

PERVERSE SHEAVES AND FINITE DIMENSIONAL ALGEBRAS

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ABSTRACT. We study finite dimensional algebras which arise from categories of perverse sheaves on complex algebraic varieties.

0. INTRODUCTION

This paper complements and expands the results in [MiV]. In that paper we studied the category of perverse sheaves on stratified spaces whose strata S satisfy $\pi_1(S) = \pi_2(S) = 0$. In particular we showed that this category is of Artin type, i.e. equivalent to the category of finitely generated modules over a finite dimensional algebra, and that it has some very special properties in that it satisfies the Bernstein-Gelfand-Gelfand reciprocity principle.

In this paper we extend the scope of [MiV]. We show that if K is a complex algebraic group acting on a complex algebraic variety X with finitely many orbits then the category of K -equivariant perverse sheaves on X is of Artin type. However this category does not satisfy the BGG-reciprocity and it is not of finite projective dimension.

We have also included a fairly extensive discussion of our main technical tool, the glueing construction of [MV], and explain how the work in [MiV] is related to the work of Cline-Parshall-Scott [CPS] (see also [PS]).

In the first section we recall the glueing construction of [MV] and then in §2 we apply it to finite dimensional algebras characterizing the ones that arise in this way.

In §3 we show that the categories of Artin type we considered in [MiV] coincide with the highest weight categories of [CPS]. This also provides a short proof of a result of Dlab and Ringel [DR].

In §4 we prove the result on equivariant perverse sheaves stated above.

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1. GLUEING OF CATEGORIES

In this section we first recall for the convenience of the reader the glueing construction of [MV] and its main properties.

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For the rest of the paper let k be a field. We say that a category is of *Artin type* if it is equivalent to the category of finitely generated modules over a finite dimensional k -algebra A . We say that an abelian category is *artinian* if every object satisfies the descending chain condition.

We start with some generalities.

Let \mathcal{A} and \mathcal{B} be the two categories $F, G: \mathcal{A} \rightarrow \mathcal{B}$ two functors and $T: F \rightarrow G$ a natural transformation. We construct a new category $\mathcal{C}(F, G; T)$ as follows. Its objects are pairs $(A, B) \in \text{Ob}(\mathcal{A}) \times \text{Ob}(\mathcal{B})$ together with a factorization $FA \xrightarrow{m} B \xrightarrow{n} GA, TA = n \circ m$. Its morphisms are pairs $(f, g) \in \text{Mor}(\mathcal{A}) \times \text{Mor}(\mathcal{B})$ which make the appropriate prism commute.

Proposition 1.1. *Let \mathcal{A} and \mathcal{B} be abelian categories, F right exact and G left exact then*

- (a) $\mathcal{C}(F, G; T)$ is abelian.
- (b) If \mathcal{A} and \mathcal{B} are artinian then $\mathcal{C}(F, G; T)$ is.
- (c) If \mathcal{A} and \mathcal{B} are of Artin type then $\mathcal{C}(F, G; T)$ is.

For proofs of (a) and (b) we refer to [MV]. Although (c) is essentially proved in [MiV] we give a proof in a moment.

As is pointed out in [MiV] we have the following functors. We have the restriction functors $|\mathcal{A}: \mathcal{C}(F, G; T) \rightarrow \mathcal{A}$ and $|\mathcal{B}: \mathcal{C}(F, G; T) \rightarrow \mathcal{B}$ which are defined as follows. Let $N \in \text{Ob}(\mathcal{C}(F, G; T))$ be given by $(A, B) \in \text{Ob}(\mathcal{A}) \times \text{Ob}(\mathcal{B})$ and factorization $FA \xrightarrow{m} B \xrightarrow{n} GA$ then we define $N|\mathcal{A} = A$ and $N|\mathcal{B} = B$.

We also have an inclusion functor $\mathcal{B} \rightarrow \mathcal{C}(F, G; T)$. The functor $|\mathcal{A}$ has a left and a right adjoints \widehat{F} and \widehat{G} which are given by

$$\begin{array}{ccc}
 FA & \xrightarrow{TA} & GA \\
 A \mapsto \text{id} \searrow & & \nearrow TA \\
 & & GA \\
 & & \nearrow \text{id}
 \end{array}$$

Finally there is a functor $\widehat{T}: \mathcal{A} \rightarrow \mathcal{C}(F, G; T)$ which is given by

$$\begin{array}{ccc}
 & FA & \xrightarrow{TA} & GA \\
 A \mapsto & \searrow & & \nearrow \\
 & & \text{Im } TA &
 \end{array}$$

All the irreducible objects in $\mathcal{C}(F, G; T)$ are either of the form $\widehat{T}(L)$ where L is irreducible in \mathcal{A} or of the form

$$\begin{array}{ccc}
 0 & \longrightarrow & 0 \\
 \searrow & & \nearrow \\
 & & L
 \end{array}$$

where L is irreducible in \mathcal{B} .

Because \widehat{F} is a left adjoint of an exact functor it preserves projectives. So to show that $\mathcal{C}(F, G; T)$ is of Artin type it suffices to show that the objects of

the form

$$\begin{array}{ccc}
 0 & \longrightarrow & 0 \\
 & \searrow & \nearrow \\
 & & L
 \end{array}$$

where L is irreducible in \mathcal{B} , have projective covers.

Proof of (c). Let \mathcal{V} be the category of finite dimensional k -vector spaces. Let $\overline{P} \rightarrow L$ be a projective cover of L in \mathcal{B} . Consider the functor $\text{Hom}_{\mathcal{B}}(\overline{P}, G_-): \mathcal{A} \rightarrow \mathcal{V}$. Because it is left exact it is representable by an $R \in \mathcal{A}$ (see e.g. [MiV, Proposition 2.4]) and so we have $\text{Hom}_{\mathcal{B}}(\overline{P}, GA) = \text{Hom}_{\mathcal{A}}(R, A)$.

