TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 341, Number 2, February 1994

ON THE HYPERBOLIC KAC-MOODY LIE ALGEBRA $HA_1^{(1)}$

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ABSTRACT. In this paper, using a homological theory of graded Lie algebras and the representation theory of $A_1^{(1)}$, we compute the root multiplicities of the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$ up to level 4 and deduce some interesting combinatorial identities.

INTRODUCTION

The hyperbolic Kac-Moody Lie algebras have been considered as the next natural objects of study after the affine case. While the affine Kac-Moody Lie algebras have been extensively studied for their close connections to areas such as combinatorics, modular forms, and mathematical physics, many basic questions regarding the hyperbolic case are still unresolved. For example, the behavior of the root multiplicities is not well understood. Feingold and Frenkel [F-F] and Kac, Moody, and Wakimoto [K-M-W] made some progress in this area. They computed the level 2 root multiplicities for the hyperbolic Kac-Moody Lie algebras $HA_1^{(1)}$ and $HE_8^{(1)}$, respectively.

In [Kang1] and [Kang2], we introduced an inductive program to study the higher level root multiplicities and the principally specialized affine characters of a certain class of Lorentzian Kac-Moody Lie algebras. More precisely, we realize these Lorentzian Kac-Moody Lie algebras. as the minimal graded Lie algebras $L = \bigoplus_{n\in Z} L_n$ with local part $V + L_0 + V^*$, where L_0 is an affine Kac-Moody Lie algebra, V is the basic representation of L_0, and V^* is the contragredient of V. Thus $L = G/I$, where $G = \bigoplus_{n\in Z} G_n$ is the maximal graded Lie algebra with local part $V+L_0+V^*$ and $I = \bigoplus_{n\in Z} I_n$ is the maximal graded ideal of G which intersects the local part trivially. By developing a homological theory, we determined the structure of homogeneous subspaces $L_n = G_n/I_n$ as modules over the affine Kac-Moody Lie algebra L_0 for certain higher levels. The Hochschild-Serre spectral sequences played an important role in determining the structure of I_n.

For the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$, we were able to determine the structure of L_n for $n = 1, 2, \ldots, 5$, and computed the principally specialized affine characters up to level 5. But the root multiplicities were computed up to level 3 only. In this paper, using the representation theory of $A_1^{(1)}$ developed

Received by the editors June 4, 1990 and, in revised form, September 12, 1991.
1991 *Mathematics Subject Classification*. Primary 17B67, 17B65; Secondary 17B10.
This work was supported by NSF grant DMS 8806371.

in [D-J-K-M-O], we compute the root multiplicities up to level 4 and deduce some interesting combinatorial identities.

1. Preliminaries

An $n \times n$ matrix $A = (a_{ij})$ is called a *generalized Cartan matrix* if it satisfies the following conditions: (i) $a_{ii} = 2$ for $i = 1, \dots, n$, (ii) a_{ij} are nonpositive integers for $i \neq j$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. A is called *symmetrizable* if DA is symmetric for some diagonal matrix $D = \operatorname{diag}(q_1, \dots, q_n)$ with $q_i > 0$, $q_i \in \mathbf{Q}$. A *realization* of an $n \times n$ matrix A of rank l is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a $(2n - l)$-dimensional complex vector space, $\Pi = \{\alpha_1, \dots, \alpha_n\}$, and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ are linearly independent indexed subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$ for $i, j = 1, \dots, n$. The *Kac-Moody Lie algebra* $\mathfrak{g}(A)$ associated with a generalized Cartan matrix A is the Lie algebra generated by the elements e_i, f_i $(i = 1, \dots, n)$ and $\mathfrak{h}$ with the following defining relations:

$$
\begin{aligned}
&[h, h'] = 0 \quad \text{for } h, h' \in \mathfrak{h}, \\
&[e_i, f_j] = \delta_{ij}\alpha_i^\vee \quad \text{for } i, j = 1, \dots, n, \\
&[h, e_j] = \alpha_j(h)e_j, \quad [h, f_j] = -\alpha_j(h)f_j \quad \text{for } j = 1, \dots, n, \ h \in \mathfrak{h}, \\
&(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0 \quad \text{for } i, j = 1, \dots, n \text{ with } i \neq j, \\
&(\operatorname{ad} f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{for } i, j = 1, \dots, n \text{ with } i \neq j.
\end{aligned}
\tag{1.1}
$$

An indecomposable generalized Cartan matrix A is said to be of *finite* type if all its principal minors are positive, of *affine* type if all its proper principal minors are positive and $\det A = 0$, and of *indefinite* type if A is of neither finite nor affine type. A is of *hyperbolic* type if it is of indefinite type and all its proper principal submatrices are of finite or affine type.

In this paper, we concentrate on the study of the structure of the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$ with Cartan matrix

$$
A = (a_{ij})_{i,j=-1,0,1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}.
$$

We construct the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$ as follows. Let $\mathfrak{g}$ be the affine Kac-Moody Lie algebra $A_1^{(1)}$ with Cartan matrix $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$, let $V = V(\Lambda_0)$ be the basic representation of $\mathfrak{g}$ with highest weight Λ_0, and let V^* be the contragredient of V. Then the space $V \oplus \mathfrak{g} \oplus V^*$ has a local Lie algebra structure. Thus there exist maximal and minimal graded Lie algebras G and L with the local part $V \oplus \mathfrak{g} \oplus V^*$. Then $L = G/I$, where I is the maximal graded ideal of G intersecting the local part trivially. One can prove that L is isomorphic to the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$ [F-F, Kang1]. We will study the structure of the homogeneous spaces $L_n = G_n/I_n$ as modules over the affine Kac-Moody Lie algebra $A_1^{(1)}$. Write $G_\pm = \bigoplus_{n \geq 1} G_{\pm n}$, $L_\pm = \bigoplus_{n \geq 1} L_{\pm n}$, and $I_\pm = \bigoplus_{n \geq 2} I_{\pm n}$. Note that $G_\pm$ is the free Lie algebra generated by $G_{\pm 1}$. We denote by $\alpha_{-1}, \alpha_0, \alpha_1$ the simple roots of $HA_1^{(1)}$. Let v_0 and v_0^* be highest and lowest weight vectors of V and V^*, respectively.

