarrow \Theta) \) is a lax functor and there is a natural transformation \( \xi: S(A \circ F) \to S(A) \circ F \) such that

\[
\begin{array}{ccc}
S(A \circ F)(k) & \xrightarrow{\eta_A F(k)} & S(A) \circ F(k) \\
A \circ F(k) & \xrightarrow{\eta_A F(k)} & A \circ F(k) \\
\end{array}
\]

commutes for all \( k \in \text{ob} \mathcal{D} \). Moreover \( \xi \) is natural with respect to lax natural transformations \( A \to B \).

(3) Given an additional functor \( G: \mathcal{C} \to \mathcal{D} \), then

\[
\begin{array}{ccc}
S(A \circ F \circ G) & \xrightarrow{\varepsilon} & S(A \circ F) \circ G \\
S(A \circ F) \circ G & \xrightarrow{\varepsilon \circ G} & S(A) \circ F \circ G \\
\end{array}
\]

commutes.

(4) If \( X: \Theta \to \mathcal{T}_\Theta \) is a \( \Theta \)-space, then \( \text{Cat}(X) \circ A: \mathcal{C} \to \text{Cat}(\mathcal{T}_\Theta) \) is a lax functor, and \( S(\text{Cat}(X) \circ A) = \text{Cat}(X) \circ SA \). The analogous result holds for theory functors \( \Theta_1 \to \Theta_2 \).

(5) If \( \alpha: d \to d' \) is a natural homotopy of lax natural transformations \( d, d': A \to B \) then there is a natural homotopy \( S\alpha: Sd \to Sd' \) of genuine natural transformations, i.e. (4.3) commutes with \( Sd(f) \) and \( Sd'(f) \) being identities. The correspondence \( \alpha \mapsto S\alpha \) preserves both compositions of natural homotopies defined in (4.3).

Proof. We give the constructions and leave the verification of the details to the reader. They are straightforward translations of Street's proofs once one knows the translation mechanism.

For \( n \in \text{ob} \mathcal{C} \) we have to define a category object \( SA(n) \) in \( \Sigma \downarrow \Theta \). We construct its nerve \( SA_*(n) \):

\[
SA_k(n) = \bigsqcup_{(f_0, \ldots, f_0)} P(f_0, \ldots, f_0)
\]

taken over all composable morphisms

\[
p_0 \overset{f_0}{\longrightarrow} p_1 \overset{f_1}{\longrightarrow} p_2 \overset{f_2}{\longrightarrow} \cdots \overset{f_{k-1}}{\longrightarrow} p_k \overset{f_k}{\longrightarrow} n
\]

in \( \mathcal{C} \) with final target \( n \). Let \( s \) and \( t \) denote the source and target morphisms of our category objects. Then \( P_{(f_0, \ldots, f_0)} \) is the canonical pullback of

\[
\begin{array}{cccccccc}
A_1(p_k) & \xrightarrow{s} & A_0(p_k) & \xrightarrow{A(f_k) = t} & A_0(p_{k-1}) & \cdots & A_1(p_1) & \xrightarrow{s} & A_0(p_1) \\
A_0(p_k) & \xrightarrow{s} & A_0(p_{k-1}) & & A_0(p_2) & & A_0(p_1) & & \xrightarrow{A(f_0) = t} & A_0(p_0)
\end{array}
\]
if $k > 0$. For $k = 0$ we have

$$\text{ob } SA(n) = SA_0(n) = \bigsqcup_{f_0 : p \to n} P_{(f_0)}$$

with $P_{(f_0)} = A_0(p)$. Source and target morphism in $SA(n)$ are induced by

$$\bar{s} = \text{proj}_1 : P_{(f_1, f_0)} \to P_{(f_1 \circ f_0)} = A_0(p_0),$$

$$\bar{t} = t \circ \text{proj}_2 : P_{(f_1, f_0)} \to A_1(p_1) \to A_0(p_1) = P_{(f_1)}$$

where $\text{proj}_1$ and $\text{proj}_2$ are the structure maps of the pullback

$$\begin{array}{ccc}
P_{(f_1, f_0)} & \xrightarrow{\text{proj}_2} & A_1(p_1) \\
\downarrow \text{proj}_1 & & \downarrow s \\
A_0(p_0) & \xrightarrow{A_0(f_0)} & A_0(p_1)
\end{array}$$

The inclusion of the identities $i : SA_0(n) \to SA_1(n)$ sends $P_f$ to $P_{(f, id)}$ by the map induced by

$$A_1(p_0) \xrightarrow{p(p_0)} P_f = A_0(p_0) \xrightarrow{id} A_0(p_0).$$

Composition $c : SA_2(n) \to SA_1(n)$ decomposes into sums

$$c : P_{(f_2, f_1, f_0)} \to P_{(f_2, f_1 \circ f_0)}, \quad p_0 \xrightarrow{f_2} p_1 \xrightarrow{f_1} p_2 \xrightarrow{f_0} n.$$

$P_{(f_2, f_1, f_0)}$ and $A_3(p_2)$ are iterated pullbacks so that we have a commutative diagram of maps and structure maps

Since $A_3(p_2)$ is the iterated pullback of the lower row, this diagram induces a morphism $r : P_{(f_2, f_1, f_0)} \to A_3(p_2)$. The solid arrow diagram...
commutes to induce the composition \( c \).

This determines the nerve of \( SA(n) \). For a morphism \( g : m \to n \) in \( \mathcal{C} \) the functor \( SA(g) : SA(m) \to SA(n) \) is determined by \( id : P(f_k, \ldots, f_0) \to P(g \circ f_k, \ldots, f_0) \).

The functor \( e(n) : SA(n) \to A(n) \) is evaluation given on objects by \( A(f) : P(f) = A(p_0) \to A(n) \) and on morphisms by

\[
P(f_k, f_0) \xrightarrow{h} A_2(n) \xrightarrow{\text{comp.}} A_1(n)
\]

where \( h \) is induced by

\[
A_1(p_1) \xleftarrow{\text{comp.}} A_0(p_0)
\]

The functor \( \eta(n) : A(n) \to SA(n) \) is given on objects by \( id : A_0(n) \to P(id) = A_0(n) \) and on morphisms by \( v : A_1(n) \to P(id, id) \) defined by

\[
A_2(n) \xleftarrow{\text{comp.}} A_1(n)
\]

where \( v' \) comes from
The functors $\rho(n) : A_0(n) \to A_1(n)$ define the back adjunction $\varepsilon(n) \circ \eta(n) \to \text{Id}$. The front adjunction $\text{Id} \to \eta(n) \circ \varepsilon(n)$ is defined by

where $i$ is the inclusion of the identities. If $d : A \to B$ is a lax natural transformation, then $Sd : SA \to SB$ is defined by the functors $Sd(n) : SA(n) \to SB(n)$ specified by

$$SA_1(n) = \bigsqcup_{(f_1, f_0)} P(f_1, f_0) \to SB_1(n) = \bigsqcup_{(f_1, f_0)} P'(f_1, f_0)$$
with $u$ defined by

\[
\begin{array}{ccc}
A_1(p_1) & \xrightarrow{d(p_1)} & B_1(p_1) \\
\downarrow & & \downarrow s \\
P(f_1, f_0) & \xrightarrow{u} & B_2(p_1) \\
\downarrow & & \downarrow \text{pullback} \\
A_0(p_0) & \xrightarrow{d(f_0)} & B_1(p_1) \\
\end{array}
\]

If $\alpha: d \to d'$ is a natural homotopy of lax natural transformations, the natural transformations $S\alpha(n): Sd(n) \to Sd'(n)$ of the natural homotopy $S\alpha: Sd \to Sd'$ are given by

\[
SA_0(n) \to SB_1(n), \quad P(f) \to P(f, id)
\]

$f: p \to n$, given by

\[
\begin{array}{ccc}
B_2(p) & \xrightarrow{u} & P_f = A_0(p) \\
\downarrow \text{comp.} & & \downarrow d(p) \\
B_1(p) & \xrightarrow{s} & B_0(p) \\
\downarrow & & \downarrow \text{pullback} \\
B_1(p) & \xrightarrow{\alpha(p)} & B_2(p) \\
\downarrow & & \downarrow \text{pullback} \\
B_0(p) & \xrightarrow{\rho(p)} & B_1(p) \\
\end{array}
\]

with $u$ from

\[
\begin{array}{ccc}
A_0(p) & \xrightarrow{d(p)} & B_0(p) \\
\downarrow & & \downarrow \rho(p) \\
B_1(p) & \xrightarrow{s} & B_0(p) \\
\end{array}
\]
Now let $F : \mathcal{D} \to \mathcal{C}$ be a functor. The natural transformation $\xi : S(A \circ F) \to SA \circ F$ is the collection of functors
\[ \xi(k) : S(A \circ F)(k) \to SA(Fk), \quad k \in \text{ob} \mathcal{D}, \]
given on objects by $id : P_{(f)} \to P_{(Ff)}$, $f : p \to k$ in mor $\mathcal{D}$, and on morphisms by $id : P_{(f_1, f_0)} \to P_{(Ff_1, Ff_0)}$.

