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ON CONVERGENCE AND CLOSEDNESS OF MULTIVALUED MARTINGALES

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ABSTRACT. In this paper, various convergence theorems and criteria of closedness of multivalued martingales, submartingales, and supermartingales are proved.

1. INTRODUCTION AND PRELIMINARIES

The study of multivalued functions has been developed extensively with applications in several areas of applied mathematics, such as mathematical economics, optimal control, and decision theory, (cf. Hildenbrand (1974), Himmelberg and Vleck (1974), Papageorgiou (1986), Clarke (1984), de Korvin and Kleyle (1985), Vovits, Foulk and Rose (1981), and Aubin and Frankowska (1990)). Four notions of convergence of multivalued functions in a Banach space, the Hausdorff distance convergence, the Kuratowski-Mosco convergence, the weak convergence, and the Wijsman convergence are particularly useful in the study. Illustrated by the works of de Korvin and Kleyle (1985) and Papageorgiou (1986), multivalued martingales, submartingales, and supermartingales are powerful tools in the study of convergence of random multivalued functions. In this paper we shall make a further study on convergence and closedness of multivalued martingales, submartingales, and supermartingales.

Throughout this paper $(\Omega, \mathscr{A}, P)$ is a complete probability space, X is a separable Banach space with the dual X^*, and 2^X is the set of all subsets of X. Let

$$P_c(X) = \{A \in 2^X : A \text{ is nonempty, closed, and convex}\},$$
$$P_{cb}(X) = \{A \in P_c(X) : A \text{ is bounded}\},$$
$$P_{wkc}(X) = \{A \in P_c(X) : A \text{ is weakly compact}\}.$$

For $A \in 2^X \backslash \phi$, we denote by $\operatorname{cl} A$ and $\overline{\operatorname{co}} A$ the closure and the closed convex hull of A respectively, and define $|A| = \sup\{\|x\| : x \in A\}$,

$$s(x^*, A) = \sup\{\langle x^*, y\rangle : y \in A\}, \quad s(x^*, \phi) = -\infty, \qquad x^* \in X^*,$$
$$d(x, A) = \inf\{\|x - y\| : y \in A\}, \quad d(x, \phi) = \infty, \qquad x \in X.$$

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$s(x^*, A)$ and $d(x, A)$ are called the support function and the distance function of A respectively. Let $\mathscr{F}$ be a sub-σ-algebra of $\mathscr{A}$. A random multivalued function F from $\Omega \to P_c(X)$ is $\mathscr{F}$-measurable, if there exist $\mathscr{F}$-measurable random variables $f_n: \Omega \to X$,

$$F(\omega) = \mathrm{cl}\{f_n(\omega), n \geq 1\}, \qquad \omega \in \Omega,$$

(cf. [4, Chapter III]). The original study of random multivalued functions goes back to Robbins (1944, 1945). Let $\mathscr{M}[\mathscr{F}]$ be the family of all random multivalued functions which are $\mathscr{F}$-measurable. Let

$$\mathscr{L}_c = \{F \in \mathscr{M}[\mathscr{A}]: F(\omega) \in P_c(X) \text{ a.s.}\},$$
$$\mathscr{L}_{wkc} = \{F \in \mathscr{L}_c: F(\omega) \in P_{wkc}(X) \text{ a.s.}\},$$
$$\mathscr{L}_c^{d1} = \{F \in \mathscr{L}_c: Ed(0, F) < \infty\}, \quad \mathscr{L}_c^1 = \{F \in \mathscr{L}_c: E|F| < \infty\}.$$

For $F \in \mathscr{L}_c$, let

$$S_F^1(\mathscr{F}) = \{f \in L_X^1: f \text{ is } \mathscr{F}\text{-measurable}, f(\omega) \in F(\omega) \text{ a.s.}\},$$

where L_X^1 is the set of all Bochner integrable r.v.'s from $\Omega \to X$. We also simplify $S_F^1(\mathscr{A})$ as S_F^1. It is easy to show that S_F^1 is nonempty if and only if $F \in \mathscr{L}_c^{d1}$. The Aumann integral EF of F is defined by $EF = \{Ef: f \in S_F^1\}$, where Ef is the Bochner integral of f. The conditional expectation of $F \in \mathscr{L}_c^{d1}$ with respect to $\mathscr{F}$, $E(F|\mathscr{F})$, is the unique (up to a P-null set) $\mathscr{F}$-measurable multivalued function in $\mathscr{L}_c^{d1}$ such that

$$S_{E(F|\mathscr{F})}^1(\mathscr{F}) = \mathrm{cl}\{E(f|\mathscr{F}): f \in S_F^1\},$$

the closure in L_X^1, (cf. [15, Theorem 5.1]). The conditional expectation of a random multivalued function behaves like the traditional single-valued conditional expectation. For example, for any $F \in \mathscr{L}_c^{d1}$, $E(F|\mathscr{F}) = F$ a.s., if F is $\mathscr{F}$-measurable; and for any sub-σ-algebras $\mathscr{F}_1, \mathscr{F}_2$ of $\mathscr{A}$,

$$E(E(F|\mathscr{F}_1)|\mathscr{F}_2) = E(F|\mathscr{F}_2) \text{ a.s.}, \quad \text{if } \mathscr{F}_2 \subset \mathscr{F}_1,$$

(cf. [15, Theorem 5.3]). For details of the definitions and properties of the measurability, the integration, and the conditional expectation of random multivalued functions, we refer the reader to the works of Castaing and Valadier (1977), Hiai and Umegaki (1977), Hiai (1985), Papageorgiou (1985a), and Aubin and Frankowska (1990). For $A, B \in 2^X$, let $h^+(A, B) = \sup\{d(a, B): a \in A\}$, $h^-(A, B) = \sup\{d(b, A): b \in B\}$. The Hausdorff metric h on $P_c(X)$ is defined by

$$h(A, B) = \max\{h^+(A, B), h^-(A, B)\}, \qquad A, B \in P_c(X).$$

For $F, G \in \mathscr{L}_c^1$, define $\Delta(F, G) = Eh(F, G)$. Then $(P_c(X), h)$ and $(\mathscr{L}_c^1, \Delta)$ are complete metric spaces (cf. Hiai and Umegaki (1977), p. 160). For $(A_n, n \geq 1, A) \subset P_c(X)$, let

$$s\text{-}\liminf A_n = \{x \in X: \lim d(x, A_n) = 0\},$$

and

$$w\text{-}\limsup A_n = \{x \in X: x_k \xrightarrow{\mathrm{w}} x, \text{ for some } x_k \in A_{n_k}\},$$

where $(n_k, k \geq 1)$ is a subsequence of $(n \geq 1)$ and $\xrightarrow{\mathrm{w}}$ means convergence in the weak topology of X. We denote by $A_n \xrightarrow{\text{K-M}} A$ the convergence of A_n

to A in the Kuratowski-Mosco sense, if $w\text{-}\limsup A_n = A = s\text{-}\liminf A_n$; by $A_n \xrightarrow{h} A$ the Hausdorff distance convergence, if $h(A_n, A) \to 0$; and by $A_n \xrightarrow{W} A$ the weak convergence, if $s(x^*, A_n) \to s(x^*, A)$ for each $x^* \in X^*$. We say that A_n is Wijsman convergent to A if $d(x, A_n) \to d(x, A)$, $x \in X$, (cf. [10 and 19]). Since $h(A, B) = \sup\{|d(x, A) - d(x, B)|, x \in X\}$, the Hausdorff distance convergence is stronger than the Wijsman convergence. However, if (and only if) X is totally bounded, these two notions of convergence coincide (cf. [8 and 33]). If restricted to $P_{cb}(X)$, then the Hausdorff distance convergence implies the Kuratowski-Mosco convergence and the weak convergence, and when X is a finite dimensional space, these three types of convergence are equivalent (cf. [13, p. 165] and [30]). Clearly, for any $H \in \mathscr{L}_c^1$, $H(\omega) \in P_{cb}(X)$ a.s.

Let $(\mathscr{F}_n, n \geq 1)$ be an increasing sequence of complete sub-σ-algebras of $\mathscr{A}$, $\mathscr{F}_\infty = \sigma(\bigcup_n \mathscr{F}_n)$. By $(F_n, \mathscr{F}_n, n \geq 1)$ we mean that $(F_n) \subset \mathscr{L}_c^{d1}$ and F_n is $\mathscr{F}_n$-measurable, $n \geq 1$. Let T be the set of bounded stopping times with respect to $(\mathscr{F}_n, n \geq 1)$ and $T(s) = \{t \in T : t \geq s\}$, $s \in T$.

Definition 1.1. $(F_n, \mathscr{F}_n, n \geq 1)$ is called a (multivalued) martingale, submartingale, or supermartingale, if

$$E(F_{n+1}|\mathscr{F}_n) = , \supset, \text{ or } \subset F_n \text{ a.s.}, \qquad n \geq 1.$$

The rest of this paper is organized as follows. In §1, we shall prove various convergence results and criteria of closedness of multivalued submartingales. In §§3 and 4, we shall focus on convergence and closedness of multivalued martingales and supermartingales.

2. On multivalued submartingales

Lemma 2.1. *For any $F \in \mathscr{L}_c^{d1}$ and sub-σ-algebra $\mathscr{F} \subset \mathscr{A}$,*

(i) $|E(F|\mathscr{F})| \leq E(|F|\,|\mathscr{F})$ *a.s.*;

(ii) [12] $d(x, E(F|\mathscr{F})) \leq E(d(x, F)|\mathscr{F})$ *a.s.*, $x \in X$;

(iii) [26] $s(x^*, E(F|\mathscr{F})) = E(s(x^*, F)|\mathscr{F})$ *a.s.*, $x^* \in X^*$.

