TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 341, Number 2, February 1994

A SINGULAR REPRESENTATION OF $\mathbf{E}_6$

B. BINEGAR AND R. ZIERAU

ABSTRACT. Algebraic properties of a singular representation of $\mathbf{E}_6$ are studied. This representation has the Joseph ideal as its annihilator and it remains irreducible when restricted to $\mathbf{F}_4$.

It is of interest in the representation theory of semisimple Lie groups to understand singular representations. For example, a detailed understanding of the metaplectic representation has been quite fruitful. There is also a particularly interesting representation of $SO(4, 4)$ studied by Kostant [11]. These are examples of "unipotent representations" which should in some sense form the building blocks for the unitary dual of a semisimple Lie group. In this paper we study a singular representation of $\mathbf{E}_6$ and show that it has many of the same algebraic properties as the metaplectic representation.

We study an irreducible unitary representation, call it V, of the real form of $\mathbf{E}_6$ with Hermitian symmetric space. We see that V is a ladder representation, has minimal Gelfand-Kirillov dimension and its annihilator is the Joseph ideal. Thus V is in some sense associated to a minimal nilpotent orbit in the dual of the real Lie algebra. We also restrict V to the rank one $\mathbf{F}_4$ and see that it remains irreducible. This reflects the fact that each minimal nilpotent real orbit in $\mathbf{E}_6$ contains a dense $\mathbf{F}_4$ orbit.

We thank J. T. Chang and David Vogan for useful conversations, especially in connection with Appendix A.

1

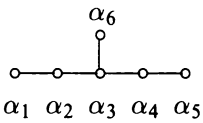
We will first set down some notation and specify the representation we will be studying.

Let G be the real form of $\mathbf{E}_6$ with an associated Hermitian symmetric space (denoted by $E\ III$ in Helgason [7, p. 534]). Assume G is contained in the simply connected complex group $\mathbf{E}_6$. The maximal compact subgroup of G is $K = SO(2) \times Spin(10)$. We fix a compact Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ be the corresponding set of roots. We choose a positive system of roots as follows: since G/K is Hermitian symmetric there is an element ζ in the center of K so that $J = \mathrm{Ad}(\zeta)$ gives the complex structure at eK. Let $\mathfrak{p}_{\pm}$ be the $\pm i$ eigenspace of J. Let Δ^+ be any positive system so

Received by the editors July 15, 1991 and, in revised form, December 11, 1991.
1991 *Mathematics Subject Classification*. Primary 22E46, 22E30.
Key words and phrases. Reductive Lie groups, unitary representations, singular representations, metaplectic representation, exceptional groups.
The second author was supported in part by NSF Grant No. DMS-8902352.

that $\Delta(\mathfrak{p}_+) \subset \Delta^+$ (where $\Delta(\mathfrak{p}_+)$ is the set of $\mathfrak{t}$ roots in $\mathfrak{p}_+$). Such positive systems are exactly those for which (nontrivial) irreducible unitary highest weight representations exist.

We label the simple roots of the Dynkin diagram of $\mathbf{E}_6$ as follows:



with α_1 the unique noncompact simple root. The simple roots may be written in coordinates as

$$\begin{aligned}
\alpha_1 &= \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right), \\
\alpha_2 &= (0, 0, 0, 0, 1, -1), \\
\alpha_3 &= (0, 0, 0, 1, -1, 0), \\
\alpha_4 &= (0, 0, 1, -1, 0, 0), \\
\alpha_5 &= (0, 1, -1, 0, 0, 0), \\
\alpha_6 &= (0, 0, 0, 0, 1, 1),
\end{aligned}$$

with the usual Euclidean inner product. If $\{e_1, \ldots, e_6\}$ is the standard basis of $\mathbb{R}^6$, then the compact and noncompact positive roots are:

$\Delta^+(\mathfrak{k}) = \{e_i \pm e_j \mid 2 \le i < j \le 6\}$,

$\Delta^+(\mathfrak{p}) = \Delta(\mathfrak{p}_+) = \left\{\left(\frac{\sqrt{3}}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \mid \text{an even number of } -\frac{1}{2}\text{'s occur}\right\}$.

If we let $\lambda_1, \ldots, \lambda_6$ denote the corresponding fundamental weights

$$(2\langle\lambda_i, \alpha_j\rangle/\|\alpha_j\|^2 \equiv \delta_{ij}),$$

then $\lambda_1 = \left(\frac{2}{\sqrt{3}}, 0, 0, 0, 0, 0\right)$. Also,

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha \;=\; (4\sqrt{3}, 4, 3, 2, 1, 0).$$

It is clear from the form of the noncompact roots that the representations of $Spin(10)$ on $\mathfrak{p}_\pm$ are the representations with highest weights $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}\right)$; these are the two "spinor representations" of $Spin(10)$.

We will study the irreducible representation of G with highest weight $-3\lambda_1$ and we will call its $(\mathfrak{g}, K)$ module V. If $\lambda \in \mathfrak{t}^*$ is Δ_c^+-dominant, we let

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} F_\lambda$$

where F_λ is the irreducible representation of $\mathfrak{k}$ with highest weight λ, and let $L(\lambda)$ denote the unique irreducible quotient. Both $M(\lambda)$ and $L(\lambda)$ have highest weight λ and $V = L(-3\lambda_1)$. The infinitesimal character of V is $-3\lambda+\rho$ which is conjugate under the Weyl group to $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6$.

Our first observation is that V occurs in the parameterization of unitary highest weight modules given by Enright, Howe, and Wallach (see [4]) as the first reduction point in the series with 1-dimensional highest K-type. Specifically, the family has highest weights $(t-11)\lambda_1$ and the modules $L((t-11)\lambda_1)$ are unitary for $t \le 8$ and $t = 11$. The $L((t-11)\lambda_1)$ are holomorphic discrete representations for $t < 0$. V occurs when $t = 8$ (the first reduction point) and the trivial representation corresponds to $t = 11$.

Theorem 1.1. *V is a ladder representation with K-types $\{-3\lambda_1 - n\alpha_1 \mid n = 0, 1, 2, \ldots\}$.*

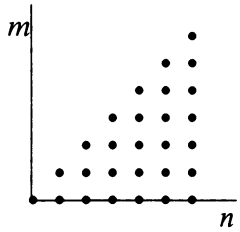
Proof. We first write down the decomposition of $U(\mathfrak{p}_-) \approx S(\mathfrak{p}_-)$ under K by applying Schmid's Theorem [12]. Let $\mu_1, \mu_2 \in \Delta(\mathfrak{p}_-)$ be defined by setting $\mu_1 = -\alpha_1$ and taking $-\mu_2$ to be the lowest root in $\Delta(\mathfrak{p}_+)$ that is strongly orthogonal to α_1. Then

$$S(\mathfrak{p}_-) = \bigoplus_{n \geq m \geq 0} F_{n\mu_1 + m\mu_2}$$

where F_μ is the finite dimensional representation of K with highest weight μ.

For $\alpha \in \Delta$, let X_α denote a nontrivial vector in the root space $\mathfrak{g}_\alpha$. By a result of Vogan (Lemma 3.4 in [15]) we know that either X_{α_1} or $X_{-\alpha_1}$ acts injectively on any infinite dimensional irreducible $(\mathfrak{g}, K)$-module. Since V has a highest weight it must be $X_{-\alpha_1}$ that acts injectively. If μ is the highest weight of a K-type in V and v_μ is the corresponding weight vector, then $(X_{-\alpha_1})^n v_\mu$ is the highest weight vector of another K-type $(\approx F_{\mu - n\alpha_1})$; for if $\alpha \in \Delta_c^+$, then $[X_\alpha, X_{-\alpha_1}] = 0$ because $\alpha - \alpha_1 \notin \Delta$, so $X_\alpha (X_{-\alpha_1})^n v_\mu = (X_{-\alpha_1})^n X_\alpha v_\mu = 0$. We conclude that the $F_{-3\lambda_1 - n\alpha_1}$, $n = 0, 1, 2, \ldots$, are K-types occurring in V.