The object

$$\begin{array}{ccc}
 FR & \longrightarrow & GR \\
 (\text{id}, 0) \searrow & & \nearrow (TR, \varepsilon) \\
 & & FR \oplus \overline{P}
 \end{array}$$

where $\varepsilon \in \text{Hom}(\overline{P}, GR) = \text{Hom}(R, R)$ corresponds to the identity map $R \rightarrow R$, is a projective cover of L in $\mathcal{E}(F, G; T)$.

Next we explore when a category \mathcal{E} is of the form $\mathcal{E}(F, G; T)$. For future reference we study the following more general situation. Let \mathcal{E} be an abelian category and assume that we have an abelian category \mathcal{A} , an exact functor $|\mathcal{A}: \mathcal{E} \rightarrow \mathcal{A}$ which has a left adjoint \widehat{F} and a right adjoint \widehat{G} such that $\widehat{F}(A)|_{\mathcal{A}} = A$ and $\widehat{G}(A)|_{\mathcal{A}} = A$ for all $A \in \text{Ob}(\mathcal{A})$. Let $\widetilde{\mathcal{B}} \rightarrow \mathcal{E}$ be a full subcategory such that the kernels and cokernels of the adjunction morphisms $\widehat{F}(X|_{\mathcal{A}}) \rightarrow X$ and $X \rightarrow \widehat{G}(X|_{\mathcal{A}})$ belong to $\widetilde{\mathcal{B}}$. Assume further that there exists an exact functor $E: \mathcal{E} \rightarrow \mathcal{B}$ where \mathcal{B} is an abelian category such that $E|_{\widetilde{\mathcal{B}}}$ is an embedding.

Let $F = E \circ \widehat{F}$, $G = E \circ \widehat{G}$ and $T = E \circ \theta$, where $\theta: \widehat{F} \rightarrow \widehat{G}$ is given by composing the adjunction morphisms for \widehat{F} and \widehat{G} and using the property that $\widehat{F}(A)|_{\mathcal{A}} = A$ and $\widehat{G}(A)|_{\mathcal{A}} = A$. This gives us an abelian category $\mathcal{E}(F, G; T)$ and a canonical functor $\widetilde{E}: \mathcal{E} \rightarrow \mathcal{E}(F, G; T) = \widetilde{\mathcal{E}}$ given by

$$\begin{array}{ccc}
 F(X|_{\mathcal{A}}) & \longrightarrow & G(X|_{\mathcal{A}}) \\
 X \mapsto \searrow & & \nearrow \\
 & & E(X)
 \end{array}$$

Proposition 1.2. *The functor \widetilde{E} is an embedding of categories.*

Proof. By construction \widetilde{E} is a faithful exact functor. It remains to show that \widetilde{E} is full. Therefore we have to show that for any $N, M \in \mathcal{E}$ $\text{Hom}_{\mathcal{E}}(N, M) = \text{Hom}_{\widetilde{\mathcal{E}}}(\widetilde{E}N, \widetilde{E}M)$.

The category $\widetilde{\mathcal{E}} = \mathcal{E}(F, G; T)$ gives us functors $\widehat{\widetilde{F}}, \widehat{\widetilde{G}}: \mathcal{A} \rightarrow \widetilde{\mathcal{E}}$ and the restriction functors $|\widetilde{\mathcal{A}}: \widetilde{\mathcal{E}} \rightarrow \mathcal{A}$ and $|\widetilde{\mathcal{B}}: \widetilde{\mathcal{E}} \rightarrow \mathcal{B}$. We see that $\widetilde{E}(\widehat{\widetilde{F}}(A)) = \widehat{\widetilde{F}}(A)$ and $\widetilde{E}(\widehat{\widetilde{G}}(A)) = \widehat{\widetilde{G}}(A)$ by using the property that $\widehat{F}(A)|_{\mathcal{A}} = A$ and $\widehat{G}(A)|_{\mathcal{A}} = A$.

(1) If $M = \widehat{G}A$ for some $A \in \text{Ob}(\mathcal{A})$ we have

$$\text{Hom}_{\mathcal{E}}(N, \widehat{G}A) = \text{Hom}_{\mathcal{A}}(N|_{\mathcal{A}}, A) = \text{Hom}_{\widetilde{\mathcal{E}}}(\widetilde{E}N, \widetilde{G}A).$$

So we have proved the theorem for M of the form $M = \widehat{G}A$.

(2) Let $M = B \in \widetilde{\mathcal{B}}$. Consider the exact sequence

$$\widehat{F}(N|_{\mathcal{A}}) \rightarrow N \rightarrow X \rightarrow 0,$$

where $X \in \text{Ob}(\mathcal{E})$ is simply the cokernel of $\widehat{F}(N|_{\mathcal{A}}) \rightarrow N$. By applying \widetilde{E} to this exact sequence we get the exact sequence

$$\widetilde{E}\widehat{F}(N|_{\mathcal{A}}) \rightarrow \widetilde{E}N \rightarrow \widetilde{E}X \rightarrow 0.$$