Proof. For any $F \in \mathscr{L}_c^{d1}$ and sub-σ-algegra $\mathscr{F} \subset \mathscr{A}$, we can choose $f_n \in S^1_{E(F|\mathscr{F})}(\mathscr{F})$ such that $E(F|\mathscr{F}) = \mathrm{cl}\{f_n, n \geq 1\}$ a.s. (cf. [15, Lemma 1.1]). It follows that for any $x \in X$, $x^* \in X^*$ and $\varepsilon > 0$, there exists $f \in S_F^1$ such that $d(x, F) \geq \|x - f\| - \varepsilon$ a.s. and $s(x^*, F) \leq \langle x^*, f\rangle + \varepsilon$ a.s., and we can easily check that

$$\begin{aligned}|E(F|\mathscr{F})| &= \operatorname*{ess\,sup}_{f \in S^1_{E(F|\mathscr{F})}(\mathscr{F})} |f| = \operatorname*{ess\,sup}_{f \in S_F^1} |E(f|\mathscr{F})| \\ &\leq E\left(\operatorname*{ess\,sup}_{f \in S_F^1} |f|\,|\mathscr{F}\right) = E(|F|\,|\mathscr{F}) \quad \text{a.s.},\end{aligned}$$

$$\begin{aligned}E(d(x, F)|\mathscr{F}) &= E\left(\operatorname*{ess\,inf}_{f \in S_F^1} \|x - f\|\,|\mathscr{F}\right) = \operatorname*{ess\,inf}_{f \in S_F^1} E(\|x - f\|\,|\mathscr{F}) \\ &\geq \operatorname*{ess\,inf}_{f \in S_F^1} \|x - E(f|\mathscr{F})\| = \operatorname*{ess\,inf}_{f \in S^1_{E(F|\mathscr{F})}(\mathscr{F})} \|x - f\| \\ &= d(x, E(F|\mathscr{F})) \text{ a.s.}, \qquad x \in X,\end{aligned}$$

and for each $x^* \in X^*$

$$\begin{aligned} s(x^*, E(F|\mathscr{F})) &= \operatorname*{ess\,sup}_{f \in S^1_{E(F|\mathscr{F})}(\mathscr{F})} \langle x^*, f\rangle = \operatorname*{ess\,sup}_{f \in S^1_F} \langle x^*, E(f|\mathscr{F})\rangle \\ &= \operatorname*{ess\,sup}_{f \in S^1_F} E(\langle x^*, f\rangle|\mathscr{F}) = E\left(\operatorname*{ess\,sup}_{f \in S^1_F} \langle x^*, f\rangle|\mathscr{F}\right) \\ &= E(s(x^*, F)|\mathscr{F}) \quad \text{a.s.} \quad \text{Q.E.D.} \end{aligned}$$

The following lemma slightly modifies Egghe's lemma [9, Lemma VIII.1.15].

Lemma 2.2. *Let I be a countable set and $I(n)$ a subset of I such that $I(n) \subset I(n+1)$ and $\bigcup_{n \geq 1} I(n) = I$.* (a) *$\{(x_n^i, \mathscr{F}_n, n \geq m), i \in I(m), m \geq 1\}$ is a uniform sequence of real-valued subpramarts, i.e.,*

$$\lim_{t \in T} \sup_{s \in T(t)} \mathrm{P}\left(\sup_{i \in I(t)} [x_t^i - E(x_s^i|\mathscr{F}_t)] > \varepsilon\right) = 0, \qquad \varepsilon > 0,$$

if and only if

$$\lim_{t \in T} \mathrm{P}\left(\sup_{i \in I(t)} (x_t^i - r_t^i) > \varepsilon\right) = 0, \qquad \varepsilon > 0, \tag{2.1}$$

where $I(t) = I(n)$ on $(t = n)$, and

$$r_n^i = \operatorname*{ess\,inf}_{t \in T(n)} E(x_t^i|\mathscr{F}_n) \text{ a.s.}, \qquad i \in I(n), n \geq 1.$$

(b) *Let $\{(x_n^i, \mathscr{F}_n, n \geq m), i \in I(m), m \geq 1\}$ be a uniform sequence of real-valued subpramarts such that*

$$\liminf_n E \sup_{i \in I(n)} |x_n^i| < \infty.$$

Then for each $m \geq 1$ and $i \in I(m)$, $(x_n^i, n \geq m)$ and $(r_n^i, n \geq m)$ converge a.s. to integrable r.v.'s x^i and r^i respectively, $x^i = r^i$ a.s., and

$$\lim_n \left(\sup_{i \in I(n)} x_n^i\right) = \sup_{i \in I} \lim_n x_n^i = \sup_{i \in I} x^i. \tag{2.2}$$

Proof. For each $m \geq 1$ and $i \in I(m)$, it is well known that $(r_n^i, \mathscr{F}_n, n \geq m)$ is a generalized submartingale with values in $[-\infty, \infty)$, and it is easy to show that for any fixed $t \in T$ and $i \in I(t)$, $r_t^i = \operatorname{ess\,inf}_{s \in T(t)} E(x_s^i|\mathscr{F}_t)$ a.s. and there exist $(s_n^i, n \geq 1) \subset T(t)$ such that $E(x_{s_n^i}^i|\mathscr{F}_t) \downarrow r_t^i$ a.s., (cf. [6, Lemma 4.1 and

Theorem 4.1]). Hence, for any $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{t\in T}\sup_{s\in T(t)} \mathrm{P}\left(\sup_{i\in I(t)}[x_t^i - E(x_s^i|\mathscr{F}_t)] > \varepsilon\right)\\
&\quad\le \lim_{t\in T}\mathrm{P}\left(\operatorname{ess\,sup}_{s\in T(t)}\left(\sup_{i\in I(t)}[x_t^i - E(x_s^i|\mathscr{F}_t)]\right) > \varepsilon\right)\\
&\quad= \lim_{t\in T}\mathrm{P}\left(\sup_{i\in I(t)}\left(\operatorname{ess\,sup}_{s\in T(t)}[x_t^i - E(x_s^i|\mathscr{F}_t)]\right) > \varepsilon\right) = \lim_{t\in T}\mathrm{P}\left(\sup_{i\in I(t)}[x_t^i - r_t^i] > \varepsilon\right)\\
&\quad= \lim_{t\in T}\mathrm{P}\left(\sup_{i\in I(t)}\left(\sup_{n}[x_t^i - E(x_{s_n^i}^i|\mathscr{F}_t)]\right) > \varepsilon\right)\\
&\quad= \lim_{t\in T}\mathrm{P}\left(\sup_{n}\left(\sup_{i\in I(t)}[x_t^i - E(x_{s_n^i}^i|\mathscr{F}_t)]\right) > \varepsilon\right)\\
&\quad= \lim_{t\in T}\sup_{n}\mathrm{P}\left(\sup_{i\in I(t)}[x_t^i - E(x_{s_n^i}^i|\mathscr{F}_t)] > \varepsilon\right)\\
&\quad\le \lim_{t\in T}\sup_{s\in T(t)}\mathrm{P}\left(\sup_{i\in I(t)}[x_t^i - E(x_s^i|\mathscr{F}_t)] > \varepsilon\right),
\end{aligned}$$

where the last equality is based on the fact that $(\sup_{i\in I(t)}[x_t^i - E(x_{s_n^i}^i|\mathscr{F}_t)],\ n \ge 1)$ is increasing, and (a) holds. Now assume that $\{(x_n^i, \mathscr{F}_n, n \ge m),\ i \in I(m),\ m \ge 1\}$ is a uniform sequence of real-valued subpramarts such that $\liminf_n E\sup_{i\in I(n)}|x_n^i| < \infty$. Since (2.1) is equivalent to

$$\sup_{i\in I(n)}|x_n^i - r_n^i| \to 0 \quad \text{a.s.} \tag{2.3}$$

(cf. [20, Theorem 5.1]), applying Millet and Sucheston's subpramart convergence theorem [20, Theorem 5.1] and (2.3), for each $i \in I(m)$, $m \ge 1$, $(x_n^i, n \ge m)$ and $(r_n^i, n \ge m)$ converge a.s. to integrable r.v.'s x^i and r^i respectively, $x^i = r^i$ a.s. To prove (2.2), first we assume that $\inf_{n\ge1, i\in I(n)} x_n^i \ge a$ for some $a \le 0$. Then $\inf_{n\ge1, i\in I(n)} r_n^i \ge a$ and $\liminf_n E\sup_{i\in I(n)}|r_n^i| \le \liminf_n E\sup_{i\in I(n)}|x_n^i| + |a| < \infty$. Applying the proof of Neveu's (1972) lemma [21, Lemma 4, or 22, p. 109], $\lim_n \sup_{i\in I(n)} r_n^i = \sup_{i\in I}\lim_n r_n^i = \sup_{i\in I} r^i$, and by (2.3),

$$\lim_n \sup_{i\in I(n)} x_n^i = \lim_n \sup_{i\in I(n)} r_n^i = \sup_{i\in I} r^i = \sup_{i\in I} x^i,$$

(2.2) holds. In a general case, for any fixed $a \le 0$, since

$$\begin{aligned}
\mathrm{P}\left(\sup_{i\in I(t)}[x_t^i \vee a - E(x_s^i \vee a|\mathscr{F}_t)] > \varepsilon\right) &\le \mathrm{P}\left(\sup_{i\in I(t)}[x_t^i \vee a - E(x_s^i|\mathscr{F}_t) \vee a] > \varepsilon\right)\\
&\le \mathrm{P}\left(\sup_{i\in I(t)}[x_t^i - E(x_s^i|\mathscr{F}_t)] > \varepsilon\right),
\end{aligned}$$

$\{(x_n^i \vee a, \mathscr{F}_n, n \ge m),\ i \in I(m),\ m \ge 1\}$ is a uniform sequence of real-valued subpramarts satisfying

$$\liminf_n E\sup_{i\in I(n)}|x_n^i \vee a| < \infty.$$

Thus

$$\sup_{i\in I} x^i \le \lim_n \sup_{i\in I(n)} x_n^i \le \lim_n \sup_{i\in I(n)} (x_n^i \vee a)$$
$$= \sup_{i\in I} \lim_n (x_n^i \vee a) = \sup_{i\in I}(x^i \vee a) \to \sup_{i\in I} x^i,$$

as $a \to -\infty$, (2.2) holds. Q.E.D.

Definition 2.1. Given $(F_n) \subset \mathscr{L}_c$, we say that (i) assumption (A) holds, if either (a) or (b) holds: (a) X has the Radon-Nikodým property, (b) for some $G \in \mathscr{L}_{wkc}$, $\liminf_n h^+(F_n, G) = 0$ a.s.; (ii) assumption (B) holds, if for some $G \in \mathscr{L}_{wkc}$, $h(F_n, G \cap F_n) \to 0$ a.s.; (iii) (F_n) is L^1-bounded, if $\sup_n E|F_n| < \infty$.