By the Gabber-Kac theorem [G-K], the ideal I is generated by the elements $(\operatorname{ad} v_0)^2(f_0)$ and $(\operatorname{ad} v_0^*)^2(e_0)$. In particular, $I_\pm$ is generated by the subspace $I_{\pm 2}$.

2. The Witt formula

Let $V = V(\Lambda)$ be an integrable irreducible representation of a Kac-Moody Lie algebra $\mathfrak{g}(A)$ and let $G = \bigoplus_{n\geq 1} G_n$ be the free Lie algebra generated by V. Then by using the Poincaré-Birkhoff-Witt theorem and the Möbius inversion, we obtain the following generalization of the Witt formula [Kang1, Kang2].

Theorem 2.1. *Let $S = \{\tau_i | i = 1, 2, 3, \dots\}$ be an enumeration of all the weights of V. For $\tau \in \mathfrak{h}^*$, set*

$$T(\tau) = \left\{(n) = (n_1, n_2, n_3, \dots) | n_i \in Z_{\geq 0}, \sum n_i \tau_i = \tau\right\}, \tag{2.1}$$

and define

$$B(\tau) = \sum_{(n)\in T(\tau)} \frac{((\sum n_i) - 1)!}{\prod (n_i!)} \prod (\dim V_{\tau_i})^{n_i}. \tag{2.2}$$

Then

$$\dim G_\lambda = \sum_{\tau|\lambda} \mu\left(\frac{\lambda}{\tau}\right) \frac{\tau}{\lambda} B(\tau), \tag{2.3}$$

where μ is the classical Möbius function and $\tau|\lambda$ if $\lambda = k\tau$ for some positive integer k, in which case $\lambda/\tau = k$ and $\tau/\lambda = 1/k$.

Remark 2.2. We call the function $B(\tau)$ the *Witt partition function* on V. If $\tau = k\Lambda + \sum_i n_i\alpha_i$, then the set $T(\tau)$ corresponds to the partition of τ into k parts. The formula (2.3) is called the *Witt formula.*

Example 2.3. Let $V = V(-\alpha_{-1})$ be the basic representation of the affine Kac-Moody Lie algebra $A_1^{(1)}$ and let $P(V)$ be the set of weights of V. The weights of V are of the form $-\alpha_{-1} - k\alpha_0 - l\alpha_1$, where k and l are nonnegative integers satisfying the inequality $k - (k-l)^2 \geq 0$ [F-L, Kac2]. So for $k = 0, 1, 2, 3, \dots, l$ ranges from $k - [\sqrt{k}]$ to $k + [\sqrt{k}]$. For a weight $\lambda = -\alpha_{-1} - k\alpha_0 - l\alpha_1$, we have by [F-L],

$$\dim V_\lambda = p(k - (k-l)^2),$$

where the $p(n)$ are defined by

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{\phi(q)} = \frac{1}{\prod_{n\geq 1}(1-q^n)}.$$

We enumerate the elements of the set $P(V)$ lexicographically:

$$
\begin{aligned}
\tau_1 &= -\alpha_{-1}, \\
\tau_2 &= -\alpha_{-1} - \alpha_0, \\
\tau_3 &= -\alpha_{-1} - \alpha_0 - \alpha_1, \\
\tau_4 &= -\alpha_{-1} - \alpha_0 - 2\alpha_1, \\
\tau_5 &= -\alpha_{-1} - 2\alpha_0 - \alpha_1, \\
\tau_6 &= -\alpha_{-1} - 2\alpha_0 - 2\alpha_1, \\
\tau_7 &= -\alpha_{-1} - 2\alpha_0 - 3\alpha_1, \\
\tau_8 &= -\alpha_{-1} - 3\alpha_0 - 2\alpha_1, \\
&\vdots
\end{aligned}
\tag{2.4}
$$

Let G be the free Lie algebra generated by the basis elements of V. Then every root of G is of the form $\alpha = -t\alpha_{-1} - k\alpha_0 - l\alpha_1$ for $t \geq 1$, and $k, l \geq 0$. By the Witt formula (2.3), we have

$$\dim G_\alpha = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B(\tau), \tag{2.5}$$

where $B(\tau)$ is given by

$$B(\tau) = \sum_{(n)\in T(\tau)} \frac{((\sum n_j) - 1)!}{\prod (n_j!)} \prod p(k_j - (k_j - l_j)^2)^{n_j}, \tag{2.6}$$

with the enumeration $\tau_1, \tau_2, \tau_3, \ldots$ defined above.

Let us compute the multiplicity of the root $\alpha = -3\alpha_{-1} - 3\alpha_0 - 3\alpha_1$ of G using the formula (2.5). Note that the only roots that can divide α are α itself and $\frac{1}{3}\alpha = -\alpha_{-1} - \alpha_0 - \alpha_1$. So (2.5) becomes

$$\dim G_\alpha = B(-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1) - \tfrac{1}{3}B(-\alpha_{-1} - \alpha_0 - \alpha_1).$$