This specifies the construction. We should point out that all pullbacks used in this construction exist by (3.8). If $X : \Theta \to \mathcal{F}_\rho$ is a $\Theta$-space, $\coprod X$ maps these pullbacks to the corresponding canonical ones in $\mathcal{F}_\rho$. Hence (4) holds. The verification of the remaining properties is a simple diagram chase.

5. Homotopy theory in $\text{Cat}(\Sigma \Theta_r)$

For a given $A_\infty$ ring $X : \Theta \to \mathcal{F}_\rho$ we associate to each diagram $D : \mathcal{H} \to \text{Cat}(\Sigma \Theta_r)$ satisfying (5.2) below a diagram $[\tilde{BD}] : \mathcal{H} \to \mathcal{F}_\rho$ such that for any subcategory $i : \mathcal{L} \subset \mathcal{H}$ there is an isomorphism $[\tilde{B}(D \circ i)] \to [\tilde{B}D] \circ i$ in $\mathcal{F}_\rho$. This correspondence is strictly functorial in $X$.

Let $\mathcal{A} \in \text{ob} \text{Cat}(\Sigma \Theta_r)$. The nerve of $A$ defines a functor $\Delta^{\text{op}} \to \Sigma \Theta_r$. Consider the pullback
\[ \begin{array}{ccc}
\mathcal{A} & \xrightarrow{H\mathcal{A}} & \Sigma \Theta \\
\downarrow \nu & & \downarrow \Sigma \Theta_0 \\
\Delta^{\text{op}} & \xrightarrow{\Delta \mathcal{A}} & \Sigma \Theta_r
\end{array} \]

The Segal pushdown induces a simplicial space
\[ N\mathcal{A} = \nu_* \left( \coprod X \circ H\mathcal{A} \right) : \Delta^{\text{op}} \to \mathcal{F}_\rho. \]

We are interested in the homotopy type of its realization $\tilde{N}\mathcal{A} = |N\mathcal{A}|$. To be able to make any statements we introduce the following convention.

5.2 Convention. The structure maps (source, target, inclusion of identities, iterated composition) of all category objects, all functors and natural transformations studied in this section are supposed to take simple operators in $\Sigma \Theta_r$ as values.

Let $\mathcal{L}_n$ denote the linear category $0 \to 1 \to \cdots \to n$. A functor $f : \mathcal{A} \to \mathcal{C}$ of category objects induces a diagram
\[ \begin{array}{ccc}
\mathcal{P}_f & \xrightarrow{Hf} & \Sigma \Theta \\
\downarrow \mu & & \downarrow \Sigma \Theta_f \\
\mathcal{L}_n \times \Delta^{\text{op}} & \xrightarrow{\Delta f} & \Sigma \Theta_r
\end{array} \]
and hence a map of simplicial spaces (for notational convenience we do not distinguish between a functor $\mathcal{L} \times \Delta^{\text{op}} \to \mathcal{F}_\rho$ and its adjoint $\mathcal{L} \to \mathcal{F}_\rho \Delta^{\text{op}}$)
\[ N_1 f = \mu_* \left( \coprod X \circ Hf \right) : \mathcal{L}_1 \times \Delta^{\text{op}} \to \mathcal{F}_\rho. \]
We denote its source by $N_{f_{\mathcal{A}}}$ and its target by $N_{f_{\mathcal{B}}}$.
By (2.8.4) we have weak equivalences $i_{\mathcal{A}} : N_{\mathcal{A}} \to N_{f_{\mathcal{A}}}$ and $i_{\mathcal{B}} : N_{\mathcal{B}} \to N_{f_{\mathcal{B}}}$. Choose any homotopy inverse of the realization of $i_{\mathcal{B}}$ to obtain a map of spaces

$$\tilde{B}f : \tilde{B}\mathcal{A} \to \tilde{B}\mathcal{B}.$$ 

It depends on the choice of the homotopy inverse but its homotopy class $[\tilde{B}f]$ is well-defined.

5.4 Remark. We can do better. From Segal's construction and elementary facts about simplicial spaces one deduces that $N_{f_{\mathcal{A}}} = N_{f_{\mathcal{B}}}$ and that $|i_{\mathcal{B}}|$ embeds $|N_{\mathcal{B}}|$ as strong deformation retract into $|N_{f_{\mathcal{B}}}|$. Since the space of deformation retractions is contractible, $\tilde{B}f$ is uniquely defined up to contractible choice.

5.5 Lemma. (1) For a commutative triangle of category objects and functors

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{B} \\
\downarrow{g} & & \downarrow{h} \\
\mathcal{C} & \xrightarrow{\tilde{B}f} & \mathcal{D}
\end{array}$$

we have $[\tilde{B}h] = [\tilde{B}g] \circ [\tilde{B}f]$.

(2) $[\tilde{B}id_{\mathcal{A}}] = id_{\tilde{B}\mathcal{A}}$.

Proof (1). The triangle defines a functor $T : \mathcal{L}_2 \to \mathcal{Cat}(\Sigma \Theta_r)$ inducing a triangle $NT : \mathcal{L}_2 \times \Delta^{op} \to \mathcal{T}_{\mathcal{P}}$ of simplicial spaces. The solid arrow diagram

$$\begin{array}{ccc}
N_{\mathcal{A}} & \xrightarrow{i} & N_{f_{\mathcal{A}}} \\
\downarrow{j} & & \downarrow{k} \\
N_{f_{\mathcal{B}}} & \xrightarrow{\tilde{B}f} & N_{f_{\mathcal{B}}}
\end{array}$$

commutes, the various $i$'s and $j$'s are weak equivalences. The dotted arrows exist after realization making the corresponding squares homotopy commutative; they represent $[\tilde{B}f]$, $[\tilde{B}g]$, and $[\tilde{B}h]$. 
Proof (2). Let \( r : \mathcal{L}_1 \times \Delta \text{op} \to \Delta \text{op} \) be the projection and \( i_0, i_1 : \Delta \text{op} \to \mathcal{L}_1 \times \Delta \text{op} \) the two inclusions. The diagram

\[
\begin{array}{ccccccc}
\mathcal{P}_d & \xrightarrow{R} & \mathcal{P}_d & \xrightarrow{H A} & \Sigma \Theta & \xrightarrow{\Delta} & \mathcal{T}_\Theta \\
\downarrow \nu & & \downarrow \mu & & \downarrow \nu & & \\
\Delta \text{op} & \xrightarrow{i_k} & \mathcal{L}_1 \times \Delta \text{op} & \xrightarrow{r} & \Delta \text{op} & \xrightarrow{\Delta} & \Sigma \Theta_r \\
\end{array}
\]

induces a diagram of simplicial spaces

\[
\begin{array}{c}
\mathcal{X}\mathcal{I}_1 \\
\downarrow \nu \\
\mathcal{N}_A \\
\end{array}
\]

where \( N_1 \text{id} \) is defined as in (5.3). It is not the identity. By naturality of Segal's construction \( r \circ i_k = \text{id} \). Hence \( r \) represents a homotopy inverse of \( |i_k| \), and we obtain

\[
[r \circ N_1 \text{id} \circ i_0] = [r \circ i_0] = [\text{id}].
\]

(5.6) In general we are given a diagram \( D : \mathcal{H} \to \text{Cat}(\Sigma \Theta_r) \) of category objects and functors. The previous construction and the proof of (5.5) extend to produce a functor

\[
[\tilde{B} D] : \mathcal{H} \to \mathcal{T}_\Theta^h
\]

which is naturally isomorphic to the composite functor

\[
\mathcal{H} \xrightarrow{N_D} \mathcal{T}_\Theta^\text{op} \xrightarrow{\Delta \text{op} \text{realiz.}} \mathcal{T}_\Theta^\text{proj} \xrightarrow{\text{proj}} \mathcal{T}_\Theta^h
\]

where \( N_D \) is obtained from \( D \) via Segal's push-down similar to (5.1) and (5.3).

5.7 Lemma. Given a product category object and its projections

\[
A \xleftarrow{\pi_1} A \times C \xrightarrow{\pi_2} C.
\]

Then \((\tilde{B} \pi_1), (\tilde{B} \pi_2)\) : \( \tilde{B} (A \times C) \to \tilde{B} A \times \tilde{B} C \) is an isomorphism in \( \mathcal{T}_\Theta^h \).

Proof. Let \( \mathcal{E} \) denote the category \( 1 \leftarrow 0 \to 1' \). The projections induce a diagram

\[
\begin{array}{ccccccc}
\mathcal{P}_d & \xrightarrow{H P} & \Sigma \Theta & \xrightarrow{\coprod X} & \mathcal{T}_\Theta \\
\downarrow \nu & & \downarrow \Sigma F & & \downarrow p=\text{proj.} \\
\mathcal{E} \times \Delta \text{op} & \xrightarrow{\Delta} & \Sigma \Theta_r & \xrightarrow{\coprod X} & \mathcal{T}_\Theta^h
\end{array}
\]

Since \( F \) is bijective on path components, \( \coprod X \) induces a sum and product preserving functor \( [\coprod X] \). By (2.8.1) there is a natural isomorphism

\[
p \circ N_F = p \circ \nu_* \left( \coprod X \circ H P \right) \cong \coprod X \circ \Delta P.
\]
Since \([\coprod X] \circ \Delta P\) is a product diagram in \(\mathcal{F}_P\), so is \(p \circ N_P\) and the result follows from (5.6).