We must check that these are all the K-types of V. By Proposition 3.9 of [4], if F_μ is a K-type in $L(\lambda)$, $\mu \neq \lambda$, and $L(\lambda)$ is unitary, then $\|\mu + \rho\| > \|\lambda + \rho\|$. An easy calculation shows that this is not the case for $\mu = -3\lambda_1 + \mu_1 + \mu_2$; so $F_{3\lambda_1 + \mu_1 + \mu_2}$ cannot occur in V. Applying $(X_{-\alpha_1})^n$ as above we conclude that the K-types $F_{-3\lambda + (n+1)\mu_1 + \mu_2}$ do not occur in V (since they occur in the maximal submodule of $M(-3\lambda_1)$ and the multiplicity is one for all the K-types of $M(-3\lambda_1)$). The picture for the K-types in $M(-3\lambda_1)$ is



To see that the first row gives exactly the K-types in V it is enough to check that $\mathfrak{p}_\pm$ sends the first row only to the first or second row. This is an easy calculation which we omit. □

Remark 1. There are several other proofs of Theorem 1. (a) By the above, $F_{-3\lambda_1 + \mu_1 + \mu_2}$ does not occur. One shows that the irreducible representation with highest weight $-3\lambda_1 + \mu_1 + \mu_2$ has exactly the remaining K-types $\{F_{-3\lambda_1 + n\mu_1 + m\mu_2} \mid n \geq m \geq 1\}$. One can see that this highest weight representation is an $A_{\mathfrak{q}}(\lambda)$ where $\mathfrak{q}$ is the θ-stable parabolic defined by $-\lambda_5$. For $\lambda = 3\lambda_5$, $A_{\mathfrak{q}}(\lambda)$ is irreducible and has highest weight $-3\lambda_1 + \mu_1 + \mu_2$ (see [21, p. 49]). One can use the Blattner formula to calculate the K-types of V. (b) From results of Enright and Joseph (see [5, Theorem 5.2]) one can show that the unique maximal subrepresentation of $M(-3\lambda_1)$ is $U(\mathfrak{p}_-) F_{-3\lambda_1 + \mu_1 + \mu_2}$ and then calculate the K-types.

Remark 2. It follows from (a) above that the maximal submodule is $L(-3\lambda_1 + \mu_1 + \mu_2)$. This occurs in the series $M((t-12)\lambda_1 + \lambda_5)$ for $t = 7$, the isolated unitary point ($t \leq 4$ and $t = 7$ are the unitaries in this series).

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. For $\lambda \in \mathfrak{t}^*$ and $\lambda \perp \Delta(\mathfrak{l})$, let $\mathscr{R}_{\mathfrak{q}}^i(\mathbb{C}_\lambda)$ be the derived functor modules as defined in [16]. Let $s = \dim(\mathfrak{u} \cap \mathfrak{k})$. If the positivity condition $\langle \lambda + \rho(\mathfrak{u}), \alpha \rangle > 0$, $\alpha \in \Delta(\mathfrak{u})$, is satisfied, then $\mathscr{R}_{\mathfrak{q}}^i(\mathbb{C}_\lambda) = 0$ for $i \neq s$ and $\mathscr{R}_{\mathfrak{q}}^s(\mathbb{C}_\lambda)$ is an irreducible unitarizable representation with infinitesimal character $\lambda + \rho(\mathfrak{u})$. When the parameter λ becomes arbitrary the vanishing, irreducibility, and unitarity do not in general hold. We set $A_{\mathfrak{q}}(\lambda) = \mathscr{R}_{\mathfrak{q}}^s(\mathbb{C}_\lambda)$ for λ arbitrary (subject of course to $\lambda \perp \Delta(\mathfrak{l})$, so that a 1-dimensional representation $\mathbb{C}_\lambda$ of L with weight λ exists).

Theorem 1.2. *V is not isomorphic to $A_{\mathfrak{q}}(\lambda)$, λ arbitrary.*

Remark. We show a little more than this: V is not a quotient of any $A_{\mathfrak{q}}(\lambda)$. In particular, if λ is in the "unitary range" (i.e., $A_{\mathfrak{q}}(\lambda)$ is unitary) then V does not occur as a constituent in $A_{\mathfrak{q}}(\lambda)$. However, we do not exclude the possibility of V being some constituent (other than a quotient) of some $A_{\mathfrak{q}}(\lambda)$ with λ so bad that $A_{\mathfrak{q}}(\lambda)$ is not irreducible or unitary.

Proof. We can calculate the Gelfand-Kirillov dimension of V using Theorem 1.2 of [14] because we know the K-types. We omit the calculation but note that $\dim(F_{-3\lambda_1 - n\alpha_1})$ is a degree 10 polynomial in n. Thus, the Gelfand-Kirillov dimension of V is 11.

Since 11 is also the minimal Gelfand-Kirillov dimension for infinite dimensional representations of G, and because the two minimal $K_{\mathbb{C}}$ orbits in $\mathfrak{p}$ are $K_{\mathbb{C}} \cdot X_\beta \subset \mathfrak{p}_+$ and $K_{\mathbb{C}} \cdot X_{-\beta} \subset \mathfrak{p}_-$ (where β is the highest root), we may conclude that the characteristic variety $\mathscr{V}(V)$ (see Appendix A) is contained in $\mathfrak{p}_+$ or $\mathfrak{p}_-$. When $\langle \lambda + \rho(\mathfrak{u}), \alpha \rangle > 0$, $\alpha \in \Delta(\mathfrak{u})$, the characteristic variety of $A_{\mathfrak{q}}(\lambda)$ is $K_{\mathbb{C}} \cdot (\mathfrak{u} \cap \mathfrak{p})$ (see the theorem in Appendix A). The proposition in Appendix A shows that for arbitrary λ, as long as $A_{\mathfrak{q}}(\lambda) \neq 0$, every quotient of $A_{\mathfrak{q}}(\lambda)$ has the same characteristic variety (and so the same Gelfand-Kirillov dimension) as an $A_{\mathfrak{q}}(\lambda)$ with $\langle \lambda + \rho(\mathfrak{u}), \alpha \rangle > 0$, $\alpha \in \Delta(\mathfrak{u})$. Thus, we need only prove that no $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ with $\mathfrak{u} \cap \mathfrak{p} \subset \mathfrak{p}_+$ (or $\mathfrak{u} \cap \mathfrak{p} \subset \mathfrak{p}_-$) has $\dim K_{\mathbb{C}} \cdot (\mathfrak{u} \cap \mathfrak{p}) = 11$.

Consider the fundamental weight λ_6, this defines a θ-stable parabolic subalgebra (by $\Delta(\mathfrak{l} + \mathfrak{u}) = \{\alpha \in \Delta \mid \langle \alpha, \lambda_6 \rangle \geq 0\}$). We consider the roots $\beta_1 = (\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $\beta_2 = (\frac{\sqrt{3}}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the corresponding root vectors X_1, X_2. Both are in $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{u} \cap \mathfrak{p}_+$. Let Y_1, Y_2 be root vectors for, respectively, $-\beta_1$, $-\beta_2$. There are two corresponding 3-dimensional subalgebras $\{X_1, Y_1, H_1\}$, $\{X_2, Y_2, H_2\}$. Since β_1 and β_2 are strongly orthogonal these two subalgebras commute and so $\{X = X_1 + X_2, Y = Y_1 + Y_2, H = H_1 + H_2\}$ is a 3-dimensional subalgebra. We claim $K_{\mathbb{C}} \cdot X \subset \mathfrak{u} \cap \mathfrak{p}$ has dimension 16. This is seen as follows. Since H corresponds to $\lambda_1 + \lambda_5$, the decomposition of $\mathfrak{g}$ under $\{X, Y, H\}$ shows that the stabilizer of $X \in \mathfrak{k}$ has dimension 30. So

$$\dim_{\mathbb{C}}(K_{\mathbb{C}} \cdot X) = \dim_{\mathbb{C}} K_{\mathbb{C}} - \dim_{\mathbb{C}}(\mathrm{stab}_{\mathfrak{k}}(X)) = 46 - 30 = 16 \quad .$$

For other parabolics $\mathfrak{l} + \mathfrak{u}$ the argument is the same because X always lies in $\mathfrak{u} \cap \mathfrak{p}$ (as is easily seen by writing β_1, β_2 as sums of simple roots). □

2

We will show that the annihilator of V is the Joseph ideal. This ideal is a maximal primitive ideal in $U(\mathfrak{g})$ associated with the minimal nilpotent complex

coadjoint orbit and was defined by Joseph in [9]. The characterization we use was formulated by Garfinkle [6] and is as follows.

Let $\mathfrak{g}$ be a simple, complex Lie algebra not of type A_n. Let β be the highest root with respect to any positive system of roots. The space of degree 2 elements in the symmetric algebra of $\mathfrak{g}$ decomposes as $S^2(\mathfrak{g}) \approx E_{2\beta} + S$, where $E_{2\beta}$ is the irreducible representation of $\mathfrak{g}$ with highest weight 2β and S is the $\mathfrak{g}$-invariant complement of $E_{2\beta}$ in $S^2(\mathfrak{g})$. If $J \subset U(\mathfrak{g})$ is an ideal, consider the graded ideal $\mathrm{gr}(J) \subset \mathrm{gr}(U(\mathfrak{g})) \approx S(\mathfrak{g})$.