Since $T(-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1)$ corresponds the partition of $-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1$ into three parts, and since

$$
\begin{aligned}
-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1 &= (-\alpha_{-1}) + (-\alpha_{-1}) + (-\alpha_{-1} - 3\alpha_0 - 3\alpha_1) \\
&= (-\alpha_{-1}) + (-\alpha_{-1} - \alpha_0) + (-\alpha_{-1} - 2\alpha_0 - 3\alpha_1) \\
&= (-\alpha_{-1}) + (-\alpha_{-1} - \alpha_0 - \alpha_1) + (-\alpha_{-1} - 2\alpha_0 - 2\alpha_1) \\
&= (-\alpha_{-1}) + (-\alpha_{-1} - \alpha_0 - 2\alpha_1) + (-\alpha_{-1} - 2\alpha_0 - 2\alpha_1) \\
&= (-\alpha_{-1} - \alpha_0) + (-\alpha_{-1} - \alpha_0 - \alpha_1) + (-\alpha_{-1} - \alpha_0 - 2\alpha_1) \\
&= (-\alpha_{-1} - \alpha_0 - \alpha_1) + (-\alpha_{-1} - \alpha_0 - \alpha_1) + (-\alpha_{-1} - \alpha_0 - \alpha_1),
\end{aligned}
$$

we have

$$
\begin{aligned}
B(-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1) &= p(0)^2 p(3) + 2p(0)p(1)^2 + 2p(0)p(1)p(2) \\
&\quad + 2p(0)^2 p(1) + 2p(0)^2 p(1) + \tfrac{1}{3}p(1)^3 \\
&= 3 + 2 + 4 + 2 + 2 + \tfrac{1}{3} = 13\tfrac{1}{3}.
\end{aligned}
$$

It is clear that $B(\alpha_{-1} - \alpha_0 - \alpha_1) = p(1) = 1$. Therefore we obtain

$$\dim G_{-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1} = 13\tfrac{1}{3} - \tfrac{1}{3} = 13.$$

Corollary 2.4. *Let $V = \bigoplus_{j\geq 1} V_j$ be an integrable Z_+-graded representation of $\mathfrak{g}(A)$, and let G be the free Lie algebra on V. For an element $v \in V_j$, we set $\deg(v) = (1, j)$. Thus every root α of G has the form $\alpha = (l, m)$ for $l \geq 1$, $m \geq l$. For $s \geq r \geq 1$, define*

$$T(r, s) = \left\{(n) \mid \sum n_j = r,\ \sum j n_j = s\right\} \tag{2.7}$$

and

$$B(r, s) = \sum_{(n)\in T(r,s)} \frac{((\sum n_j) - 1)!}{\prod(n_j!)} \prod(\dim V_j)^{n_j}. \tag{2.8}$$

Then

$$\dim G_{(l,m)} = \sum_{(r,s)|(l,m)} \mu\left(\frac{(l, m)}{(r, s)}\right) \frac{(r, s)}{(l, m)} B(r, s). \tag{2.9}$$

Note that the set $T(r, s)$ corresponds to the set of partitions of s into r parts.

Proof. The result follows directly from Theorem 2.1.

Let $V = \bigoplus_{m\geq 0} V_m$ be a principally graded representation of a Kac-Moody Lie algebra $\mathfrak{g}(A)$, and let $G = \bigoplus_{l\geq 1} G_l$ be the free Lie algebra on V. Then each G_l is also a principally graded representation of $\mathfrak{g}(A)$ induced by the principal gradation of V:

$$G_l = \bigoplus_{m\geq 0} G_{(l,m)} \quad \text{for } l \geq 1.$$

We give a new gradation to V by setting

$$\widehat{V}_{m+1} = V_m \quad \text{for } m \geq 0. \tag{2.10}$$

Thus we have $V = \bigoplus_{j\geq 1} \widehat{V}_j$. Then this induces a new gradation on G_l:

$$G_l = \bigoplus_{j\geq l} \widehat{G}_{(l,j)},$$

where $\widehat{G}_{(l,j)}$ are defined by

$$\widehat{G}_{(l,m+l)} = G_{(l,m)} \quad \text{for } m \geq 0. \tag{2.11}$$

Therefore by Corollary 2.4, we obtain

$$\begin{aligned} \dim G_{(l,m)} &= \dim \widehat{G}_{(l,m+l)} \\ &= \sum_{(r,s)|(l,m+l)} \mu\left(\frac{(l, m+l)}{(r, s)}\right) \frac{(r, s)}{(l, m+l)} \widehat{B}(r, s), \end{aligned} \tag{2.12}$$

where $\widehat{B}(r, s)$ is defined by

$$\begin{aligned} \widehat{B}(r, s) &= \sum_{(n)\in T(r,s)} \frac{((\sum n_j) - 1)!}{\prod(n_j!)} \prod(\dim \widehat{V}_j)^{n_j}, \\ &= \sum_{(n)\in T(r,s)} \frac{((\sum n_j) - 1)!}{\prod(n_j!)} \prod(\dim V_{j-1})^{n_j}. \end{aligned} \tag{2.13}$$

Hence the principally specialized character of G_l over $A_1^{(1)}$ is given by

$$\begin{aligned} \mathrm{ch}_q\, G_l &= \sum_{m\geq 0} (\dim G_{(l,m)}) q^m \\ &= \sum_{m\geq 0} \left(\sum_{(r,s)|(l,m+l)} \mu\left(\frac{(l,m+l)}{(r,s)}\right) \frac{(r,s)}{(l,m+l)} \widehat{B}(r,s) \right) q^m . \end{aligned} \tag{2.14}$$

Example 2.5. Let $V = \bigoplus_{n\geq 0} V_n$ be the basic representation of the affine Kac-Moody Lie algebra $A_1^{(1)}$ with the principal gradation and let $F(q) = \sum_{n=0}^{\infty} f(n) q^n$ be the principally specialized character of V. By [L-M], we have

$$F(q) = \frac{\phi(q^2)}{\phi(q)} = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + \cdots ,$$

so $f(n)$ equals the number of partitions of n into odd parts. Let $G = \bigoplus_{l\geq 1} G_l$ be the free Lie algebra on V. We will compute $\dim G_{(3,3)}$. By (2.12), we have

$$\begin{aligned} \dim G_{(3,3)} = \dim \widehat{G}_{(3,6)} &= \sum_{(r,s)|(3,6)} \mu\left(\frac{(3,6)}{(r,s)}\right) \frac{(r,s)}{(3,6)} \widehat{B}(r,s) \\ &= \widehat{B}(3,6) - \frac{1}{3}\widehat{B}(1,2). \end{aligned}$$