5.8 Lemma. Let \(\omega: f \to g\) be a natural transformation of functors \(f, g: \mathcal{A} \to \mathcal{C}\). Then \([Bf] = [Bg]\).

Proof. \(\omega\) defines a functor \(\omega: \mathcal{A} \times \mathcal{L}_1 \to \mathcal{C}\) such that

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{i_0} & \mathcal{A} \\
\downarrow f & & \downarrow g \\
\mathcal{C} & \xleftarrow{i_1} & \mathcal{A} \times \mathcal{L}_1
\end{array}
\]

commutes, where \(i_0\) and \(i_1\) are the obvious inclusion functors. We obtain

\([Bf] = [B\omega] \circ [Bi_0] \quad \text{and} \quad [Bg] = [B\omega] \circ [Bi_1].\)

Let \(r: \mathcal{A} \times \mathcal{L}_1 \to \mathcal{A}\) and \(q: \mathcal{A} \times \mathcal{L}_1 \to \mathcal{L}_1\) be the projection. Then \([B\omega]\) is left inverse to \([Bi_1]\). Hence we are done, if \([Br]\) is an isomorphism. Since

\([(Br), (Bq)]: B(\mathcal{A} \times \mathcal{L}_1) \to B\mathcal{A} \times B\mathcal{L}_1\)

is an isomorphism in \(\mathcal{F}_P\), it suffices to show that \(B\mathcal{L}_1\) is contractible. Since \(\mathcal{L}_1 \in \text{ob}\mathcal{C} \subset \text{ob}\mathcal{Cat}(\Sigma \Theta)\), the functor \(\Delta \mathcal{L}_1: \Delta \mathcal{P} \to \Sigma \Theta\) factors through \(\text{Sets}\). In

\[
\begin{array}{ccc}
\Delta \mathcal{P} & \xrightarrow{\Delta \mathcal{L}_1} & \text{Sets} \\
\downarrow \text{id pullback} & & \downarrow \text{id} \\
\Delta \mathcal{P} & \xrightarrow{\Sigma \Theta} & \text{Sets} \subset \Sigma \Theta
\end{array}
\]

the square \(I\) is a pullback because \(0 \in \Theta\) is a terminal object. Hence \(N\mathcal{L}_1 = \text{id}_*(\coprod X \circ j \circ \Delta \mathcal{L}_1)\), which by (2.8.2) is weakly equivalent to \(\coprod X \circ j \circ \Delta \mathcal{L}_1\) which in turn is the usual nerve of \(\mathcal{L}_1\). Hence \(|N\mathcal{L}_1|\) is equivalent to the unit interval and contractible.

5.9 Corollary. Given a pair of adjoint functors \(f: \mathcal{A} \to \mathcal{C}\) and \(g: \mathcal{C} \to \mathcal{A}\), then \([Bf]: B\mathcal{A} \cong B\mathcal{C}\) with inverse \([Bg]\) in \(\mathcal{F}_P\).

6. The algebraic \(K\)-theory of \(A_\infty\) rings

As indicated in §3 we now give a formalized definition of the permutative category (for a definition see [M4]) of finitely generated free \(R\)-modules. We rectify the associated lax functor by Street's construction of §4. The methods of §5 then provide a functor \(\mathcal{F} \to \mathcal{F}_P\) which we use to define the algebraic \(K\)-theory of an \(A_\infty\) ring.

The permutative category object \(\mathcal{M}\) in \(\Sigma \Theta\). The idea is to define \(\mathcal{M}\) in such a way that for any genuine ring \(R: \Theta \to \text{Sets}\) the category object \(\text{Cat}(R)(\mathcal{M}) \in \text{Cat}(\text{Sets}) = \text{Cat}\) is the permutative category of finitely generated free \(R\)-modules \(R^n, n = 0, 1, \ldots\), and linear maps. So we have one
object \( R^n \) for each \( n \in \mathbb{N} \) and the monoids \( M_n^R \) of \((n \times n)\)-matrices as sets of morphisms.

Consequently, \( \mathcal{M}_0 = \text{ob}\mathcal{M} \) is the object \((\mathbb{N}, (0))\) of \( \Sigma \mathcal{I} \mathcal{O}_r \) and \( \mathcal{M}_1 = \text{mor}\mathcal{M} = (\mathbb{N}, (n^2)_{n \in \mathbb{N}}) \). Source and target morphisms
\[
s, t : (\mathbb{N}, (n^2)) \to (\mathbb{N}, (0))
\]
are given by the pairs consisting of the set map \( id_{\mathbb{N}} \) and the projection operations \( n^2 \to 0 \) in \( \Theta_r \). In dimension \( p \), \( \mathcal{M} \) is given by \( \mathcal{M}_p = (\mathbb{N}, (p \cdot n^2)) \), where \( p \cdot n^2 \) should be considered as a \( p \)-tuple of \((n \times n)\)-matrices. Hence \( p \)-fold composition
\[
\text{comp} : (\mathbb{N}, (pn^2)) \to (\mathbb{N}, (n^2))
\]
consists of \( id_{\mathbb{N}} \) and the sequence of \( p \)-fold matrix multiplications
\[
M_n(x_1 \cdots x_p) : pn^2 \to n^2
\]
in the terminology of [SV4, §3]. The unit map
\[
\overline{u} : \mathcal{M}_0 = (\mathbb{N}, (0)) \to \mathcal{M}_1 = (\mathbb{N}, (n^2))
\]
consists of \( id_{\mathbb{N}} \) and the sequence of \( M_n(1) \) of unit matrices.

This defines \( \mathcal{M} \) as simplicial object with the boundaries induced by suitable projections and compositions, and degeneracies by \( \overline{u} \). It remains to codify the permutative structure under direct sum.

\[
\oplus : \mathcal{M}_1 \times \mathcal{M}_1 = (\mathbb{N} \times \mathbb{N}, (k^2 + l^2)_{(k,l)}) \to \mathcal{M}_1 = (\mathbb{N}, (n^2))
\]
is the pair consisting of
\[
f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (k, l) \mapsto k + l
\]
and the sequence of operations \( k^2 + l^2 \to (k + l)^2 \) given by block sum of matrices
\[
(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.
\]

For the matrix notation we use the following convention: In the category \( \mathcal{S} \) of finite sets we identify \( \langle k \rangle \times \langle l \rangle \) with \( \langle k \cdot l \rangle \) by lexicographical ordering and \( \langle k \rangle \cup \langle l \rangle \) with \( \langle k + l \rangle \) in blocks. Hence we can identify an operation \( k^2 + l^2 \to (k + l)^2 \) with an operation
\[
(\langle k \rangle \times \langle k \rangle) \cup (\langle l \rangle \times \langle l \rangle) \to (\langle k \rangle \cup \langle l \rangle) \times (\langle k \rangle \cup \langle l \rangle).
\]
This operation has \( (k + l)^2 \) components, the entries of a \([(k + l) \times (k + l)]\)-matrix, in \( k^2 + l^2 \) variables, the entries of an ordered pair of matrices \( (A, B) \) with \( A \) a \((k \times k)\)-matrix and \( B \) an \((l \times l)\)-matrix.

Clearly, \( \oplus \) is associative. The commuting natural equivalence
\[
c : \oplus \to \oplus \circ T,
\]
where \( T \) is the switch map, is the morphism
\[
\mathcal{M}_0 \times \mathcal{M}_0 = (\mathbb{N} \times \mathbb{N}, (0)) \to \mathcal{M}_1 = (\mathbb{N}, (n^2))
\]
given by the set map \( \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (k, l) \mapsto k + l \) and the family of operations
\[
c_{k, l} : 0 \to (k + l)^2
\]
\[
c_{k, l} = \begin{pmatrix} 0 & E_k \\ E_l & 0 \end{pmatrix}.
\]
where $E_n$ stands for the $n$-squared unit matrix.