By applying $\text{Hom}_{\mathcal{E}}(_, B)$ and $\text{Hom}_{\widetilde{\mathcal{E}}}(_, \widetilde{E}B)$ to the above exact sequences and observing that $\widetilde{E}(\widehat{F}(N|_{\mathcal{A}})) = \widehat{F}(N|_{\mathcal{A}})$ we get the following exact sequences:

$$\begin{array}{ccccccc} 0 \rightarrow & \text{Hom}(X, B) & \rightarrow & \text{Hom}(N, B) & \rightarrow & \text{Hom}(\widehat{F}(N|_{\mathcal{A}}), B) = \text{Hom}_{\mathcal{A}}(N|_{\mathcal{A}}, B|_{\mathcal{A}}) & \\ & \downarrow \parallel & & \downarrow & & \downarrow & & \downarrow \parallel \\ 0 \rightarrow & \text{Hom}(\widetilde{E}X, \widetilde{E}B) & \rightarrow & \text{Hom}(\widetilde{E}N, \widetilde{E}B) & \rightarrow & \text{Hom}(\widetilde{E}\widehat{F}(N|_{\mathcal{A}}), \widetilde{E}B) = \text{Hom}(N|_{\mathcal{A}}, B|_{\mathcal{A}}) & \end{array}$$

The map $\text{Hom}(N, B) \rightarrow \text{Hom}(\widetilde{E}N, \widetilde{E}B)$ is therefore an isomorphism. Thus we have proved the theorem for $M = B \in \widetilde{\mathcal{B}}$.

We next claim that $\text{Ext}^1(N, B) \rightarrow \text{Ext}^1(\widetilde{E}N, \widetilde{E}B)$ for $B \in \widetilde{\mathcal{B}}$ is an injection. Let $0 \rightarrow B \rightarrow X \rightarrow N \rightarrow 0$ be an exact sequence in \mathcal{E} and assume that $0 \rightarrow \widetilde{E}(B) \rightarrow \widetilde{E}(X) \rightarrow \widetilde{E}(N) \rightarrow 0$ splits. Because $\text{Hom}(\widetilde{E}X, \widetilde{E}B) = \text{Hom}(X, B)$, the sequence $0 \rightarrow B \rightarrow X \rightarrow N \rightarrow 0$ is also split.

(3) Let us consider a general M in \mathcal{E} . Split the exact sequence $0 \rightarrow K \rightarrow M \rightarrow \widehat{G}(M|_{\mathcal{A}}) \rightarrow X \rightarrow 0$ into two exact sequences

$$0 \rightarrow K \rightarrow M \rightarrow I \rightarrow 0, \quad 0 \rightarrow I \rightarrow \widehat{G}(M|_{\mathcal{A}}) \rightarrow X \rightarrow 0.$$

We get from (1) and (2) that $\text{Hom}(N, I) = \text{Hom}(\widetilde{E}N, \widetilde{E}I)$ and the diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow & \text{Hom}(N, K) & \rightarrow & \text{Hom}(N, M) & \rightarrow & \text{Hom}(N, I) = \text{Ext}^1(N, K) & \\ & \downarrow \wr & & \downarrow & & \downarrow \parallel & & \downarrow \parallel \\ 0 \rightarrow & \text{Hom}(\widetilde{E}N, \widetilde{E}K) & \rightarrow & \text{Hom}(\widetilde{E}N, \widetilde{E}M) & \rightarrow & \text{Hom}(\widetilde{E}N, \widetilde{E}I) \rightarrow \text{Ext}^1(\widetilde{E}N, \widetilde{E}K) & \end{array}$$

implies the result. \square

2. GLUEING OF CATEGORIES OF ARTIN TYPE

In this section we investigate when a category of Artin type can be built up from simpler components by the glueing construction.

Let \mathcal{E} be a category of Artin type and let L_r be an irreducible object in \mathcal{E} . Let $\widetilde{\mathcal{B}}$ be the full subcategory of \mathcal{E} consisting of objects whose Jordan-Hölder series contain L_r only.

It is not too difficult to check that $S = \{f \in \text{Mor}(\mathcal{E}) \mid \text{Ker}(f), \text{Coker}(f) \in \widetilde{\mathcal{B}}\}$ is a multiplicative set (with calculus of fractions) and so we can form the localized category $\mathcal{A} = \mathcal{E}_S$. We denote the exact functor $\mathcal{E} \rightarrow \mathcal{A}$ by $|\mathcal{A}$.

Remark. If P_1, \dots, P_{r-1}, P_r are the projective covers of the irreducible objects L_1, \dots, L_r of \mathcal{E} then the category \mathcal{A} is equivalent to the module category of $\text{End}(\bigoplus_{i=1}^{r-1} P_i)$ and the functor $\mathcal{E} \rightarrow \mathcal{A}$ corresponds to the inclusion

$$\text{End} \left(\bigoplus_{i=1}^{r-1} P_i \right) \rightarrow \text{End} \left(\bigoplus_{i=1}^r P_i \right)$$

of rings.

Lemma 2.1. *The restriction functor $\mathcal{E} \rightarrow \mathcal{A}$ has a left adjoint \widehat{F} and a right adjoint \widehat{G} . These adjoints have the property that $\widehat{F}(A)|_{\mathcal{A}} = A$ and $\widehat{G}(A)|_{\mathcal{A}} = A$.*

Proof. We will show that \widehat{G} exists. We have to construct a functor $\widehat{G}: \mathcal{A} \rightarrow \mathcal{E}$ such that $\text{Hom}_{\mathcal{A}}(M, \widehat{G}A) = \text{Hom}_{\mathcal{A}}(M|_{\mathcal{A}}, A)$. Because $\text{Hom}_{\mathcal{A}}(|_{\mathcal{A}}, A)$ is left exact it is representable by an object $\widehat{G}A$. It is not hard to check that this defines a functor \widehat{G} .

To prove that $\widehat{F}(A)|_{\mathcal{A}} = A$ we first observe that it suffices to prove it for projectives. Therefore it suffices to observe that the exact functor $|\mathcal{A}: \mathcal{E} \rightarrow \mathcal{A}$ takes the projective covers P_1, \dots, P_{r-1} of the irreducibles L_1, \dots, L_{r-1} to the projective covers $\overline{P}_1, \dots, \overline{P}_{r-1}$ of the irreducibles $\overline{L}_1, \dots, \overline{L}_{r-1}$ ($\overline{L}_i = L_i|_{\mathcal{A}}$). The functor \widehat{F} on the other hand takes the projectives $\overline{P}_1, \dots, \overline{P}_{r-1}$ to P_1, \dots, P_{r-1} .