Let $\mathscr{N}$ be the set of all positive integers, $\mathscr{N}^2 = \{(m, k): m \in \mathscr{N}, k \in \mathscr{N}\}$, and $\mathscr{N}^2(l) = \{(m, k) \in \mathscr{N}^2: m \le l\}$, $l \ge 1$.

Lemma 2.3. *Let $(F_n, \mathscr{F}_n, n \ge 1)$ be a submartingale. Then there exists a family of adapted sequences $\{(f_n^{(m,k)}, \mathscr{F}_n, n \ge m), (m, k) \in \mathscr{N}^2\} \subset L_X^1$ such that*

(a) *for each $n \ge 1$,*

$$F_n(\omega) = \mathrm{cl}\{f_n^{(n,k)}(\omega), k \ge 1\} = \mathrm{cl}\{f_n^{(m,k)}(\omega), 1 \le m \le n, k \ge 1\} \quad a.s.,$$

$$E\|f_n^{(m,k)} - E(f_{n+1}^{(m,k)}|\mathscr{F}_n)\| \le \frac{1}{2^{m+k+n+1}}, \qquad n \ge m, k \ge 1,$$

and $\{(\|f_n^{(m,k)}\|, \mathscr{F}_n, n \ge l), (m, k) \in \mathscr{N}^2(l), l \ge 1\}$ is a uniform sequence of subpramarts.

(b) *under assumption* (A), *if* $\sup_n E|F_n| < \infty$, *then there exist r.v.'s* $(f^{(m,k)}, (m, k) \in \mathscr{N}^2) \subset L_X^1$ *such that*

$$f_n^{(m,k)} \to f^{(m,k)} \quad a.s., \tag{2.4}$$

$$\limsup_n d(x, F_n) \le d(x, F) \ a.s., \qquad x \in X, \tag{2.5}$$

and (2.9) *and* (2.10) *in Theorem* 2.1 *below hold, where*

$$F = \overline{\mathrm{co}}\{f^{(m,k)}, (m, k) \in \mathscr{N}^2\} \in \mathscr{L}_c^1. \tag{2.6}$$

Proof. Choose $\{f_n^{(n,k)}, k \ge 1\} \subset S_{F_n}^1(\mathscr{F}_n)$ such that

$$F_n(\omega) = \mathrm{cl}\{f_n^{(n,k)}(\omega), k \ge 1\}.$$

By the definition of multivalued submartingales and conditional expectations,

$$S_{F_n}^1(\mathscr{F}_n) \subset S_{E(F_{n+1}|\mathscr{F}_n)}^1(\mathscr{F}_n) = \mathrm{cl}\{E(f|\mathscr{F}_n): f \in S_{F_{n+1}}^1(\mathscr{F}_{n+1})\}.$$

Since $f_n^{(n,k)} \in S_{F_n}^1(\mathscr{F}_n)$, there is $f_{n+1}^{(n,k)} \in S_{F_{n+1}}^1(\mathscr{F}_{n+1})$,

$$E\|f_n^{(n,k)} - E(f_{n+1}^{(n,k)}|\mathscr{F}_n)\| \le \frac{1}{2^{n+k+n+1}}.$$

For each $k \ge 1$ and $m \ge 1$, by induction, we can get $f_n^{(m,k)} \in S_{F_n}^1(\mathscr{F}_n)$, $n \ge m$, such that

$$E\|f_n^{(m,k)} - E(f_{n+1}^{(m,k)}|\mathscr{F}_n)\| \le \frac{1}{2^{m+k+n+1}}.$$

Hence, for each $(m, k) \in \mathcal{N}^2$, $(f_n^{(m,k)}, \mathcal{F}_n, n \geq m)$ is a quasi-martingale i.e., $\sum_{n \geq m} E\|f_n^{(m,k)} - E(f_{n+1}^{(m,k)}|\mathcal{F}_n)\| < \infty$. For any $j \geq 1$, $t \in T(j)$ and $s \in T(t)$, let $K = \max_{\omega \in \Omega} s(\omega)$, then $(s = K) \in \mathcal{F}_{K-1}$, and for $n \geq j$ and $m \leq n$,

$$\begin{aligned}
&E\|f_n^{(m,k)} - E(f_s^{(m,k)}|\mathcal{F}_n)\| I(t = n) \\
&\quad = E\|f_n^{(m,k)} - E[f_{s\wedge(K-1)}^{(m,k)} - (f_{K-1}^{(m,k)} - f_K^{(m,k)})I(s = K)|\mathcal{F}_n]\| I(t = n) \\
&\quad \leq E\|f_n^{(m,k)} - E(f_{s\wedge(K-1)}^{(m,k)}|\mathcal{F}_n)\| I(t = n) \\
&\qquad + E\|f_{K-1}^{(m,k)} - E(f_K^{(m,k)}|\mathcal{F}_{K-1})\| I(t = n) \\
&\quad \leq \cdots \leq \sum_{i \geq n} E\|f_i^{(m,k)} - E(f_{i+1}^{(m,k)}|\mathcal{F}_i)\| I(t = n),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{P}\left(\sup_{(m,k)\in\mathcal{N}^2(t)} (\|f_t^{(m,k)}\| - E(\|f_s^{(m,k)}\| \,|\mathcal{F}_t)) > \varepsilon\right) \\
&\quad \leq P\left(\sup_{(m,k)\in\mathcal{N}^2(t)} (\|f_t^{(m,k)} - E(f_s^{(m,k)}|\mathcal{F}_t)\|) > \varepsilon\right) \\
&\quad \leq \sum_{n\geq j}\sum_{k\geq 1}\sum_{m\leq n} P(\|f_n^{(m,k)} - E(f_s^{(m,k)}|\mathcal{F}_n)\| I(t = n) > \varepsilon) \\
&\quad \leq \sum_{n\geq j}\sum_{k\geq 1}\sum_{m\leq n} E\|f_n^{(m,k)} - E(f_s^{(m,k)}|\mathcal{F}_n)\| I(t = n)/\varepsilon \\
&\quad \leq \sum_{n\geq j}\sum_{k\geq 1}\sum_{m\leq n}\sum_{i\geq n} E\|f_i^{(m,k)} - E(f_{i+1}^{(m,k)}|\mathcal{F}_i)\| I(t = n)/\varepsilon \\
&\quad \leq \sum_{i\geq j}\sum_{k\geq 1}\sum_{m\leq i}\sum_{n\geq j} E\|f_i^{(m,k)} - E(f_{i+1}^{(m,k)}|\mathcal{F}_i)\| I(t = n)/\varepsilon \\
&\quad \leq \sum_{i\geq j}\sum_{k\geq 1}\sum_{m\leq i} E\|f_i^{(m,k)} - E(f_{i+1}^{(m,k)}|\mathcal{F}_i)\|/\varepsilon \\
&\quad \leq \sum_{(m,k)\in\mathcal{N}^2}\sum_{i=j}^{\infty} \frac{1}{2^{m+k+i+1}\varepsilon} = \frac{1}{2^j\varepsilon} \to 0,
\end{aligned} \tag{2.7}$$

as $j \to \infty$. Hence, $\{(\|f_n^{(m,k)}\|, \mathcal{F}_n, n \geq l), (m, k) \in \mathcal{N}^2(l), l \geq 1\}$ is a uniform sequence of subpramarts and (a) holds. If $\sup_n E|F_n| < \infty$, then

$$\sup_n E \sup_{(m,k)\in\mathcal{N}^2(n)} \|f_n^{(m,k)}\| = \sup_n E|F_n| < \infty. \tag{2.8}$$

Since $(f_n^{(m,k)}, \mathcal{F}_n, n \geq m)$ is a quasi-martingale, it is a uniform amart, i.e.,

$$\lim_{t\in T} \sup_{s\in T(t)} E\|f_t^{(m,k)} - E(f_s^{(m,k)}|\mathcal{F}_t)\| = 0$$

(Bellow (1978)). If X has the Radon-Nikodým property, (2.4) follows from Bellow's (1978) uniform amart convergence theorem. Now we assume that for some $G \in \mathcal{L}_{wkc}$ $\liminf_n h^+(F_n, G) = 0$ a.s. By the Riesz decomposition theorem for uniform amarts (cf. [3 and 11]), $f_n^{(m,k)} = h_n^{(m,k)} + z_n^{(m,k)}$, where $(h_n^{(m,k)}, \mathcal{F}_n, n \geq m)$ is a martingale and $\lim_n z_n^{(m,k)} = 0$ a.s. and in L^1.

