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OPERATOR SEMIGROUPS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

W. M. RUESS AND W. H. SUMMERS

ABSTRACT. We show that a strongly continuous operator semigroup can be associated with the functional differential delay equation

$$\text{(FDE)} \qquad \begin{cases} x'(t) + \alpha x(t) + Bx(t) \ni F(x_t), & t \geq 0 \\ x|_{\mathbf{R}^-} = \varphi \in E \end{cases}$$

under local conditions which give wide latitude to those subsets of the state space X and initial data space E, respectively, where (a) the (generally multivalued) operator $B \subseteq X \times X$ is defined and accretive, and (b) the history-responsive function $F: D(F) \subseteq E \to X$ is defined and Lipschitz continuous. The associated semigroup is then used to investigate existence and uniqueness of solutions to (FDE). By allowing the domain of the solution semigroup to be restricted according to specific local properties of B and F, moreover, our methods automatically lead to assertions on flow invariance. We illustrate our results through applications to the Goodwin oscillator and a single species population model.

The utility of operator semigroups in the context of the abstract Cauchy problem is well established (e.g., see [2, 14, 19]), and so it is natural that efforts have been made to also bring the theory of strongly continuous semigroups to bear on the qualitative study of functional differential equations with delay. The basic direction for this line of investigation was set in a series of articles initiated in the mid seventies by G. F. Webb [23, 24, 25], but a number of other authors (e.g., see [4–7, 13, 20]) have also substantially contributed to the development. Here, we essentially adopt the form of the problem as treated by D. W. Brewer [5, 6, 7] since it explicitly allows for the damping present in any physical process.

For a Banach (state) space X and $I = \mathbb{R}^-$ or $I = [-R, 0]$, $R > 0$, let E be a "suitably" chosen Banach space of functions from I into X. Given $\varphi \in E$, the problem is then to find (if possible) a function $x: I \cup \mathbb{R}^+ \to X$ which satisfies

$$\text{(FDE)} \qquad \begin{cases} x'(t) + \alpha x(t) + Bx(t) \ni F(x_t), & t \geq 0 \\ x(t) = \varphi(t), & t \in I, \end{cases}$$

where $\alpha \in \mathbb{R}$, $B \subseteq X \times X$ is a (possibly multivalued) accretive operator, $F: D(F) \subseteq E \to X$, and the function $x_t: I \to X$ defined by $x_t(r) = x(t+r)$, $r \in I$, belongs to E for each $t \geq 0$.

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Several complications arise. The first, but perhaps not the least, is the selection of the initial data space E, which is somewhat delicate in the case of infinite delays (cf. [16, 17]). Roughly speaking, the idea is then to transform (FDE) into a Cauchy problem in E and, under "suitable" restrictions, apply the Crandall-Liggett generation theorem to obtain a strongly continuous operator semigroup $(S(t))_{t\geq 0}$ in E such that $x(t) = S(t)\varphi(0)$ for $t \geq 0$ yields a viable candidate for a solution of (FDE). The most serious difficulty now comes from the questions as to when this approach actually leads to a classical solution of (FDE) and, if so, whether this is the only solution.

In order to apply the Crandall-Liggett theorem, it has been common to require that $F: E \to X$ be (globally) Lipschitz continuous, which precludes the type of operators that arise, for example, in Lotka-Volterra based models of population dynamics (see §1). On the other hand, Brewer [7] has given a semigroup approach to (FDE) in which F is only assumed to be locally Lipschitz continuous, but his techniques require that $F(0) = 0$ as well as that B is a (single valued) m-accretive operator with $B(0) = 0$, and only lead to the generation of semigroups in norm balls of E, which still excludes population models where a truncated cone, say, would be the appropriate setting.

Our objective in this paper is to provide the framework for a local approach to semigroup representations of solutions to (FDE) which is broad enough to apply in a variety of settings. As motivation for the form in which we place our results, we begin in §1 by considering two representative examples that provide insight into the direction we have chosen to take. In §2, we establish conditions under which a strongly continuous semigroup can be associated with (FDE) (Theorem 2.1), and then proceed to demonstrate that this semigroup leads to a representation of the (unique) solution to (FDE) under certain circumstances (Theorem 2.5). While our results synthesize much of the existing literature relating to the semigroup approach to (FDE)—and we also correct a few misconceptions along the way, the local context in which we work significantly extends the range of application by placing fewer restrictions on both (a) the history-responsive function $F: D(F) \subseteq E \to X$, and (b) the state-responsive operator $B \subseteq X \times X$. In particular, we not only show that greater latitude can be given to the subsets of the initial data space E where F is defined and Lipschitz continuous, but that it is neither necessary for B to be m-accretive nor even to have dense domain in the state space X (see Theorem 2.1, Theorem 2.5, and Proposition 2.7). As an added feature, our methods automatically lead to assertions on flow invariance by allowing the domain of the solution semigroup to be restricted according to specific local properties of F and B (compare §4). Furthermore, our existence result (Theorem 2.5(a)) for the special situation where X^* is uniformly convex and the operator $B \subseteq X \times X$ is multivalued and m-accretive extends the previous work in the global case to the multivalued setting.

Following a brief description of an explicit class of initial data spaces for which our results hold (§3), we conclude by returning to the examples of the first section to illustrate how our results apply in concrete situations.

1. Two typical examples

While relatively simple in concept, the following models give some indication as to the variety of conditions that can be encountered in practice. We start by considering the autonomous case of a single species population model treated by Bardi [3].

1.1 **Single species populations with infinite delay.** The model is based on the delay logistic equation

$$(1) \qquad \begin{cases} x'(t) = x(t)\left(a - bx(t) - \int_{-\infty}^{0} k(-s)x_t(s)\,ds\right), & t \geq 0 \\ x(t) = \varphi(t), \quad t \leq 0. \end{cases}$$

Here, $a, b > 0$ and $k \in L^1(\mathbb{R}^+, X) \cap C(\mathbb{R}^+, X)$, where $X = \mathbb{R}$, $I = \mathbb{R}^-$, and the initial condition φ belongs to the Banach space $E = BUC(\mathbb{R}^-, X)$ of all bounded uniformly continuous functions from $(-\infty, 0]$ into $\mathbb{R}$ under the usual supremum norm $\|\cdot\|_\infty$. As considered by Bardi [3], the integral operator on E is replaced by a more general operator $H: E \to \mathbb{R}$ subject to conditions which include the following:

(H1) H is locally Lipschitz continuous; i.e., given $\beta > 0$, there exists $m(\beta) \geq 0$ such that

$$|H(\varphi) - H(\psi)| \leq m(\beta)\|\varphi - \psi\|_\infty$$

whenever $\varphi, \psi \in E$ with $\|\varphi\|_\infty, \|\psi\|_\infty \leq \beta$;

(H2) There exists a continuous nondecreasing function $l: \mathbb{R}^+ \to \mathbb{R}^+$ with $l(0) = 0$ such that

$$|H(\varphi)| \leq l(\|\varphi\|_\infty) \quad \text{for all } \varphi \in E.$$

Thus, given $\varphi \in E$, the initial value problem in question takes the form

$$(2) \qquad \begin{cases} x'(t) = x(t)(a - bx(t) - H(x_t)), & t \geq 0 \\ x(t) = \varphi(t), \quad t \leq 0. \end{cases}$$

In order to consider (2) in the context of (FDE), fix $\alpha \geq 0$, put $Bx = bx^2 - \alpha x$, $x \in \mathbb{R}$, and define $F: E \to \mathbb{R}$ by $F(\varphi) = \varphi(0)(a - H(\varphi))$ for $\varphi \in E$. Though F is certainly locally Lipschitz continuous under (H1) and (H2), the above mentioned result by Brewer [7, Theorem 2, p. 375] does not apply. Since (2) is a population model, moreover, the domain of the associated semigroup should at least be limited to a truncated cone $\widehat{E} = \{\varphi \in E: 0 \leq \varphi \leq \beta\}$ for some $\beta > 0$ rather than $E_\beta = \{\varphi \in E: \|\varphi\|_\infty \leq \beta\}$. More to the point, only the restriction of B to $D(B) = [\alpha/(2b), \infty)$ is accretive, and this operator is neither m-accretive nor do we generally have that $0 \in D(B)$. On the other hand, even to establish the existence of (global) solutions to (2), Bardi [3] required further restrictions on the operator H. As we will show in the sequel, however, information on existence and asymptotic behavior of solutions to (FDE) can nonetheless be obtained via an associated operator semigroup in a context that includes (2) when H is only assumed to satisfy (H1) and (H2).