We next construct a functor $E: \mathcal{E} \rightarrow \mathcal{B}$ such that $E|_{\widetilde{\mathcal{B}}}$ is an embedding.

Let \mathcal{B} be the category of finitely generated right $\text{Hom}(P_r, P_r)$ -modules where P_r is the indecomposable projective cover of L_r . We get an exact functor $E: \mathcal{E} \rightarrow \mathcal{B}$ by $E(M) = \text{Hom}(P_r, M)$. Let $F = E \circ \widehat{F}$, $G = E \circ \widehat{G}$ and $T = E \circ \theta$, where $\theta: \widehat{F} \rightarrow \widehat{G}$ is gotten by composing two adjunction morphisms as in §1. This gives us a category $\mathcal{E}(F, G; T)$ and an exact functor $\widetilde{E}: \mathcal{E} \rightarrow \mathcal{E}(F, G; T)$. We have

Proposition 2.2. *The functor \widetilde{E} is an embedding.*

Proof. Follows immediately from Proposition 1.2.

To describe the image of \widetilde{E} we perform the following construction.

Consider a family of functors $I_\alpha: G \rightarrow F$, $\alpha \in \Lambda$, such that $T \circ I_\alpha = 0$ and $I_\alpha \circ T = 0$. Let $\widetilde{\mathcal{E}} \subset \mathcal{E}(F, G; T)$ be the full subcategory consisting of factorizations $FA \xrightarrow{m} B \xrightarrow{n} GA$ such that $m \circ I_\alpha(A) \circ n = 0$ for all $\alpha \in \Lambda$.

Proposition 2.3. *The category $\widetilde{\mathcal{E}}$ is of Artin type if \mathcal{A} and \mathcal{B} are.*

Proof. As in the proof of Proposition 1.1(c) it suffices to construct the projective cover of an irreducible object $L_r \in \mathcal{B} \subset \widetilde{\mathcal{E}}$. The projective cover of L_r in

$\mathcal{E}(F, G; T)$ is given by

$$\begin{array}{ccc} FR & \xrightarrow{TR} & GR \\ & \searrow & \nearrow \\ & FR \oplus \bar{P}_r & \end{array}$$

where $R = P_r|_{\mathcal{A}}$ and \bar{P}_r is the projective cover of L_r in \mathcal{B} . Let $Q = \text{Span}((m \circ I_\alpha(R) \circ n)(FR \oplus \bar{P}_r)) \subset FR$ and consider

$$\begin{array}{ccc} FR & \longrightarrow & GR \\ & \searrow & \nearrow \\ & (FR/Q) \oplus \bar{P}_r & \end{array}$$

Clearly it belongs to $\tilde{\mathcal{E}}$ and by construction it is projective.

We now impose some rather strong restrictions on our category \mathcal{E} so that we can characterize the image of $\tilde{E}: \mathcal{E} \rightarrow \mathcal{E}(F, G; T)$. We assume (for convenience) that $\text{End}(L_r) = k$ and $\text{Ext}^1(L_r, L_r) = 0$. We also drop the $\text{End}(P_r)$ -action in the definition of the category \mathcal{B} and take $\mathcal{B} = \mathcal{V}$, where \mathcal{V} is the category of finite dimensional k -vector spaces.

The fact that $\text{Ext}^1(L_r, L_r) = 0$ implies that $\hat{F}R \rightarrow P_r \rightarrow L_r \rightarrow 0$ is exact, where $R = P_r|_{\mathcal{A}}$. Let us now consider $\tilde{E}(P_r)$. Because $\tilde{E}: \mathcal{E} \rightarrow \mathcal{E}(F, G; T)$ is an embedding the projective $FR \rightarrow FR \oplus k \rightarrow GR$ is a cover of $\tilde{E}(P_r)$ and in particular we have a canonical surjection $\theta: FR \oplus k \rightarrow \text{Hom}(P_r, P_r)$.

Lemma 2.4. *We have $\text{Ker}(\theta) \subset FR \subset FR \oplus k$.*

Proof. The map $\theta|_k: k \rightarrow \text{Hom}(P_r, P_r)$ is the one mapping k to the multiples of the identity map. When we apply $\text{Hom}(P_r, -)$ to the exact sequence $\hat{F}R \rightarrow P_r \rightarrow L_r \rightarrow 0$ we get the exact sequence $FR \rightarrow \text{Hom}(P_r, P_r) \rightarrow k \rightarrow 0$. Therefore $\text{Ker}(\theta) \subset FR$.

Let $v_\alpha \in \text{Ker}(\theta)$, $\alpha \in \Lambda$, be a set of generators. For each α we construct a natural transformation $I_\alpha: G \rightarrow F$ as follows: $I_\alpha(A)(\tau) = F(\tau)(v_\alpha)$, where $A \in \mathcal{A}$ and $\tau \in G(A) = \text{Hom}(R, A)$. These natural transformations I_α define a subcategory $\tilde{\mathcal{E}} \subset \mathcal{E}(F, G; T)$.

Proposition 2.5. *The functor $\tilde{E}: \mathcal{E} \rightarrow \mathcal{E}(F, G; T)$ induces an equivalence between \mathcal{E} and $\tilde{\mathcal{E}}$.*

Proof. By construction \tilde{E} takes the projectives of \mathcal{E} to the projectives of $\tilde{\mathcal{E}}$ therefore inducing an equivalence of categories.