Hence $(h_n^{(m,k)}, \mathscr{F}_n, n \geq m)$ is an L^1-bounded martingale and for a.s. $\omega \in \Omega$, $(h_n^{(m,k)}(\omega))$ has a weakly accumulative point in $G(\omega)$. Applying [5, Proposition 4.4], $h_n^{(m,k)}$, and hence $f_n^{(m,k)}$, strongly converges to $f^{(m,k)} \in L_X^1$ and (2.4) holds. For each $x \in X$,

$$\begin{aligned}
&\limsup_n d(x, F_n) \\
&\quad = \limsup_n \left\{ \inf_{(m,k),(m',k') \in \mathscr{N}^2(n), a \in [0,1]} d(x, af_n^{(m,k)} + (1-a) f_n^{(m',k')}) \right\} \\
&\quad \leq \inf_{(m,k),(m',k') \in \mathscr{N}^2, a \in [0,1]} \lim_n d(x, af_n^{(m,k)} + (1-a) f_n^{(m',k')}) \\
&\quad = \inf_{(m,k),(m',k') \in \mathscr{N}^2, a \in [0,1]} d(x, af^{(m,k)} + (1-a) f^{(m',k')}) = d(x, F) \quad \text{a.s.},
\end{aligned}$$

(2.5) holds. By Lemma 2.2,

$$\begin{aligned}
\lim_n |F_n| &= \lim_n \sup_{(m,k) \in \mathscr{N}^2(n)} \|f_n^{(m,k)}\| \\
&= \sup_{(m,k) \in \mathscr{N}^2} \lim_n \|f_n^{(m,k)}\| = \sup_{(m,k) \in \mathscr{N}^2} \|f^{(m,k)}\| \\
&= \sup_{(m,k),(m',k') \in \mathscr{N}^2, a \in [0,1]} \|af^{(m,k)} + (1-a) f^{(m',k')}\| = |F| \quad \text{a.s.},
\end{aligned}$$

(2.9) holds, and by Fatou's lemma, $E|F| \leq \liminf_n E|F_n| < \infty$, $F \in \mathscr{L}_c^1$. For any fixed $x^* \in X^*$ and $(m, k) \in \mathscr{N}^2$, let

$$x_n^{(m,k)} = \langle x^*, f_n^{(m,k)} \rangle, \qquad n \geq m.$$

Since $|x_n^{(m,k)}| \leq \|x^*\| \, \|f_n^{(m,k)}\|$ and

$$|x_t^{(m,k)} - E(x_s^{(m,k)} | \mathscr{F}_t)| \leq \|x^*\| \, \|f_t^{(m,k)} - E(f_s^{(m,k)} | \mathscr{F}_t)\|,$$

by (2.7) and (2.8), $\{(x_n^{(m,k)}, \mathscr{F}_n, n \geq l), (m, k) \in \mathscr{N}^2(l), l \geq 1\}$ is a uniform sequence of subpramarts satisfying

$$\liminf_n E \sup_{(m,k) \in \mathscr{N}^2(n)} |x_n^{(m,k)}| < \infty.$$

Applying Lemma 2.2 again,

$$\begin{aligned}
\lim_n s(x^*, F_n) &= \lim_n \sup_{(m,k) \in \mathscr{N}^2(n)} x_n^{(m,k)} \\
&= \sup_{(m,k) \in \mathscr{N}^2} \lim_n x_n^{(m,k)} = \sup_{(m,k) \in \mathscr{N}^2} \langle x^*, f^{(m,k)} \rangle \\
&= \sup_{(m,k),(m',k') \in \mathscr{N}^2, a \in [0,1]} \langle x^*, af^{(m,k)} + (1-a) f^{(m',k')} \rangle \\
&= s(x^*, F) \quad \text{a.s.},
\end{aligned}$$

(2.10) holds. Q.E.D.

We denote by B^* the closed unit ball of X^* and by M^* a countable subset of B^* which is dense in the Mackey topology.

Lemma 2.4. *Assume that* $(F_n, \mathscr{F}_n, n \geq 1)$ *is an* L^1*-bounded submartingale and assumption* (A) *holds. Let* F *be the random multivalued function constructed in* (2.6). *If* $F \in \mathscr{L}_{wkc}$, *particularly, if for some* $G \in \mathscr{L}_{wkc}$ $\liminf_n h^+(F_n, G) = 0$ *a.s., then for a.s.* $\omega \in \Omega$, $F_n(\omega)$ *is Wijsman convergent to* $F(\omega)$.

Proof. If $G \in \mathscr{L}_{wkc}$ and $\liminf_n h^+(F_n, G) = 0$ a.s., then

$$G \supset \overline{\mathrm{co}}\{f^{(m,k)}, (m, k) \in \mathscr{N}^2\} = F \quad \text{a.s.}$$

(since $F_n \ni f_n^{(m,k)} \to f^{(m,k)}$ a.s. as $m \leq n \to \infty$), and $F \in \mathscr{L}_{wkc}$. It is easy to see that for any $x \in X$ and $A \in 2^X$,

$$d(x, A) \geq \sup_{x^* \in B^*} [\langle x^*, x\rangle - s(x^*, A)],$$

and if $A \in \mathscr{L}_{wkc}$,

$$d(x, A) = \sup_{x^* \in M^*} [\langle x^*, x\rangle - s(x^*, A)].$$

Hence, by (2.10), which has been proved in Lemma 2.3,

$$\begin{aligned}\liminf_n d(x, F_n) &\geq \liminf_n \sup_{x^* \in B^*} [\langle x^*, x\rangle - s(x^*, F_n)] \\ &\geq \sup_{x^* \in M^*} \liminf_n [\langle x^*, x\rangle - s(x^*, F_n)] \\ &= \sup_{x^* \in M^*} [\langle x^*, x\rangle - s(x^*, F)] = d(x, F) \quad \text{a.s.},\end{aligned}$$

and the a.s. Wijsman convergence follows from (2.5), the equicontinuity of $\{d(\cdot, A), A \in 2^X \backslash \phi\}$, and the separability of X. Q.E.D.

Let $\mathbf{L}_c^1$ be the closure of the set of all simple functions in $(\mathscr{L}_c^1, \Delta)$. It is easy to show that $G \in \mathbf{L}_c^1$ if and only if G a.s. takes values in a separable subset of $(P_c(X), h)$. When X is a finite dimensional space, $\mathbf{L}_c^1 = \mathscr{L}_c^1$.

Theorem 2.1. *Assume that* $(F_n, \mathscr{F}_n, n \geq 1)$ *is an* L^1*-bounded submartingale and assumption* (A) *holds. Let* F *be the random multivalued function constructed in* (2.6). *Then* $F \in \mathscr{L}_c^1$ *and the following hold*:

(a)

$$|F_n| \to |F| \quad \textit{a.s.} \tag{2.9}$$

and

$$\lim_n s(x^*, F_n) = s(x^*, F) \quad \textit{a.s.}, \qquad x^* \in X^*. \tag{2.10}$$

(b) *If* X^* *is strongly separable or assumption* (B) *holds, then*

$$F_n \xrightarrow{\mathrm{W}} F \quad \textit{a.s.} \tag{2.11}$$

and

$$F_n \xrightarrow{\text{K-M}} F \quad \textit{a.s.} \tag{2.12}$$

(c) *If* $F \in \mathscr{L}_{wkc}$, *particularly, if for some* $G \in \mathscr{L}_{wkc}$ $\liminf_n h^+(F_n, G) = 0$ *a.s., then for a.s.* $\omega \in \Omega$, $F_n(\omega)$ *is Wijsman convergent to* $F(\omega)$.

(d) *If* $F \in \mathbf{L}_c^1$, *then*

(d1) (2.12) *holds*;

(d2) *if* (i), (ii), *or* (iii) *holds*: (i) $(|F_n|)$ *is uniformly integrable,* (ii) X^* *is strongly separable,* (iii) $(F_n) \subset \mathbf{L}_c^1$ *a.s., then for a.s.* $\omega \in \Omega$, $F_n(\omega)$ *is Wijsman convergent to* $F(\omega)$ *and* $h^+(F_n, F) \to 0$ *a.s.*

Proof. (a) and (c) have been proved in Lemmas 2.3 and 2.4.

Proof of (b). If X^* is strongly separable, let X_d^* be a dense and countable subset of the unit ball B^*. By (2.9) and (2.10), there is a P-null set N such that for each $\omega \in \Omega \backslash N$ and $x^* \in X_d^*$, $\sup_n |F(\omega)| < \infty$ and

$$\lim_n s(x^*, F_n(\omega)) = s(x^*, F(\omega)). \tag{2.13}$$

Then, by a routine dense method, (2.13) holds for each $\omega \in \Omega \backslash N$ and $x^* \in X^*$, and $F_n \xrightarrow{\mathrm{W}} F$ a.s. If assumption (B) holds: $h(F_n, G \cap F_n) \to 0$ a.s. for some $G \in \mathscr{L}_{wkc}$. Then, $F \subset G$ a.s. and, by (2.10), there is a P-null set N such that for each $\omega \in \Omega \backslash N$ and $x^* \in M^*$,

$$F(\omega) \subset G(\omega), \quad \lim_n s(x^*, F_n(\omega)) = \lim_n s(x^*, F_n(\omega) \cap G(\omega)) = s(x^*, F(\omega)). \tag{2.14}$$

Since $(s(x^*, F_n(\omega) \cap G(\omega)), n \geq 1)$ is equicontinuous for the Mackey topology, hence, (2.14) holds for each $x^* \in X^*$, and (2.11) holds. Since F_n is convex, it is easy to see that

$$F = \overline{\mathrm{co}}\{f^{(m,k)}, (m, k) \in \mathscr{N}^2\} \subset s\text{-}\liminf_n F_n \quad \text{a.s.}, \tag{2.15}$$

and by (2.11),

$$w\text{-}\limsup_n F_n \subset F_n \subset F \quad \text{a.s.}, \tag{2.16}$$

(cf. [14, Lemma 1.1]), (2.12) holds.

Proof of (d). We assume that $F \in \mathbf{L}_c^1$. To prove (2.12), by (2.15), we need only to show that (2.16) is true. Choose $(X_n, n \geq 1) \subset P_c(X)$ such that for a.s. $\omega \in \Omega$, $F(\omega) \in \mathrm{cl}\{X_n, n \geq 1\}$, the closure in $(P_c(X), h)$. For each $i \geq 1$, let $(x_{i,j}, j \geq 1)$ be a dense subset in $X \backslash X_i$. For $i, j \geq 1$, by the separation theorem, there exists $x_{i,j}^* \in B^*$ such that

$$\langle x_{i,j}^*, x_{i,j} \rangle \geq s(x_{i,j}^*, X_i) + d(x_{i,j}, X_i). \tag{2.17}$$

Choose a P-null set N such that for each $\omega \in \Omega \backslash N$, (X_n) is dense in the set $\{F(\omega), \omega \in \Omega \backslash N\}$, and

$$\lim_n s(x_{i,j}^*, F_n(\omega)) \to s(x_{i,j}^*, F(\omega)), \qquad i, j \geq 1.$$

If (2.16) is false, then there exist $\omega \in \Omega \backslash N$, $x \in \omega\text{-}\limsup_n F_n(\omega) \backslash F(\omega)$, and $c > 0$ such that $d(x, F(\omega)) > 5c$. Choose $i \geq 1$ and $j \geq 1$ such that $h(X_i, F(\omega)) < c$ and $d(x_{i,j}, x) < c$. Then $d(x_{i,j}, X_i) \geq d(x_{i,j}, F(\omega)) - h(X_i, F(\omega)) \geq d(x, F(\omega)) - d(x, x_{i,j}) - h(X_i, F(\omega)) > 3c$. Since $x \in w\text{-}\limsup_n F_n(\omega)$, by (2.17),

$$\begin{aligned}
\limsup_n s(x_{i,j}^*, F_n(\omega)) &\geq \langle x_{i,j}^*, x \rangle \geq \langle x_{i,j}^*, x_{i,j} \rangle - d(x_{i,j}, x) \\
&\geq s(x_{i,j}^*, X_i) + d(x_{i,j}, X_i) - c \\
&\geq s(x_{i,j}^*, F(\omega)) - h(X_i, F(\omega)) + 2c > s(x_{i,j}^*, F(\omega)) + c,
\end{aligned}$$

which is a contradiction, and (d1) holds. Now we prove (d2). By Theorem 3.1 below, $h(E(F|\mathscr{F}_n, F)) \to 0$ a.s.