1.2 **Feedback in the Goodwin oscillator.** The Goodwin oscillator with infinite delay (cf. [18]) is a model for biochemical reaction sequences with end product inhibition described for $t \geq 0$ by the system

$$(1) \qquad \begin{cases} x_1'(t) + a_1 x_1(t) = b_1 \left[1 + \left(\int_{-\infty}^{0} k(-s)x_n(t+s)\,ds\right)^m\right]^{-1} \\ x_i'(t) + a_i x_i(t) = b_i \int_{-\infty}^{0} k(-s)x_{i-1}(t+s)\,ds, \qquad i = 2, \ldots, n, \end{cases}$$

where $a_i, b_i > 0$ for $i = 1, \ldots, n$, $k \in L^1(\mathbb{R}^+) \cap C(\mathbb{R}^+)$ with $k \geq 0$, $m \in \mathbb{N}$, and $x_i|_{\mathbb{R}^-} \in BUC(\mathbb{R}^-, \mathbb{R}^+)$, $i = 1, \ldots, n$. Since the pattern remains the same when $n > 2$, however, we shall hereafter restrict attention to chains of two reactions $(n = 2)$.

Taking $X = \mathbb{R}^2$, $I = \mathbb{R}^-$, and E to be the sup-normed Banach space $BUC(\mathbb{R}^-, X)$, we now view 1.2(1) in the context of (FDE) by choosing $\alpha = \min\{a_1, a_2\}$ and

$$B = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} - \alpha I.$$

Here, of course, $D(F)$ is the cone E^+ of all

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in E$$

such that $\varphi_i \geq 0$ for $i = 1, 2$, and

$$F(\varphi) = \begin{pmatrix} f_1(\varphi_2) \\ f_2(\varphi_1) \end{pmatrix} \quad \text{for } \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in E^+,$$

where

$$f_1(\varphi_2) = b_1 \left[1 + \left(\int_{-\infty}^{0} k(-s) \varphi_2(s)\, ds \right)^m \right]^{-1}$$

and

$$f_2(\varphi_1) = b_2 \int_{-\infty}^{0} k(-s) \varphi_1(s)\, ds.$$

Thus, except perhaps for the fact that $D(F) = E^+$, the situation is nice enough since B is an m-accretive bounded linear operator on $X = \mathbb{R}^2$ and $F: E^+ \to X$ is even globally Lipschitz continuous. On the other hand, if some feedback is introduced into the model 1.2(1) by, say, redefining F so that, given

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in E^+,$$

we have

$$F(\varphi) = \begin{pmatrix} f_1(\varphi_2) \\ \varphi_2(0) f_2(\varphi_1) \end{pmatrix},$$

then we are not only back to the case of Lipschitz continuity on truncated cones, but as well have that $F(0) \neq 0$. We will return to this and the preceding example in §4.

2. The solution semigroup to (FDE)

In this section, we derive our principal results on solution semigroups for (FDE). With a view to including as wide a range of applications as possible, we set the problem in the general context outlined below. Specific instances will be considered throughout the sequel.

2.A **The initial history space.** Letting $I = (-\infty, 0]$ or $I = [-R, 0]$ for some $R > 0$, the initial history space is assumed to be a Banach space $(E, \|\cdot\|)$ of continuous functions $\varphi: I \to X$ with the following properties:

(E1) (a) For all $\varphi \in E$,
(i) $\|\varphi(0)\| \leq \|\varphi\|$, and
(ii) $\tilde{\phi} \in E$, where $\tilde{\phi}(r) = \phi(0)$, $r \in I$.
(b) For φ, $(\varphi_n)_n$ in E, if $\|\varphi - \varphi_n\| \to 0$, then $\|\varphi_n(s) - \varphi(s)\| \to 0$ for each $s \in I$.

(E2) If $\lambda > 0$, $x \in X$, $\psi \in E$, and $\varphi \in C^1(I, X)$ is the solution to

$$\varphi - \lambda\varphi' = \psi, \qquad \varphi(0) = x,$$

then $\varphi \in E$ and $\|\varphi\| \leq \max\{\|x\|, \|\psi\|\}$.

(E3) (a) If $x\colon I \cup [0, \infty) \to X$ is continuous and $x|_I \in E$, then
(i) $x_t \in E$ for all $t \geq 0$, and
(ii) the map $t \to x_t$ is continuous from $\mathbb{R}^+$ into E.
(b) There exist $M_0 \geq 1$, and a locally bounded function $M_1\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, given $x\colon I \cup [0, \infty) \to X$ as in (a) above,

$$\|x_t\| \leq M_0\|x_0\| + M_1(t) \max_{0 \leq s \leq t} \|x(s)\| \quad \text{for all } t \geq 0.$$

(Concerning these axioms, compare [7, 16, and 17].)

2.B **The framework for (FDE).** Given an initial history space E as in 2.A, we make the following assumptions:

(A1) (i) $\widehat{X}$ is a closed subset of X;
(ii) $\widehat{E}$ is a closed and convex subset of E;
(iii) $B \subseteq X \times X$ is an accretive operator;
(iv) $F\colon \widehat{E} \to X$ is Lipschitz continuous with Lipschitz constant $M \geq 0$;
(v) $\alpha \in \mathbb{R}$, and $\gamma = \max\{0, M - \alpha\}$.

Further, we assume that the following conditions are fulfilled:

(A2) If $x \in \widehat{X}$, $\psi \in \widehat{E}$, $\lambda > 0$ with $\lambda\gamma < 1$, and φ_x is the solution to

$$\varphi - \lambda\varphi' = \psi, \qquad \varphi(0) = x,$$

then $\varphi_x \in \widehat{E}$.

(A3) If $\psi \in \widehat{E}$ and $\lambda > 0$ with $\lambda\gamma < 1$, then

$$\frac{1}{1 + \lambda\alpha}(\psi(0) + \lambda F(\varphi_x)) \in \left(I + \frac{\lambda}{1 + \lambda\alpha} B\right)(D(B) \cap \widehat{X})$$

for each $x \in \widehat{X}$.

2.1 **Theorem.** *In the context of* (E1)–(E3) *and* (A1)–(A3), *consider the operator A in E defined by*

$$D(A) = \{\varphi \in \widehat{E}\colon \varphi' \in E,\ \varphi(0) \in D(B),\ \varphi'(0) \in F(\varphi) - \alpha\varphi(0) - B(\varphi(0))\}$$

and $A\varphi = -\varphi'$, $\varphi \in D(A)$. Then

(a) *$R(I + \lambda A) \supseteq \widehat{E}$ for all $\lambda > 0$ with $\lambda\gamma < 1$, and $-A$ generates a strongly continuous semigroup $(S(t))_{t \geq 0}$ of type γ on* $\operatorname{cl} D(A)$ (*in the sense of Crandall-Liggett* [9, *Theorem* I]); *i.e.,*

$$\|S(t)\varphi - S(t)\psi\| \leq e^{\gamma t}\|\varphi - \psi\| \quad \textit{for all } t \geq 0 \textit{ and all } \varphi, \psi \in \operatorname{cl} D(A);$$

(b) *if $\varphi \in \operatorname{cl} D(A)$, then $S(t)\varphi = (x_\varphi)_t$ for all $t \geq 0$, where*

$$x_\varphi(t) = \begin{cases} \varphi(t), & t \in I \\ S(t)\varphi(0), & t \geq 0. \end{cases}$$

Although carried forth in a more general context, our proof will closely parallel those for the corresponding, mostly global ($\widehat{X} = X$ and $\widehat{E} = E$) results treated in [4–7, 13, 20, 23–25].

Proof of 2.1(a). According to [9], it will suffice to show the following:

(a1) $A \in \mathscr{A}(\gamma)$; i.e.,

$$\|\varphi_1 - \varphi_2\| \leq (1 - \lambda\gamma)^{-1} \|(\varphi_1 - \lambda\varphi_1') - (\varphi_2 - \lambda\varphi_2')\|$$

for all φ_1, $\varphi_2 \in D(A)$ and all $\lambda > 0$ with $\lambda\gamma < 1$;

(a2) $R(I + \lambda A) \supseteq \widehat{E}$ for all $\lambda > 0$ with $\lambda\gamma < 1$. (Note that $\mathrm{cl}\, D(A) \subseteq \widehat{E}$ since $\widehat{E}$ is assumed to be closed.)