We choose $(\{0\}, (0)) \in \Sigma \Theta_r$ as canonical terminal object. Then the unit of the permutative structure is the morphism

$$0 : (\{0\}, (0)) \to M_0 = (N, (0))$$

consisting of the set map $\{0\} \to N$, $0 \mapsto 0$, and the operation $id_0$. As in the classical case we obtain

6.1 Proposition. $M = (M, \oplus, 0, c)$ is a permutative category object in $\Sigma \Theta_r$.

In the commutative case, $M$ can be extended to a bipermutative (for a definition see [M4]) category object $M_c$ in $\Sigma \Theta_{cr}$ using the tensor product of matrices:

$$\otimes : M_1 \times M_1 = (N \times N)(k^2 + l^2)_{(k, l)} \to M_1 = (N, (n^2))$$

consists of the set map $N \times N \to N$, $(k, l) \mapsto k \cdot l$ and the operations $k^2 + l^2 \to (k \cdot l)^2$ defined by

$$(A, B) \to A \otimes B.$$

(If we identify $(k \cdot l)^2$ with $(k) \times (l) \times (k) \times (l)$, the $(i, p, j, q)$th component of the tensor product is the monomial $a_{ij} \cdot b_{pq}$, where $A = (a_{ij})$ and $B = (b_{pq})$.) The tensor product is strictly associative. The commuting equivalence $c' : \otimes \to \otimes \circ T$ is the morphism

$$c' : M_0 \times M_0 = (N \times N, (0)) \to M_1 = (N, (n^2))$$

given by the set map $N \times N \to N$, $(k, l) \mapsto k \cdot l$ and the family of operations $c'_{k, l} : 0 \to (k \cdot l)^2$ given by the entries of the permutation matrix which maps the $(i, p)$th unit vector of $R^{k \cdot l}$ to the $(p, i)$th unit vector under the identifications

$$(k) \times (l) \cong (k \cdot l) \cong (l) \times (k).$$

The unit of $\otimes$ is the morphism

$$1 : (\{0\}, (0)) \to M_0 = (N, (0))$$

consisting of $\{0\} \to N$, $0 \mapsto 1$, and the operation $id_0$. Again, as in the classical case, we have

6.2 Proposition. $M_c = (M, \oplus, 0, c, \otimes, 1, c')$ is a bipermutative category object in $\Sigma \Theta_{cr}$.

The definition of $K$-theory. We proceed in accordance with the classical case (e.g. [M7, §§3, 4] or [T1]). We use the terminology of [M7]. The permutative category object $M$ determines a lax functor

$$A : \mathcal{F} \to \text{Cat}(\Sigma \Theta_r)$$

defined on objects by $A(n) = M^n$ with $M^0$ the trivial category object $(\{0\}, (0))$, and on morphisms $f : m \to n$ by $A(f) : M^m \to M^n$ given by the $n$-tuple of morphisms

$$M^m \xrightarrow{\text{proj}} \prod_{i \in f^{-1}(j)} M \xrightarrow{\oplus} M, \quad j = 1, \ldots, n,$$

where the empty direct sum is the unit 0 in $M$. 

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We apply Street’s rectification of §4 to obtain a genuine functor

\[ SA : \mathcal{T} \to \mathcal{C}at(\Sigma \Theta_r). \]

Given an \( A_\infty \) ring \( \Theta : X \to \mathcal{T}_\infty \) with \( F_\Theta : \Theta \to \Theta_r \) an \( A_\infty \) ring theory, we form the diagram

\[
\begin{array}{cccc}
\mathcal{P} & \xrightarrow{HSA} & \Sigma \Theta & \xrightarrow{\coprod X} & \mathcal{T}_\infty \\
\downarrow \nu & \text{pullback} & \downarrow \Sigma F & \Downarrow \pi_0 \\
\mathcal{P} \times \Delta^{op} & \xrightarrow{\Delta S \Sigma} & \Sigma \Theta_r & \xrightarrow{\coprod \pi_0 X} & \mathcal{T}_{\mathcal{P} h}
\end{array}
\]

where \( \pi_0 X \) is the space of path components of \( X \) with the quotient topology (of course, \( \pi_0 X \) is discrete if \( X \) is of the homotopy type of a CW-complex). Since \( F \) is bijective on path components, \( \coprod \pi_0 X \) is defined by \( \coprod X \). A check through the definition of \( \mathcal{M} \), the rectification process and the nerve construction shows that \( \Delta S \Sigma \) takes simple morphisms as values. Hence \( \nu \) is an equivalence of categories. Hence Segal pushdown defines a functor

\[ N_{SA}(X) = \nu_* \left( \coprod X \circ HSA \right) : \mathcal{P} \times \Delta^{op} \to \mathcal{T}_\infty \]

together with weak equivalences of \( \mathcal{P} \)-spaces

\[ N_{SA}(X) \circ \nu \cong id_* \left( \coprod X \circ HSA \right) \to \coprod X \circ HSA. \]

6.4 Lemma. The topological realization \( |N_{SA}(X)| : \mathcal{T} \to \mathcal{T}_{\mathcal{P} h} \) of \( N_{SA}(X) \) is a special \( \mathcal{T} \)-space.

Proof. For simplicity we drop \( (X) \) from the notation. We have to show that the maps \( \pi_i : N_{SA}(n) \to N_{SA}(1) \) of (2.6) induce a homotopy equivalence \( N_{SA}(n) \to (N_{SA}(1))^n \) after realization. Let \( Q \) be the composite

\[ Q : \mathcal{T} \xrightarrow{N_{SA}} \mathcal{T}_{\mathcal{P} h}^{\Delta^{op}} \xrightarrow{\text{realiz.}} \mathcal{T}_{\mathcal{P} h} \xrightarrow{\text{proj.}} \mathcal{T}_{\mathcal{P} h}. \]

By (5.6), \( Q \) is naturally equivalent to the functor \( [\tilde{BS}A] : \mathcal{T} \to \mathcal{T}_{\mathcal{P} h} \). If \( \eta : A \to SA \) is the lax natural transformation of (4.4.1) then for each \( \pi = \pi_i \) the diagram

\[
\begin{array}{cc}
A(n) & \xrightarrow{\eta(n)} & SA(n) \\
\downarrow A(\pi) & & \downarrow SA(\pi) \\
A(1) & \xrightarrow{\eta(1)} & SA(1)
\end{array}
\]

commutes up to the natural transformation \( \eta(\pi) \). In \( \mathcal{T}_{\mathcal{P} h} \) we hence obtain a commutative diagram

\[
\begin{array}{c}
\tilde{BA}(n) \xrightarrow{[\tilde{BA}(n)]} \tilde{BS}A(n) \cong Q(n) \\
\downarrow [\tilde{BA}(\pi)] & \downarrow [\tilde{BS}A(\pi)] & \downarrow Q(\pi) \\
\tilde{BA}(1) \xrightarrow{[\tilde{BA}(1)]} \tilde{BS}A(1) \cong Q(1)
\end{array}
\]
Since the $\eta(n)$ have adjoints, $[\tilde{B}\eta(n)]$ is an isomorphism. Combining these diagrams we get a commutative diagram in $\mathcal{T}_{n,h}$

$$\begin{array}{ccc}
\tilde{B}A(n) & \cong & Q(n) \\
(\tilde{B}A\pi_1, \ldots, \tilde{B}A\pi_n) & \downarrow & (Q\pi_1, \ldots, Q\pi_n) \\
\tilde{B}A(1)^n & \cong & Q(1)^n
\end{array}$$

Since $A(n) = A(1)^n$ and the $\pi_i$ are the projections, the left arrow is an isomorphism by (5.7).

For the same reasons $[\tilde{B}\eta(0)]$: $\tilde{B}A(0) \cong Q(0)$. Since $\Delta A(0)$ is the constant simplicial object on $(\{0\}, (0))$ in $\Sigma \Theta_r$ and since $X$ is product preserving, $\coprod X \circ HA(0)$ is the constant functor on a point. Hence $\tilde{B}A(0)$ is contractible by (2.8). □

$|NSA(X)|$ is the $A_\infty$ analogue of the $\mathcal{F}$-space associated with the permutative category of finitely generated free $R$-modules $R^n$ and all linear maps. For $K$-theory we have to restrict to linear isomorphisms, i.e. invertible matrices in our model. The $A_\infty$ analogue of the general linear group $Gl_nR$ is the space of homotopy invertible matrices over the $A_\infty$ ring $X$, denoted by $\tilde{Gl}_nX$ [W1]. It is the pullback

$$\begin{array}{ccc}
\tilde{Gl}_nX & \longrightarrow & M_nX = X(n^2) \\
\downarrow & & \downarrow \\
Gl_n(\pi_0X) & \subset & \pi_0(M_nX) = M_n(\pi_0X)
\end{array}$$

where $M_nX$ is considered as space of all $(n \times n)$-matrices over $X$. Since $\pi_0X$ is a genuine semiring, $Gl_n(\pi_0X)$ is the classical general linear group.

For the passage from all matrices to homotopy invertible ones we have to analyze the composite functor $\pi_0 \circ NSA(X): \mathcal{F} \times \Delta^{op} \to \mathcal{Sets}$. Since

$$(6.5) \quad \pi_0 \circ NSA(X) \circ \nu \cong \pi_0 \circ \coprod X \circ HSA = \coprod \pi_0X \circ \Delta SA \circ \nu$$

and since $\nu$ is an equivalence, it suffices to consider

$$\coprod \pi_0X \circ \Delta SA: \mathcal{F} \times \Delta^{op} \to \Sigma \Theta_r \to \mathcal{Sets},$$

which is the nerve of the functor $(4.4.4)$

$$\text{Cat}(\pi_0X) \circ SA = S(\text{Cat}(\pi_0X) \circ A): \mathcal{F} \to \text{Cat}(\Sigma \Theta_r) \to \text{Cat}.$$
6.6 **Lemma.** \(|G(X)|: \mathcal{T} \to \mathcal{T}_{\text{top}} \) is a special \(\mathcal{T}\)-space.

6.7 **Definition.** The algebraic \(K\)-theory of the \(A_\infty\) ring \(X: \Theta \to \mathcal{T}_{\text{top}}\) is the 0th space of the \(\Omega\)-spectrum \(E|G(X)|\) associated with the special \(\mathcal{T}\)-space \(|G(X)|\). (In particular \(KX\) is an infinite loop space.)