(i) If $(|F_n|)$ is uniformly integrable, then, by Theorem 2.2(ii) below, $F_n \subset E(F|\mathscr{F}_n)$ a.s., and $h^+(F_n, E(F|\mathscr{F}_n)) = 0$ a.s. Hence,

$$h^+(F_n, F) \le h^+(F_n, E(F|\mathscr{F}_n)) + h(E(F|\mathscr{F}_n), F) \to 0 \quad \text{a.s.} \tag{2.18}$$

(ii) If X^* is strongly separable, let X^*_{d1} be a dense and countable subset of B^*. Since for any $(A, B) \subset P_{cb}(X)$,

$$h^+(A, B) = \sup_{x^* \in B^*} [s(x^*, A) - s(x^*, B)] = \sup_{x^* \in X^*_{d1}} [s(x^*, A) - s(x^*, B)],$$

(cf. [4, Theorem II-18]), by (2.10) and Lemmas 2.1 and 2.2, $(s(x^*, \mathscr{F}_n) - s(x^*, E(F|\mathscr{F}_n)), \mathscr{F}_n, n \ge 1)$ is a real-valued submartingale,

$$h^+(F_n, E(F|\mathscr{F}_n)) = \sup_{x^* \in X^*_{d1}} [s(x^*, F_n) - s(x^*, E(F|\mathscr{F}_n))] \to 0 \quad \text{a.s.},$$

and (2.18) holds.

(iii) If $(F_n) \subset \mathbf{L}^1_c$ a.s., then F_n and $E(F|\mathscr{F}_n)$ a.s. take values in a separable subset of $(P_c(X), h)$, since $F \in \mathbf{L}^1_c$ implies $E(F|\mathscr{F}_n) \in \mathbf{L}^1_c$. Choose $(X_n, n \ge 1) \subset P_{cb}(X)$ such that there is a P-null set N, (X_n) is dense in the set $\{F_n(\omega), E(F|\mathscr{F}_n)(\omega), \omega \in \Omega \backslash N, n \ge 1\}$. Then we can choose a countable subset X^*_{d2} of B^* such that

$$h^+(X_i, X_j) = \sup_{x^* \in X^*_{d2}} [s(x^*, X_i) - s(x^*, X_j)], \qquad i, j \ge 1,$$

and

$$h^+(F_n, E(F|\mathscr{F}_n)) = \sup_{X^*_{d2}} [s(x^*, F_n) - s(x^*, E(F|\mathscr{F}_n))] \to 0 \quad \text{a.s.},$$

(2.18) holds. Since $d(x, F_n) \ge d(x, F) - h^+(F_n, F)$,

$$\liminf_n d(x, F_n) \ge d(x, F) \quad \text{a.s.},$$

and the a.s. Wijsman convergence follows from (2.5). Q.E.D.

Theorem 2.2. *Assume that* $(F_n, \mathscr{F}_n, n \ge 1)$ *is an* L^1*-bounded submartingale and assumption* (A) *holds. Then the following are equivalent*:

(i) $(|F_n|)$ *is uniformly integrable*;

(ii) $(F_n, \mathscr{F}_n, 1 \le n \le \infty)$ *is a submartingale, where* $F_\infty = F$ *is constructed in* (2.6);

(iii) *for some* $H \in \mathscr{L}^1_c$, $F_n \subset E(H|\mathscr{F}_n)$ *a.s.,* $n \ge 1$; *and if* X^* *is strongly separable,*

(iv) *for some* $H \in \mathscr{L}^1_c$, $s(x^*, F_n) \xrightarrow{L^1} s(x^*, H)$, $x^* \in X^*$.

Proof. (i) $\Rightarrow$ (ii). We need only to show that for each $n \ge 1$,

$$S^1_{F_n}(\mathscr{F}_n) \subset S^1_{E(F|\mathscr{F}_n)}(\mathscr{F}_n), \tag{2.19}$$

where F is constructed in (2.6). For any $n \ge 1$, $f \in S^1_{F_n}(\mathscr{F}_n)$ and $\varepsilon > 0$, choose $l \ge n$, $1/2^l < \varepsilon$. Since $F_n \subset E(F_l|\mathscr{F}_n)$, there is $g \in S^1_{F_l}(\mathscr{F}_l)$ such that $E\|E(g|\mathscr{F}_n) - f\| < \varepsilon$. Recall that $F_l = \mathrm{cl}\{f_l^{l,k}, k \ge 1\}$ a.s., by [14, Lemma 1.3], there exist $K \ge 1$ and an $\mathscr{F}_l$-measurable partition $\{A_k, 1 \le k \le K\}$ of Ω such that

$$E\left\| g - \sum_{1 \le k \le K} f_l^{(l,k)} I(A_k) \right\| < \varepsilon.$$

Let $h = \sum_{1\le k\le K} f^{(l,k)} I(A_k)$. Then $h \in S_F^1$, $E(h|\mathscr{F}_n) \in S^1_{E(F|\mathscr{F}_n)}(\mathscr{F}_n)$ and

$$\begin{aligned}
E\|f - E(h|\mathscr{F}_n)\| &\le E\|f - E(g|\mathscr{F}_n)\| \\
&\quad + E\left\| E(g|\mathscr{F}_n) - E\left(\sum_{1\le k\le K} f_l^{(l,k)} I(A_k)|\mathscr{F}_n\right)\right\| \\
&\quad + E\left\| E\left(\sum_{1\le k\le K} f_l^{(l,k)} I(A_k)|\mathscr{F}_n\right) - E(h|\mathscr{F}_n)\right\| \\
&\le \varepsilon + E\left\| g - \sum_{1\le k\le K} f_l^{(l,k)} I(A_k)\right\| \\
&\quad + E\left\| \sum_{1\le k\le K} (E(f_l^{(l,k)} I(A_k)|\mathscr{F}_n) - E(f^{(l,k)} I(A_k)|\mathscr{F}_n))\right\| \\
&\le 2\varepsilon + \sum_{1\le k\le K} E\|f_l^{(l,k)} - E(f^{(l,k)}|\mathscr{F}_l)\| \\
&\le 2\varepsilon + \sum_{1\le k\le K} \sum_{i=l}^{l+L} E\|f_i^{(l,k)} - E(f_{i+1}^{(l,k)}|\mathscr{F}_i)\| \\
&\quad + \sum_{1\le k\le K} E\|f_{l+L+1}^{(l,k)} - E(f^{(l,k)}|\mathscr{F}_{l+L+1})\| \\
&\le 2\varepsilon + \sum_{1\le k\le K} \sum_{i=l}^{\infty} \frac{1}{2^{i+l+k+1}} + \sum_{1\le k\le K} E\|f_{l+L+1}^{(l,k)} - E(f^{(l,k)}|\mathscr{F}_{l+L+1})\| \\
&\le 2\varepsilon + \frac{1}{2^{2l}} + \sum_{1\le k\le K} E\|f_{l+L+1}^{(l,k)} - E(f^{(l,k)}|\mathscr{F}_{l+L+1})\| \\
&\le 3\varepsilon + \sum_{1\le k\le K} E\|f_{l+L+1}^{(l,k)} - E(f^{(l,k)}|\mathscr{F}_{l+L+1})\| \to 3\varepsilon,
\end{aligned}$$

as $L \to \infty$, since $(|F_n|)$ is uniformly integrable and $f_i^{(l,k)}$ and $E(f^{(l,k)}|f_i)$ L^1-converge to $f^{(l,k)}$. Therefore, $f \in S^1_{E(F|\mathscr{F}_n)}(\mathscr{F}_n)$, (2.19) holds. (ii) $\Rightarrow$ (iii) is clear. (iii) $\Rightarrow$ (i): For each $n \ge 1$, by Lemma 2.1, $|F_n| \le |E(H|\mathscr{F}_n)| \le E(|H|\,|\mathscr{F}_n)$ a.s., (i) holds. By (2.10), (i) implies (iv). (iv) $\Rightarrow$ (iii): By (iv) and Lemma 2.1, $(s(x^*, F_n), \mathscr{F}_n, n \ge 1)$ is a closed real-valued submartingale,

$$s(x^*, F_n) \le E(s(x^*, H)|\mathscr{F}_n) = s(x^*, E(H|\mathscr{F}_n)) \quad \text{a.s.}, \qquad x^* \in X^*,$$

which implies (iii) if X^* is strongly separable. Q.E.D.

Remark 2.1. For an L^1-bounded martingale $(F_n, \mathscr{F}_n, n \ge 1)$, Papageorgiou (1989) proved that $|F_n|$ converges a.s. In Theorem 2.1, we identify the limit under assumption (A).

Remark 2.2. Under additional conditions: X is reflexive and $\{s(x^*, F_n), n \ge 1\}$ is a.s. equi-lower-semicontinuous, Papageorgiou (1985b) proved that if sub-

martingale $(F_n, \mathscr{F}_n, n \geq 1)$ is L^1-bounded, then $F_n \overset{\text{K-M}}{\to} H$ a.s. for some $H \in \mathscr{L}_c^1$; and if $(|F_n|)$ is uniformly integrable, then the submartingale is closed.

3. On multivalued martingales

Under different assumptions, Van Cutsem (1969), Neveu (1972), Daures (1973), Hiai and Umegaki (1977), and Papageorgiou (1989) proved convergence and closedness theorems for multivalued martingales. In this section we continue this study. We begin with a multivalued martingale convergence theorem in Lévy's type.