Proof of (a1). Given $\lambda > 0$ with $\lambda\gamma < 1$ and $\varphi_1, \varphi_2 \in D(A)$, $\psi = \varphi_1 - \varphi_2 \in E$ is the solution of

$$\begin{cases} \psi - \lambda\psi' = (\varphi_1 - \lambda\varphi_1') - (\varphi_2 - \lambda\varphi_2') \\ \psi(0) = \varphi_1(0) - \varphi_2(0), \end{cases}$$

and so, by (E2),

$$\|\varphi_1 - \varphi_2\| = \|\psi\| \leq \max\{\|\varphi_1(0) - \varphi_2(0)\|, \|(\varphi_1 - \lambda\varphi_1') - (\varphi_2 - \lambda\varphi_2')\|\}.$$

In case the second term is the maximum, we are done. Otherwise, using (a) of (E1), we have $\|\varphi_1 - \varphi_2\| = \|\varphi_1(0) - \varphi_2(0)\|$. Moreover since $\varphi_i \in D(A)$, we have

$$\varphi_i(0) = \left(I + \frac{\lambda}{1 + \lambda\alpha} B\right)^{-1} [(1+\lambda\alpha)^{-1}(\varphi_i(0) - \lambda\varphi_i'(0) + \lambda F(\varphi_i))], \qquad i \in \{1, 2\}.$$

Thus, by (E1)(a), (A1)(iii), and (A1)(iv),

$$\|\varphi_1 - \varphi_2\| = \|\varphi_1(0) - \varphi_2(0)\| \leq (1+\lambda\alpha)^{-1}(\|(\varphi_1 - \lambda\varphi_1') - (\varphi_2 - \lambda\varphi_2')\| + \lambda M\|\varphi_1 - \varphi_2\|),$$

and so

$$(1 + \lambda(\alpha - M))\|\varphi_1 - \varphi_2\| \leq \|(\varphi_1 - \lambda\varphi_1') - (\varphi_2 - \lambda\varphi_2')\|,$$

whereby we are done in any event.

Proof of (a2). Let $\lambda > 0$ with $\lambda\gamma < 1$, and fix $\psi \in \widehat{E}$. In view of (A1)(iii) and (A3), if we put

$$Tx = \left(I + \frac{\lambda}{1 + \lambda\alpha} B\right)^{-1} \left[\frac{1}{1 + \lambda\alpha}(\psi(0) + \lambda F(\varphi_x))\right], \qquad x \in \widehat{X},$$

then T is a well defined mapping from $\widehat{X}$ into $\widehat{X}$. Moreover, if $x, y \in \widehat{X}$, then

$$\|Tx - Ty\| \leq \frac{\lambda M}{1 + \lambda\alpha}\|\varphi_x - \varphi_y\| \leq \frac{\lambda M}{1 + \lambda\alpha}\|x - y\|,$$

where the last estimate follows from (E2) since $\varphi_x - \varphi_y$ is the solution of

$$\rho - \lambda\rho' = 0, \qquad \rho(0) = x - y.$$

Thus, by our choice of γ and λ, T is a strict contraction from $\widehat{X}$ into $\widehat{X}$, and so there exists a unique $x \in \widehat{X}$ such that

$$x = Tx = \left(I + \frac{\lambda}{1+\lambda\alpha}B\right)^{-1}[(1+\lambda\alpha)^{-1}(\psi(0) + \lambda F(\varphi_x))].$$

In particular, $\varphi_x(0) = x \in D(B) \cap \widehat{X}$, $\varphi_x - \lambda\varphi_x' = \psi$, and $\varphi_x'(0) = \lambda^{-1}(x - \psi(0)) \in F(\varphi_x) - \alpha x - B(x)$, whereby $\varphi_x \in D(A)$ and $(I + \lambda A)\varphi_x = \psi$. This completes the proof of (a2).

To establish 2.1(b), we begin with a lemma, and adopt the following notation in order to state this result:

1. $E_0 = \{\varphi \in E : \varphi(0) = 0\}$;
2. For $\lambda > 0$, the function $e_\lambda : I \to \mathbb{R}$ is defined by $e_\lambda(s) = \exp(s/\lambda)$, $s \in I$.

2.2 **Lemma.** *Under the assumptions of Theorem* 2.1, *the following propositions hold*:

(a) *For* $t \geq 0$, *define* $S_0(t) : E_0 \to E_0$ *by*

$$S_0(t)\varphi(s) = \begin{cases} 0, & -t \leq s \leq 0 \\ \varphi(t+s), & s \leq -t, \end{cases}$$

$s \in I$. *Then* $(S_0(t))_{t\geq 0}$ *is a* (*linear*) C_0-*semigroup of contractions on* E_0 *generated by* $-A_0$, *where*

$$D(A_0) = \{\varphi \in E_0 : \varphi' \in E_0\} \quad \textit{and} \quad A_0\varphi = -\varphi', \ \varphi \in D(A_0).$$

(b) *If* A_1 *is the operator in* E *defined by*

$$D(A_1) = \{\varphi \in E_0 : \varphi' \in E\} \quad \textit{and} \quad A_1\varphi = -\varphi', \ \varphi \in D(A_1),$$

then

(b1) A_1 *is accretive and* $R(I + \lambda A_1) = E$ *for all* $\lambda > 0$;

(b2) $(I + \lambda A_1)^{-1}\psi = (I + \lambda A_0)^{-1}(\psi - \psi(0)) + (1 - e_\lambda)\psi(0)$ *for all* $\psi \in E$ *and all* $\lambda > 0$;

(b3) $(I + \lambda A)^{-1}\psi = (I + \lambda A_1)^{-1}\psi + e_\lambda((I + \lambda A)^{-1}\psi)(0)$ *for all* $\psi \in \widehat{E}$ *and all* $\lambda > 0$ *with* $\lambda\gamma < 1$.

Proof of Lemma 2.2. For (a), the fact that $(S_0(t))_{t\geq 0}$ is a uniformly bounded C_0-semigroup on E_0 follows from (E3) by considering the function $x(\varphi) : I \cup [0, \infty) \to X$ defined by

$$x(\varphi)(s) = \begin{cases} 0, & s \geq 0 \\ \varphi(s), & s \in I \end{cases}$$

for any given $\varphi \in E_0$. Taking $\widetilde{A}_0$ to denote the generator of $(S_0(t))_{t\geq 0}$, if $\varphi \in E_0$, we have that

$$(\lambda I - \widetilde{A}_0)^{-1}\varphi = \int_0^\infty e^{-\lambda t} S_0(t)\varphi \, dt$$

if $\lambda > 0$ is large enough. Together with the continuity of the evaluation maps $\varphi \to \varphi(s)$, $s \in I$ (see (E1)(b)), this formula and standard arguments for translation semigroups imply that, in fact, $\widetilde{A}_0 = -A_0$. By (E2), moreover, A_0 is accretive and $R(I + \lambda A_0) = E_0$ for all $\lambda > 0$. Hence, according to [19, Chapter I, Theorem 4.3], $(S_0(t))_{t\geq 0}$ is actually a contraction semigroup.

Turning to (b), (b1) and (b2) are direct consequences of (E2) and (E1)(a)(ii). As for (b3), given $\psi \in \widehat{E}$ and $\lambda > 0$ with $\lambda\gamma < 1$, let $\varphi = (I + \lambda A)^{-1}\psi$, $\varphi_0 = (I + \lambda A_1)^{-1}\psi$, and $\rho = \varphi_0 + e_\lambda\varphi(0)$. (Note that $e_\lambda\varphi(0) \in E$ by (E2).)

Since ρ satisfies

$$\rho - \lambda \rho' = \psi, \qquad \rho(0) = \varphi(0),$$

$\rho = \varphi$, and the proof is thus complete.

Proof of Theorem 2.1(b). Starting from the assertions of Lemma 2.2 above, and using (E1)(b), we can now follow the line of proof for the special case $E = C([-R, 0], X)$ as given in [13] to conclude that, if $\varphi \in \mathrm{cl}\, D(A)$, then $S(t)\varphi(s) = \varphi(t+s)$ for all $t \geq 0$ and $s \leq -t$ in case $I = (-\infty, 0]$, respectively, for all $t \in [0, R]$, and $-R \leq s \leq -t$ in case $I = [-R, 0]$. From here, some elementary computation yields 2.1(b). This completes the proof of Theorem 2.1.