Since the set $T(3,6)$ corresponds to the partition of 6 into three parts, and since

$$6 = 1+1+4 = 1+2+3 = 2+2+2,$$

the formula (2.13) yields

$$\begin{aligned} \widehat{B}(3,6) &= \frac{2!}{2!1!} f(0)^2 f(3) + \frac{2!}{1!1!1!} f(0) f(1) f(2) + \frac{2!}{3!} f(1)^3 \\ &= 2 + 2 + \frac{1}{3} = \frac{13}{3}. \end{aligned}$$

Similarly, $\widehat{B}(1,2) = (1!/1!) f(1) = 1$. Therefore we obtain

$$\dim G_{(3,3)} = \tfrac{13}{3} - \tfrac{1}{3} = 4.$$

3. The symmetric square and the antisymmetric square

Let α be a weight of an integrable representation V of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}(A)$. For a positive integer t, we say that α is *divisible* by t if $\alpha = t\beta$ for some weight β of V. We define the *signum function* $\varepsilon(\alpha, t)$ by

$$\begin{aligned} \varepsilon(\alpha, t) &= 1 \quad \text{if } \alpha \text{ is divisible by } t, \\ &= 0 \quad \text{otherwise}. \end{aligned} \tag{3.1}$$

Propositon 3.1. *Let V be an integrable irreducible representation of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}(A)$ and let $B(\tau)$ be the Witt partition function on V defined by* (2.2). *Then we have*

$$\dim S^2(V)_\alpha = B(\alpha) + \tfrac{1}{2}\varepsilon(\alpha, 2) \dim V_{\alpha/2}, \tag{3.2}$$

$$\dim \Lambda^2(V)_\alpha = B(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2) \dim V_{\alpha/2}, \tag{3.3}$$

and hence

$$\dim(V \otimes V)_\alpha = 2B(\alpha). \tag{3.4}$$

Proof (G. B. Seligman). Note that since V is an integrable module over $\mathfrak{g}(A)$, so are $V \otimes V$, $S^2(V)$, and $\Lambda^2(V)$, and hence are completely reducible over $\mathfrak{g}(A)$ [Corollary 10.7, Kac2]. Let $\mathfrak{B} = \{b_i\}$ be an ordered basis of V consisting of weight vectors. Consider $\Lambda^2(V)$ as a submodule of $V \otimes V$ consisting of skew-symmetric tensors. Then $\Lambda^2(V)$ has a basis consisting of elements

$$\tfrac{1}{2}(b_i \otimes b_j - b_j \otimes b_i), \qquad i < j.$$

The linear map sending this element to $\frac{1}{2}(b_i \otimes b_j + b_j \otimes b_i)$ is an injective $\mathfrak{h}$-module homomorphism of $\Lambda^2(V)$ into $S^2(V)$. Let M be a complementary $\mathfrak{h}$-submodule to the image of this linear map in $S^2(V)$. Then as $\mathfrak{h}$-modules, we have

$$S^2(V) \cong \Lambda^2(V) \oplus M. \tag{3.5}$$

But M may be taken to have a basis consisting of elements $b_i \otimes b_i$ which are weight vectors of weight $2\lambda_i$, where the λ_i is the weight of b_i. Thus we have

$$\dim M_\alpha = \varepsilon(\alpha, 2) \dim V_{\alpha/2}. \tag{3.6}$$

Since $G_2 \cong \Lambda^2(V)$, the Witt formula gives

$$\dim \Lambda^2(V)_\alpha = B(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2)B(\alpha/2) = B(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2) \dim V_{\alpha/2}.$$

Hence by (3.5), we obtain

$$\dim S^2(V)_\alpha = \dim \Lambda^2(V)_\alpha + \dim M_\alpha = B(\alpha) + \tfrac{1}{2}\varepsilon(\alpha, 2) \dim V_{\alpha/2}. \quad \square$$

Example 3.2. Let $V = V(-2\alpha_{-1} - \alpha_0)$ be an irreducible representation of $A_1^{(1)}$. The weights of V are of the form $-2\alpha_{-1} - k\alpha_0 - l\alpha_1$ with $k \geq 1$, $l \geq 0$, and $2k - (k-l)^2 - 1 \geq 0$. For $\lambda = -2\alpha_{-1} - k\alpha_0 - l\alpha_1$, by [F-L], we have

$$\dim V_\lambda = E(2k - (k-l)^2 - 1),$$

where the function $E(n)$ is defined by

$$\sum_{n=0}^{\infty} E(n)q^n = \frac{\phi(q^2)}{\phi(q)\phi(q^4)} = 1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + 5q^6 + \cdots. \tag{3.7}$$

We also enumerate the weights of V lexicographically. Now for $\alpha = -4\alpha_{-1} - 4\alpha_0 - 4\alpha_1$, let us compute $\dim S^2(V)_\alpha$. By (3.2), we have

$$\begin{aligned}\dim S^2(V)_\alpha &= B(\alpha) + \tfrac{1}{2}\varepsilon(\alpha, 2) \dim V_{\alpha/2} = B(-4\alpha_{-1} - 4\alpha_0 - 4\alpha_1) + \tfrac{1}{2}E(3) \\ &= B(-4\alpha_{-1} - 4\alpha_0 - 4\alpha_1) + 1.\end{aligned}$$

Since

$$\begin{aligned}-4\alpha_{-1} - 4\alpha_0 - 4\alpha_1 &= (-2\alpha_{-1} - \alpha_0) + (-2\alpha_{-1} - 3\alpha_0 - 4\alpha_1) \\ &= (-2\alpha_{-1} - \alpha_0 - \alpha_1) + (-2\alpha_{-1} - 3\alpha_0 - 3\alpha_1) \\ &= (-2\alpha_{-1} - \alpha_0 - 2\alpha_1) + (-2\alpha_{-1} - 3\alpha_0 - 2\alpha_1) \\ &= (-2\alpha_{-1} - 2\alpha_0 - \alpha_1) + (-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1) \\ &= (-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1) + (-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1),\end{aligned}$$

we have

$$B(-4\alpha_{-1} - 4\alpha_0 - 4\alpha_1) = E(0)E(4) + E(1)E(5) + E(0)E(4) + E(2)^2 + \tfrac{1}{2}E(3)^2$$
$$= 3 + 4 + 3 + 1 + \tfrac{1}{2} \cdot 2^2 = 13.$$