Theorem 2.1 (Garfinkle). *If J is an ideal of infinite codimension in $U(\mathfrak{g})$, then J is the Joseph ideal if and only if* $\mathrm{gr}(J) \cap S^2(\mathfrak{g}) = S$.

Let us return to the case where $\mathfrak{g} = \mathbf{E}_6$.

Lemma 2.2. $S^2(\mathfrak{g}) \approx E_{2\beta} \oplus E_{(\sqrt{3},1,0,0,0,0)} \oplus \mathbb{C}$.

Proof. $\beta = (\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the highest root. The possible highest weights of subrepresentations occurring in $\mathfrak{g} \otimes \mathfrak{g}$ are of the form $\beta + \alpha$ where $\alpha \in \Delta \cup \{0\}$ and the multiplicity of such a subrepresentation is one (or six in case $\alpha = 0$). The possible subrepresentations along with their dimensions are:

$$\begin{aligned} E_{2\beta} &= E_{(\sqrt{3},1,1,1,1,1)}; && 2430, \\ & E_{(\sqrt{3},1,1,1,0,0)}; && 2925, \\ & E_{(\sqrt{3},1,0,0,0,0)}; && 650, \\ E_{\beta} &= E_{(\sqrt{3}/2,1/2,1/2,1/2,1/2,1/2)}; && 78, \\ & E_{(0,0,0,0,0,0)}; && 1. \end{aligned}$$

Also $\dim S^2(\mathfrak{g}) = \frac{78 \cdot 79}{2} = 3081$. The only way to get the dimensions to add up correctly is to take $3081 = 2430 + 650 + 1$. □

Remark. Although it is not necessary, one can check using Helgason's theorem (see [8, p. 535]) that $E_{(\sqrt{3},1,0,0,0,0)} = E_{\lambda_1+\lambda_5}$ is spherical.

Let Ω_{E_6}, Ω_K be the Casimirs of, respectively, E_6 and $K = SO(2) \times Spin(10)$, and let H_1 correspond to λ_1 by the Killing form of $\mathbf{E}_6$.

Lemma 2.3. $\Omega_K - 2(H_1)^2 - \frac{5}{9}\Omega_{E_6}$ *annihilates* V.

Proof. $(H_1)^2$ acts on $F_{-3\lambda_1 - n\alpha_1}$ by

$$((-3\lambda_1 - n\alpha_1)(H_1))^2 = (-3\langle \lambda_1, \lambda_1\rangle - n\langle \alpha_1, \lambda_1\rangle) = (n+4)^2,$$

Ω_K acts on $F_{-3\lambda_1 - n\alpha_1}$ by

$$\| -3\lambda_1 - n\alpha_1 + \rho_c\|^2 - \|\rho_c\|^2 = 2n^2 + 16 + 12,$$

Ω_{E_6} acts on $F_{-3\lambda_1 - n\alpha_1}$ by

$$\| -3\lambda_1 + \rho\|^2 - \|\rho\|^2 = -36.$$

Thus,

$$(\Omega_K - 2(H_1)^2 - \tfrac{5}{9}\Omega_{E_6})F_{-3\lambda_1 - n\alpha_1} = 0.$$

Theorem 2.4. *The annihilator, J, of V in $U(\mathfrak{g})$ is the Joseph ideal.*

Proof. $(X_{-\beta})^2 \in E_{-2\beta}$ and $X_{-\beta}$ acts injectively on V so $\mathrm{gr}(J) \cap S^2(\mathfrak{g}) \subset S$. However, there are two linearly independent K-fixed vectors in S which kill

$V : (\Omega_{E_6} + 36)$, and $(\Omega_K - 2(H_1)^2 - \frac{5}{9}\Omega_{E_6})$. Since the multiplicity of a K-fixed vector in an irreducible finite dimensional representation of G is at most one, we see that each summand in S meets the annihilator. Since the annihilator is $\mathfrak{g}$-invariant, $S = \mathrm{gr}(J) \cap S^2(\mathfrak{g})$. □

Remark. Kostant used this argument when studying a special representation of $SO(4, 4)$ (see [11, §3]).

3

The real form of $\mathbf{E}_6$ under consideration contains the rank one real form of $\mathbf{F}_4$ as the fixed points of an outer automorphism. We will restrict V to $\mathbf{F}_4$ and see that it remains irreducible. We also identify $V|_{\mathbf{F}_4}$.

First we describe the automorphism which has $\mathbf{F}_4$ as the fixed points. Recall that there is a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$ and that $\mathfrak{p}_\pm$ are the 16-dimensional spin representations. Let σ_o be the automorphism of $K \approx SO(2) \times Spin(10)$ defined by multiplying by $-I_2$ on the $\mathfrak{so}(2)$ and conjugating by

$$\begin{pmatrix} I_9 & 0 \\ 0 & -1 \end{pmatrix}$$

on $\mathfrak{so}(10)$.

Lemma 3.1. *If (τ, F) is a finite dimensional irreducible representation of K with highest weight $(\lambda_1, \dots, \lambda_6)$ then the representation $(\tau^{\sigma_o}, F^{\sigma_o})$ defined by $\tau^{\sigma_o}(k) = \tau(\sigma_o(k))$ on $F^{\sigma_o} = F$ is an irreducible representation with highest weight*

$$\sigma_o(\lambda) = (-\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, -\lambda_6).$$

In particular, $(\mathfrak{p}_+)^{\sigma_0} \approx \mathfrak{p}_-$ and $(\mathfrak{p}_-)^{\sigma_o} \approx \mathfrak{p}_+$.

Proof. This is easy and is omitted. □

We conclude from this lemma that there is an intertwining map $f\colon \mathfrak{p}_+ \to (\mathfrak{p}_-)^{\sigma_o}$; that is, a linear map $f : \mathfrak{p}_+ \to \mathfrak{p}_-$ such that

$$f(\mathrm{Ad}(k) \cdot Y) = \mathrm{Ad}(\sigma_o(k)) \cdot f(Y), \quad \text{for all } k \in K \text{ and all } Y \in \mathfrak{p}_+. \tag{3.1}$$

We now extend $\sigma_o\colon \mathfrak{k} \to \mathfrak{k}$ to a map $\sigma : \mathfrak{g} \to \mathfrak{g}$ by $\sigma(X + Y + Z) = \sigma_o(X) + f(Y) + f^{-1}(Z)$ for all $X \in \mathfrak{k}$, all $Y \in \mathfrak{p}_+$ and all $Z \in \mathfrak{p}_-$.

Proposition 3.2. *When f is scaled properly, σ is an involution of $\mathfrak{g}$ which commutes with the conjugation of $\mathfrak{g}$ over $\mathfrak{g}_o$ and commutes with Cartan involution θ. The fixed point set of σ in $\mathfrak{g}$ is the rank 1 real form of $\mathbf{F}_4$.*

Proof. We must show that σ is a Lie algebra homomorphism and σ commutes with the complex conjugation. Let $\langle\ ,\ \rangle$ be the Killing form (or any other nondegenerate K-invariant pairing between $\mathfrak{p}_+$ and $\mathfrak{p}_-$).

We first prove that for $Y \in \mathfrak{p}_+$, $Z \in \mathfrak{p}_-$,

$$\langle f(Y), f^{-1}(Z)\rangle = \langle Y, Z\rangle. \tag{3.2}$$

Since $Y, Z \to \langle f(Y), f^{-1}(Z)\rangle$ is a nondegenerate K-invariant pairing between the irreducible K-modules $\mathfrak{p}_+$ with $\mathfrak{p}_-$, we know that $\langle f(Y), f^{-1}(Z)\rangle = c\langle Y, Z\rangle$ for some nonzero constant c. It is enough to show that $\langle f(Y), f^{-1}(Z)\rangle = \langle Y, Z\rangle \neq 0$ for some choice of $Y \in \mathfrak{p}_+$ and $Z \in \mathfrak{p}_-$. Let $\beta = (\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ denote the highest root and set $\gamma = -\sigma_o(\beta) =$

$(\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, and let X_β, X_γ be corresponding root vectors. Consider also the compact roots $\xi_1 = (0, 1, 1, 0, 0, 0)$ and $\xi_2 = (0, 0, 0, 1, 1, 0)$. We see that $\beta = \gamma + \xi_1 + \xi_2$, so for appropriately chosen root vectors, X_{ξ_1}, X_{ξ_2},

(i) $\operatorname{ad}(X_{\xi_1})\operatorname{ad}(X_{\xi_2})X_\gamma = X_\beta$,

(ii) $\sigma_o(X_{\xi_1}) = X_{\xi_1}$ and $\sigma_o(X_{\xi_2}) = X_{\xi_2}$,

(iii) $[X_{\xi_1}, X_{\xi_2}] = 0$.