Proposition 2.6. *The functor $\tilde{E}: \mathcal{E} \rightarrow \mathcal{E}(F, G; T)$ is an equivalence of categories if and only if in addition to $\text{End}(L_r) = k$ and $\text{Ext}^1(L_r, L_r) = 0$ we have $\text{Ext}^2(L_r, L_r) = 0$.*

Proof. For \mathcal{E} and $\mathcal{E}(F, G; T)$ to have the same projectives we must have $\text{Hom}(P_r, P_r) \cong \text{Hom}(P_r, \hat{F}R) \oplus k$. Consider the exact sequence

$$0 \rightarrow N_r \rightarrow \hat{F}R \rightarrow P_r \rightarrow L_r \rightarrow 0$$

where $N_r = \bigoplus_{i=1}^l L_r$ and split it into two exact sequences

$$0 \rightarrow N_r \rightarrow \widehat{F}R \rightarrow I_r \rightarrow 0, \quad 0 \rightarrow I_r \rightarrow P_r \rightarrow L_r \rightarrow 0.$$

We then see that $\text{Ext}^2(L_r, L_r) \cong \text{Ext}^1(I_r, L_r) \cong \text{Hom}(N_r, L_r)$ because

$$\text{Hom}(\widehat{F}R, L_r) = \text{Ext}^1(\widehat{F}R, L_r) = 0.$$

Therefore $\text{Ext}^2(L_r, L_r) = 0 \Leftrightarrow \text{Hom}(N_r, L_r) = 0 \Leftrightarrow N_r = 0 \Leftrightarrow 0 \rightarrow \widehat{F}R \rightarrow P_r \rightarrow L_r \rightarrow 0$ is exact $\Leftrightarrow \text{Hom}(P_r, P_r) = \text{Hom}(P_r, \widehat{F}R) \oplus k$.

We will next introduce two classes of categories of Artin type.

Let \mathcal{E} be a category of Artin type. Let Λ be the set of irreducible objects of \mathcal{E} and denote the irreducible object corresponding to $\mu \in \Lambda$ by L_μ , its projective cover by P_μ and its injective hull by I_μ . Choose a partial ordering of the set Λ and let M_λ be the largest quotient of P_λ such that the decomposition series of M_λ consists of irreducibles L_μ with $\mu \leq \lambda$. Let M_λ^\vee be the maximal sub of I_λ s.t. the decomposition series of M_λ^\vee consists of irreducibles L_μ s.t. $\mu \leq \lambda$.

Definition 2.7. (i) A category \mathcal{E} of Artin type is said to be of type A if for all irreducible objects L we have $\text{End}(L) = k$ and $\text{Ext}^1(L, L) = 0$.

(ii) \mathcal{E} is of type B if it is type A and it has a partial ordering Λ such that $\text{Ext}^2(M_\mu, M_\mu^\vee) = 0$ for all $\mu \in \Lambda$.

Theorem 2.8. (i) A category \mathcal{E} is of type A if and only if it can be constructed by iterating the glueing construction starting with $\mathcal{A} = \mathcal{V}$ and always using $\mathcal{B} = \mathcal{V}$ with some relation functors I_α .

(ii) A category \mathcal{E} is of type B if and only if it can be constructed by iterating the glueing construction starting with $\mathcal{A} = \mathcal{V}$ and always using $\mathcal{B} = \mathcal{V}$.

Proof. (i) Follows directly from 2.5.

(ii) Note first that if \mathcal{E} is constructed by iterating the glueing construction the irreducible objects will be naturally partially ordered. Also note that if we have M_μ and M_μ^\vee in \mathcal{A} then after applying the glueing construction we get the corresponding objects \widehat{M}_μ and \widehat{M}_μ^\vee in \mathcal{E} by $\widehat{M}_\mu = \widehat{F}M_\mu$ and $\widehat{M}_\mu^\vee = \widehat{G}M_\mu^\vee$.

Also note that if \mathcal{E} is of type B then for a lowest element λ we have $L_\lambda = M_\lambda = M_\lambda^\vee$ and so \mathcal{E} is gotten by the glueing construction from some \mathcal{A} .

Putting this together we see that it suffices to show that if \mathcal{E} is constructed from \mathcal{A} by the glueing construction then

$$\text{Ext}_{\mathcal{A}}^2(M_\mu, M_\mu^\vee) \cong \text{Ext}_{\mathcal{E}}^2(\widehat{F}M_\mu, \widehat{G}M_\mu^\vee).$$

Consider the exact sequence $0 \rightarrow K_\mu \rightarrow P_\mu \rightarrow M_\mu \rightarrow 0$. Then $\text{Ext}^2(M_\mu, M_\mu^\vee) \cong \text{Ext}^1(K_\mu, M_\mu^\vee)$ and by applying $\text{Hom}(K_r, \)$ to the exact sequence $0 \rightarrow M_\mu^\vee \rightarrow I_\mu \rightarrow C_\mu \rightarrow 0$ we get the exact sequence

$$(*) \quad \begin{aligned} 0 \rightarrow \text{Hom}(K_\mu, M_\mu^\vee) \rightarrow \text{Hom}(K_\mu, I_\mu) \\ \rightarrow \text{Hom}(K_\mu, C_\mu) \rightarrow \text{Ext}^1(K_\mu, M_\mu^\vee) \rightarrow 0. \end{aligned}$$