Therefore

$$\dim S^2(V)_\alpha = 13 + 1 = 14. \quad \square$$

Corollary 3.3. *Let $V = V(\lambda)$ be an integrable irreducible representation of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}(A)$. We write*

$$F(e(\mu) : \mu \leq \lambda) = \operatorname{ch} V = \sum_{\mu \leq \lambda} (\dim V_\mu) e(\mu).$$

Then

$$\operatorname{ch} S^2(V) = \tfrac{1}{2}((F(e(\mu) : \mu \leq \lambda))^2 + F(e(2\mu) : \mu \leq \lambda)) \tag{3.8}$$

and

$$\operatorname{ch} \Lambda^2(V) = \tfrac{1}{2}((F(e(\mu) : \mu \leq \lambda))^2 - F(e(2\mu) : \mu \leq \lambda)). \tag{3.9}$$

In particular, if $F(q) = \sum_{n \geq 0} f(n) q^n$ is the principally specialized character of V, then

$$\operatorname{ch}_q S^2(V) = \tfrac{1}{2}(F(q)^2 + F(q^2)) \tag{3.10}$$

and

$$\operatorname{ch}_q \Lambda^2(V) = \tfrac{1}{2}(F(q)^2 - F(q^2)). \tag{3.11}$$

Proof (cf. [K-P]). By (3.5), we have

$$\operatorname{ch} M = \operatorname{ch} S^2(V) - \operatorname{ch} \Lambda^2(V).$$

Since

$$\dim M_\alpha = \varepsilon(\alpha, 2) \dim V_{\alpha/2},$$

we get

$$\operatorname{ch} M = \sum_\alpha \dim M_\alpha e(\alpha) = \sum_{\alpha = 2\mu} \dim V_{\alpha/2} e(\alpha) = \sum_{\mu \leq \lambda} \dim V_\mu e(2\mu)$$
$$= F(e(2\mu) : \mu \leq \lambda).$$

Hence

$$\operatorname{ch} M = F(e(2\mu) : \mu \leq \lambda) = \operatorname{ch} S^2(V) - \operatorname{ch} \Lambda^2(V).$$

Combining this with the obvious identity

$$(F(e(\mu) : \mu \leq \lambda))^2 = \operatorname{ch} S^2(V) + \operatorname{ch} \Lambda^2(V)$$

yields the desired formulas. $\square$

4. The combinatorial Identities

Let $A = (a_{ij})_{i,j=-1,0,1}$ be the generalized Cartan matrix for the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$ given in §1. Define a bilinear form $(\ |\ )$ on $\mathfrak{h}$ by $(\alpha_i | \alpha_j) = a_{ij}$ for $i, j = -1, 0, 1$. Let $V = V(-\alpha_{-1})$ be the basic

representation of the affine Kac-Moody Lie algebra $A_1^{(1)}$. Then, by Proposition 3.1, we have

$$\dim S^2(V)_\alpha = B(\alpha) + \frac{1}{2}\varepsilon(\alpha, 2)p\left(1 - \frac{(\alpha|\alpha)}{8}\right), \tag{4.1}$$

$$\dim \Lambda^2(V)_\alpha = B(\alpha) - \frac{1}{2}\varepsilon(\alpha, 2)p\left(1 - \frac{(\alpha|\alpha)}{8}\right), \tag{4.2}$$

and

$$\dim(V \otimes V)_\alpha = 2B(\alpha). \tag{4.3}$$

By [Fe], we have

$$V \otimes V \cong \sum_{m\geq 0}(a_{2m}V(-2\alpha_{-1} - m\delta) + a_{2m+1}V(-2\alpha_{-1} - \alpha_0 - m\delta)),$$

where $\delta = \alpha_0 + \alpha_1$ and the coefficients a_n are given by

$$\sum_{n\geq 0} a_n q^n = \prod_{n\geq 1}(1 + q^{2n-1}) = \frac{\phi(q^2)^2}{\phi(q)\phi(q^4)}. \tag{4.4}$$

In [F-F], using the vertex operators, Feingold and Frenkel showed that

$$S^2(V) \cong \sum_{m\geq 0} a_{2m}V(-2\alpha_{-1} - m\delta),$$

and

$$\Lambda^2(V) \cong \sum_{m\geq 0} a_{2m+1}V(-2\alpha_{-1} - \alpha_0 - m\delta).$$

Hence by [F-L] (see Example 3.2), we have

$$\dim S^2(V)_\alpha = \sum_{m\geq 0} a_{2m}E\left(4 - 2m - \frac{(\alpha|\alpha)}{2}\right), \tag{4.5}$$

$$\dim \Lambda^2(V)_\alpha = \sum_{m\geq 0} a_{2m+1}E\left(3 - 2m - \frac{(\alpha|\alpha)}{2}\right), \tag{4.6}$$

and therefore

$$\begin{aligned}\dim(V \otimes V)_\alpha &= \sum_{m\geq 0}\left\{a_{2m}E\left(4 - 2m - \frac{(\alpha|\alpha)}{2}\right)\right. \\ &\qquad \left. + a_{2m+1}E\left(3 - 2m - \frac{(\alpha|\alpha)}{2}\right)\right\} \\ &= \sum_{m\geq 0} a_m E\left(4 - m - \frac{(\alpha|\alpha)}{2}\right),\end{aligned} \tag{4.7}$$

where the $E(n)$ are given by (3.7). Therefore we obtain the following combinatorial identities.