We now turn to the problem of when, given $\varphi \in \mathrm{cl}\, D(A)$, the function x_φ defined in Theorem 2.1(b) is actually a solution to (FDE), and the only such.

2.3 **Definition.** A continuous function $x: I \cup [0, \infty) \to X$ is called a solution to (FDE) if

(i) $x|_I = \varphi$;
(ii) $x|_{\mathbb{R}^+}$ is locally absolutely continuous and differentiable a.e.;
(iii) $x_t \in \widehat{E}$, $x(t) \in D(B)$, and $x'(t) + \alpha x(t) - F(x_t) \in -B(x(t))$ for a.e. $t \in \mathbb{R}^+$.

We first consider the uniqueness problem.

2.4 **Proposition.** *Under the assumptions of Theorem* 2.1, *(FDE) has at most one solution corresponding to each* $\varphi \in E$.

Proof. Given $\varphi \in E$, let $x, y: I \cup [0, \infty) \to X$ be solutions to (FDE) with $x|_I = y|_I = \varphi$. Then, fixing $T > 0$, the function $(x - y)|_{[0,T]}$ is absolutely continuous and differentiable almost everywhere (Definition 2.3), and the same is true for both the function $f: [0, T] \to \mathbb{R}$ defined by $f(s) = \|x(s) - y(s)\|$, $s \in [0, T]$, and its square. Assume that $E \subset [0, T]$ is a Lebesgue nullset such that $(x - y)|_{[0,T]}$, f, and f^2 are all differentiable and x and y both fulfill (FDE) on $[0, T]\backslash E$. Then, given $t \in [0, T]\backslash E$, since B is accretive, there exists $x^*(t) \in J(x(t) - y(t))$ (where J denotes the duality map of X) such that

$$\langle (x'(t) + \alpha x(t) - F(x_t)) - (y'(t) + \alpha y(t) - F(y_t)), x^*(t)\rangle \leq 0.$$

This implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|x(t) - y(t)\|^2 &= \langle x'(t) - y'(t), x^*(t)\rangle \\ &\leq \langle F(x_t) - F(y_t), x^*(t)\rangle - \alpha\langle x(t) - y(t), x^*(t)\rangle \\ &\leq M\|x(t) - y(t)\|\,\|x_t - y_t\| - \alpha\|x(t) - y(t)\|^2\end{aligned}$$

for a.e. $t \in [0, T]$. We conclude that

$$\begin{aligned}&\tfrac{1}{2}\|x(t_1) - y(t_1)\|^2 - \tfrac{1}{2}\|x(t_0) - y(t_0)\|^2 \\ &\quad \leq M\int_{t_0}^{t_1} \|x(s) - y(s)\|\,\|x_s - y_s\|\,ds - \alpha\int_{t_0}^{t_1} \|x(s) - y(s)\|^2\,ds\end{aligned} \tag{2.1}$$

for all $0 \leq t_0 \leq t_1 \leq T$. Now, suppose that $x \neq y$. Noting that $x(0) = y(0)$, put

$$t_0 = \sup\{t \in [0, \infty): x(s) = y(s) \text{ for all } 0 \leq s \leq t\},$$

in which case $t_0 < \infty$ and $x(t_0) = y(t_0)$. Further, referring to (E3)(b), put $K = \sup\{M_1(s)\colon 0 \le s \le t_0 + 1\}$, let $\delta = \min\{(3(MK + |\alpha|) + 1)^{-1}, 1\}$, and choose $t_1 \in (t_0, t_0 + \delta]$ such that

$$\|x(t_1) - y(t_1)\| \ge \|x(s) - y(s)\| \quad \text{for all } s \in [t_0, t_0 + \delta]$$

(whence $\|x(t_1) - y(t_1)\| > 0$). From (2.1) with $T = t_0 + \delta$ and (E3)(b), however, we have that

$$\begin{aligned} \tfrac{1}{2}\|x(t_1) - y(t_1)\|^2 &\le MK \int_{t_0}^{t_1} \|x(s) - y(s)\| \left(\max_{0 \le r \le s} \|x(r) - y(r)\| \right) ds \\ &\quad + |\alpha| \int_{t_0}^{t_1} \|x(s) - y(s)\|^2 \, ds \\ &\le (MK + |\alpha|)\delta \|x(t_1) - y(t_1)\|^2, \end{aligned}$$

which contradicts our choice of δ and K. This completes the proof of Proposition 2.4.

In order to state our main result on semigroup solutions to (FDE), we recall the definition of $\widehat{D}(A)$ from [8]: $\widehat{D}(A) = \{\varphi \in \operatorname{cl} D(A)\colon |A\varphi| < \infty\}$, where

$$|A\varphi| = \lim_{\lambda \to 0^+} \|A_\lambda \varphi\|, \quad \text{and} \quad A_\lambda = \lambda^{-1}(I - (I + \lambda A)^{-1}), \qquad \lambda > 0.$$

2.5 **Theorem.** *In the context of* (E1)–(E3) *and* (A1)–(A3), *given* $\varphi \in \operatorname{cl} D(A)$, *let*

$$x_\varphi(t) = \begin{cases} \varphi(t), & t \le 0 \\ (S(t)\varphi)(0), & t \ge 0 \end{cases}$$

as in Theorem 2.1. *Then* x_φ *is the unique solution to* (*FDE*) *in each of the following situations*:

(a) X *is reflexive and the norm of* X *is Fréchet differentiable at any* $x \in X\backslash\{0\}$ (*for instance, if* X^* *is uniformly convex*), $B \subseteq X \times X$ *is maximal accretive, and* $\varphi \in \widehat{D}(A)$.

(b) X *has the Radon-Nikodym property* (*RNP*) (*for instance, if* X *is either reflexive or a separable dual space*), $D(B)$ *is closed,* B *is single valued with* $B\colon D(B) \to X$ *norm-weakly continuous, and* $\varphi \in \widehat{D}(A)$.

(c) X *is any Banach space,* $D(B)$ *is closed,* B *is single valued with* $B\colon D(B) \to X$ *continuous, and either* (c1) $\varphi \in \widehat{D}(A)$ *or* (c2) $\varphi \in \operatorname{cl} D(A)$ *and* B *maps bounded sets into bounded sets.*

(d) X *is reflexive,* $B\colon D(B) \to X$ *is single valued and demiclosed* (*i.e., the graph of* B *is norm-weakly closed in* $X \times X$), *and* $\varphi \in \widehat{D}(A)$. *In particular, this is the case when* X *is reflexive,* $B\colon D(B) \to X$ *is a single valued closed linear operator, and* $\varphi \in \widehat{D}(A)$.

Remarks. 1. Even for the special cases heretofore considered in the global setting (cf. [25]), Theorem 2.5(a) extends the corresponding results by allowing for a multivalued operator $B \subseteq X \times X$.

2. In the direction of a converse, if $\varphi \in D(A)$ and $u\colon I \cup \mathbb{R}^+ \to X$ is a corresponding solution to (FDE) which additionally satisfies

(i) $u|_{\mathbb{R}^+} \in C^1(\mathbb{R}^+, X)$ and (ii) $u'(0) = \varphi'(0)$,

then $u_t = S(t)\varphi$, $t \ge 0$. (This follows along the same lines as [4, Lemma 2.3] and [6, Lemma 3].)

Proof of Theorem 2.5. As is standard, we let $J_\lambda = (I + \lambda A)^{-1}$ and $A_\lambda = \lambda^{-1}(I - J_\lambda)$ for $\lambda > 0$ with $\lambda\gamma < 1$.