Note that $f(X_\beta)$ is a root vector for $\sigma_o(\beta) = -\gamma$. Thus,

$$\begin{aligned}
0 \neq \langle f(X_\beta), X_\gamma\rangle &= \langle f\left(\operatorname{ad}(X_{\xi_1})\operatorname{ad}(X_{\xi_2})X_\gamma\right), X_\gamma\rangle \\
&= \langle \operatorname{ad}\left(\sigma_o(X_{\xi_1})\right)\operatorname{ad}\left(\sigma_o(X_{\xi_2})f(X_\gamma), X_\gamma\right)\rangle \quad \text{(by (3.1))} \\
&= \langle \operatorname{ad}(X_{\xi_1})\operatorname{ad}(X_{\xi_2})f(X_\gamma), X_\gamma\rangle \quad \text{(by (ii))} \\
&= \langle f(X_\gamma), \operatorname{ad}(X_{\xi_2})\operatorname{ad}(X_{\xi_1})X_\gamma\rangle \quad \text{(by } K\text{-inv.)} \\
&= \langle f(X_\gamma), \operatorname{ad}(X_{\xi_1})\operatorname{ad}(X_{\xi_2})X_\gamma\rangle \quad \text{(by (iii))} \\
&= \langle f(X_\gamma), X_\beta\rangle \quad \text{(by (i))} \\
&= \langle X_\beta, f(X_\gamma)\rangle
\end{aligned}$$

Thus, (3.2) is proved.

We check that σ is a Lie algebra homomorphism. Let $X, X' \in \mathfrak{k}$, $Y \in \mathfrak{p}_+$, and $Z \in \mathfrak{p}_-$. Now,

$$\begin{aligned}
\sigma\left([X, X']\right) &= \sigma_o\left([X, X']\right) = [\sigma_o(X), \sigma_o(X')], \\
\sigma([X, Y]) &= f([X, Y]) = [\sigma_o(X), f(Y)] = [\sigma(X), \sigma(Y)], \\
\sigma([X, Z]) &= f^{-1}([X, Z]) = [\sigma_o(X), f^{-1}(Z)] = [\sigma(X), \sigma(Z)].
\end{aligned}$$

Finally, we need to confirm

$$\sigma([Y, Z]) = [\sigma(Y), \sigma(Z)]. \tag{3.3}$$

For any $X \in \mathfrak{k}$,

$$\begin{aligned}
\langle X, \sigma([Y, Z])\rangle &= \langle X, \sigma_o([Y, Z])\rangle = \langle \sigma_o(X), [Y, Z]\rangle = \langle [\sigma_o(X), Y], Z\rangle \\
&= \langle f([\sigma_o(X), Y]), f^{-1}(Z)\rangle \quad \text{(by (3.2))} \\
&= \langle [X, f(Y)], f^{-1}(Z)\rangle = \langle X, [f(Y), f^{-1}(Z)]\rangle.
\end{aligned}$$

Since the Killing form is nondegenerate on $\mathfrak{k}$, (3.3) follows.

Let $^-$ denote conjugation of $\mathfrak{g}$ over $\mathfrak{g}_o$. We now show that $\sigma(\bar{U}) = \overline{\sigma(U)}$, for all $U \in \mathfrak{g}$. Since $\langle Y', \bar{Y}\rangle$ and $\langle Y', f(\overline{f(Y)})\rangle$ are both $\mathfrak{k}$-invariant nondegenerate Hermitian forms on $\mathfrak{p}_+$,

$$\langle Y', \bar{Y}\rangle = r\langle Y', f(\overline{f(Y)})\rangle$$

for some $r \in \mathbb{R}^+$. Replacing f by $\frac{1}{\sqrt{r}}f$, we get $\langle Y', \bar{Y}\rangle = \langle Y', f(\overline{f(Y)})\rangle$ for all $Y' \in \mathfrak{p}_+$. Since the pairing is nondegenerate, it follows that $\bar{Y} = f(\overline{f(Y)})$ or $\overline{f(Y)} = f^{-1}(\bar{Y})$. Thus, if f is properly scaled $\overline{f(Y)} = f^{-1}(\bar{Y})$ for all $Y \in \mathfrak{p}_+$. For this f, one also has $\overline{f^{-1}(Z)} = f(\bar{Z})$ for all $Z \in \mathfrak{p}_-$.

Let $X \in \mathfrak{k}$, $Y \in \mathfrak{p}_+$, and $Z \in \mathfrak{p}_-$. Then

$$\begin{aligned}
\sigma(\overline{X + Y + Z}) &= \sigma_o(\bar{X}) + f^{-1}(\bar{Y}) + f(\bar{Z}) \\
&= \overline{\sigma_o(X)} + \overline{f(Y)} + \overline{f^{-1}(Z)} \\
&= \overline{\sigma(X + Y + Z)},
\end{aligned}$$

so (3.3) is proved.

Set

$$\mathfrak{g}_1 = \{X \in \mathfrak{g} \mid \sigma(X) = X\}, \quad \mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{g}_1, \quad \mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1.$$

So $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$, $\mathfrak{k}_1 = \mathfrak{so}(9)$, and $\mathfrak{p}_1 = \{Y + f(Y) : Y \in \mathfrak{p}_+\} = \{f^{-1}(Z) + Z : Z \in \mathfrak{p}_-\}$. The roots are as follows:

$$\Delta(\mathfrak{k}_1) = \{\pm e_i \pm e_j \ , \ \pm e_k \mid 1 \le i < j \le 4 \ , \ 1 \le k \le 4\},$$
$$\Delta(\mathfrak{p}_1) = \{(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2})\}.$$

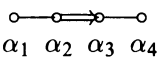
$\mathfrak{g}_1$ is thus of type $\mathbf{F}_4$.

As σ commutes with conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_o$, $\mathfrak{g}_1 \cap \mathfrak{g}_o$ is a real form of $\mathbf{F}_4$ and $\theta|_{\mathbf{F}_4}$ is a Cartan involution. It follows, for example, from the tables in [7] that $\mathfrak{g}_1 \cap \mathfrak{g}_o$ is the rank one real form of $\mathbf{F}_4$. □

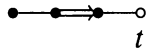
We fix a positive root system as follows:

$$\begin{aligned}\Delta^+(\mathfrak{g}_1) &= \Delta^+(\mathfrak{k}_1) \cup \Delta^+(\mathfrak{p}_1)\\ &= \{e_i \pm e_j \ , \ e_k \mid 1 \le i < j \le 4 \ , \ 1 \le k \le 4\} \cup \{(\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2})\}.\end{aligned}$$

The simple roots are $\gamma_1 = e_2 - e_3$, $\gamma_2 = e_3 - e_4$, $\gamma_3 = e_4$, $\gamma_4 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and the corresponding fundamental weights are $\varpi_1 = (1, 1, 0, 0)$, $\varpi_2 = (2, 1, 1, 0)$, $\varpi_3 = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $\varpi_4 = (1, 0, 0, 0)$. The Dynkin diagram is



and $\rho(\mathfrak{g}_1) = \rho(\mathfrak{k}_1) + \rho(\mathfrak{p}_1) = (\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$. The corresponding Satake diagram is



Here e_1 restricts to $2t$ and e_2, e_3, e_4 restrict to 0. Let $\Sigma^+(\mathfrak{a}, \mathfrak{g}) = \{t, 2t\}$ where $\mathfrak{a}$ is an appropriate maximal abelian subspace of $\mathfrak{p}$, $\rho(\mathfrak{a}, \mathfrak{g}) = \frac{23}{2}t$.

Theorem 3.3. *$V|_{\mathbf{F}_4}$ is irreducible and has infinitesimal character $(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$.*

Proof. Every K-type remains irreducible when restricted to $K_1 = Spin(9)$. This follows either from the branching law (see [Ze, §129]) or from a dimension count using the Weyl dimension formula. The K_1-types have highest weights $(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2})$ with $n \in \mathbb{Z}^+$.

The highest noncompact root in $\mathfrak{g}_1$ is $\gamma_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. By [15, Lemma 3.4], X_{γ_1} acts injectively in any irreducible representation of $\mathbf{F}_4$. Since V is unitary, and because the highest weights of the K_1-types of V lie along a single ray, we must have $V|_{\mathbf{F}_4} \approx E \oplus V_1$ where E is finite dimensional and V_1 is irreducible. By unitarity, we may also conclude that E must correspond to the trivial representation. If $E \neq \{0\}$, we may choose a nonzero element v_o of E. Then v_o is a highest weight vector of V (the K-type is necessarily the one corresponding to $n = 0$). So $\mathfrak{p}_+ \cdot v_o = 0$. Since v_o is fixed by $\mathfrak{p}_1 = \{f^{-1}(Z) + Z \mid Z \in \mathfrak{p}_-\}$,

$$0 = (f^{-1}(Z) + Z) \cdot v_o = Z \cdot v_o$$

for all $Z \in \mathfrak{p}_-$; which contradicts the irreducibility of V with respect to $\mathfrak{g}$. Thus, $V|_{F_4}$ is irreducible.