From the exact sequence $0 \rightarrow L^1 \widehat{F}M_\mu \rightarrow \widehat{F}K_\mu \rightarrow \widehat{F}P_\mu \rightarrow \widehat{F}M_\mu \rightarrow 0$ we get two exact sequences

$$0 \rightarrow L^1 \widehat{F}M_\mu \rightarrow \widehat{F}K_\mu \rightarrow D \rightarrow 0, \quad 0 \rightarrow D \rightarrow \widehat{F}P_\mu \rightarrow \widehat{F}M_\mu \rightarrow 0$$

and we get that $\text{Ext}^2(\widehat{F}M_\mu, \widehat{G}M_\mu^\vee) \cong \text{Ext}^1(D, \widehat{G}M_\mu^\vee) \cong \text{Ext}^1(\widehat{F}K_\mu, \widehat{G}M_\mu^\vee)$. By applying $\text{Hom}(\widehat{F}K_\mu, \)$ to the exact sequence $0 \rightarrow \widehat{G}M_\mu^\vee \rightarrow \widehat{G}I_\mu \rightarrow \widehat{G}C_\mu \rightarrow R^1\widehat{G}M_\mu^\vee \rightarrow 0$ we get the exact sequence

$$\begin{aligned} 0 &\rightarrow \text{Hom}(\widehat{F}K_\mu, \widehat{G}M_\mu^\vee) \rightarrow \text{Hom}(\widehat{F}K_\mu, \widehat{G}I_\mu) \\ &\rightarrow \text{Hom}(\widehat{F}K_\mu, \widehat{G}C_\mu) \rightarrow \text{Ext}^1(\widehat{F}K_\mu, \widehat{G}M_\mu^\vee) \rightarrow 0. \end{aligned}$$

The first three terms are the same as in the exact sequence (*) proving the proposition.

3. HIGHEST WEIGHT CATEGORIES

In this section we consider the highest weight categories introduced in [CPS]. We show that these categories are the same as those considered in [MiV, §3]. In this process we recover a result of Dlab and Ringel [DR].

Let \mathcal{C} be a category of Artin type and Λ a partial order on the irreducible objects. We continue to use the notation from §2.

We say that an object N has a p -filtration if it has a filtration N_i such that the quotients $N_i/N_{i-1} \cong M_{\mu_i}$ for some μ_i . Dually we have a notion of a p^\vee -filtration.

Definition 3.1 [CPS]. A category of Artin type is called a highest weight category if there exists a partial ordering on Λ s.t. every projective P_λ has a p -filtration $F_1 \subset \dots \subset F_k = P_\lambda$ s.t. $F_k/F_{k-1} = M_\lambda$ and for $i < k$, $F_i/F_{i-1} \cong M_\mu$ for some $\mu > \lambda$.

Remark. Our definition of a highest weight category is a special case of [CPS] because we have taken the category \mathcal{C} to be of Artin type.

Lemma 3.2. *If \mathcal{C} is a highest weight category then every injective I_λ has a p^\vee -filtration $0 = F_0 \subset F_1 \subset \dots \subset F_k = I_\lambda$ such that $F_i = \check{M}_\lambda$ and for $i > 1$ $F_i/F_{i-1} \cong \check{M}_\mu$ for some $\mu > \lambda$.*

Proof. See [CPS].

Lemma 3.3. *In a highest weight category \mathcal{C} we have*

- (a) *If $\text{Ext}^i(M_\lambda, L_\lambda) \neq 0$ for $i > 0$ then $\mu > \lambda$.*
- (b) *$\text{Ext}^i(L_\lambda, L_\lambda) = 0$ for $i > 0$ if λ minimal.*
- (c) *$\text{Ext}^i(M_\lambda, M_\mu^\vee) = 0$ for $i > 0$ for all μ, λ .*

Proof. (a) Consider the exact sequence $0 \rightarrow K_\lambda \rightarrow P_\lambda \rightarrow M_\lambda \rightarrow 0$ and proceed by induction.

(b) Follows from (a) because $M_\lambda = L_\lambda$.

(c) If λ is not less than μ use the exact sequence $0 \rightarrow K_\lambda \rightarrow P_\lambda \rightarrow M_\lambda \rightarrow 0$ and induction. If μ not less than λ use the exact sequence $0 \rightarrow M_\mu^\vee \rightarrow I_\mu \rightarrow C_\mu \rightarrow 0$ and induction. It is crucial to observe that for maximal λ M_λ is projective and M_λ^\vee is injective. This guarantees that the induction will end.

Let \mathcal{C} be a category of Artin type which is also of type B of the last section. This gives a partial (actually total) ordering for the irreducible objects and it also gives us the objects M_μ and M_μ^\vee . If we look at one iteration of our construction with objects M_μ , $\mu \in \Lambda$, in the category \mathcal{A} then the corresponding objects \widehat{M}_μ , $\mu \in \Lambda \cup \{\lambda\}$, for the category $\mathcal{C}(F, G; T)$ are given by $\widehat{M}_\mu = \widehat{F}M_\mu$ for $\mu \in \Lambda$ and $M_\lambda = L_\lambda$. Similarly for the objects M_μ^\vee .

Definition 3.4. A category \mathcal{E} is a BGG category if it is of type B and at every stage of its construction we have $L^1FM_\mu = 0$ and $R^1GM_\mu^\vee = 0$ for all μ .

This definition was given (in the presence of duality) in §3 of [MiV]. Let us again assume (for simplicity) as at the end of §2 that the highest weight category \mathcal{E} satisfies the property that $\text{End}(L) = k$ for all simple objects L in \mathcal{E} . Then we have

Proposition 3.5. *A category \mathcal{E} is a highest weight category if and only if it is a BGG category.*

Proof. Assume that \mathcal{E} is a highest weight category. By Lemma 3.3(b) \mathcal{E} is of type B . The representing object R for the functor G is a restriction of the projective cover of the new irreducible and therefore R has a p -filtration. Because $R^1GM_\mu^\vee = \text{Ext}^1(R, M_\mu^\vee)$ we see by induction and using Lemma 3.3(c) that $R^1GM_\mu^\vee = 0$. Similarly we see that $L^1FM_\mu = 0$.