All our constructions are natural with respect to homomorphisms of \(\Theta\)-spaces and preserve weak equivalences of \(\Theta\)-spaces. Hence

6.8 **Proposition.** The correspondence \(X \mapsto KX\) is functorial with respect to homomorphisms of \(\Theta\)-spaces and with respect to hammocks of \(\Theta\)-spaces.

6.9 **Remark.** A homotopy homomorphism \(f: X \to Y\) in the sense of [SV4; Definition 2.5] induces a map \(Kf: KX \to KY\) (see [SV2] for its construction and the functorial behaviour of \(K\) with respect to homotopy homomorphisms). If \(\mathcal{U}\) is the universal \(A_\infty\) ring theory derived from Steiner's canonical operad pair [SV1, §5], the \(A_\infty\) structures of \(X\) and \(Y\) pullback uniquely up to homotopy through \(A_\infty\) rings, i.e., a homotopy through continuous product preserving functors, to \(\mathcal{U}\)-structures making \(f\) a homotopy homomorphism of \(\mathcal{U}\)-spaces. This \(f\) "decomposes" canonically into a hammock

\[
\begin{array}{ccc}
X & \xrightarrow{r(X)} & UX \\
\downarrow & & \downarrow Uf \\
Y & & \\
\end{array}
\]

i.e. \(r(X)\) and \(Uf\) are homomorphisms of \(\mathcal{U}\)-spaces and \(r(X)\) is an equivalence. Hence our construction defines a map

\[
KX \simeq K(UX) \to KY
\]

which is uniquely determined by \(f\) up to homotopy and homotopy equivalence of source and target (see (6.10) below). This passage to hammocks makes the study of homotopy homomorphisms and their unpleasant functorial properties redundant.

In view of this remark the following result is worth noticing for coherence investigations.

6.10 **Proposition.** Let \(X_t: \Theta \to \mathcal{T}_{\text{top}}, t \in [0, 1]\), be a homotopy through \(A_\infty\) rings. Then there is an infinite loop space \(K(X_t)\) and infinite loop inclusions \(i_t: K(X_t) \to K(X_1)\) as strong deformation retracts. The \(i_t\) and the retractions are natural with respect to homotopies of homomorphisms \(X_t \to Y_t\) of \(\Theta\)-spaces. In particular, \(K(X_0) \simeq K(X_1)\) as infinite loop spaces.

**Proof.** Define \(X_t\) by \(X_t(n) = X(n) \times [0, 1]\).

7. The equivalence with Steiner’s definition

While our approach to \(KX\) for an \(A_\infty\) ring \(X\) is in the spirit of [T1 and M7], based on Street’s rectification of lax functors, Steiner’s construction [St2] follows Segal’s passage from permutative categories to special \(\mathcal{T}\)-spaces [Se]. Segal’s method can be viewed as an alternative rectification of the lax functor \(A: \mathcal{T} \to \text{Cat}\) defined by the given permutative category. The formalization of this rectification in [M5] cannot be generalized to our situation because it explicitly uses the isomorphisms in the categories \(A(m)\) which are not accessible in the abstract context. In the special case of the permutative category object \(\mathcal{M}\) of §6 one can get around this problem by using the isomorphisms defined
by permutation matrices. This is the idea behind Steiner’s construction of a functor \( \mathcal{F} \times \Delta^{op} \to \Sigma \Pi \Theta_r \).

We start with the definition of the functor

\[
C = \text{StA} : \mathcal{F} \to \text{Cat}(\Sigma \Pi \Theta_r)
\]

implicit in Steiner’s set-up, where \( A \) is the lax functor of §6.

Recall, that \( \ell \) denotes the based set \( \{0, 1, \ldots, r\} \). Let \( V(m) \) be the set of all families \( (r_1, \ldots, r_m; (\pi_T)) \) where \( r_i \in \mathbb{N} \) and \( (\pi_T) \) is a family of bijections

\[
\pi_T : \ell_{t_1} \vee \cdots \vee \ell_{t_p} \to r_{t_1} + \cdots + r_{t_p}
\]

indexed by all subsets \( T = \{0, t_1, \ldots, t_p\} \subseteq m \), where \( \emptyset \) stands for the empty wedge. We assume that \( \pi_T = \text{id} \) if \( T = \{0, t\} \). Let \( E(m) \) be the set of all families \( (r_1, \ldots, r_m; (\pi_T), (\pi_T')) \) where \( (\pi_T) \) and \( (\pi_T') \) are both families of bijections of type (7.1). The category object \( C(m) \) is defined by

\[
C_0(m) = (V(m); (0)), \quad C_1(m) = (E(m); (r_1^2 + \cdots + r_m^2)(r_1, \ldots, r_m; (\pi), (\pi'))).
\]

The source and target maps are determined by

\[
E(m) \to V(m); \quad (r_1, \ldots, r_m; (\pi), (\pi')) \mapsto (r_1, \ldots, r_m; (\pi)),
\]

resp. \( (r_1, \ldots, r_m; (\pi')) \).

Composition is defined by \( (r_1, \ldots, r_m; (\pi), (\pi'), (\pi'')) \) and matrix multiplication \( M_{r_1} \times M_{r_m} \to M_{r_i} \).

To define the \( \mathcal{F} \)-structure let \( \phi : m \to n \) be in \( \mathcal{F} \). We associate a map

\[
V(m) \to V(n); \quad (r_1, \ldots, r_m; (\pi)) \mapsto (s_1, \ldots, s_n; (\rho_U))
\]

with \( s_i = \sum_{\phi(j) = i} r_j \) and for \( U = \{0, \overline{u_1}, \ldots, \overline{u_p}\} \subseteq n \)

\[
(7.2) \quad \rho_U : \bigvee_{i \in U^*} S_i \bigvee_{i \in U^*} \ell_j \bigvee_{\phi(j) = i} \ell_j \mapsto \bigvee_{\phi(j) \in U} \pi^{-1}_{s-1(u)} s_{\overline{u_1}} + \cdots + s_{\overline{u_p}}
\]

with \( U^* = U \setminus \{0\} \). Analogously we define a map \( E(m) \to E(n) \).

The functor \( C(\phi) : C(m) \to C(n) \) is determined by \( C_1(m) \to C_1(n) \), given by the map \( E(m) \to E(n) \) and the following operations indexed by \( (r_1, \ldots, r_m; (\pi), (\pi')) \)

\[
r_1^2 + \cdots + r_m^2 \rightarrow s_1^2 + \cdots + s_n^2.
\]

Its \( i \)-th component sends the \( m \)-tuple of matrices \( (A_1, \ldots, A_m) \) to the \( s_i \times s_i \)-matrix associated to the linear map of free \( R \)-modules

\[
(7.3) \quad R^{s_i} \xrightarrow{P(\pi_{s-1(u)})^{-1}} \bigoplus_{\phi(j) = i} R^{r_j} \oplus A_j \xrightarrow{\oplus A_j} \bigoplus_{\phi(j) = i} R^{r_j} \xrightarrow{P(\pi'_{s-1(u)})} R^{s_i}
\]

where \( P(\pi) \) are the permutation matrices determined by \( \pi \) (here we have to identify \( \bigvee_{\phi(j) = i} \ell_j \) with an object of \( \mathcal{F} \) in our canonical way to obtain an ordered base of \( \bigoplus R^{r_j} \)).

Following May [M7, Appendix] we construct lax natural transformations \( \delta : C \to A \) and \( \nu : A \to C \). The functor \( \delta(m) : C_1(m) \to A_1(m) = M_1^m \) is given by

\[
E(m) \to N^m; \quad (r_1, \ldots, r_m; (\pi), (\pi')) \mapsto (r_1, \ldots, r_m)
\]
and the operations \( id : r_1^2 + \cdots + r_m^2 \to r_1^2 + \cdots + r_m^2 \). For \( \phi : m \to n \) the natural transformation \( \delta(\phi) : A(\phi) \circ \delta(m) \to \delta(n) \circ C(\phi) \) is determined by composition with permutation matrices according to (7.3).

The functor \( \nu(m) : \text{A}(m) \to \text{C}(m) \) sends \((r_1, \ldots, r_m)\) to \((r_1, \ldots, r_m; (\kappa))\) where \((\kappa)\) is the family of canonical identification. (Here, as in (7.3) we have to order the wedge which we do by the canonical ordering of the indices.)

The natural transformation \( \nu(\phi) : \text{C}(\phi) \circ \nu(m) \to \nu(n) \circ \text{A}(\phi) \) is given by composition with the permutation matrix determined by the reordering of the double wedge in (7.2).

(7.4) \( \delta \circ \nu = \text{Id} \), and for each \( m \) there is a natural isomorphism \( \xi(m) : \text{Id} \to \nu(m) \circ \delta(m) \). The \( \xi(m) \) combine to a natural homotopy \( \xi : \text{Id} \to \nu \circ \delta \).

The \( \xi(m) \) are induced by the maps \( V(m) \to V(m), (r_1, \ldots, r_m; (\pi)) \to (r_1, \ldots, r_m; (\kappa)) \) and the associated permutation matrices \( P(\pi) \). The statements of (7.4) are easily checked.