To calculate the infinitesimal character of $V|_{\mathbf{F}_4}$ we use the following trick. The diagram automorphism ϕ with respect to the positive system of $\mathbf{E}_6$ given in §1 is not the automorphism of Proposition 3.5. However, the fixed points of the two automorphisms are conjugate. Thus, it is enough to calculate the infinitesimal character of V restricted to the fixed points of ϕ. This representation is an irreducible highest weight representation of $\mathbf{F}_4$ (not a $(\mathfrak{g}_1, K_1)$-module, but that is okay). We calculate the highest weight.

Let ϕ, interchangeably, denote the automorphism of $\mathfrak{g}$ and the roots Δ. Let $\mathfrak{g}^\phi$, $\mathfrak{t}^\phi$ denote the elements of $\mathfrak{g}$, respectively, $\mathfrak{t}$ fixed by ϕ. The roots of $\mathfrak{g}^\phi$ with respect to $\mathfrak{t}^\phi$ are as follows:

(i) Short roots: $\left\{\alpha \in \Delta \mid \frac{\alpha+\phi(\alpha)}{2}\Big|_{\mathfrak{t}^\phi} = \alpha|_{\mathfrak{t}^\phi}\ ,\ \phi(\alpha) \neq \alpha\right\}$. The corresponding root vectors are $X_\alpha + \phi(X_\alpha)$. Note that we have listed each root vector twice, once for α and once for $\phi(\alpha)$.

(ii) Long roots: $\{\alpha \in \Delta \mid \phi(\alpha) = \alpha\}$. The corresponding root vectors are X_α.

Remark. $\frac{\alpha+\phi(\alpha)}{2}$ corresponds to an element in $\mathfrak{t}^\phi$ (via the Killing form), also $\alpha \perp \phi(\alpha)$ when $\phi(\alpha) \neq \alpha$. Thus the length of $\frac{\alpha+\phi(\alpha)}{2}$ in $\mathbf{F}_4$ is its length in $\mathbf{E}_6$ which is $\|\alpha\|/\sqrt{2}$.

The positive system for which $V|_{\mathfrak{g}^\phi}$ is a highest weight representation consists of those roots in (i) and (ii) with $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t})$. The simple roots, in terms of the simple roots of $\mathbf{E}_6$, are $\gamma_1 = \alpha_6$, $\gamma_2 = \alpha_3$, $\gamma_3 = \frac{1}{2}(\alpha_2 + \alpha_4)$, $\gamma_4 = \frac{1}{2}(\alpha_1 + \alpha_5)$ and the connection between the corresponding fundamental weights is $\varpi_1 = \lambda_6$, $\varpi_2 = \lambda_3$, $\varpi_3 = \frac{1}{2}(\lambda_2 + \lambda_4)$, $\varpi_4 = \frac{1}{2}(\lambda_1 + \lambda_5)$. The restriction of $-3\lambda_1$ (the highest weight of V as an $\mathbf{E}_6$ representation) to $\mathfrak{t}^\phi$ is $-3\varpi_4$. In terms of the fundamental weights the infinitesimal character is $\varpi_1 + \varpi_2 + \varpi_3 - 2\varpi_4 = (\frac{5}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$ in the coordinates appearing after Lemma 3.2. Now note that, if w_γ, $\gamma \in \mathfrak{t}^*$, is the reflection through the hyperplane orthogonal to γ,

$$\mathrm{w}_{\gamma_3}\mathrm{w}_{\gamma_4}(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}) = (\tfrac{7}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \quad .$$

Thus the infinitesimal character of $V|_{\mathbf{F}_4}$ is $(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$. □

Working in $\mathbf{F}_4$ and the coordinates given above, set $\Lambda_o = (1, 1, 0, 0)$. This defines a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ by

$$\begin{aligned}\Delta(\mathfrak{q}) &= \{\alpha \in \Delta \mid \langle \alpha, \Lambda_o\rangle \geq 0\}\ ,\\ \Delta(\mathfrak{l}) &= \{\alpha \in \Delta \mid \langle \alpha, \Lambda_o\rangle = 0\}\ ,\\ \Delta(\mathfrak{u}) &= \{\alpha \in \Delta \mid \langle \alpha, \Lambda_o\rangle > 0\}\ .\end{aligned}$$

Let $\lambda = a\Lambda_o$. $A_{\mathfrak{q}}(\lambda) = \mathscr{R}^s_{\mathfrak{q}}(\mathbb{C}_\lambda)$, the cohomologically induced representation, has infinitesimal character $\lambda + \rho$.

Theorem 3.4. *$V|_{\mathbf{F}_4} \approx A_{\mathfrak{q}}(\lambda)$ with $\lambda = (-2, -2, 0, 0)$. This is the Langlands quotient of $\mathrm{Ind}_{MAN}^G(1 \otimes \frac{5}{2}t \otimes 1)$, where $P = MAN$ is the minimal parabolic with $\mathfrak{n}$ corresponding to $\Sigma^+(\mathfrak{a}, \mathfrak{g})$.*

Proof. We show that $A_{\mathfrak{q}}(\lambda)$ has the right infinitesimal character, the correct K_1-types (is spherical) and that this data determines the representation.

The infinitesimal character is $(-2, -2, 0, 0) + (\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}) = (\frac{7}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2})$ which is conjugate to $(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ as in Theorem 3.3. To determine the K_1-types, we use the Blattner formula (see [16, Theorem 6.3.12]).

Lemma 3.5. *The degree m elements of the symmetric algebra $S(\mathfrak{u} \cap \mathfrak{p})$ form an irreducible $L \cap K_1$ representation with highest weight $(\frac{m}{2}, \frac{m}{2}, \frac{m}{2}, \frac{m}{2})$.*

Proof. Since

$$\Delta(\mathfrak{u} \cap \mathfrak{p}_1) = \{(\tfrac{1}{2}, \tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2})\},$$
$$\Delta(\mathfrak{l} \cap \mathfrak{k}_1) = \{\pm(e_1 - e_2), \pm e_3 \pm e_4, \pm e_3, \pm e_4\},$$

we see that $\mathfrak{l} \cap \mathfrak{k}_1 \approx \mathfrak{so}(3) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(5)$. The $\mathfrak{so}(3)$ acts trivially on $\mathfrak{u} \cap \mathfrak{p}_1$, while the $\mathfrak{so}(2)$ acts by $(\frac{1}{2}, \frac{1}{2}, 0, 0)$. Also

$$\dim S^m(\mathfrak{u} \cap \mathfrak{p}_1) = \dim S^m(\mathbb{C}^4) = \binom{m+3}{3}.$$

The highest $\mathfrak{so}(5)$ weight occurring in $S^m(\mathfrak{u} \cap \mathfrak{p}_1)$ is $(\frac{m}{2}, \frac{m}{2})$ and the corresponding irreducible representation of $\mathfrak{so}(5)$ also has dimension $\binom{m+3}{3}$. □

If F_μ denotes the irreducible representation of K_1 with highest weight μ, then if $a \geq -2$

$$H^k(\mathfrak{l} \cap \mathfrak{k}_1, F_{(a+2+m/2, a+2+m/2, m/2, m/2)}) = \begin{cases} E_{(a+2+m/2, a+2+m/2, m/2, m/2)} & \text{if } k = 0, \\ 0 \quad \text{if } k > 0, \end{cases}$$

where $E_{(a+2+m/2, a+2+m/2, m/2, m/2)}$ is the irreducible highest weight representation of $\mathfrak{u} \cap \mathfrak{k}_1$ with highest weight $(a + 2 + \frac{m}{2}, a + 2 + \frac{m}{2}, \frac{m}{2}, \frac{m}{2})$. This is just Kostant's Borel-Weil Theorem (see [16, Theorem 3.2.3]). Thus for $\lambda = a\Lambda_o$, $\mathscr{R}_{\mathfrak{q}}^j(\lambda) = 0$, $j \neq s = \dim(\mathfrak{u} \cap \mathfrak{k}_1)$, and $a \geq -2$ (in fact, $a \geq -5$). $\mathscr{R}_{\mathfrak{q}}^s(\lambda)$ is irreducible for $a \geq -1$ by §5 of [19]. We shall show that $\mathscr{R}_{\mathfrak{q}}^s(\lambda)$ is also irreducible for $a = -2$ and trivial if $a \leq -3$. The K_1-types of $\mathscr{R}_{\mathfrak{q}}^s(\lambda)$ are $\{(a + 2 + \frac{m}{2}, a + 2 + \frac{m}{2}, \frac{m}{2}, \frac{m}{2}) \mid m \in \mathbb{Z}^+\}$ (note: $\lambda + 2\rho(\mathfrak{u} \cap \mathfrak{k}_1) = (a+2, a+2, 0, 0)$). Thus, $A_{\mathfrak{q}}(\lambda)$ has the same K_1-types as $V|_{\mathbf{F}_4}$ when $a = -2$.