Theorem 3.1. *Suppose that* $F \in \mathbf{L}_c^1$. *Let* $F_n = E(F|\mathscr{F}_n)$, $n \geq 1$. *Then* $F_n \overset{\mathbf{h}}{\to} F_\infty$ *a.s. and* $\Delta(F_n, F_\infty) \to 0$, *where* $F_\infty = E(F|\mathscr{F}_\infty)$.

Proof. Without loss of generality, we may assume that F is $\mathscr{F}_\infty$-measurable. For any $\varepsilon > 0$, pick a simple function $H \in \mathscr{L}_c^1$ such that H is $\mathscr{F}_\infty$ measurable and $\Delta(F, H) < \varepsilon^2$. Assume that $H = \sum_{k=1}^K H_k I(A_k)$, where $\bigcup_{k=1}^K A_k = \Omega$, $A_k A_j = \varnothing$, $k \neq j$, and $H_k \in P_{cb}(X)$. Pick $\delta > 0$ such that

$$\delta < \varepsilon^2 / \left(2K \max_{1 \leq i \leq K} |H_i|\right).$$

Choose $n_1 < n_2 < \cdots < n_K$ and $B_k \in \mathscr{F}_{n_k}$ such that $\mathrm{P}(B_k \backslash A_k) + \mathrm{P}(A_k \backslash B_k) < \delta/3K$. Let

$$C_k = B_k \backslash \left(\bigcup_{1 \leq j < k} B_j\right), \qquad 1 \leq k < K, \quad C_K = \Omega \backslash \left(\bigcup_{1 \leq k < K} C_k\right),$$

and $G = \sum_{k=1}^K H_k I(C_k)$. Then $\mathrm{P}(C_k \backslash A_k) + \mathrm{P}(A_k \backslash C_k) < \delta$, and

$$\Delta(G, F) \leq \Delta(G, H) + \Delta(H, F) \leq 2K\delta \max_{1 \leq i \leq K} |H_i| + \varepsilon^2 < 2\varepsilon^2.$$

For any $n \geq n_K$, by [14, Lemma 2.6],

$$h(F_n, G) = h(E(F|\mathscr{F}_n), E(G|\mathscr{F}_n)) \leq E(h(F, G)|\mathscr{F}_n) \equiv h_n.$$

Let $t = \inf\{n \geq n_K, h_n > \varepsilon\}$ $(\inf\{\phi\} = \infty)$. Then

$$\mathrm{P}\left(\sup_{n \geq n_K} h_n > \varepsilon\right) \leq E h_t I(t < \infty)/\varepsilon \leq \Delta(F, G)/\varepsilon < 2\varepsilon,$$

and

$$\begin{aligned}\mathrm{P}\left(\sup_{n \geq n_K} h(F_n, F) > 2\varepsilon\right) &\leq \mathrm{P}\left(\sup_{n \geq n_K} h(F_n, G) > \varepsilon\right) + \mathrm{P}(h(G, F) > \varepsilon) \\ &\leq \mathrm{P}\left(\sup_{n \geq n_K} h_n > \varepsilon\right) + 2\varepsilon < 4\varepsilon.\end{aligned}$$

Hence $F_n \overset{\mathbf{h}}{\to} F$ a.s. Since $h(F_n, F) \leq |F_n| + |F| \leq E(|F|\,|\mathscr{F}_n) + |F|$,

$$\Delta(F_n, F) = E h(F_n, F) \to 0. \quad \text{Q.E.D.}$$

Theorem 3.2. *Assume that* $(F_n, \mathscr{F}_n, n \geq 1)$ *is an* L^1*-bounded martingale and assumption* (A) *holds. Then the following are equivalent*:

(i) $(|F_n|)$ *is uniformly integrable*;

(ii) $(F_n, \mathscr{F}_n, 1 \leq n \leq \infty)$ *is a martingale, where* $F_\infty = F$ *is constructed in* (2.6);

(iii) *for some* $H \in \mathscr{L}_c^1$, $F_n = E(H|\mathscr{F}_n)$ *a.s.*, $n \geq 1$;
and if X^* *is strongly separable,*

(vi) *for some* $H \in \mathscr{L}_c^1$, $s(x^*, F_n) \xrightarrow{L^1} s(x^*, H)$, $x^* \in X^*$.

Proof. By Theorem 2.2 and its proof, we need only to show that (i) implies $E(F|\mathscr{F}_n) \subset F_n$ a.s. For any $n \geq 1$, $\mathscr{F}_\infty$-measurable partition $(A_k, 1 \leq k \leq K)$ of Ω, and $\bar{k} \in \mathscr{N}^2$, $1 \leq k \leq K$, there is $L \geq 1$ such that $\bar{k} \in \mathscr{N}^2(L)$, $1 \leq k \leq K$. Let $f = \sum_{1 \leq k \leq K} f^{\bar{k}} I(A_k)$ and $f_l = \sum_{1 \leq k \leq K} \mathrm{P}(A_k|\mathscr{F}_{n+l}) f_{n+l}^{\bar{k}}$, $l \geq L$, where $f^{\bar{k}}$ and $f_{n+k}^{\bar{k}}$ are defined in Lemma 2.3. Then $E(f|\mathscr{F}_n) \in S^1_{E(F|\mathscr{F}_n)}(\mathscr{F}_n)$ and $E(f_l|\mathscr{F}_n) \in S^1_{E(F_{n+l}|\mathscr{F}_n)}(\mathscr{F}_n) = S^1_{F_n}(\mathscr{F}_n)$, since F and F_{n+l} are convex. By (i),

$$\begin{aligned} &E\|E(f|\mathscr{F}_n) - E(f_l|\mathscr{F}_n)\| \\ &\quad \leq E\|f - f_l\| \leq \sum_{1 \leq k \leq K} E\|f^{\bar{k}} I(A_k) - f_{n+l}^{\bar{k}} \mathrm{P}(A_k|\mathscr{F}_{n+l})\| \to 0 \end{aligned}$$

as $l \to \infty$, $E(f|\mathscr{F}_n) \in S^1_{F_n}(\mathscr{F}_n)$, and applying [15, Lemma 1.3], $S^1_{E(F|\mathscr{F}_n)}(\mathscr{F}_n) \subset S^1_{F_n}(\mathscr{F}_n)$, which implies $E(F|\mathscr{F}_n) \subset F_n$ a.s. Q.E.D.

For martingales, in addition to the convergence results in Theorems 2.1, 3.1, and 4.1 in the next section, we have the following Hausdorff convergence result.

Theorem 3.3. *Assume that* $(F_n, \mathscr{F}_n, n \geq 1)$ *is an* L^1*-bounded martingale, assumption* (A) *holds, and* F *constructed in* (2.6) *is in* $\mathbf{L}_c^1$. *If* (i), (ii), *or* (iii) *holds*: (i) $(|F_n|)$ *is uniformly integrable*; (ii) X^* *is strongly separable*; (iii) $(F_n) \subset \mathbf{L}_c^1$ *a.s., then*

$$F_n \xrightarrow{h} F \text{ a.s.} \tag{3.1}$$

Proof. (3.1) follows from the proof of (d2) in Theorem 2.1, noticing that $(s(x^*, F_n) - s(x^*, E(F|\mathscr{F}_n)), \mathscr{F}_n, n \geq 1)$ now is a martingale. Q.E.D.

Remark 3.1. Theorem 3.1 was obtained by Hiai and Umegaki (1977) under an additional assumption that X is reflexive or $F(\omega)$ is compact for a.s. ω. They also constructed an example [14, Example 6.6], showing that the condition $F \in \mathbf{L}_c^1$ cannot be weakened by $F \in \mathscr{L}_c^1$ even if X is reflexive.

Remark 3.2. (1) (i) $\Rightarrow$ (iii) in Theorem 3.2 was obtained by Hiai and Umegaki (1977) (under an additional condition that X^* is strongly separable) and Papageorgiou (1989). (2) When X is the separable dual space of a Banach space, Neveu (1972) proved (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (vi) in Theorem 3.2.

Remark 3.3. When X is the separable dual of a Banach space, Neveu (1972) proved (2.10) for L^1-bounded martingales and proved (3.1) when $F \in \mathbf{L}_c^1$; Daures (1973) proved (3.1) under conditions: F_n is a.s. compact, $n \geq 1$, and X is reflexive; and under the condition $E \sup_n |F_n| < \infty$, Papageorgiou (1989) proved $F_n \xrightarrow{\text{K-M}} H$ a.s. for some $H \in \mathscr{L}_c^1$.

4. On multivalued supermartingales

In Van Cutsem (1972), the author proved convergence theorems for multivalued supermartingales when X is a finite dimensional space. When X is

a general Banach space, Papageorgiou (1987) proved that if a supermartingale $(F_n, \mathscr{F}_n, n \geq 1)$ is contained in $G \in \mathscr{L}_{wkc} \cap \mathscr{L}_c^1$: $\bigcup_n F_n \subset G$ a.s., and if X^* is strongly separable, then $F_n \xrightarrow{\mathrm{W}} F$ a.s. for some $F \in \mathscr{L}_c^1$. Recently, Hess (1991) developed a truncation argument in Van Cutsem (1972), and proved a Kuratowski-Mosco convergence theorem for supermartingales with unbounded values when $\bigcup_n F_n \subset G$ for some $G \in \mathscr{L}_{lwkc}$, where

$$\mathscr{L}_{lwkc} = \{G \in \mathscr{L}_c \colon G \cap B(0, r) \in \mathscr{L}_{wkc}, r > 0\},$$

and $B(0, r)$ is the closed ball of radius r, centered at 0.