Theorem 4.1.

$$\sum_{m\geq 0} a_{2m} E\left(4-2m-\frac{(\alpha|\alpha)}{2}\right) = B(\alpha)+\frac{1}{2}\varepsilon(\alpha,2)p\left(1-\frac{(\alpha|\alpha)}{8}\right), \tag{4.8}$$

$$\sum_{m\geq 0} a_{2m+1} E\left(3-2m-\frac{(\alpha|\alpha)}{2}\right) = B(\alpha)+\frac{1}{2}\varepsilon(\alpha,2)p\left(1-\frac{(\alpha|\alpha)}{8}\right), \tag{4.9}$$

$$\sum_{m\geq 0} a_{m} E\left(4-m-\frac{(\alpha|\alpha)}{2}\right) = 2B(\alpha). \tag{4.10}$$

Let V be an integrable irreducible representation of a Kac-Moody Lie algebra $\mathfrak{g}(A)$ and let $F(q)=\sum_{n=0}^{\infty} f(n)q^n$ be the principally specialized character of V. By Proposition 3.1, we have

$$\mathrm{ch}_q\, S^2(V) = \sum_{m\geq 0}\left(\widehat{B}(2,m+2)+\frac{1}{2}\varepsilon(m,2)f\left(\frac{m}{2}\right)\right)q^m,$$

$$\mathrm{ch}_q\, \Lambda^2(V) = \sum_{m\geq 0}\left(\widehat{B}(2,m+2)+\frac{1}{2}\varepsilon(m,2)f\left(\frac{m}{2}\right)\right)q^m,$$

and

$$\mathrm{ch}_q(V\otimes V) = 2\sum_{m\geq 0}\widehat{B}(2,m+2)q^m,$$

where the function $\widehat{B}(r,s)$ is defined by (2.13).

On the other hand, Corollary 3.3 yields

$$\mathrm{ch}_q\, S^2(V) = \tfrac{1}{2}(F(q)^2+F(q^2)), \quad \mathrm{ch}_q\, S^2(V) = \tfrac{1}{2}(F(q)^2-F(q^2)),$$

and

$$\mathrm{ch}_q(V\otimes V) = F(q)^2.$$

Therefore we obtain the following combinatorial identities.

Theorem 4.2.

$$\sum_{m\geq 0}\left(\widehat{B}(2,m+2)+\frac{1}{2}\varepsilon(m,2)f\left(\frac{m}{2}\right)\right)q^m = \frac{1}{2}(F(q)^2+F(q^2)), \tag{4.11}$$

$$\sum_{m\geq 0}\left(\widehat{B}(2,m+2)-\frac{1}{2}\varepsilon(m,2)f\left(\frac{m}{2}\right)\right)q^m = \frac{1}{2}(F(q)^2-F(q^2)), \tag{4.12}$$

and

$$2\sum_{m\geq 0}\widehat{B}(2,m+2)q^m = F(q)^2. \tag{4.13}$$

5. Homological approach

We briefly recall the homological theory for the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$ developed in [Kang1] and [Kang2]. Let $J_- = I_-/[I_-, I_-]$ and consider the exact sequence

$$0\to K \xrightarrow{\psi} U(L_-)\otimes I_{-2}\to J_-\to 0,$$

where ψ is the usual bracket mapping, and K is the kernel of ψ. From the long exact sequence

$$\begin{aligned}\cdots &\to H_1(L_-, K) \to H_1(L_-, U(L_-)\otimes I_{-2}) \to H_1(L_-, J_-)\\ &\to H_0(L_-, K) \to H_0(L_-, U(L_-)\otimes I_{-2}) \to H_0(L_-, J_-) \to 0,\end{aligned}$$

we get an $A_1^{(1)}$-module isomorphism

$$H_1(L_-, J_-) \cong H_0(L_-, K) \cong K/L_-\cdot K. \tag{5.1}$$

On the other hand, by the Poincaré-Birkhoff-Witt theorem, we have the following exact sequence of L_--modules [Kac2, Kang1, Kang2]:

$$0 \to J_- \xrightarrow{\alpha} U(L_-)\otimes V \xrightarrow{\beta} U(L_-) \xrightarrow{\gamma} C \to 0. \tag{5.2}$$

In [Kang1] and [Kang2, Theorem 5.2], we proved that there is an isomorphism of $A_1^{(1)}$-modules

$$H_1(L_-, J_-) \cong H_3(L_-). \tag{5.3}$$

Therefore combining with (5.1) yields

$$K/(L_-\cdot K) \cong H_3(L_-). \tag{5.4}$$

In particular, we have

$$H_3(L_-)_{-3} \cong K_{-3}. \tag{5.5}$$

Therefore we get

$$L_{-3} \cong (V\otimes I_{-2})/H_3(L_-)_{-3}. \tag{5.6}$$

To study the higher levels, for $j \geq 2$, let $I_-^{(j)} = \sum_{n\geq j} I_{-n}$ and consider the quotient Lie algebra $L_-^{(j)} = G_-/I_-^{(j)}$. Then the same homological argument shows that

$$I_{-(j+1)} \cong (V\otimes I_{-j})/H_3(L_-^{(j)})_{-(j+1)}. \tag{5.7}$$

By Hochschild-Serre spectral sequences and the Kostant formula, we have the following theorem.

Theorem 5.1. *We have the following information on the $A_1^{(1)}$-module structure of the maximal graded ideal I.*

$$\begin{aligned}&I_{-3} \cong V\otimes I_{-2},\\ &I_{-4} \cong (V\otimes I_{-3})/(V(-4\alpha_{-1}-3\alpha_0-\alpha_1)\oplus S^2(I_{-2})),\\ &I_{-5} \cong (V\otimes I_{-4})/(V\otimes \Lambda^2(I_{-2})).\end{aligned}$$