Assume now that \mathcal{E} is a BGG category. Then Lemma 3.3 in [MiV] claims that N has a p -filtration if and only if $\text{Ext}^1(N, M_\mu^\vee) = 0$ for all μ . In particular all projectives have p -filtrations. By the proof of that lemma we see that the p -filtrations satisfy the condition for a highest weight category.

We now state the following immediate corollary which gives one more characterization of highest weight categories.

Corollary 3.6. *\mathcal{E} is a highest weight category if and only if it is of type B and at every stage of its constructions the representing object of G has a p -filtration and the representing object of F has a p^\vee -filtration.*

This corollary is essentially a result of [DR].

4. EQUIVARIANT PERVERSE SHEAVES

Let X be a complex algebraic variety and let K be an algebraic group acting on X . Let $P_K(X)$ denote the category of K -equivariant perverse sheaves of k -vector spaces on X . An element of $P_K(X)$ is a perverse sheaf A on X together with an isomorphism $\alpha: p^*A \rightarrow m^*A$ s.t. $\alpha|_{X \times \{0\}} = \text{id}$ where $p: K \times X \rightarrow X$ is the projection and $m: K \times X \rightarrow X$ is the multiplication map.

If K is connected then $P_K(X)$ is a full subcategory of $P(X)$. To simplify the arguments we assume that K is connected. We have the following easy lemma.

Lemma 4.1. *If K acts on a smooth variety Y then a local system on Y is K -equivariant if and only if the image of $\pi_1(K)$ in $\pi_1(Y)$ acts trivially. In particular if Y is a K -orbit then the K -equivariant local systems are given by representations of K_y/K_y^0 , where $y \in Y$.*

Assume now that K acts with finitely many orbits on a variety X . This gives us a Whitney stratification \mathcal{S} of X . We will now study $P_K(X)$ using the glueing construction of [MV].

Let S be a closed K -orbit, $S = K/K_x$ for $x \in S$, and let $\mathcal{E} = P_K(X)$, $\mathcal{A} = P_K(X - S)$, $\widehat{\mathcal{B}} = P_K(S) = \{\text{representations of } K_x/K_x^0\}$, $\widehat{F} = {}^p j_! : P_K(X - S) \rightarrow P_K(X)$, $\widehat{G} = {}^p j_* : P_K(X - S) \rightarrow P_K(X)$. To complete the

situation of Proposition 1.2 we let

$$\tilde{\Lambda}_S = T_S^* X - \bigcup_{\substack{s' \neq s \\ s' \in \mathcal{S}}} \overline{T_S^*, X}$$

and choose $\mathcal{B} = \{K\text{-equivariant local systems on } \tilde{\Lambda}_S\}$. It is not very hard to see that the construction of the vanishing cycle functor Φ of [MV] passes to the equivariant situation giving us an exact functor $E: P_K(X) \rightarrow \mathcal{B}$ and so we get $F = E \circ \hat{F}$, $G = E \circ \hat{G}$ and by Proposition 1.2 an embedding $P_K(X) \rightarrow \mathcal{E}(F, G; T)$. Let $x \in S$. We have the following exact sequence

$$\pi_1(\pi^{-1}x) \xrightarrow{\tau} \pi_1(\tilde{\Lambda}_S) \rightarrow \pi_1(S) \rightarrow 0$$

where $\pi: \tilde{\Lambda}_S \rightarrow S$. We also have natural transformations $I_\alpha: G \rightarrow F$ for any $\alpha \in \pi_1(\pi^{-1}(x))$ as defined in [MV, §5]. We get a full subcategory $\tilde{\mathcal{E}} \rightarrow \mathcal{E}(F, G; T)$ by requiring that the factorizations $F(A) \xrightarrow{m} B \xrightarrow{n} G(A)$ satisfy the following condition. For any $\alpha \in \pi_1(\pi^{-1}(x))$ we require that the action $\mu_{\tau(\alpha)}$ of $\tau(\alpha)$ on B be given by $\mu_{\tau(\alpha)} = 1 + m \circ I_\alpha(A) \circ n$.

Proposition 4.2. *The functor $\tilde{E}: P_K(X) \rightarrow \tilde{\mathcal{E}}$ is an equivalence of categories.*

Proof. If we forget the K -equivariance condition we get by Theorem 5 in [MV] the equivalence $P_{\mathcal{S}}(X) \cong \tilde{\mathcal{E}}(\bar{F}, \bar{G}; \bar{T})$ where the bar indicates that we forget the K -action and $P_{\mathcal{S}}(X)$ denotes the category of perverse sheaves constructible with respect to \mathcal{S} . We now have the following situation:

$$\begin{array}{ccc} P_{\mathcal{S}}(X) & \xrightarrow[\cong]{\bar{E}} & \tilde{\mathcal{E}}(\bar{F}, \bar{G}; \bar{T}) \\ \cup & & \cup \\ P_K(X) & \xrightarrow{\tilde{E}} & \mathcal{E}(F, G; T) \end{array}$$

and we have to show that \tilde{E} is essentially surjective. If we take an object $(A, B) \in \tilde{\mathcal{E}}(F, G; T)$ we see that it gives rise to a perverse sheaf $Q \in P_{\mathcal{S}}(X)$. Now we have to show that $Q \in P_K(X)$, i.e. that there is an isomorphism $p^*Q \xrightarrow{\cong} m^*Q$ on $K \times X$. Considering the stratification $\mathcal{S}' = \{K \times S'\}_{S' \in \mathcal{S}}$ on $K \times X$ with closed stratum $K \times S$ gives us again functors F', G' and an equivalence $P_{\mathcal{S}'}(K \times X) \simeq \tilde{\mathcal{E}}(F', G'; T')$. Because $\bar{E}(Q) \in \tilde{\mathcal{E}}(F, G; T)$, i.e. its data is K -equivariant we see that the data of p^*Q is isomorphic to that of m^*Q in $\tilde{\mathcal{E}}(F', G'; T')$. Therefore $p^*Q \cong m^*Q$.