Definition 4.1. Given $(F_n) \subset \mathscr{L}_c^1$, we say that (i) assumption (A′) holds, if X^* is strongly separable and there is a subsequence $\{n_k\}$ such that $F_{n_k} \in \mathscr{L}_{lwkc}$ and either (a) or (b) in the following holds: (a) X has the Radon-Nikodým property, (b) $\liminf_n h^+(F_n, G) = 0$ a.s. for some $G \in \mathscr{L}_{lwkc}$; (ii) assumption (B′) holds, if $F_n \in \mathscr{L}_{lwkc}$ and $h(F_n, G \cap F_n) \to 0$ a.s. for some $G \in \mathscr{L}_{lwkc}$; (iii) assumption (C′) holds, if there is a subsequence $\{n_k\}$ such that $F_{n_k} \in \mathscr{L}_{lwkc}$ and either (a) or the following (c) holds: (c) $\liminf_k h^+(F_{n_k}, G) = 0$ a.s. for some $G \in \mathscr{L}_{lwkc}$.

In this section we prove the following convergence and closedness results for multivalued supermartingales.

Theorem 4.1. *Suppose that* $(F_n, \mathscr{F}_n, n \geq 1)$ *is a supermartingale such that* $\sup_n Ed(0, F_n) < \infty$ *and assumption* (A′) *or* (B′) *holds. Then there exists* $F \in \mathscr{L}_c^{d1}$ *such that*

(i)

$$F_n \xrightarrow{\text{K-M}} F \quad a.s.; \tag{4.1}$$

(ii) *if* $\liminf_n |F_n| < \infty$,

$$\lim_n s(x^*, F_n) = s(x^*, F) \quad a.s., \qquad x^* \in X^*, \tag{4.2}$$

and if $\limsup_n |F_n| < \infty$ *a.s., then*

$$F_n \xrightarrow{\mathrm{W}} F; \tag{4.3}$$

(iii) *if for some* $G \in \mathscr{L}_{lwkc}$, $\liminf_n h^+(F_n, G) = 0$ *a.s., then for a.s.* $\omega \in \Omega$, $F_n(\omega)$ *is Wijsman convergent to* $F(\omega)$.

Theorem 4.2. *Let* $(F_n, \mathscr{F}_n, n \geq 1)$ *be a supermartingale. If assumption* (C′) *holds, then there exists* $F \in \mathscr{L}_c^{d1}$ *such that* $E(F|\mathscr{F}_n) \subset F_n$ *a.s. if and only if* $(d(0, F_n), n \geq 1)$ *is uniformly integrable.*

Remark 4.1. Under the assumption $\bigcup_n F_n \subset G$ for some $G \in \mathscr{L}_{lwkc}$, Hess (1991) proved that if $\sup_n Ed(0, F_n) < \infty$, then (4.1) holds; and if $(d(0, F_n))$ is uniformly integrable, then supermartingale $(F_n, \mathscr{F}_n, n \geq 1)$ is closed. Under the assumption $\bigcup_n F_n \subset G$ for some $G \in \mathscr{L}_{wkc}$, Hess (1991) also proved (4.3) and the a.s. Wijsman convergence of F_n to F.

To prove Theorems 4.1 and 4.2 we need the following lemmas.

Lemma 4.1. *Suppose that* $(A_n, n \geq 1) \subset P_c(X)$, $A_1 \supset A_2 \supset \cdots$, *and* $h(A_n, G \cap A_n) \to 0$ *for some* $G \in P_{wkc}(X)$. *Then* $A \equiv \bigcap_n A_n \in P_{wkc}(X)$ *and*

$$\lim_n s(x^*, A_n) = s(x^*, A), \qquad x^* \in X^*. \tag{4.4}$$

Proof. It is easy to show that $A \in P_{wkc}(X)$, and $\lim_n s(x^*, A_n) \geq s(x^*, A)$, $x^* \in X^*$. Now for any fixed $x^* \in X^*$, choose $y_n \in A_n \cap G$, $s(x^*, y_n) > s(x^*, A_n) - 1/n - \|x^*\| h(A_n, A_n \cap G)$. Then there is a subsequence $(y_{n_k}, k \geq 1)$ such that $y_{n_k} \xrightarrow{w} y$. It is easy to see that $y \in A$ and

$$\lim_n s(x^*, A_n) = \lim_k \langle x^*, y_{n_k} \rangle = \langle x^*, y \rangle \leq s(x^*, A),$$

and (4.4) holds. Q.E.D.

Lemma 4.2. *Suppose that* $(F_n, n \geq 1) \subset \mathscr{L}_c^{d1}$, $F_1 \supset F_2 \supset \cdots$ *a.s., and* $h(F_n, G \cap F_n) \to 0$ *a.s. for some* $G \in \mathscr{L}_{wkc}$. *Then* $F \equiv \bigcap_n F_n \in \mathscr{L}_{wkc}$ *and for any sub-σ-algebra* $\mathscr{F} \subset \mathscr{A}$,

$$s(x^*, E(F|\mathscr{F})) = s\left(x^*, \bigcap_n E(F_n|\mathscr{F})\right) \quad a.s., \qquad x^* \in X^*. \tag{4.5}$$

And if $\bigcap_n E(F_n|\mathscr{F}) \in \mathscr{L}_{wkc}$ *or* X^* *is strongly separable, then*

$$E(F|\mathscr{F}) = \bigcap_n E(F_n|\mathscr{F}) \quad a.s. \tag{4.6}$$

Proof. By Lemma 4.1, $F \in \mathscr{L}_{wkc}$, and for any $x^* \in X^*$, by Lemma 2.1 and the monotone convergence theorem,

$$\begin{aligned} s(x^*, E(F|\mathscr{F})) &= E(s(x^*, F)|\mathscr{F}) = E\left(\lim_n s(x^*, F_n)|\mathscr{F}\right) \\ &= \lim_n E(s(x^*, F_n)|\mathscr{F}) = \lim_n s(x^*, E(F_n|\mathscr{F})) \\ &\geq s\left(x^*, \bigcap_n E(F_n|\mathscr{F})\right) \geq s(x^*, E(F|\mathscr{F})) \quad \text{a.s.}, \end{aligned}$$

(4.5) holds. If X^* is strongly separable, then (4.6) follows from (4.5). If $\bigcap_n E(F_n|\mathscr{F}) \in \mathscr{L}_{wkc}$ a.s., then $\bigcap_n E(F_n|\mathscr{F}) \supset E(F|\mathscr{F}) \in \mathscr{L}_{wkc}$ a.s., and by the continuity in the Mackey topology, (4.5) implies (4.6). Q.E.D.

Definition 4.2. Given $(F_n) \subset \mathscr{L}_c^{d1}$, we say that (i) assumption (A_1') holds, if X^* is strongly separable and if there is a subsequence $\{n_k\}$ such that $F_{n_k} \in \mathscr{L}_{wkc}$, $k \geq 1$; (ii) assumption (B_1') holds, if $F_n \in \mathscr{L}_{wkc}$, $n \geq 1$.

Given a supermartingale $(F_n, \mathscr{F}_n, n \geq 1)$, we define $G_n = \bigcap_{m \geq n} E(F_m|\mathscr{F}_n)$.

Lemma 4.3. *Let* $(F_n, \mathscr{F}_n, n \geq 1)$ *be a supermartingale satisfying assumption* (A_1') *or* (B_1'). *Then* $(G_n, \mathscr{F}_n, n \geq 1)$ *is a martingale.*

Proof. By the definition of supermartingales, we have

$$F_n \supset E(F_{n+1}|\mathscr{F}_n) \supset E(F_{n+2}|\mathscr{F}_n) \supset \cdots \text{ a.s.}, \qquad n \geq 1.$$

If assumption (B_1') holds, then, by Lemma 4.2, $G_n \in \mathscr{L}_{wkc}$ and

$$\begin{aligned} E(G_{n+1}|\mathscr{F}_n) &= E\left(\bigcap_{m \geq n+1} E(F_m|\mathscr{F}_{n+1})|\mathscr{F}_n\right) \\ &= \bigcap_{m \geq n} E(F_m|\mathscr{F}_n) = G_n \quad \text{a.s.} \end{aligned}$$

In the following we assume that X^* is strongly separable and there exists a subsequence (n_k) such that $F_{n_k} \in \mathscr{L}_{wkc}$, $k \geq 1$. Then, for any $k \geq 1$, $\bigcap_{m \geq n_k} E(F_m|\mathscr{F}_{n_k}) \in \mathscr{L}_{wkc}$. For any $n \geq 1$, choose $n_{k-1} \geq n$, then, by Lemma 4.2,

$$G_n = \bigcap_{m \geq n} E(F_m|\mathscr{F}_n) = \bigcap_{m \geq n_{k-1}} E(E(F_m|\mathscr{F}_{n_{k-1}})|\mathscr{F}_n)$$
$$= E\left(\bigcap_{m \geq n_{k-1}} E(F_m|\mathscr{F}_{n_{k-1}})|\mathscr{F}_n\right) \in \mathscr{L}_c^{d1},$$

and

$$E(G_{n+1}|\mathscr{F}_n) = E\left(\bigcap_{m \geq n+1} E(F_m|\mathscr{F}_{n+1})|\mathscr{F}_n\right)$$
$$= E\left(\bigcap_{m \geq n_k} E(E(F_m|\mathscr{F}_{n_k})|\mathscr{F}_{n+1})|\mathscr{F}_n\right)$$
$$= E\left(E\left(\bigcap_{m \geq n_k} E(F_m|\mathscr{F}_{n_k})|\mathscr{F}_{n+1}\right)|\mathscr{F}_n\right)$$
$$= E\left(\bigcap_{m \geq n_k} E(F_m|\mathscr{F}_{n_k})|\mathscr{F}_n\right) = \bigcap_{m \geq n_k} E(E(F_m|\mathscr{F}_{n_k})|\mathscr{F}_n)$$
$$= \bigcap_{m \geq n_k} E(F_m|\mathscr{F}_n) = G_n \quad \text{a.s.}$$

$(G_n, \mathscr{F}_n, n \geq 1)$ is a martingale. Q.E.D.

For a supermartingale $(F_n, \mathscr{F}_n, n \geq 1)$, by Lemma 2.1, $(d(x, F_n), \mathscr{F}_n, n \geq 1)$ is a real-valued submartingale. Let $v_n = \sup_{m \geq n} E(d(0, F_m)|\mathscr{F}_n)$. It is well known that $(v_n, n \geq 1)$ is the martingale part in the Krickeberg decomposition of submartingale $(d(0, F_n), \mathscr{F}_n, n \geq 1)$, $\lim_n Ev_n = \lim_n Ed(0, F_n)$, and $(v_n, n \geq 1)$ is uniformly integrable if and only if $(d(0, F_n), n \geq 1)$ is uniformly integrable (cf. [6 and 22]).