Step 1. According to (A1)(ii) and Theorem 2.1(a), $\operatorname{clco} D(A) \subseteq \widehat{E} \subseteq R(I + \lambda A)$ for all $\lambda > 0$ with $\lambda\gamma < 1$. Thus, by [10, §4], if $(S_\lambda(t))_{t\geq 0}$ is the semigroup on $\operatorname{clco} D(A)$ generated by $-A_\lambda$, $\lambda > 0$, then, given $\varphi \in \operatorname{cl} D(A)$,

$$\lim_{\lambda\to 0^+} S_\lambda(t)\varphi = S(t)\varphi \text{ uniformly on bounded } t\text{-intervals}, \tag{2.2}$$

and

$$\dot{S}_\lambda(t)\varphi + A_\lambda S_\lambda(t)\varphi = 0 \quad \text{for all } t \geq 0. \tag{2.3}$$

Moreover, by (2.2) and [10, Lemma 1.1],

$$\lim_{\lambda\to 0^+} \|J_\lambda S_\lambda(\xi)\varphi - S(\xi)\varphi\| = 0 \quad \text{for all } \xi \geq 0 \text{ and all } \varphi \in \operatorname{cl} D(A). \tag{2.4}$$

Thus, given $T > 0$ and $\lambda_n \to 0$ with $\lambda_n > 0$ and $\lambda_n\gamma \leq \gamma_0 < 1$, (2.2) and (2.4) (and again [10, Lemma 1.1]) imply that

$$\begin{gathered}(\|S_{\lambda_n}(\cdot)\varphi\|)_n,\ (\|J_{\lambda_n}S_{\lambda_n}(\cdot)\varphi\|)_n, \quad \text{and} \quad (\|F(J_{\lambda_n}S_{\lambda_n}(\cdot)\varphi\|)_n \\ \text{are uniformly bounded on } [0, T].\end{gathered} \tag{2.5}$$

Moreover, (2.3) in conjunction with (E1)(b) and the definition of $D(A)$ imply

$$(S_\lambda(t)\varphi)(0) = \varphi(0) - \int_0^t (AJ_\lambda S_\lambda(s)\varphi)(0)\,ds \tag{2.6}$$

for all $\varphi \in \operatorname{cl} D(A)$ and all $t \geq 0$. In case B is single valued, this leads to

$$\begin{aligned}&(S_\lambda(t)\varphi)(0) \\ &= \varphi(0) + \int_0^t [F(J_\lambda S_\lambda(s)\varphi) - \alpha(J_\lambda S_\lambda(s)\varphi)(0) - B((J_\lambda S_\lambda(s)\varphi)(0))]\,ds\end{aligned} \tag{2.7}$$

for all $\varphi \in \operatorname{cl} D(A)$ and all $t \geq 0$. Finally, if $\varphi \in \widehat{D}(A)$, then Lemma 1.3 and Lemma 4.1 of [10] imply that, given $T > 0$, there exists $M_T > 0$ such that

$$\|A_{\lambda_n}S_{\lambda_n}(\xi)\varphi\| \leq M_T|A\varphi| < \infty \quad \text{for all } n \in \mathbb{N} \text{ and all } \xi \in [0, T]. \tag{2.8}$$

Step 2 (proof of Proposition (a)). Given $T > 0$, define
$x\colon [0, T] \to X$ by $x(t) = (S(t)\varphi)(0)$;
$x_n\colon [0, T] \to X$ by $x_n(t) = (J_{\lambda_n}S_{\lambda_n}(t)\varphi)(0)$;
$y_n\colon [0, T] \to X$ by

$$y_n(t) = (A_{\lambda_n}S_{\lambda_n}(t)\varphi)(0) + F(J_{\lambda_n}S_{\lambda_n}(t)\varphi) - \alpha(J_{\lambda_n}S_{\lambda_n}(t)\varphi)(0).$$

From (2.4), (2.5), and (2.8), $(x_n)_n$, $(y_n)_n \subset C([0, T], X)$ are uniformly bounded, $\|x_n(t) - x(t)\| \to 0$ for all $t \in [0, T]$, and $[x_n(t), y_n(t)] \in B$ for all $n \in \mathbb{N}$. Thus, under the assumptions of Proposition (a), Lemma 2.1 of [21] implies the existence of a function $y \in L^\infty([0, T], X)$ and a subsequence $(y_{n_k})_k$ of $(y_n)_n$—which, for the following, we can assume to be $(y_n)_n$ itself—such that

$$\begin{gathered}(y_n)_n \text{ converges weakly in } L^1([0, T], X) \text{ to } y, \text{ and } [x(t), y(t)] \in B \\ \text{for a.e. } t \in [0, T].\end{gathered} \tag{2.9}$$

From (2.6), we have

$$\begin{gathered}(S_{\lambda_n}(t)\varphi)(0) = \varphi(0) - \int_0^t y_n(\tau)\,d\tau + \int_0^t [F(J_{\lambda_n}S_{\lambda_n}(\tau)\varphi) - \alpha x_n(\tau)]\,d\tau \\ \text{for each } t \in [0, T].\end{gathered} \tag{2.10}$$

Fixing $0 \le t \le T$, evaluating at $x^* \in X^*$, and letting n tend to infinity, (2.10) in conjunction with (2.2), (2.4), (2.5), and (2.9) yields that

$$(S(t)\varphi)(0) = \varphi(0) - \int_0^t y(\tau)\,d\tau + \int_0^t [F(S(\tau)\varphi) - \alpha(S(\tau)\varphi)(0)]\,d\tau$$

for each $t \in [0, T]$. This shows that $x_\varphi|_{\mathbb{R}^+} = (S(\cdot)\varphi)(0)$ is locally absolutely continuous, differentiable almost everywhere, and, using (2.9),

$$x'_\varphi(t) + \alpha x_\varphi(t) - F((x_\varphi)_t) = -y(t) \in -B(x_\varphi(t)) \quad \text{for a.e. } t \in \mathbb{R}^+.$$

Hence, x_φ is a solution to (FDE).

Step 3 (proof of Proposition (b)). Under the assumptions of (b), we conclude from (2.7) together with (2.2), (2.4), (2.5), and (2.8) that

$$\begin{aligned} &\langle x_\varphi(t), x^*\rangle = \langle \varphi(0), x^*\rangle + \int_0^t \langle F((x_\varphi)_\tau) - \alpha x_\varphi(\tau) - B(x_\varphi(\tau)), x^*\rangle\,d\tau \\ &\text{for all } 0 \le t \le T \text{ and all } x^* \in X^*. \end{aligned} \tag{2.11}$$

Using our continuity assumptions, there exists a separable closed linear subspace X_0 of X such that $\{x_\varphi(t) : t \in [0, T]\} \subseteq X_0$, and $\{F((x_\varphi)_t) - \alpha x_\varphi(t) - B(x_\varphi(t)) : t \in [0, T]\} \subseteq X_0$. Also, since $\varphi \in \widehat{D}(A)$, the function $x_\varphi = S(\cdot)\varphi(0) : [0, T] \to X$ is Lipschitz continuous [8, Corollary 1], and thus, since X is assumed to have RNP, differentiable for a.e. $t \in [0, T]$. Now, consider $(x_n^*)_n \subseteq B_X^*$ such that $(x_n^*|_{X_0})_n$ is w^*-dense in $B_{X_0}^*$, and let $E \subset [0, T]$ be a Lebesgue nullset such that, for each $t \in [0, T]\backslash E$, $x'_\varphi(t)$ exists and

$$\frac{d}{dt}\int_0^t \langle F((x_\varphi)_\tau) - \alpha x_\varphi(\tau) - B(x_\varphi(\tau)), x_n^*\rangle\,d\tau = \langle F((x_\varphi)_t) - \alpha x_\varphi(t) - B(x_\varphi(t)), x_n^*\rangle$$

for all $n \in \mathbb{N}$. From (2.11), we conclude that

$$\langle x'_\varphi(t), x_n^*\rangle = \langle F((x_\varphi)_t) - \alpha x_\varphi(t) - B(x_\varphi(t)), x_n^*\rangle$$

for each $n \in \mathbb{N}$ and $t \in [0, T]\backslash E$. Thus, by our choice of X_0 and the sequence $(x_n^*)_n$,

$$x'_\varphi(t) + \alpha x_\varphi(t) + B(x_\varphi(t)) = F((x_\varphi)_t) \quad \text{for a.e. } t \in [0, T],$$

which completes the proof.