Since our $\mathbf{F}_4$ has rank one, all irreducible spherical representations occur in some $\mathrm{Ind}_{MAN}^G(1 \otimes \nu \otimes 1)$ with $\mathrm{Re}(\nu) \geq 0$. The infinitesimal character is $\rho_m + \nu = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \nu)$. This has length $\frac{35}{4} + \nu^2$, the length of the infinitesimal character of $V|_{\mathbf{F}_4}$ is 15. Therefore, $\nu = \frac{5}{2}$ and there is just one spherical representation with infinitesimal character $(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$. We conclude that $A_{\mathfrak{q}}(\lambda)$, $\lambda = (-2, -2, 0, 0)$, is irreducible and equivalent to $V|_{F_4}$ (and to the Langlands quotient of $\mathrm{Ind}_{MAN}^G(1 \otimes \frac{5}{2} \otimes 1)$). □

Remark. There is a "dual pair" picture here. Let (π, V) be our unitary highest weight representation and σ the automorphism of Proposition 3.2. Let $\tilde{G} = \mathbb{Z}_2 \ltimes G$, where $\mathbb{Z}_2 = \{1, \sigma\}$. (π, V) does not extend to a representation of $\tilde{G}$. However, $(\pi \oplus \pi^\sigma, V \oplus V^\sigma)$ does extend:

$$(1, g) \cdot (v_1, v_2) = (\pi(g)v_1, \pi(\sigma(g))v_2),$$
$$(\sigma, g) \cdot (v_1, v_2) = (\pi(\sigma(g))v_2, \pi(g)v_1).$$

Call this representation $\tilde{V}$. This is an irreducible representation of $\tilde{G}$. For the connected subgroup $G_1 \subset \tilde{G}$ with Lie algebra $\mathfrak{g}_1$, there is a dual pair $(\mathbb{Z}_2, \tilde{G}_1)$

in $\tilde{G}$, $\tilde{G}_1 = \mathbb{Z}_2 \times G_1$. (The centralizer of $\mathbb{Z}_2$ is clearly $\tilde{G}_1$, as G_1 is the group of fixed points of σ. The centralizer of $\tilde{G}_1$ is $\mathbb{Z}_2 \ltimes Z_{G_1} = \mathbb{Z}_2$. Note that the center $\mathbb{Z}_{G_1}$ of G_1 is $\{e\}$ as $\zeta \in Z_{G_1}$ satisfies $\zeta \in K_1$ and $\mathrm{Ad}(\zeta) = \mathrm{Id}$, but $\mathrm{Ad}\colon K \to GL(\mathfrak{p}_1)$ is the spin representation of $K_1 = Spin(9)$ and has no kernel.)

$$\tilde{V}|_{\tilde{G}_1} = \{(v, v) \mid v \in V\} \oplus \{(v, -v) \mid v \in V\} \approx (1 \otimes \pi_1) \oplus (\mathrm{sign} \otimes \pi_1).$$

So there is a one-to-one correspondence between representations of $\mathbb{Z}_2$ and representations of $\tilde{G}_1$ occurring in $\tilde{V}$.

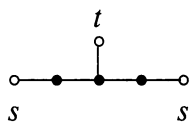
4

We now give several additional properties of V.

(a) We determine the Langlands parameters of V by applying Proposition 2.8 of [20]. We obtain the following:

$$V \hookrightarrow \mathrm{Ind}_{MAN}^{G}(\sigma \otimes \mathrm{e}^{\nu} \otimes 1)$$

where MAN is a minimal parabolic, $\sigma \in \hat{M}$ is the 1-dimensional representation of $M = \mathbb{Z}_2 \cdot U(1) \times SU(4)$ given by $\sigma(\epsilon, \theta, \kappa) = \mathrm{sign}(\epsilon)\mathrm{e}^{-3i\theta} \cdot 1$, and $\nu = -8\epsilon_1 - 2\epsilon_2$ (where $\Delta(\mathfrak{n}) = \{\epsilon_1 \pm \epsilon_2, \epsilon_1, \epsilon_2, 2\epsilon_1, 2\epsilon_2\}$ as in [20]). Note that $\nu = -10s + 8t$ where s, t are the simple roots on the Satake diagram pictured below.



(b) The Harish-Chandra module V is special unipotent in the sense of Vogan (see [17, Chapter 12]). This can be easily checked by comparing the infinitesimal character of V with the tables in [3]. We will now confirm for V the K-type formula given in Conjecture 12.1 of [18].

Let β denote the maximal root in $\mathfrak{p}_+$, and $K_{\mathbb{C}}(\beta)$ the stabilizer of X_β in $K_{\mathbb{C}}$. The $K_{\mathbb{C}}$ orbit $K_{\mathbb{C}} \cdot X_\beta \approx K_{\mathbb{C}}/K_{\mathbb{C}}(\beta)$ is admissible in the sense of Definition 7.13 of [18] (see also [13]); i.e.,

$$\chi(k) = [\det(\mathrm{Ad}(k))|_{(\mathfrak{k}/\mathfrak{k}(\beta))^*}]^{1/2}$$

is a well-defined character of $K_{\mathbb{C}}(\beta)$. To see this, one must check that

$$\frac{1}{2}\left(\frac{11\sqrt{3}}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$$

integrates to the torus $H_{\mathbb{C}}(\beta) = H_{\mathbb{C}} \cap K_{\mathbb{C}}$.

Proposition 4.1. *χ defines an admissible character of $K_{\mathbb{C}}(\beta)$ and*

$$V \approx \mathrm{Ind}_{K_{\mathbb{C}}(\beta)}^{K_{\mathbb{C}}}(\chi)$$

as $K_{\mathbb{C}}$-representations.

Proof. $K_{\mathbb{C}}(\beta)$ is connected by Appendix B. $K_{\mathbb{C}}(\beta)$ is nearly a parabolic subgroup of $K_{\mathbb{C}}$ in the sense that there is a parabolic subgroup K' such that $K_{\mathbb{C}}(\beta) \subset K' \subset K_{\mathbb{C}}$ and

$$\mathfrak{k}' = \mathfrak{k}(\beta) + (\mathbb{C} \cdot h_\beta) = \mathfrak{u}(5) \oplus \mathfrak{so}(2) \oplus \mathfrak{n}' = \mathfrak{m}' \oplus \mathfrak{n}',$$

where $\Delta(\mathfrak{u}(5)) = \{\epsilon_i - \epsilon_j \mid 2 \le i, j \le 6\}$, $\Delta(\mathfrak{n}') = \{\epsilon_i + \epsilon_j \mid 2 \le i < j \le 6\}$, and h_β is the element of $\mathfrak{t}$ corresponding to β via the Killing form.

We use induction in stages. $\mathrm{Ind}_{K(\beta)}^{K'}(\chi) = \bigoplus_{n\in\mathbb{Z}} \sigma_n$, where σ_n is the 1-dimensional representation of $K' = M'N'$ with weight $-3\lambda_1 - n\lambda_6$ on M' and trivial on N'. An irreducible finite dimensional $K_{\mathbb{C}}$ representation F occurs in $\mathrm{Ind}_{K'}^{K}(\sigma_n)$ with multiplicity $\dim \mathrm{Hom}_{K'}(F, \sigma_n) = \dim \mathrm{Hom}_{M'}(F/n'F, \sigma_n)$. This is zero unless F is the representation with lowest weight $-3\lambda_1 - n\lambda_6$, i.e., highest weight $-3\lambda_1 - n\alpha_1$ (which is dominant for $n \ge 0$). These are precisely the K-types of V.

Appendix A

The following theorem seems to be well known, but we were unable to find a proof written down in one place. Below we will give a very brief sketch of the proof, indicating references for the key steps.

Theorem A. *Assume $\langle \lambda + \rho(\mathfrak{u}), \alpha\rangle > 0$ for $\alpha \in \Delta(\mathfrak{u})$. Then the characteristic variety of $A_{\mathfrak{q}}(\lambda)$ is $K_{\mathbb{C}} \cdot (\mathfrak{u} \cap \mathfrak{p})$. Thus, the Gelfand-Kirillov dimension of $A_{\mathfrak{q}}(\lambda)$ is $\dim_{\mathbb{C}}(K_{\mathbb{C}} \cdot (\mathfrak{u} \cap \mathfrak{p}))$.*

Proof. First note that the Gelfand-Kirillov dimension of a representation M is equal to the complex dimension of its characteristic variety $\mathscr{V}(M)$ (see, e.g., [14 and 9, Lemma 10.1]).