6. The principally specialized characters and the root multiplicities

In this section, we compute the principally specialized affine characters and the root multiplicities for the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$ for the levels 1 to 5 and for the levels 1 to 4, respectively. Let $F(q)$, $X(q)$, $Y(q)$, and $Z(q)$ be the principally specialized characters of the integrable irreducible

representations of $A_1^{(1)}$ of level 1, 2, 3, and 4 given in [Kang1] and [L-M]. We introduce the following functions

$$W_2(q) = \mathrm{ch}_q G_{-2} = \sum_{m\geq 0}\left(\widehat{B}(2, m+2) - \frac{1}{2}\varepsilon(m, 2) f\left(\frac{m}{2}\right)\right) q^m, \tag{6.1}$$

$$W_3(q) = \mathrm{ch}_q G_{-3} = \sum_{m\geq 0}\left(\widehat{B}(3, m+3) - \frac{1}{3}\varepsilon(m, 3) f\left(\frac{m}{3}\right)\right) q^m, \tag{6.2}$$

$$W_4(q) = \mathrm{ch}_q G_{-4} = \sum_{m\geq 0}\left(\widehat{B}(4, m+4) - \frac{1}{2}\varepsilon(m, 2)\widehat{B}\left(2, \frac{m}{2}+2\right)\right) q^m, \tag{6.3}$$

$$W_5(q) = \mathrm{ch}_q G_{-5} = \sum_{m\geq 0}\left(\widehat{B}(5, m+5) - \frac{1}{5}\varepsilon(m, 5) f\left(\frac{m}{5}\right)\right) q^m. \tag{6.4}$$

Theorem 6.1. *Let $L = \bigoplus_{n\in Z} L_n$ be the realization of the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$. Then the principally specialized characters of L_{-n}, $n = 1, 2, \dots, 5$, are given by the following formulas:*

$$\begin{aligned}
\mathrm{ch}_1 L_{-1} &= F(q),\\
\mathrm{ch}_q L_{-2} &= W_2(q) - X(q),\\
\mathrm{ch}_q L_{-3} &= W_3(q) - F(q)X(q),\\
\mathrm{ch}_q L_{-4} &= W_4(q) - F(q)^2X(q) + Z(q) + \tfrac{1}{2}(X(q)^2 + X(q^2)),\\
\mathrm{ch}_q L_{-5} &= W_5(q) - F(q)^3X(q) + F(q)Z(q) + F(q)X(q)^2.
\end{aligned}$$

Proof. The results follow directly from Theorem 5.1 and formula (2.14). □

Now we compute the root multiplicities. We recall the representation theory of $A_1^{(1)}$ developed in [D-J-K-M-O]. Let $\varepsilon_{2n} = (1, 0)$ and $\varepsilon_{2n+1)} = (0, 1)$ for $n \geq 0$. Fix a positive integer l. A *path* is a sequence $\eta = (\eta(k))_{k\geq 0}$ consisting of elements $\eta(k) \in Z^2$ of the form $\varepsilon_{\mu_1(k)} + \cdots + \varepsilon_{\mu_l(k)}$ with $\mu_i(k) \in Z$. With a level l dominant weight $\Lambda = \omega_{i_1} + \cdots + \omega_{i_l}$ we associate a path $\eta_\Lambda = (\eta_\Lambda(k))_{k\geq 0}$, where

$$\eta_\Lambda(k) = \varepsilon_{i_1+k} + \cdots + \varepsilon_{i_l+k}.$$

A path η is called a Λ-*path* if $\eta(k) = \eta_\Lambda(k)$ for $k \gg 0$. Let $\mathscr{P}(\Lambda)$ denote the set of Λ-paths. We define the weight $\mathrm{wt}(\eta)$ of η by

$$\mathrm{wt}(\eta) = \Lambda - \sum_{k\geq 0} \pi(\eta(k) - \eta_\Lambda(k)) - \omega(\eta)\delta, \tag{6.5}$$

$$\omega(\eta) = \sum_{k\geq 1} k(H(\eta(k-1), \eta(k)) - H(\eta_\Lambda(k-1), \eta_\Lambda(k))), \tag{6.6}$$

where π is defined by $\pi(\varepsilon_\mu) = \omega_{\mu+1} - \omega_\mu$ $(\omega_2 = \omega_0)$. The function H is

given as follows: for $\alpha = \varepsilon_{\mu_1} + \cdots + \varepsilon_{\mu_l}$ and $\beta = \varepsilon_{\nu_1} + \cdots + \varepsilon_{\nu_l}$ $(0 \le \mu_i, \nu_i < 2)$,

$$H(\alpha, \beta) = \min_{\sigma} \sum_{i=1}^{l} \theta(\mu_i - \nu_{\sigma(i)}), \tag{6.7}$$

where σ runs over the permutation group on l letters, and

$$\begin{aligned} \theta(\mu) &= 1 \quad \text{if } \mu \ge 0, \\ &= 0 \quad \text{otherwise}. \end{aligned} \tag{6.8}$$

Let

$$\mathscr{P}(\Lambda)_\mu = \{\eta \in \mathscr{P}(\Lambda) | \operatorname{wt}(\eta) = \mu\},$$

and define

$$P(\Lambda)_\mu = \#\mathscr{P}(\Lambda)_\mu. \tag{6.9}$$

Theorem 6.2 [D-J-K-M-O]. $\dim V(\Lambda)_\mu = P(\Lambda)_\mu$.

Let $S_1 = \{\tau_i | i = 1, 2, 3, \dots\}$ and $S_2 = \{\mu_i | i = 1, 2, 3, \dots\}$ be enumerations of all the weights of $V(-\alpha_{-1})$ and $V(-2\alpha_{-1} - \alpha_0)$, respectively. Then the Witt partition functions on $V(-\alpha_{-1})$ and $V(-2\alpha_{-1} - \alpha_0)$ are given by

$$B_1(\tau) = \sum_{(n) \in T(\tau)} \frac{\sum((n_i) - 1)!}{\prod(n_i!)} \prod_i P(-\alpha_{-1})_{\tau_i}^{n_i} \tag{6.10}$$

and

$$B_2(\tau) = \sum_{(n) \in T(\tau)} \frac{\sum((n_i) - 1)!}{\prod(n_i!)} \prod_i P(-2\alpha_{-1} - \alpha_0)_{\mu_i}^{n_i}. \tag{6.11}$$

By the Witt formula, we have

$$\dim(G_{-2})_\alpha = B_1(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2) P(-\alpha_{-1})_{\alpha/2}, \tag{6.12}$$

$$\dim(G_{-3})_\alpha = B_1(\alpha) - \tfrac{1}{3}\varepsilon(\alpha, 3) P(-\alpha_{-1})_{\alpha/3}, \tag{6.13}$$

and

$$\dim(G_{-4})_\alpha = B_1(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2) B_1(\alpha/2), \tag{6.14}$$