Steiner proceeds with \( \text{StA} \) in the same way as we did with \( \text{SA} \) in §6 to define a \( K \)-functor \( K_{\text{St}X} \). Consider the diagram

\[
\begin{array}{ccc}
\text{SA}(m) & \xrightarrow{S\nu(m)} & S(\text{StA})(m) \\
\eta_s(m) \uparrow & & \eta_{\text{StA}}(m) \uparrow \\
A(m) & \xrightarrow{\nu(m)} & \text{StA}(m)
\end{array}
\]

By (4.4.1) the square commutes and, in the notation of §5.65, the maps \( \tilde{B}\nu(m), \tilde{B}\eta_s(m), \tilde{B}\eta_{\text{StA}}(m), \) and \( \tilde{B}\varepsilon(m) \) are homotopy equivalences. We obtain a natural transformation

\( \varepsilon \circ S\nu : \text{SA} \to \text{StA} \)

which induces a weak equivalence of the associated \( \mathcal{F} \)-spaces

\[ |\mathcal{N}_{\text{SA}}(X)| \to |\mathcal{N}_{\text{StA}}(X)|. \]

Restriction to “invertible components” preserves this equivalence, and we obtain

7.5 **Proposition.** There is an infinite loop equivalence \( KX \to K_{\text{St}X} \), natural with respect to homomorphisms and hammocks of \( \Theta \)-spaces.

8. **KX as plus construction**

An important result of the algebraic \( K \)-theory \( KR \) of a ring \( R \) is that it can be expressed as the plus construction on \( B\text{G}l R = \text{colim} B\text{G}l_n R \). It “reduces” the calculation of the homology of \( KR \) to the one of \( B\text{G}l R \). For strictly product preserving \( A_\infty \) rings \( X \) May has given a plus construction version \( K_M X \) of algebraic \( K \)-theory [M6]. We will show that \( KX \) is equivalent to \( K_M X \), which establishes this result for \( A_\infty \) rings.

Steiner in [St2, Remark 3.5] and in private conversations expected that a comparison of the telescope implicit in his construction with the telescope used by May in [M6] will yield the proof but did not provide the details for technical reasons. The problem is to define a reasonable map \( KX \to K_M X \) (which then almost automatically is a homotopy equivalence). We start with an explicit description of the telescope defined by a special \( \mathcal{F} \)-space \( Y \) following [Se, §4].
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\(\mu: Y(1) \times Y(1) \xrightarrow{\varphi_{12}} Y(2) \xrightarrow{\beta_2} Y(1)\).

In our cases \(\pi_0 Y(1) \cong \mathbb{N}\) as abelian monoid, and we denote the path component corresponding to \(n \in \mathbb{N}\) by \(Z_n\). Choose \(z \in Z_1\) and define \(i_n: Z_n \rightarrow Z_{n+1}\) by

\[
i_n: Z_n \xrightarrow{(id, z)} Z_n \times Z_1 \xrightarrow{\eta(1) \times \eta(1)} Z_{n+1}.
\]

**8.1 Lemma.** Let \(\tilde{B}Y\) denote the infinite loop space associated with \(Y\). Then

\[Z \times (\text{Tel} Z_n)^+ \simeq \tilde{B}Y\]

May's proof [M2, §15] applies to our situation although \(\mu\) is not strictly associative.

Although \(\text{Tel} Z_n\) depends on the particular choice of \(z \in Z_1\) and of the homotopy inverse of \((\pi_1, \pi_2)\) in the definition of \(\mu\), its homotopy type is uniquely determined by \(Y\).

In the case of \(Y = |\tilde{N}_{SA}(X)|\) recall the diagram

\[
\begin{array}{c}
SA(1) \times SA(1) \xrightarrow{(SA\pi_1, SA\pi_2)} SA(2) \xrightarrow{SA{\beta_2}} SA(1) \\
\uparrow \eta(1) \times \eta(1) \quad \uparrow \eta(2) \quad \uparrow \eta(1) \\
A(1) \times A(1) \xrightarrow{(A\pi_1, A\pi_2) = \text{id}} A(2) \xrightarrow{A{\beta_2}} A(1)
\end{array}
\]

which after application of the functor \([\tilde{B}]\) of §5 becomes a commutative diagram in \(\mathcal{T}_{\varphi}^{op}\). The top row induces the multiplication \(\mu\) on \(\tilde{BSA}(1)\). Hence the telescope associated with \(|\tilde{N}_{SA}(X)|\) is equivalent to the telescope obtained from the lower row of (8.2): By (5.7) we have a morphism

\[
\tilde{BA}(1) \times \tilde{BA}(1) \simeq \tilde{BA}(2) \xrightarrow{[\tilde{B}\Theta]} \tilde{BA}(1)
\]

in \(\mathcal{T}_{\varphi}^{op}\). By [M1, 11.11]

\[
\pi_0 \tilde{B}C = \pi_0 |N C^r| = \pi_0 |\pi_0 N C^r|, \quad C \in \mathcal{C}at(\Sigma \Theta_r).
\]

Hence \(\pi_0 [\tilde{B} \Theta]: \pi_0 \tilde{BA}(1) \times \pi_0 \tilde{BA}(1) \rightarrow \pi_0 \tilde{BA}(1)\) is equivalent to \(\pi_0\) applied to the realization of

\[
\mathcal{L} \times \Delta^{op} \xrightarrow{F} \Sigma \Theta_r \xrightarrow{\prod \pi_0 X} \mathcal{T}_{\varphi}^{op}
\]

by (2.8.2). Here \(F\) is defined by the functor \(A(\hat{\beta}_2)\) in (8.2). By construction, this is

\[
\left(\bigcoprod_{n \in \mathbb{N}} BM_n(\pi_0 X)\right) \times \left(\bigcoprod_{n \in \mathbb{N}} BM_n(\pi_0 X)\right) \xrightarrow{\tilde{B}\Theta, \bigcoprod_n} \bigcoprod_n BM_n(\pi_0 X).
\]

Hence the telescope can be described as follows: Define \(T: \mathbb{N} \times \Delta^{op} \rightarrow \Theta_r\) by \(T(k, [n]) = n \cdot k^2\) considered as an \(n\)-tuple of \(k \times k\)-matrices \((A_1, \ldots, A_n)\).
The boundary maps $d^i$ are given by matrix multiplication $A_i \cdot A_{i+1}$ for $0 < i < n$ and by the projections deleting $A_1$ resp. $A_n$ for $i = 0$ and $n$. The degeneracies insert unit matrices. For $k < l$, the operation $n \cdot k^2 \rightarrow n \cdot l^2$ substitutes each matrix $A_i$ by

$$
\begin{pmatrix}
A_i & 0 \\
0 & E_{l-k}
\end{pmatrix}.
$$

(8.3) The telescope obtained from $\mathbb{N} \xrightarrow{N_T} \Theta_{op} \xrightarrow{\Delta^{op}} \Theta_{op}$ where $N_T$ comes from the Segal pushdown of

$$
\begin{array}{ccc}
\varnothing_T & \xrightarrow{T'} & \Theta \\
\downarrow{\nu} & \quad & \downarrow{F} \\
\mathbb{N} \xrightarrow{T} \Theta_r
\end{array}
$$

is equivalent to the telescope associated with $SA$ (i.e. the telescope of $|N_{SA}(X)|$).

For the construction of $K_M X$ we use the reformulation in [SV4, §4]. Let $\Theta_m$ be the theory of monoids. There is a canonical functor $i: \Delta^{op} \rightarrow \Theta_m$ sending $[n]$ to $n$, boundaries $d^i: [n] \rightarrow [n-1]$ to the operations $(x_1, \ldots, x_i \cdot x_{i+1}, \ldots, x_n)$ for $0 < i < n$, and to the projections deleting $x_1$ resp. $x_n$ if $i = 0$ and $n$ etc. The functor $T$ factorizes as

$$
\begin{array}{ccc}
\mathbb{N} \xrightarrow{id \times i} \mathbb{N} \xrightarrow{T_m} \Theta_r
\end{array}
$$

and $A_* \Theta$ is an $\mathbb{N}$-coloured $A_\infty$ monoid theory. If $W(\mathbb{N} \times \Theta_m)$ is the universal $\mathbb{N}$-coloured $A_\infty$ monoid theory of [BV], there is a theory functor $H: W(\mathbb{N} \times \Theta_m) \rightarrow A_* \Theta$, unique up to homotopy through functors, such that $\rho \circ H = \varepsilon: W(\mathbb{N} \times \Theta_m) \rightarrow \Theta_m$, the universal augmentation [BV, 3.20]. In [SV4] we apply the rectification process $M$ of [BV, 4.49] to obtain an $\mathbb{N}$-indexed sequence of monoids and homomorphisms. Restriction to the invertible components corresponding to the groups $Gl_n(\pi_0 X)$ gives a sequence

$$
R_0 X \rightarrow R_1 X \rightarrow R_2 X \rightarrow \cdots
$$

of monoids and cofibration homomorphisms such that $R_n X$ contains the $A_\infty$ monoid $\widetilde{Gl}_n X$ as strong deformation retract. Then