Step 4 (proof of Proposition (c)). In view of (2.8), it is enough to note that, assuming either (c1) or (c2), (2.7) implies that

$$x_\varphi(t) = \varphi(0) + \int_0^t [F((x_\varphi)_\tau) - \alpha x_\varphi(\tau) - B(x_\varphi(\tau))]\,d\tau \quad \text{for all } 0 \le t \le T.$$

Step 5 (proof of Proposition (d)). Noting that

$$B(J_{\lambda_n}S_{\lambda_n}(t)\varphi(0)) = (A_{\lambda_n}S_{\lambda_n}(t)\varphi)(0) + F(J_{\lambda_n}S_{\lambda_n}(t)\varphi) - \alpha(J_{\lambda_n}S_{\lambda_n}(t)\varphi)(0)$$

for $n \in \mathbb{N}$ and $0 \le t \le T$, and using (2.8), the assumptions of (d) imply that there exist a weakly compact subset C and closed separable linear subspace X_0 of X such that $\{B(J_{\lambda_n}S_{\lambda_n}(t)\varphi(0)) : n \in \mathbb{N},\ 0 \le t \le T\} \subseteq C \subseteq X_0$. Hence, given $t \in [0, T]$, every subsequence of $(B(J_{\lambda_n}S_{\lambda_n}(t)\varphi(0)))_n$ has a further subsequence that is weakly convergent. Together with (2.4) and the fact that B is demiclosed, this implies that $(B(J_{\lambda_n}S_{\lambda_n}(t)\varphi(0)))_n$ converges weakly to $B(x_\varphi(t))$ for each $t \in [0, T]$. Thus, starting from (2.7), we again reach the conclusion (2.11), and the proof can now be completed by repeating the arguments used in Step 3.

We next characterize the closure of $D(A)$ for the operator A of Theorem 2.1. In so doing, it will be convenient to let

$$\mathscr{D}_B = \bigcup_{\kappa>0}\left(\bigcap_{0<\lambda<\kappa} R(I+\lambda B)\right).$$

2.6 **Proposition.** *Under the assumptions of Theorem* 2.1,

$$\{\varphi \in \widehat{E}\colon \varphi(0) \in \operatorname{cl}(D(B)\cap\mathscr{D}_B)\} \subseteq \operatorname{cl} D(A) \subseteq \{\varphi \in \widehat{E}\colon \varphi(0) \in \operatorname{cl} D(B)\}.$$

In particular, if there exists $\kappa > 0$ *such that* $R(I+\lambda B) \supseteq D(B)$ *for all* $0 < \lambda < \kappa$, *then*

$$\operatorname{cl} D(A) = \{\varphi \in \widehat{E}\colon \varphi(0) \in \operatorname{cl} D(B)\}.$$

Proof. The second inclusion is obvious from (E1)(b). In order to prove the first, let $\varphi \in \widehat{E}$, and assume $\varphi(0) \in \operatorname{cl}(D(B)\cap\mathscr{D}_B)$. Then, by [10, Lemma 1.2],

$$\lim_{\lambda\to0^+} \|(I+\lambda B)^{-1}\varphi(0) - \varphi(0)\| = 0. \tag{2.12}$$

From Theorem 2.1 (and its proof), given $\lambda > 0$ with $\lambda\gamma < 1$, $\varphi_\lambda = (I+\lambda A)^{-1}\varphi \in D(A)$, and

$$\varphi_\lambda(0) = \left(I + \frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\left[\frac{1}{1+\lambda\alpha}(\varphi(0)+\lambda F(\varphi_\lambda))\right].$$

Hence

$$\begin{aligned}\|\varphi_\lambda(0)-\varphi(0)\| &\le \left\|\frac{-\lambda\alpha}{1+\lambda\alpha}\varphi(0) + \frac{\lambda}{1+\lambda\alpha}F(\varphi_\lambda)\right\| \\ &\quad + \left\|\left(I+\frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\varphi(0)-\varphi(0)\right\| \\ &\le \frac{\lambda|\alpha|}{1+\lambda\alpha}\|\varphi(0)\| + \frac{\lambda M}{1+\lambda\alpha}\|\varphi_\lambda-\varphi\| + \frac{\lambda}{1+\lambda\alpha}\|F(\varphi)\| \\ &\quad + \left\|\left(I+\frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\varphi(0)-\varphi(0)\right\|,\end{aligned} \tag{2.13}$$

and so, by Lemma 2.2(b),

$$\begin{aligned}\|\varphi_\lambda-\varphi\| &= \|(I+\lambda A_0)^{-1}(\varphi-\varphi(0)) + (1-e_\lambda)\varphi(0) + e_\lambda\varphi_\lambda(0) - \varphi\| \\ &\le \|(I+\lambda A_0)^{-1}(\varphi-\varphi(0)) - (\varphi-\varphi(0))\| + \|\varphi_\lambda(0)-\varphi(0)\|.\end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14),

$$\begin{aligned}\frac{1+\lambda(\alpha-M)}{1+\lambda\alpha}\|\varphi_\lambda-\varphi\| &\le \|(I+\lambda A_0)^{-1}(\varphi-\varphi(0)) - (\varphi-\varphi(0))\| \\ &\quad + \left\|\left(I+\frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\varphi(0)-\varphi(0)\right\| \\ &\quad + \frac{\lambda}{1+\lambda\alpha}[|\alpha|\,\|\varphi(0)\| + \|F(\varphi)\|].\end{aligned}$$

Thus, by Lemma 2.2(a) and (2.12), $\lim_{\lambda\to 0^+} \|\varphi_\lambda - \varphi\| = 0$, whereby $\varphi \in \operatorname{cl} D(A)$, and the proof is complete.

For future reference and to provide perspective, we proceed to list some particular instances when conditions (A1)–(A3) are automatically satisfied. Further special cases will be illustrated by the examples in §4.

2.7 **Proposition.** *Assume X is a Banach space and E is an initial data space satisfying* (E1)–(E3). *Then we have the following*:

(a) *If $F: E \to X$ is (globally) Lipschitz continuous and $B \subseteq X \times X$ is m-accretive, then Theorems* 2.1 *and* 2.5 *hold with $\widehat{E} = E$ and $\widehat{X} = X$.*

(b) *Assume $\alpha \geq 0$, and that there exist $\beta > 0$, $M(\beta) \geq 0$, and a nondecreasing function $m: [0, \beta] \to \mathbb{R}^+$ such that*

(i) $\|F(\varphi_1) - F(\varphi_2)\| \leq M(\beta)\|\varphi_1 - \varphi_2\|$ *for all $\varphi_i \in E$ with $\|\varphi_i\| \leq \beta$, $i \in \{1, 2\}$, and*

(ii) $\|F(\varphi)\| \leq m(\|\varphi\|)$ *for all $\varphi \in E$ with $\|\varphi\| \leq \beta$.*

Then, setting $\widehat{E} = \{\varphi \in E: \|\varphi\| \leq \beta\}$, $\widehat{X} = \{x \in X: \|x\| \leq \beta\}$, and $M = M(\beta)$, Theorems 2.1 *and* 2.5 *apply if $m(\beta) \leq \alpha\beta$ and either*

(b1) $(I + \lambda B)(D(B) \cap \widehat{X}) \supseteq \widehat{X}$ *for all $\lambda > 0$ with $\lambda\gamma < 1$, or*

(b2) *B is m-accretive and $0 \in B(0)$.*

(c) *In case $X = \mathbb{R}^n$ with the Euclidean norm for some $n \in \mathbb{N}$, $\alpha \geq 0$, $a_1, \ldots, a_n \in \mathbb{R}$, and $\beta > 0$, consider $\widehat{E} = \{\varphi \in E: \|\varphi\| \leq \beta,\ \varphi_i \geq a_i,\ i \in \{1, \ldots, n\}\}$, and assume that $F: \widehat{E} \to \mathbb{R}^n$ is a Lipschitz continuous map with Lipschitz constant $M(\beta) > 0$ satisfying $(F(\varphi))_i \geq \alpha a_i$ for each $\varphi \in \widehat{E}$ and $i \in \{1, \ldots, n\}$. Furthermore, assume that there exists a function $m: [0, \beta] \to \mathbb{R}^+$ as in* (b) *above such that* (b)(ii) *is fulfilled for all $\varphi \in \widehat{E}$. Then, setting $\widehat{X} = \{x \in X: \|x\| \leq \beta;\ x_i \geq a_i,\ i \in \{1, \ldots, n\}\}$, Theorems* 2.1 *and* 2.5 *both apply if $m(\beta) \leq \alpha\beta$ and either*

(c1) $(I + \lambda B)(D(B) \cap \widehat{X}) \supseteq \widehat{X}$ *for all $\lambda > 0$ with $\lambda\gamma < 1$, or*

(c2) *$n = 1$, B is m-accretive, $0 \in B(0)$, and $a_1 = 0$.*

Remarks. 1. Assertion 2.7(a) treats the situation considered for a single valued operator B in [4–6, 23, and 25].