Since $I_{-2} \cong V(-2\alpha_{-1} - \alpha_0)$, by Theorem 6.2, we have

$$\dim(I_{-2})_\alpha = P(-2\alpha_{-1} - \alpha_0)_\alpha. \tag{6.15}$$

By Theorem 5.1 and [Fe], we have

$$\begin{aligned} I_{-3} &\cong V \otimes I_{-2} \cong V(-\alpha_{-1}) \otimes V(-2\alpha_{-1} - \alpha_0) \\ &\cong \sum_{m \ge 0} (a_m V(-3\alpha_{-1} - \alpha_0 - m\delta) + b_m V(-3\alpha_{-1} - (m+1)\delta)), \end{aligned}$$

where the coefficients a_m and b_m are given by

$$a_m = \sum_{j \in Z} (p(m - j(20j + 3)) - p(m - (4j + 3)(5j + 3)))$$

and

$$b_m = \sum_{j \in Z} (p(m - (20j^2 + 11j + 1)) - p(m - (20j^2 + 19j + 4))).$$

Therefore, by Theorem 6.2, we have

$$\begin{aligned} N_1(\alpha) &= \dim(I_{-3})_\alpha \\ &= \sum_{m\geq 0}(a_m P(-3\alpha_{-1}-\alpha_0-m\delta)_\alpha + b_m P(-3\alpha_{-1}-(m+1)\delta)_\alpha). \end{aligned} \tag{6.16}$$

For I_{-4}, Theorem 5.1 gives

$$\begin{aligned} I_{-4} &\cong (V\otimes I_{-3})/(V(-4\alpha_{-1}-3\alpha_0-\alpha_1)\oplus S^2(I_{-2})) \\ &\cong (V\otimes V\otimes I_{-2})/(V(-4\alpha_{-1}-3\alpha_0-\alpha_1)\oplus S^2(I_{-2})). \end{aligned}$$

Define

$$\begin{aligned} A_n &= \sum_{j\in Z}(p(n-3j(10j+1))-p(n-3j(10j+7))), \\ B_n &= \sum_{j\in Z}(p(n-j(30j+7))-p(n-(30j^2+11j-6))), \\ C_n &= \sum_{j\in Z}(p(n-(30j^2+13j+1))-p(n-(30j^2+31j+7))), \\ D_n &= \sum_{j\in Z}(p(n-j(30j+1))-p(n-j(30j+7))), \\ E_n &= \sum_{j\in Z}(p(n-3j(10j-3))-p(n-(30j^2-3j-2))), \\ F_n &= \sum_{j\in Z}(p(n-(30j^2-19j+2))-p(n-(30j^2-13j-2))). \end{aligned}$$

By [Fe], we get

$$\begin{aligned} V\otimes V\otimes I_{-2} \cong \sum_{k\geq 0}(&p_k V(-4\alpha_{-1}-\alpha_0-k\delta)+q_k V(-4\alpha_{-1}-2\alpha_0-k\delta) \\ &+r_k V(-4\alpha_{-1}-(k+1)\delta)), \end{aligned}$$

where the coefficients p_k, q_k, and r_k are given by

$$\begin{aligned} p_k &= \sum_{m\geq 0}(a_m A_{k-m}+b_m E_{k-m-1}), \\ q_k &= \sum_{m\geq 0}(a_m B_{k-m}+b_m F_{k-m-1}), \\ r_k &= \sum_{m\geq 0}(a_m C_{k-m}+b_m D_{k-m}). \end{aligned}$$

It follows that

$$\begin{aligned} \dim(V\otimes I_{-3})_\alpha = \sum_{k\geq 0}(&p_k P(-4\alpha_{-1}-\alpha_0-k\delta)_\alpha + q_k P(-4\alpha_{-1}-2\alpha_0-k\delta)_\alpha \\ &+r_k P(-4\alpha_{-1}-(k+1)\delta)_\alpha). \end{aligned} \tag{6.17}$$

By Proposition 3.1, we have

$$\dim S^2(I_{-2})_\alpha = B_2(\alpha)+\tfrac{1}{2}\varepsilon(\alpha,2)P(-2\alpha_{-1}-\alpha_0)_{\alpha/2}.$$

Therefore we obtain

(6.18)

$$\begin{aligned} N_2(\alpha) &= \dim(I_{-4})_\alpha \\ &= \sum_{k\geq 0}(p_k P(-4\alpha_{-1}-\alpha_0-k\delta)_\alpha + q_k P(-4\alpha_{-1}-2\alpha_0-k\delta)_\alpha \\ &\qquad\qquad + r_k P(-4\alpha_{-1}-(k+1)\delta)_\alpha) \\ &\quad - P(-4\alpha_{-1}-3\alpha_0-\alpha_1)_\alpha - B_2(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2)P(-2\alpha_{-1}-\alpha_0)_{\alpha/2}. \end{aligned}$$

We summarize these results in the following theorem.

Theorem 6.3. *Let $L = \bigoplus_{n\in Z} L_n$ be the realization of the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$. Then we have the following root multiplicity formulas:*

$$\begin{aligned} \dim(L_{-1})_\alpha &= P(-\alpha_{-1})_\alpha, \\ \dim(L_{-2})_\alpha &= B_1(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2)P(-\alpha_{-1})_{\alpha/2} - P(-2\alpha_{-1}-\alpha_0)_\alpha, \\ \dim(L_{-3})_\alpha &= B_1(\alpha) - \tfrac{1}{3}\varepsilon(\alpha, 3)P(-\alpha_{-1})_{\alpha/3} - N_1(\alpha), \\ \dim(L_{-4})_\alpha &= B_1(\alpha) - \tfrac{1}{2}\varepsilon(\alpha, 2)B_1(\alpha/2) - N_2(\alpha). \end{aligned}$$

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