Now we are ready to state

Theorem 4.3. *The category $P_K(X)$ is of Artin type.*

Proof. We prove the result by induction. Let S be a closed K -orbit. By the previous proposition we have $P_K(X) \cong C(F, G; T)$ and so it remains to prove that a K -equivariant local system L on S has a projective cover.

Let us consider the geometric situation more closely. Let $x \in S$. We have the exact sequence

$$\begin{array}{ccccccc} \pi_1(\pi^{-1}(x)) & \xrightarrow{\tau} & \pi_1(\tilde{\Lambda}_S) & \longrightarrow & \pi_1(S) & \longrightarrow & 0 \\ \parallel & & \parallel & & \parallel & & \\ N & & \Gamma & & H & & \end{array}$$

Because we have an action by K on S and on $\tilde{\Lambda}_S$ we have maps

$$\begin{array}{ccc} & \nearrow \pi_1(\tilde{\Lambda}_S) & \\ \pi_1(K) & & \downarrow \\ & \searrow \pi_1(S) & \end{array}$$

and the image of $\pi_1(K)$ is central in $\pi_1(\tilde{\Lambda}_S)$ and $\pi_1(S)$. Consider the following diagram with columns and the two bottom rows exact:

$$\begin{array}{ccccccc} N \cap \tau^{-1}(\pi_1 K) & \longrightarrow & \pi_1(K) & \xlongequal{\quad} & \pi_1(K) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ N & \longrightarrow & \Gamma & \longrightarrow & H & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ N' = N / (N \cap \tau^{-1}(\pi_1 K)) & \longrightarrow & \Gamma' & \longrightarrow & H' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Note that here $H' = K_x / K_x^0$, i.e. the representations of H' give all the K -equivariant local systems on S . In our construction $F, G: P_K(X - S) \rightarrow \{k[\Gamma]\text{-modules}\}$. We now forget the $k[\Gamma]$ -module structure and we get

$$\overline{F}, \overline{G}: P_K(X - S) \rightarrow \{k\text{-vector spaces}\},$$

and we get an exact functor $\overline{\text{Res}}: \mathcal{E}(F, G; T) \rightarrow \tilde{\mathcal{E}}(\overline{F}, \overline{G}; \overline{T})$. Because our objects are K -equivariant we know that $\pi_1(K)$ acts trivially and so we must have that $N \cap \tau^{-1}(\pi_1(K))$ acts trivially. This gives us a full subcategory $\tilde{\mathcal{E}}(\overline{F}, \overline{G}; \overline{T})$ given by the conditions $m \circ I_\alpha(A) \circ n = 0$ for $\alpha \in N \cap \tau^{-1}(\pi_1(K))$ on objects $FA \xrightarrow{m} B \xrightarrow{n} GA$ of $\tilde{\mathcal{E}}(\overline{F}, \overline{G}; \overline{T})$. By Proposition 2.3 the category $\tilde{\mathcal{E}}(\overline{F}, \overline{G}; \overline{T})$ is of Artin type.

We now have exact functor $\text{Res}: \tilde{\mathcal{E}}(F, G; T) \rightarrow \tilde{\mathcal{E}}(\overline{F}, \overline{G}; \overline{T})$.

Lemma 4.4. *The functor Res has a left adjoint Ind.*

This lemma immediately implies Theorem 4.3 as follows. Let L be irreducible in $\tilde{\mathcal{E}}(F, G; T)$ and let \overline{P} be a projective cover of $\text{Res}(L)$. Then $\text{Ind } \overline{P}$ is a projective cover of L .

Proof of Lemma 4.4. Given an object

$$\begin{array}{ccc}
 FA & \xrightarrow{TA} & GA \\
 X: & m \searrow & \nearrow n \\
 & & B
 \end{array}$$

in $\tilde{\mathcal{C}}(\bar{F}, \bar{G}; \bar{T})$ all the spaces have an action by $k[N]$ which by definition descends to an action by $k[N']$. Furthermore FA, GA have an action by $k[\Gamma]$ descending to an action by $k[\Gamma']$.

Form the diagram

$$\begin{array}{ccc}
 FA & \xrightarrow{TA} & GA \\
 m'=1 \otimes m \searrow & & \nearrow n' \\
 & & k[\Gamma'] \otimes_{k[N']} B
 \end{array}$$

where n' is defined by $n'(\gamma \otimes b) = \gamma n'(b)$. All morphisms here are $\mathbb{C}[\Gamma']$ -linear except m' . So we form a subspace $R \subset k[\Gamma'] \otimes_{k[N']} B$ generated by $\gamma \otimes m(a) - 1 \otimes m(\gamma a)$, $\gamma \in \Gamma, a \in FA$. Clearly $n'|R = 0$ and so we can form

$$\text{Ind}(X): FA \rightarrow (k[\Gamma'] \otimes_{k[N']} B)/R \rightarrow Ga.$$

By definition $\text{Ind}(X)$ belongs in $\tilde{\mathcal{C}}(F, G; T) \cong P_K(X)$ and it is easy to check that $\text{Hom}(\text{Ind } X, Y) \cong \text{Hom}(X, \text{Res } Y)$.

Remarks. 1. In the same way as in Lemma 4.4 one can prove that Res has a right adjoint as well.

2. The results in this section can be extended to disconnected K also.

3. These categories $P_K(X)$ are usually not highest weight categories. It is not clear whether these categories satisfy some nice properties in general.

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