2. If $F(0) = 0$ and (i) of 2.7(b) holds, then (ii) of 2.7(b) is automatically satisfied if m is defined by $m(s) = M(\beta)s$, $s \in [0, \beta]$. Thus, the case (b2) of 2.7(b) includes the situation considered in the context of a single valued operator B by Brewer [7, Theorem 2]. On the other hand, as Example 4.1 (in §4) illustrates, it is not necessary to have $M(\beta) \leq \alpha$ in order for Proposition 2.7(b) to apply.

3. The case (c1) of 2.7(c) has an obvious extension to the context where X is any Banach lattice.

Proof of Proposition 2.7. We only need verify (A1)–(A3), and these obviously hold under the assumptions of (a). Turning to (b) and (c), we first note that, given $\psi \in E$, $x \in X$, and $\lambda > 0$, the unique solution φ to

$$\varphi - \lambda\varphi' = \psi, \qquad \varphi(0) = x$$

is given by

$$\varphi(s) = e^{s/\lambda}x + \frac{1}{\lambda}e^{s/\lambda}\int_s^0 e^{-\xi/\lambda}\psi(\xi)\,d\xi, \qquad s \in I, \tag{2.15}$$

and $\|\varphi\| \leq \max\{\|x\|, \|\psi\|\}$ by (E2). From this, we readily see that (A2) is fulfilled under the assumptions of both (b) and (c). Under these same assumptions, given $\psi \in \widehat{E}$, $x \in \widehat{X}$, and $\lambda > 0$ with $\lambda\gamma < 1$, it is also easy to see that

$$\frac{1}{1+\lambda\alpha}(\psi(0) + \lambda F(\varphi_x)) \in \widehat{X},$$

whereby (A3) follows from either (b1) or (c1). In case (b2) holds, on the other hand, choose $y \in D(B)$ such that

$$\frac{1}{1+\lambda\alpha}(\psi(0) + \lambda F(\varphi_x)) \in \left(I + \frac{\lambda}{1+\lambda\alpha}B\right)y. \tag{2.16}$$

Then, since $0 \in B(0)$,

$$\begin{aligned}\|y\| &= \left\|\left(I + \frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\left[\frac{1}{1+\lambda\alpha}(\psi(0) + \lambda F(\varphi_x))\right]\right\| \\ &\leq \left\|\frac{1}{1+\lambda\alpha}(\psi(0) + \lambda F(\varphi_x))\right\| \leq \beta,\end{aligned}$$

and so (A3) also holds in this setting. Finally, if (c2) holds and $y \in D(B)$ is chosen as in (2.16), then $\|y\| \leq \beta$; it remains to show that $y \geq 0$. To this end, we have already observed that

$$z = \frac{1}{1+\lambda\alpha}(\psi(0) + \lambda F(\varphi_x)) \in \widehat{X};$$

i.e., $z \geq 0$. Also,

$$z = y + y_1 \quad \text{for some } y_1 \in \frac{1}{1+\lambda\alpha}By. \tag{2.17}$$

In case $y_1 \leq 0$, we have that $y = z - y_1 \geq 0$. Otherwise, since B is accretive, and $0 \in B(0)$, $y_1 y \geq 0$, and so $y \geq 0$ in either event. This completes the proof of 2.7.

2.8 *Remark.* In a different direction, given a problem described by (FDE) where an initial data space E satisfying (E1)–(E3) has been selected, the following general restrictions on the relationship between $\widehat{E}$ and $\widehat{X}$ are necessary in order that (A1)–(A3) also hold:

(a) If (A2) is satisfied for any (nonempty) subsets $\widehat{E} \subseteq E$ and $\widehat{X} \subseteq X$, then

$$\widehat{X} \subseteq \widehat{E}(0) = \{\varphi(0) \colon \varphi \in \widehat{E}\}. \tag{2.18}$$

(b) Moreover, if both (A2) and (A3) are satisfied for (nonvoid) subsets $\widehat{E} \subseteq E$ and $\widehat{X} \subseteq X$, and if

$(*)$ $R(I + \lambda B) \supseteq D(B)$ when $0 < \lambda < \lambda_0$ for some $\lambda_0 > 0$ and $\widehat{E}(0) \subseteq \operatorname{cl} D(B)$,

then

$$\widehat{E}(0) \subset \operatorname{cl} \widehat{X}. \tag{2.19}$$

In particular, if $(*)$ is fulfilled and $\widehat{E}(0)$ is closed in X, then $\widehat{E}(0)$ is the only possible choice for $\widehat{X}$.

Assertion 2.8(a) is obvious. As for 2.8(b), let $x \in \widehat{X}$, and choose any sequence $0 < \lambda_n \to 0$ such that $\lambda_n \gamma < 1$ and $\gamma_n = \lambda_n(1+\lambda_n\alpha)^{-1} < \lambda_0$ for all $n \in \mathbb{N}$. Then, given $\psi \in \widehat{E}$, (A2) yields that the unique solution $(\varphi_x)_n$ to

$$\varphi - \lambda_n \varphi' = \psi, \qquad \varphi(0) = x$$

belongs to $\widehat{E}$ with $\|(\varphi_x)_n\| \leq \max\{\|\psi\|, \|x\|\}$, $n \in \mathbb{N}$. By (A3), there exists $x_n \in D(B) \cap \widehat{X}$ such that

$$y_n = \frac{1}{1+\lambda_n\alpha}(\psi(0) + \lambda_n F(\varphi_x)_n) \in (I + \gamma_n B)x_n, \qquad n \in \mathbb{N}.$$

Since the sequence $(\|(\varphi_x)_n\|)_n$ is bounded, $\|y_n - \psi(0)\| \to 0$, and hence, according to the assumptions on B and $\widehat{E}(0)$, $\|(I+\gamma_n B)^{-1}\psi(0) - \psi(0)\| \to 0$. We conclude that

$$\|x_n - (I+\gamma_n B)^{-1}\psi(0)\| \leq \|y_n - \psi(0)\| \to 0,$$

whereby $\|x_n - \psi(0)\| \to 0$, and 2.8(b) is thus established.

We bring this section to a close by collecting some (mostly obvious) facts concerning fixed points of the solution semigroup $(S(t))_{t\geq 0}$ for (FDE). To this end, assuming the context of Theorem 2.1, we put

$$\mathscr{F}(S) = \{\varphi \in \operatorname{cl} D(A) \colon S(t)\varphi = \varphi,\ t \geq 0\}.$$

Then (a) $A^{-1}(0) \subseteq \mathscr{F}(S)$, and (b) the constant function $x_\varphi = (S(\cdot)\varphi)(0)$ corresponding to each $\varphi \in A^{-1}(0)$ is a solution to (FDE).

Notation. As a matter of notational convenience in the sequel, given $x \in X$, we hereafter let $\tilde{x} \colon I \to X$ denote the constant function defined by $\tilde{x}(s) = x$, $s \in I$.

2.9 **Proposition.** *In the context of* (E1)–(E3) *and* (A1)–(A3), *the following assertions hold:*

1. *If* (i) $B \subseteq X \times X$ *is closed and* (ii) *the topology on* E *is stronger than that of uniform convergence on the compact subsets of* I, *then* $A^{-1}(0) = \mathscr{F}(S)$.

2. $A^{-1}(0) \neq \varnothing$ *if and only if there exists* $x \in D(B)$ *such that* $\tilde{x} \in \widehat{E}$ *and* $F(\tilde{x}) \in (\alpha I + B)x$. *In particular, if* (i) $\alpha > 0$, (ii) $\tilde{x} \in \widehat{E}$ *for each* $x \in \widehat{X}$, *and* (iii) $\frac{1}{\alpha}F(\tilde{x}) \in (I + \frac{1}{\alpha}B)(D(B) \cap \widehat{X})$ *for all* $x \in \widehat{X}$, *then* $\tilde{x} \in A^{-1}(0)$ *whenever* $x \in \widehat{X}$ *is a fixed point of the map* $T \colon \widehat{X} \to \widehat{X}$ *defined by*

$$Tx = \left(I + \frac{1}{\alpha}B\right)^{-1}\left(\frac{1}{\alpha}F(\tilde{x})\right).$$

3. *If* 2(i)–2(iii) *are satisfied and, additionally,* $\|\tilde{x}\| = \|x\|$ *for each* $x \in \widehat{X}$, *then* $A^{-1}(0) \neq \varnothing$ *in case* $M < \alpha$ *or* $\widehat{X}$ *is a compact convex subset of* X.