In the following we assume that $(F_n, \mathscr{F}_n, n \geq 1)$ is a supermartingale satisfying $\sup_n Ed(0, F_n) < \infty$, and use the following Hess' (1991) truncation:

$$F_n^k = F_n \cap B(0, v_n + k), \qquad k \geq 1,$$

(since $v_n \geq d(0, F_n)$, F_n^k is not empty).

Lemma 4.4 (Hess (1991)). *$(F_n^k, \mathscr{F}_n, n \geq 1)$ is a supermartingale such that* (i) $\sup_n E|F_n^k| \leq \sup_n Ev_n + k < \infty$; (ii) $\sup_n |F_n^k| \leq \sup_n v_n + k < \infty$ *a.s.*; (iii) *if* $F_n^k \overset{\text{K-M}}{\to} F^k$, $k \geq 1$, *then* $F_n \overset{\text{K-M}}{\to} \bigcup_k F^k$ *a.s.*

Lemma 4.5. (i) *If* $d(x, F_n^k) \to d(x, F^k)$ *a.s.*, $k \geq 1$, *then*

$$d(x, F_n) \to d\left(x, \bigcup_k F^k\right) \quad \textit{a.s.}$$

(ii) *If* $\liminf_n |F_n| < \infty$ *a.s., and if* $s(x^*, F_n^k) \to s(x^*, F^k)$ *a.s.,* $k \geq 1$, *then* $s(x^*, F_n) \to s(x^*, \bigcup_k F^k)$ *a.s.*

(iii) *If* $\limsup_n |F_n| < \infty$ *a.s., and if* $F_n^k \xrightarrow{\mathrm{W}} F^k$ *a.s.,* $k \geq 1$, *then* $F_n \xrightarrow{\mathrm{W}} \bigcup_k F^k$ *a.s.*

Proof. (i) It is easy to show that

$$\lim_n d(x, F_n) \leq \inf_k \lim_n d(x, F_n^k) = \inf_k d(x, F^k) = d\left(x, \bigcup_k F^k\right) \quad \text{a.s.}$$

On the other hand, we can choose $x_n \in F_n$ (x_n is a function of ω) such that $d(x, F_n) \geq \|x - x_n\| - 1/n$. Then $(\|x - x_n\|, n \geq 1)$ is bounded and

$$\lim_n d(x, F_n) = \lim_n \|x - x_n\| \geq \inf_k \lim_n d(x, F_n^k) = d\left(x, \bigcup_k F^k\right) \text{ a.s.}$$

(ii) Since $Es(x^*, F_n)^- \leq Es(x^*, F_n^k)^- \leq \|x^*\| E(v_n + k)$, by Lemma 2.1, $(s(x^*, F_n), n \geq 1)$ and $(s(x^*, F_n^k), n \geq 1)$ are supermartingales and converge almost surely. If $\liminf_n |F_n| < \infty$ a.s., then for a.s. $\omega \in \Omega$, there exists $k = k(\omega) \geq 1$ such that for infinitely many $n \geq 1$ $F_n(\omega) = F_n^k(\omega)$, and

$$\begin{aligned}\lim_n s(x^*, F_n) &= \sup_k \lim_n s(x^*, F_n^k) = \sup_k s(x^*, F^k) \\ &= s\left(x^*, \bigcup_k F^k\right) \text{ a.s.}, \qquad x^* \in X^*,\end{aligned}$$

(ii) holds. (iii) If $\limsup_n |F_n| < \infty$ a.s. and if $F_n^k \xrightarrow{\mathrm{W}} F^k$, then the null set in the proof of (ii) can be independent of x^*, and (iii) holds. Q.E.D.

Proof of Theorem 4.1. By Lemmas 4.4 and 4.5, we may assume that $\sup_n E|F_n| < \infty$, $\sup_n |F_n| < \infty$ a.s. and the following (1) or (2) is satisfied: (1) X^* is strongly separable and assumption (A) holds; (2) assumption (B) holds. We may also assume that for some subsequence (n_k), $F_{n_k} \in \mathscr{L}_{wkc}$, $k \geq 1$, and in proof of (iii) $\liminf_n h^+(F_n, G) = 0$ a.s. for some $G \in \mathscr{L}_{wkc}$. Let $G_n = \bigcap_{m \geq n} E(F_m | \mathscr{F}_n)$. Then $(G_n, \mathscr{F}_n, n \geq 1)$ is an L^1-bounded martingale satisfying (1) or (2). Applying Theorem 2.1, there is $F \in \mathscr{L}_c^1$ such that

$$G_n \xrightarrow{\mathrm{W}} F \text{ a.s.} \quad \text{and} \quad G_n \xrightarrow{\mathrm{K-M}} F \text{ a.s.} \tag{4.7}$$

For any $n \geq 1$, choose $k \geq 1$ such that $n_k \geq n$. Then, by Lemmas 2.1, 4.1, 4.2 and the monotone convergence theorem for conditional expectations,

$$\begin{aligned}s(x^*, G_n) &= s\left(x^*, \bigcap_{m \geq n_k} E(E(F_m | \mathscr{F}_{n_k}) | \mathscr{F}_n)\right) \\ &= s\left(x^*, E\left(\bigcap_{m \geq n_k} E(F_m | \mathscr{F}_{n_k}) | \mathscr{F}_n\right)\right) = E\left(s\left(x^*, \bigcap_{m \geq n_k} E(F_m | \mathscr{F}_{n_k})\right) | \mathscr{F}_n\right) \\ &= E\left(\lim_m s(x^*, E(F_m | \mathscr{F}_{n_k})) | \mathscr{F}_n\right) = \lim_m E(s(x^*, E(F_m | \mathscr{F}_{n_k})) | \mathscr{F}_n) \\ &= \lim_m s(x^*, E(E(F_m | \mathscr{F}_{n_k}) | \mathscr{F}_n)) = \lim_m s(x^*, E(F_m | \mathscr{F}_n)) \\ &= \lim_m E(s(x^*, F_m) | \mathscr{F}_n) \quad \text{a.s.},\end{aligned}$$

and $(s(x^*, F_n), \mathscr{F}_n, n \geq 1)$ is a real-valued supermartingale satisfying

$$\sup_n E|s(x^*, F_n)| \leq \|x^*\| \sup_n E|F_n| < \infty.$$

Hence $(s(x^*, F_n), \mathscr{F}_n, n \geq 1)$ is a amart, of course, subpramart (cf. Millet and Sucheston (1980)), and

$$s(x^*, G_n) = \lim_m E(s(x^*, F_m)|\mathscr{F}_n) = \operatorname{ess} \inf_{t \in T(n)} E(s(x^*, F_t)|\mathscr{F}_n) \quad \text{a.s.}$$

Applying (4.7) and Lemma 2.2, we have

$$s(x^*, F) = \lim_n s(x^*, G_n) = \lim_n s(x^*, F_n) \quad \text{a.s.}, \qquad x^* \in X^*,$$

(4.2) holds. As the proof of (b) in Theorem 2.1, we get (4.1) and (4.3) from (4.2), noticing that

$$s\text{-}\liminf_n F_n \supset s\text{-}\liminf_n G_n = F \quad \text{a.s.}$$

Proof of (iii). Since there is $G \in \mathscr{L}_{wkc}$ such that $\liminf_n h^+(G_n, G) \leq \liminf_n h^+(F_n, G) = 0$ a.s., $G \supset F \in \mathscr{L}_{wkc}$ a.s. (see the proof of Lemma 2.4). By Lemma 2.4 and its proof, $\lim_n d(x, F_n) \leq \lim_n d(x, G_n) = d(x, F)$ a.s. and

$$\begin{aligned}\lim_n d(x, F_n) &\geq \limsup_n \sup_{x^* \in B^*} (\langle x^*, x\rangle - s(x^*, F_n)) \\ &\geq \sup_{x^* \in M^*} (\langle x^*, x\rangle - s(x^*, F)) = d(x, F) \quad \text{a.s.},\end{aligned}$$

(iii) holds. Q.E.D.

Proof of Theorem 4.2. Assume that for some subsequence (n_k), $F_{n_k} \in \mathscr{L}_{lwkc}$, $k \geq 1$. If $(d(0, F_n))$ is uniformly integrable, then (v_n), and hence $(|F_n^j|)$, is uniformly integrable, $j \geq 1$. Let $G_{n_k}^j = \bigcap_{m \geq n_k} E(F_m^j|\mathscr{F}_{n_k})$. Then, by Lemmas 4.4 and 4.3, $(G_{n_k}^j, \mathscr{F}_{n_k}, k \geq 1)$ is a uniformly integrable martingale satisfying assumption (A), and by Theorem 3.2, there is $F^j \in \mathscr{L}_c^1$ such that $G_{n_k}^j = E(F^j|\mathscr{F}_{n_k})$ a.s. For each $n \geq 1$ choose $k \geq 1$ such that $n_k \geq n$, then

$$\begin{aligned}F_n^j &\supset E(F_{n_k}^j|\mathscr{F}_n) \supset E(G_{n_k}^j|\mathscr{F}_n) \\ &= E(E(F^j|\mathscr{F}_{n_k})|\mathscr{F}_n) = E(F^j|\mathscr{F}_n) \quad \text{a.s.}\end{aligned}$$

Let $F = \bigcup_j F^j$. Then $F \in \mathscr{L}_c^{d1}$, and, by [14, Theorem 2.1],

$$F_n = \bigcup_j F_n^j \supset \operatorname{cl}\left(\bigcup_j E(F^j|\mathscr{F}_n)\right) = E\left(\bigcup_j F^j|\mathscr{F}_n\right) = E(F|\mathscr{F}_n) \quad \text{a.s.} \tag{4.9}$$

On the other hand, if (4.9) holds for some $F \in \mathscr{L}_c^{d1}$, then, by Lemma 2.1,

$$d(0, F_n) \leq d(0, E(F|\mathscr{F}_n)) \leq E(d(0, F)|\mathscr{F}_n),$$

$(d(0, F_n))$ is uniformly integrable. Q.E.D.

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