Consider the generalized flag variety X of all parabolics conjugate to $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. The moment map $\mu : T^*X \to \mathfrak{g}^*$ is defined by identifying T^*X with $G \times_{\mathfrak{q}} \mathfrak{u}$ and setting $\mu(g, \eta) = \mathrm{Ad}(g)\eta \in \mathfrak{g} \approx \mathfrak{g}^*$. If M is a Harish-Chandra module and $\mathscr{M}$ is the localization of M on X, then

$$V(M) = \mu(\mathrm{Ch}(\mathscr{M}))$$

where $\mathrm{Ch}(\mathscr{M})$ is the characteristic variety of $\mathscr{M}$ in T^*X. (See [1, §1, particularly, 1.6, 1.8, and 1.9].)

Now consider $M = A_{\mathfrak{q}}(\lambda)$. It follows (under the positivity condition) from [2, Theorem 2.4], that $M = \Gamma(X, \mathscr{M})$ where $\mathscr{M}$ is a $\mathscr{D}_\lambda$-module on X supported in the closed orbit $Z = K_{\mathbb{C}} \cdot \mathfrak{q}$. The characteristic variety is thus T_Z^*X, the conormal bundle. Thus,

$$\begin{aligned}\mathrm{Ch}(\mathscr{M}) &\approx T_Z^*X \approx K_{\mathbb{C}} \times_{Q\cap K_{\mathbb{C}}} (\mathfrak{g}/(\mathfrak{k} + \mathfrak{q})) \\ &\approx K_{\mathbb{C}} \times_{Q\cap K_{\mathbb{C}}} (\bar{\mathfrak{u}} \cap \mathfrak{p})^* \approx K_{\mathbb{C}} \times_{Q\cap K_{\mathbb{C}}} (\mathfrak{u} \cap \mathfrak{p}).\end{aligned}$$

So,

$$V(A_{\mathfrak{q}}(\lambda)) = \mu(\mathrm{Ch}(\mathscr{M})) = \mu(K_{\mathbb{C}} \times (\mathfrak{u} \cap \mathfrak{p})) = K_{\mathbb{C}} \cdot (\mathfrak{u} \cap \mathfrak{p}). \quad \square$$

For the following proposition we consider arbitrary $A_{\mathfrak{q}}(\lambda)$ representations (assuming the parameters λ satisfy $\lambda \perp \Delta(\mathfrak{l})$).

Proposition. *Let λ_o be any parameter so that $\langle \lambda_o + \rho(\mathfrak{u}), \alpha\rangle > 0$, $\alpha \in \Delta(\mathfrak{u})$, and let λ be arbitrary. Then $V(A_{\mathfrak{q}}(\lambda)) = V(A_{\mathfrak{q}}(\lambda_o))$ as long as $A_{\mathfrak{q}}(\lambda) \neq 0$.*

Proof. It is clear from the definition of characteristic variety that for any admissible $(\mathfrak{g}, K)$-module X, $V(X \otimes F) \subseteq V(X)$. If Y is a quotient of $X \otimes F$ then $V(Y) \subseteq V(X \otimes F) \subseteq V(X)$. Since $\mathrm{Hom}(X \otimes F, Y) = \mathrm{Hom}(X, Y \otimes F^*)$, for X irreducible, we get $X \subseteq Y \otimes F^*$. Therefore, $V(X) \subseteq V(Y \otimes F^*) \subseteq V(Y)$ and we conclude that $V(Y) = V(X)$. We will show that for appropriate λ_o and F, $A_{\mathfrak{q}}(\lambda)$ is a quotient of $A_{\mathfrak{q}}(\lambda_o) \otimes F$; the proposition will follow.

Recall that the derived functor modules are defined by

$$\mathscr{R}_{\mathfrak{q}}^{i}(W) = \Gamma^{i}\left(\mathrm{Hom}_{U(\mathfrak{q})}\left(U(g),\, W \oplus \bigwedge^{\mathrm{top}} \mathfrak{u}\right)\right)$$

where W is an $(\mathfrak{q}, L \cap K)$-module and Γ^i is the right derived functor of the "K-finite" functor Γ (see [16, 6.31]).

Let λ be arbitrary, by the lemma below there is a finite dimensional $\mathfrak{g}$-module F, a λ_o so that $\langle \lambda_o + \rho(\mathfrak{u}), \alpha\rangle > 0$, $\alpha \in \Delta(\mathfrak{u})$, and an exact sequence

$$0 \to \mathrm{Ker} \to \mathbb{C}_{\lambda_o} \otimes F \to \mathbb{C}_{\lambda} \to 0$$

(as $\mathfrak{q}$-modules), hence a long exact sequence

$$\cdots \to \mathscr{R}_{\mathfrak{q}}^{s}(\mathbb{C}_{\lambda_o}) \otimes F \to \mathscr{R}_{\mathfrak{q}}^{s}(\mathbb{C}_{\lambda}) \to \mathscr{R}_{\mathfrak{q}}^{s+1}(\mathrm{Ker}) \to \cdots .$$

Here we have used $\mathscr{R}_{\mathfrak{q}}^{s}(\mathbb{C}_{\lambda_o} \otimes F) \approx \mathscr{R}_{\mathfrak{q}}^{s}(\mathbb{C}_{\lambda_o}) \otimes F$ (F is a representation of $\mathfrak{g}$). The vanishing theorem for degrees greater than s [16, 6.3.21]) along with a standard filtration argument shows $\mathscr{R}_{\mathfrak{q}}^{s+1}(\mathrm{Ker}) = 0$. Thus we get

$$\cdots \to A_{\mathfrak{q}}(\lambda_o) \otimes F \to A_{\mathfrak{q}}(\lambda) \to 0 .$$

The theorem follows by noting that if $\langle \lambda_o + \rho(\mathfrak{u}), \alpha\rangle > 0$, $\alpha \in \Delta(\mathfrak{u})$, then $V(A_{\mathfrak{q}}(\lambda_o)) = V(A_{\mathfrak{q}}(\lambda_o'))$ by the standard tensoring (within a chamber) argument (see, for example, [16, 7.2.9 and 7.2.22]). □

Lemma. *Given λ, there is a λ_o so that $\langle \lambda_o + \rho(\mathfrak{u}), \alpha\rangle > 0$, $\alpha \in \Delta(\mathfrak{u})$, a finite dimensional $\mathfrak{g}$-module F, and a surjection $\mathbb{C}_{\lambda_o} \otimes F \to \mathbb{C}_{\lambda}$ as $\mathfrak{q}$-modules.*

Proof. The $\mathfrak{u}$-invariants $(\mathbb{C}_{\lambda} \otimes F_o)^{\mathfrak{u}} = \mathbb{C}_{\lambda} \otimes F_o^{\mathfrak{u}}$ contains a highest weight vector (for $\mathfrak{l}$). Choose λ_o so that $\lambda_o - \lambda$ is $\Delta^+(\mathfrak{g})$-dominant then take F_o to be the irreducible finite dimensional $\mathfrak{g}$-representation with highest weight $\lambda_o - \lambda$. Thus we have arranged for a nonzero map $\mathbb{C}_{\lambda_o} \to \mathbb{C}_{\lambda} \otimes F_o$ as $\mathfrak{q}$-modules. Thus,

$$\mathrm{Hom}(\mathbb{C}_{\lambda_o}, \mathbb{C}_{\lambda} \otimes F_o) \approx \mathrm{Hom}(\mathbb{C}_{\lambda_o} \otimes F_o^*, \mathbb{C}_{\lambda})$$

is nonzero. Take $F = F_o^*$ in the lemma. □

Appendix B

In this appendix we prove a fact about stabilizers for a minimal nilpotent orbit of $K_{\mathbb{C}}$ in $\mathfrak{p}$ and then apply this to our $\mathbf{E}_6$ example.

Let G be a complex, connected, simply connected, semisimple group. Let θ be the complexification of a Cartan involution for some real form of G, and let $\mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the $\mathfrak{g} = \mathrm{Lie}(G)$. Assume $\mathrm{rank}\, K = \mathrm{rank}\, G$. In this case, θ is an inner automorphism, so $K_{\mathbb{C}} = G^{\theta}$ is connected. Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$ (and let H be the corresponding Cartan subgroup) and a positive system of roots Δ^+. Let β be the highest root in $\mathfrak{p}$, and let x_β be a root vector for β. There is a triple $\{x_\beta, h_\beta, y_\beta\}$ spanning a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ with $h_\beta \in \mathfrak{t}$ and y_β a root vector for $-\beta$. h_β is normalized such that $\beta(h_\beta) = 2$.