$$
K_M X = K_0(\pi_0 X) \times (\widetilde{Bgl} X)^+,
$$

where $\widetilde{Bgl} X = \text{colim} \ B R_n$. So we have to compare the nerve of the $M$-construction $M(X \circ T'_M \circ H)$, which is $M(X \circ T'_M \circ H) \circ (id \times i)$, with $\nu_*(X \circ T')$. Consider
By (2.8.4) we have weak equivalences of \( N \times \Delta^{op}\)-spaces
\[
\nu_*(X \circ T') \rightarrow \rho_*(X \circ T'_M) \circ (id \times i) \leftarrow (\rho \circ H)_*(X \circ T'_M \circ H) \circ (id \times i).
\]
Hence it suffices to compare \((\rho \circ H)_*(X \circ T'_M \circ H)\) with the \( M \)-construction. So let \( Y = X \circ T'_M \circ H : W(\mathbb{N} \times \Theta_m) \rightarrow \mathbb{N} \times \Theta_m \). The \( M \)-construction comes equipped with a universal homotopy homomorphism \( Y \rightarrow e^*MY \). By [BV, 4.23] and (2.8.2) we have a sequence of \( W(\mathbb{N} \times \Theta_m) \)-homomorphisms
\[
e^*e_*Y \leftarrow id^*id_*Y \rightarrow Y \leftarrow UY \rightarrow e^*MY
\]
which are weak equivalences. Apply the Segal pushdown \( e_* \) to transform them into weak \( N \times \Delta^{op} \)-equivalences. Since there are weak equivalences
\[
e_*Y \leftarrow e_*e^*e_*Y, \quad e_*e^*MY \rightarrow MY
\]
by (2.8.3), we are done.

8.4 Proposition. There is a chain of homotopy equivalences \( KX \simeq \mathbb{Z} \times (BGLX)^+ \), natural with respect to homomorphisms of \( \Theta \)-spaces. In particular,
\[
\pi_iKX \cong \pi_iK_MX \quad \text{for } i > 0.
\]

9. Morita invariance

In [SV4] we showed that the space \( M_kX \) of \( k \times k \)-matrices over a strictly product preserving \( A_\infty \) ring \( X \) has a canonical \( A_\infty \) ring structure. This proof literally goes through with the present more general definition of an \( A_\infty \) ring. In this section we prove

9.1 Morita invariance. There is a homotopy equivalence of infinite loop spaces \( K(M_kX) \simeq XX \), natural with respect to homomorphisms of \( \Theta \)-spaces and hammocks of \( \Theta \)-spaces.

We construct a permutative category object \( \text{Mat} \) in \( \Sigma \Theta_r \), which corresponds to the permutative category of free \( M_kR \)-modules \( (M_kR)^n \) for a genuine ring \( R \), by putting
\[
\text{Mat}_0 = (\mathbb{N}, (0)), \quad \text{Mat}_1 = (\mathbb{N}, (kn)_{n \in \mathbb{N}}^2).
\]
We have to think of \( \text{Mat}_1 \) as the union over \( n \in \mathbb{N} \) of all \( n \times n \)-matrices with \( k \times k \)-matrices as entires. Since a matrix of matrices is a matrix, there is an inclusion \( J : \text{Mat} \rightarrow \text{Mat} \). The permutative category structure on \( \text{Mat} \) is induced from the one on \( \text{Mat} \), so that \( J \) is a permutative functor.
If \( M_n : \Theta_r \to \Theta_r \) is the product preserving functor of [SV4, §3], then \( \text{Mat} \) is the image of \( \mathcal{M} \) under \( \Sigma l M_n \), and \( J \) is induced by the inclusion of the image. From [SV4] we obtain a diagram

\[
\begin{array}{ccc}
\phi & \xrightarrow{N'} & \Sigma l M_n \Theta \\
\downarrow & & \downarrow \Sigma l F M_n \\
\mathcal{T} \times \Delta^{op} & \xrightarrow{\text{Nerve} \circ S\Lambda} & \Sigma l \Theta_r
\end{array}
\]

and Street's rectification of the lax functor \( A_{\text{mat}} \) arising from \( \text{Mat} \) composed with the nerve is nothing but the lower row of (9.2). Since \( K(M_k X) \) is defined from the left square of (9.2) and since \( J \) is a functor of permutative category objects, we obtain a natural transformation of functors \( S\Lambda_{\text{mat}} \to S\Lambda : \mathcal{T} \to \mathcal{Cat}(\Sigma l \Theta_r) \), which induces an infinite loop map \( K(M_k X) \to KX \). The induced map of associated telescopes maps the \( n \)th space \( \widetilde{Gl}_n(M_k X) \) to the \( (k \cdot n) \)th space \( \widetilde{Gl}_{n+k}(X) \). Hence, for cofinality reasons, the map of associated telescopes is a homotopy equivalence. The result now follows from (8.4).

10. THE \( E_\infty \) RING CASE

We make extensive use of [M8]. Throughout this section let \( X : \Theta \to \mathcal{T}_{op} \) be an \( E_\infty \) ring.

The bipermutative category object \( \mathcal{M}_c \) of (6.2) gives rise to a lax functor (compare [M8, §3])

\[
P : \mathcal{T} \times \mathcal{T} \to \mathcal{Cat}(\Sigma l \Theta_{cr})
\]

such that

\[
A : \mathcal{T} \xrightarrow{J} \mathcal{T} \xrightarrow{P} \mathcal{Cat}(\Sigma l \Theta_{cr})
\]

is the functor described in §6. Here \( J \) is the inclusion functor sending \( n \) to \((1; (n))\) and \( \varphi : m \to n \) to \((id_1; (\varphi))\). We apply Street's rectification and obtain a functor

\[
\mathcal{T} \times \Delta^{op} \xrightarrow{\text{Nerve} \circ S\Lambda_{SP}} \Sigma l \Theta_{cr}.
\]

As in §6 it takes simple morphisms as values. Segal's pushdown provides us with a functor (depending on \( X \))

\[
N_{SP} = N_{SP}(X) : \mathcal{T} \to \mathcal{T}_{op} \Delta^{op}.
\]

Using the results of §5 we obtain

\[(10.1) \quad \text{The realization } |N_{SP}(X)| \text{ is a special } \mathcal{T} \times \mathcal{T}-\text{space. We restrict } |N_{SP}(X)| \text{ to invertible components by restricting the monoids } M_n(\pi_0 X) \text{ to the monoids } Gl_n(\pi_0 X) \text{ as in §6 and obtain a special } \mathcal{T} \times \mathcal{T}-\text{space}
\]

\[
|G_c(X)| : \mathcal{T} \times \mathcal{T} \to \mathcal{T}_{op}.
\]

By [M8, §4], the spectrum \( E(|G_c(X)| \circ J) \) is an \( E_\infty \) ring spectrum and its 0th space \( E_0(|G_c(X)| \circ J) \) an \( E_\infty \) ring whose structure is codified by Steiner's canonical operad pair [St1]. Since \( A = P \circ J \), there is a weak equivalence of special \( \mathcal{T} \)-spaces \( |G(X)| \to |G_c(X)| \circ J \) by (2.8.4), and hence an infinite loop equivalence.

\[(10.2) \quad KX \xrightarrow{=} E_0(|G_c(X)| \circ J).
\]
By construction, the infinite loop structure on $KX$ is codified as an additive $E_\infty$ monoid by the additive operad of Steiner's operad pair. By homotopy invariance of $E_\infty$ ring structures [SV1, 4.8] we can extend the infinite loop structure on $KX$ to an $E_\infty$ ring structure making (10.2) an $E_\infty$ ring equivalence. In particular,

10.3 **Theorem.** If $X$ is an $E_\infty$ ring, $KX$ is an $E_\infty$ ring, too.

11. The monomial map

Let $X: \Theta \to \mathcal{T}_0 \rho$ be an $A_\infty$ ring and $X^* = \widetilde{GL}_1 X \subset X(1)$ its homotopy units. To simplify the argument we assume that $X$ is a strictly product preserving $A_\infty$ ring. Then $X^*$ is an $A_\infty$ monoid under multiplication. In [M6] May constructed a map $Q((BX^*)_+) \to KX$, where $Q = \Omega^\infty S^\infty$ and $Z_+ = Z \cup \{\text{point}\}$, called monomial map because it is induced by the inclusion of the homotopy invertible monomial $n \times n$-matrices into $\widetilde{GL}_n X$, but as he pointed out in [M8, Appendix B.6.5] there was a flaw in his construction. The monomial map is of importance because it relates stable homotopy to $K$-theory and it occurs in the construction of the Waldhausen splitting [W2]

$$A(Y) \simeq Q(Y_+) \times \text{Wh} Y.$$  

In [SV4] we constructed the monomial map following the suggestions of May, but it can more easily be described in our present context.

A matrix is called monomial if it has at most one nonzero entry in each row and column.