A θ-stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is defined by $\Delta(\mathfrak{l}) = \{\alpha \in \Delta \mid \alpha(h_\beta) = 0\}$ and $\Delta(\mathfrak{u}) = \{\alpha \in \Delta \mid \alpha(h_\beta) > 0\}$. Let $Q = LU$ be the corresponding parabolic subgroup in G. Q is connected. The stabilizer $G(\beta)$ of x_β is contained in LU by Theorem 3.6 of [10]. We may conclude that $K_{\mathbb{C}}(\beta) \subset K_{\mathbb{C}} \cap LU \subset (K_{\mathbb{C}} \cap L) \cdot (K_{\mathbb{C}} \cap U)$; furthermore, $K_{\mathbb{C}} \cap L$ is the Levi component of the parabolic $(K_{\mathbb{C}} \cap LU) \subset K_{\mathbb{C}}$ and is connected.

Proposition. $K_{\mathbb{C}}(\beta)/K_{\mathbb{C}}(\beta)_e = H(\beta)/H(\beta)_e$, *where* $H(\beta) = \mathrm{Stab}_H(x_\beta)$.

Proof. Let HN be a Borel subgroup of $K_{\mathbb{C}} \cap L$ and let W be the Weyl group of $K_{\mathbb{C}} \cap L$. The Bruhat decomposition is

$$K_{\mathbb{C}} \cap L = \bigcup_{w \in W} NHwN.$$

The following lemma shows that N and W stabilize x_β.

Lemma. *The semisimple part of* $K_{\mathbb{C}} \cap L$ *is contained in* $K_{\mathbb{C}}(\beta)$ *and* $W \subset K_{\mathbb{C}}(\beta)$.

Proof. Since $K_{\mathbb{C}} \cap L$ is connected, we need only check this for the corresponding Lie algebras. If $\alpha \in \Delta^+(\mathfrak{k} \cap \mathfrak{l})$ then $(\alpha, \beta) = 0$. But $\beta + \alpha \notin \Delta$ since β is the highest root; thus also $\beta - \alpha \notin \Delta$. So all root vectors in $\mathfrak{k} \cap \mathfrak{l}$ kill x_β and thus the semisimple part of $K_{\mathbb{C}} \cap L$ is contained in $K_{\mathbb{C}}(\beta)$. It follows that $W \subset K_{\mathbb{C}}(\beta)$. $\square$

Now suppose $k = n_1 h w n_2 \in K_{\mathbb{C}} \cap L$ fixes x_β, then $x_\beta = k \cdot x_\beta = n_1 h w n_2 x_\beta$ and $x_\beta = h \cdot x_\beta$, so $h \in H(\beta)$. Therefore,

$$K_{\mathbb{C}}(\beta) \cap L = \bigcup_{w \in W} NH(\beta)wN.$$

Let F be a set of representatives of distinct cosets in $H(\beta)/H(\beta)_e$, such that

$$K_{\mathbb{C}}(\beta) \cap L = F \cdot (K_{\mathbb{C}}(\beta) \cap L)_e .$$

We claim that F is a set of representatives of distinct cosets in $(K_{\mathbb{C}}(\beta) \cap L) / (K_{\mathbb{C}}(\beta) \cap L)_e$. Suppose $f \in F$ is in $(K_{\mathbb{C}}(\beta) \cap L)_e$. Then, since f commutes with $H(\beta)_e$, f is in $H(\beta)_e$ because $H(\beta)_e$ is a Cartan subgroup of the connected group $(K_{\mathbb{C}}(\beta) \cap L)_e$. The proposition is proved.

For our situation with $\mathbf{E}_6$, we calculate the component group $H(\beta)/H(\beta)_e$. Let $L_{\mathrm{root}} = \mathrm{span}_{\mathbb{Z}}\{h_\alpha \mid \alpha \in \Delta\}$ where h_α is the element of $\mathfrak{t}$ corresponding to $\frac{2\alpha}{(\alpha,\alpha)} \in \mathfrak{t}^*$ via the Killing form. Let $e(h) = \exp(2\pi i h)$, $h \in \mathfrak{t}$. Since G is simply connected, $L_{\mathrm{root}} = \mathrm{Ker}(e)$. Write $h \in \mathfrak{t}$ as $h_1 + t h_\beta$, with $h_1 \perp h_\beta$. Then

$$\mathrm{Ad}\,(e(h))\, x_\beta = e^{2\pi i (2t)} x_\beta$$

so $e(h) \in H(\beta)$ if $t \in \frac{1}{2}\mathbb{Z}$. Thus,

$$\begin{aligned} H(\beta) &= \exp\big(2\pi i (h_\beta^\perp + \tfrac{1}{2}\mathbb{Z} h_\beta)\big) \\ &= \{e, \exp(\pi i h_\beta)\} \exp(2\pi i h_\beta^\perp) = \{e, \exp(\pi i h_\beta)\} H(\beta)_e . \end{aligned}$$

We now show that $\exp(\pi i h_\beta) \in H(\beta)_e$; i.e., $\exp(\pi i h_\beta) \in \exp(h_\beta^\perp)$. For this it is sufficient to find an element $h_1 \in h_\beta^\perp$ such that $\frac{1}{2}h_\beta - h_1 \in \mathrm{Ker}(e) = L_{\mathrm{root}}$. Choose $h_1 = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_5)$. One easily checks that $\frac{1}{2}h_\beta - h_1$ is integral on all fundamental weights; i.e., $\frac{1}{2}h_\beta - h_1 \in L_{\mathrm{root}}$. Thus, in our $\mathbf{E}_6$ example $K_{\mathbb{C}}(\beta)$ is connected.

References

1. W. Borho and J.-L. Brylinski, *Differential operators on homogeneous spaces.* III, Invent. Math. **80** (1985), 1–68.
2. J.T. Chang, *Remarks on localization and standard modules: the duality theorem on a generalized flag variety*, M.S.R.I preprint, 1988.

3. G. Elkington, *Centralizers of unipotent elements in semisimple algebraic groups*, J. Algebra **23** (1972), 137–163.

4. T. Enright, R. Howe, and N. Wallach, *A classification of unitary highest weight modules*, Representation Theory of Reductive Groups, Progr. Math., vol. 40, Birkhäuser, Boston, Mass., 1983, pp. 97–143.

5. T. Enright and A. Joseph, *An intrinsic analysis of unitary highest weight modules*, Math. Ann. **288** (1990), 571–594.

6. D. Garfinkle, *A new construction of the Joseph ideal*, Ph.D. Thesis, M.I.T., 1982.

7. S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.

8. ——, *Groups and geometric analysis*, Academic Press, New York, 1984.

9. A. Joseph, *The minimal orbit in a simple Lie algebra and its associated maximal ideal*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 1–30.

10. B. Kostant, *The principle three-dimensional subgroup and the Betti numbers of a complex semisimple Lie group*, Amer. J. Math. **81** (1959), 973–1032.

11. ——, *The vanishing of scalar curvature and the minimal representation of* $SO(4, 4)$, M.I.T. preprint.

12. W. Schmid, *Die Randwerte Holomorphic Functionen auf Hermitesch Symmetrischen Räumen*, Invent. Math. **9** (1969), 61–80.

13. J. Schwartz, *The determination of the admissible orbits in the real classical groups*, Ph.D. dissertation, MIT, Cambridge, Mass., 1987.

14. D. Vogan, Jr., *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. **48** (1978), 75–98.

15. ——, *Singular unitary representations*, Non-Commutative Harmonic Analysis, (Proc., 1980), Lecture Notes in Math., vol. 880, Springer-Verlag, Berlin, Heidelberg, and New York, 1981.

16. ——, *Representations of real reductive Lie groups*, Birkhäuser, Boston, Mass., 1981.

17. ——, *Unitary representations of reductive Lie groups*, Princeton Univ. Press, Princeton, NJ, 1987.

18. ——, *Associated varieties and unipotent representations*, preprint.

19. D. Vogan, Jr. and G. Zuckerman, *Unitary representations with non-zero cohomology*, Compositio Math. **53** (1984), 51–90.

20. N. Wallach, *The asymptotic behavior of holomorphic representations*, Mém. Soc. Math. France No. 15 (1984), 291–305.

21. R. Zierau, *Geometric constructions of unitary highest weight representations*, Ph.D. dissertation, Univ. of California, Berkeley, 1984.

DEPARTMENT OF MATHEMATICS OKLAHOMA STATE UNIVERSITY STILLWATER, OKLAHOMA 74078-0613

E-mail address: `binegar@math.okstate.edu`

E-mail address: `zierau@math.okstate.edu`