Let $\mathcal{M}_1$ be the trivial category object in $\Sigma_1 \Theta_\rho$, defined by $\text{ob} \mathcal{M}_1 = \{\{1\}, 0\}$ and $\text{mor} \mathcal{M}_1 = \{\{1\}, 1\}$. The inclusion $\{1\} \subset \mathbb{N}$ and the operation $id_1 \in \Theta_\rho$ define an inclusion functor $J: \mathcal{M}_1 \to \mathcal{M}$ into the category object $\mathcal{M}$ of §6. If $\mathcal{E} \subset \mathcal{F}$ denotes the subcategory of all permutations we can form the category object $\mathcal{E} \downarrow \mathcal{M}_1$ as in [M8, Remark 3.5]. It is permutative in the $A_\infty$ case and bipermutative in the $E_\infty$ case and $J$ extends to a strict permutative respectively bipermutative functor

$$J: \mathcal{E} \downarrow \mathcal{M}_1 \to \mathcal{M}.$$  

We treat the $E_\infty$ ring case. The arguments in the $A_\infty$ ring case are analogous and considerably simpler. $J$ induces a strict transformation of lax functors $j: U \to A: \mathcal{F} \downarrow \mathcal{F} \to \text{Cat}(\Sigma_1 \Theta_{cr})$ and Street's rectification then gives a natural transformation $S_j: SU \to SA$. Since all data consist of simple morphisms, Segal pushdown yields a diagram of $(\mathcal{F} \downarrow \mathcal{F} \times \Delta^\text{op})$-spaces

$$N_{SU} \xrightarrow{\alpha} N_{SjSU} \xrightarrow{N_{Sj}} N_{SjSA} \xleftarrow{\beta} N_{SA}$$  

where $N_{Sj}$ is induced by $Sj$ and $\alpha$, $\beta$ are the weak equivalences given by (2.8.4).

The telescope implicit in $N_{SU}$ is equivalent to the one constructed from

$$U(1; 1) \times U(1; 1) = U(1; 2) \xrightarrow{(id: \bar{\mu})} U(1; 1).$$  

If $R$ is a genuine ring, $\text{mor} U(1; 1)$ is mapped to $\bigvee \Sigma_n \downarrow R$ under $\bigvee \Sigma_n \Theta_{cr} \to \mathcal{T}_0 \rho$ . Here $\Sigma_n$ denotes the symmetric group and $\Sigma_n \downarrow R$ the wreath product of $\Sigma_n$ and $R$ with

$$\cdot \sigma; r_1, \ldots, r_n \cdot \tau; r'_1, \ldots, r'_n \cdot \rho = (\sigma \circ \tau; r_{\tau(1)} \cdot r'_{\sigma'1}, \ldots, r_{\tau(n)} \cdot r'_{\sigma'n}).$$  

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If we identify \((a; r_1, \ldots, r_n)\) with the monomial \((n \times n)\)-matrix having \(r_k\) as \((a; r_1, \ldots, r_n)\)th entry, then (11.2) is matrix multiplication. Hence the telescope is equivalent to the stabilization telescope \(B(\Sigma_\infty \iota X)\) of the \(B(\Sigma_\infty \iota X)\) (compare §8), where \(\Sigma_\infty \iota X\) is the \(A_\infty\) monoid of monomial matrices over \(X\) constructed in [SV4].

Restriction to invertible matrices and topological realization transforms (11.1) into a diagram of special \(\mathcal{F} \otimes \mathcal{F}\)-spaces

\[
\begin{array}{ccc}
|GSU(X)| & \xrightarrow{\alpha} & |G_jSU(X)| & \xrightarrow{G_j} & |G_j(X)| & \xrightarrow{\beta} & |G_c(X)|.
\end{array}
\]

Passage to the 0th space of the associated \(E_\infty\) ring spectra defines a map of \(E_\infty\) rings

\[
E_0|GSU(X)| \rightarrow KX
\]

uniquely up to homotopy (depending on choices of a homotopy inverse of \(\beta\) and the choice of the homotopy inverse of \(|G(X)| \rightarrow |G_c(X)| \circ J\) in §8). Moreover,

\[(11.3)\quad E_0|GSU(X)| \simeq \mathbb{Z} \times B(\Sigma_\infty \iota X^*)^+,
\]

where \(B(\Sigma_\infty \iota X^*)\) is the stabilization telescope of the classifying spaces \(B(\Sigma_\infty \iota X^*)\) of homotopy invertible monomial matrices, i.e. monomial matrices with nonzero entries in \(X^*\).

In the \(A_\infty\) ring case, these results were established by Steiner in his framework [St2, Theorem 3.6].

By an extension of the Barratt-Priddy-Quillen-Theorem there is a homotopy equivalence \(\mathbb{Z} \times B(\Sigma_\infty \iota X^*)^+ \simeq Q(BX^*_+)\) [M6, Theorem 8.3], so that \(E_0|GSU(X)| \simeq Q(BX^*_+)\). This result can be improved: \(Q(BX^*_+)\) has a canonical \(E_\infty\) ring structure (resp. infinite loop structure in the \(A_\infty\) ring case), and we want to show that \(E_0|GSU(X)| \simeq Q(BX^*_+)\) as \(E_\infty\) ring (resp. infinite loop space). Let us recall the structure of \(Q(BX^*_+)\). Consider

\[
\begin{array}{cccccccc}
\emptyset & \xrightarrow{\mathcal{Q}} & \mathcal{A}_\Theta & \xrightarrow{\mathcal{H}_m} & \Theta & \xrightarrow{X} & \mathcal{F}_p & \xrightarrow{\pi_0} \text{Sets},
\end{array}
\]

where \(m\) is the inclusion of the multiplicative commutative monoid structure. Then \(\mathcal{A}_\Theta\) is an \(E_\infty\) monoid theory. Restriction of \(X \circ \mathcal{H}_m\) to the components of the units of \(\pi_0X\) defines \(X^*\) as a grouplike \(E_\infty\) monoid. There are various equivalent notions of \(BX^*\) with an \(E_\infty\) monoid structure:

Define \(Y : \mathcal{A} \rightarrow \mathcal{F}_p\) to be the topological realization of

\[
\begin{array}{cccccccc}
\mathcal{F} \times \Delta^{op} & \xrightarrow{id \times i} & \mathcal{F} \times \mathcal{F} & \xrightarrow{\text{smash}} & \mathcal{F} & \xrightarrow{\nu^*(X^* \circ \mathcal{H}_m)} & \mathcal{F}_p.
\end{array}
\]

Let \((\mathcal{H}, \mathcal{L})\) denote Steiner’s canonical operad pair [St1]. The category of operators \(\mathcal{D}\) associated to \(\mathcal{L}\) augments over \(\mathcal{F}\) so that \(Y\) is an \(\mathcal{D}\)-space. \(Y\) gives rise to an \(\mathcal{D}\)-space in the sense of [M8], and application of [M8, 4.3] transforms \(Y\) into an \((\Pi, \mathcal{L})\)-space \(VY\). We define \(BX^*\) to be the \(\mathcal{L}\)-space

\[
BX^* : \mathcal{L} \subset \mathcal{L} \prod^{R^{VY}} \mathcal{F}_p.
\]
in the notation of [M8, §4]. By [M8, 4.3; [MT], and [SV3], \(BX^*\) is homotopy equivalent to the classifying spaces of \(X^*\) in the sense of [M1] and [BV].

Ring multiplication defines both composition and tensor product in the category object \(\mathcal{M}\). Hence the functor

\[
\mathcal{F} \times \Delta^{op} \xrightarrow{id \times i} \mathcal{F} \times \mathcal{F} \xrightarrow{\text{smash}} \mathcal{F} \xrightarrow{\text{m.o.M}_\mathcal{O}} \Theta_{cr} \subset \Sigma \Theta_{cr}
\]

coincides with the nerve of

\[
\mathcal{F} \xrightarrow{\mu} \mathcal{F} \xrightarrow{\mathcal{F} \rightarrow \text{Cat}(\Sigma \Theta_{cr})}
\]

where \(\mu\) is determined by \(n \mapsto (n; (1, \ldots, 1))\). Since \(U \circ \mu\) is a genuine functor, naturality of Street's construction provides natural transformations

\[
U \circ \mu \xrightarrow{\epsilon(U \circ \mu)} S(U \circ \mu) \xrightarrow{\xi} SU \circ \mu
\]

inducing weak equivalences of \((\mathcal{F} \times \Delta^{op})\)-spaces

\[
N_{U \circ \mu} \xleftarrow{} N_{SU \circ \mu} \rightarrow N_{SU} \circ (\mu \times id_{\Delta^{op}}).
\]

Restriction to invertible components and topological realization transforms this sequence into a sequence of weak equivalences of \(\mathcal{F}\)-spaces

\[
Y \xleftarrow{} |GU \mu(X)| \xleftarrow{} |GS(U \circ \mu)(X)| \rightarrow |G(SU \circ \mu)(X)| \rightarrow |GSU(X)| \circ \mu
\]

which puts us into the situation of the proof of [M8, 8.6]. Following the proof we obtain maps of \(\mathcal{F}\)-spectra

\[
\Sigma^{\infty}(BX^*) \cong \Sigma^{\infty}V|GS(U \circ \mu)(X)|(1) \rightarrow E|GSU(X)|.
\]

The composite map is an equivalence on the 0th space by (11.3). Hence we have shown

11.4 Proposition. There is an infinite loop equivalence \(E_0|GSU(X)| \cong Q(BX^*_+),\) which is an equivalence of \(E_\infty\) rings if \(X\) is an \(E_\infty\) ring.

11.5 Corollary. The monomial map induces an infinite loop map \(Q(BX^*_+) \rightarrow K(X),\) which is a map of \(E_\infty\) rings if \(X\) is an \(E_\infty\) ring.

References


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