Proof. For 1, since 1(i) and 1(ii) imply that A is closed, the conclusion can be read from [2, Chapter III, Theorem 1.5, p. 115]. The second assertion is obvious, while 3 follows from 2 together with the contraction mapping principle, respectively, the Schauder-Tychonov fixed point theorem.

Remarks. 1. Proposition 2.9 specifies sufficient conditions for the existence of fixed points of the solution semigroup $(S(t))_{t\geq 0}$ which, in turn, lead to constant solutions of (FDE). On the other hand, as Example 4.1 in §4 illustrates, (FDE) can have constant solutions without the corresponding initial conditions being fixed points of $(S(t))_{t\geq 0}$.

2. In the context of Theorem 2.1, if $\gamma = 0$ (i.e., $M \leq \alpha$), then the existence of a constant solution to (FDE) implies that all solutions are bounded.

3. A class of initial data spaces

In this section, we consider a particular class of initial history spaces satisfying (E1)–(E3), and then proceed to specify $\widehat{D}(A)$ for spaces within this class.

3.1 **Definition.** Assume that $v: \mathbb{R}^- \to (0, 1]$ is a function with the following properties:

(v1) v is continuous, nondecreasing, and $v(0) = 1$;

(v2) $\lim_{u\to 0^-} v(s+u)/v(s) = 1$ uniformly over $s \in \mathbb{R}^-$.

We then put $E_v = \{\varphi \in C(\mathbb{R}^-, X): v\varphi$ is bounded and uniformly continuous$\}$, and equip E_v with the weighted sup-norm $\|\varphi\|_v = \sup_{s\leq 0}\{v(s)\|\varphi(s)\|\}$, $\varphi \in E_v$.

In the present context, these spaces are sometimes called UC_g-spaces ($g = 1/v$), and have been considered by various authors; e.g., see [1, 15], and the further references listed therein. However, we have chosen to view them within the more familiar framework associated with the general theory of weighted sup-norm spaces.

3.2 *Remark.* Assume that $v: \mathbb{R}^- \to (0, 1]$ satisfies (v1) and (v2). Then

(a) E_v fulfills (E1)–(E3), as do

(b) $E_{v_l} = \{\varphi \in E_v: \lim_{s\to -\infty} v(s)\varphi(s)$ exists$\}$, and

(c) $E_{v_0} = \{\varphi \in E_v: \lim_{s\to -\infty} v(s)\varphi(s) = 0\}$ in case $\lim_{s\to -\infty} v(s) = 0$.

(For (a), compare [1, 15]. Assertions (b) and (c) follow by elementary arguments.)

Depending on the context, application of Theorem 2.5 to equations of the form (FDE) requires a description of either cl $D(A)$ or $\widehat{D}(A)$ for the operator A specified in Theorem 2.1. While not in the full generality of our characterization of cl $D(A)$ (Proposition 2.6), we now describe $\widehat{D}(A)$ in the E_v-space setting.

3.3 **Proposition.** *In the context of Theorem* 2.1, *assume that* $E = E_v$, *where the weight function* $v: \mathbb{R}^- \to (0, 1]$ *satisfies* (v1) *and the following, more restrictive, version of* (v2):

(v2*) *there exists a constant* $M_v \geq 0$ *such that*

$$\left|\frac{v(s+u)}{v(s)} - 1\right| \leq M_v|u| \quad \text{for all } s, u \leq 0.$$

Then

$$\widehat{D}(A) = \{\varphi \in \text{cl } D(A): v\varphi \text{ is Lipschitz continuous on } \mathbb{R}^- \text{ and } \varphi(0) \in \widehat{D}(B)\}.$$

Proof. Our argument parallels that of [12, Theorem 10] for the case $E = C([-R, 0), X)$.

Step 1. We first show that if $\varphi \in \widehat{D}(A)$, then $\varphi(0) \in \widehat{D}(B)$. Let $\varphi \in \mathrm{cl}\, D(A)$, $\lambda > 0$ with $\lambda\gamma < 1$, and choose $\varphi_\lambda \in D(A)$ such that $(I+\lambda A)\varphi_\lambda = \varphi$. Recalling (from the proof of Theorem 2.1) that

$$\varphi_\lambda(0) = \left(I + \frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\left[\frac{1}{1+\lambda\alpha}(\varphi(0) + \lambda F(\varphi_\lambda))\right],$$

we have

$$\begin{aligned}
&\left\|\frac{1}{\lambda}\left(\varphi(0) - \left(I + \frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\varphi(0)\right)\right\| \\
&\quad\le \frac{1}{\lambda}\|\varphi(0) - \varphi_\lambda(0)\| + \left\|\frac{\alpha}{1+\lambda\alpha}\varphi(0) - \frac{1}{1+\lambda\alpha}F(\varphi_\lambda)\right\| \\
&\quad\le \|A_\lambda\varphi\| + \frac{1}{1+\lambda\alpha}[|\alpha|\,\|\varphi(0)\| + \|F(\varphi)\|] + \frac{M}{1+\lambda\alpha}\|\varphi_\lambda - \varphi\| \\
&\quad= \left(1 + \frac{M\lambda}{1+\lambda\alpha}\right)\|A_\lambda\varphi\| + \frac{1}{1+\lambda\alpha}[|\alpha|\,\|\varphi(0)\| + \|F(\varphi)\|].
\end{aligned}$$

Thus, for $\rho = \lambda/(1+\lambda\alpha)$,

$$\|B_\rho\varphi(0)\| \le (1+\lambda\alpha)\left[\left(1 + \frac{M\lambda}{1+\lambda\alpha}\right)\|A_\lambda\varphi\| + \frac{1}{1+\lambda\alpha}[|\alpha|\,\|\varphi(0)\| + \|F(\varphi)\|]\right].$$

This shows that $\varphi(0) \in \widehat{D}(B)$ in case $\varphi \in \widehat{D}(A)$.

Step 2. Recalling the notation of Lemma 2.2, if $\varphi \in \widehat{D}(A)$, then it readily follows from Lemma 2.2(b) that $\psi = \varphi - \varphi(0) \in \widehat{D}(A_0)$. This being the case, the results of [8] show that

$$L(\psi) = \lim_{\tau\to 0^+}\frac{1}{\tau}(\|S_0(\tau)\psi - \psi\|) = |A_0\psi| < \infty.$$

In view of (v2), we can thus choose constants $C > 0$ and $\tau_0 > 0$ such that

$$\|S_0(\tau)\psi - \psi\| \le \tau C \quad \text{for all } 0 \le \tau \le \tau_0, \tag{3.1}$$

and

$$\left|\frac{v(s)}{v(s+u)} - 1\right| \le 1 \quad \text{for all } u \in [-\tau_0, 0] \text{ and all } s \in \mathbb{R}^-. \tag{3.2}$$

If $s_2 < s_1 \le 0$, choose $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $0 \le \tau < \tau_0$ such that $s_1 = s_2 + k\tau_0 + \tau$. Then

$$\begin{aligned}
\psi(s_2) - \psi(s_1) &= \psi(\tau_0 + (s_2 - \tau_0)) - \psi((k+1)\tau_0 + \tau + (s_2 - \tau_0)) \\
&= (S_0(\tau_0)\psi)(s_2 - \tau_0) - (S_0((k+1)\tau_0 + \tau)\psi)(s_2 - \tau_0).
\end{aligned}$$

By using (v1), (v2*), (3.1), and (3.2), we conclude that

$$\begin{aligned}
&\|v(s_2)\psi(s_2) - v(s_1)\psi(s_1)\| \\
&\quad\le \frac{v(s_2)}{v(s_2 - \tau_0)}\|S_0(\tau_0)\psi - S_0((k+1)\tau_0 + \tau)\psi\| + \left|\frac{v(s_2)}{v(s_1)} - 1\right|\|\psi\| \\
&\quad\le 2\|S_0(\tau_0)\psi - S_0((k+1)\tau_0 + \tau)\psi\| + M_v\|\psi\|\,|s_2 - s_1| \\
&\quad\le 2[k\|\psi - S_0(\tau_0)\psi\| + \|\psi - S_0(\tau)\psi\|] + M_v\|\psi\|\,|s_2 - s_1| \\
&\quad\le (2C + M_v\|\psi\|)|s_2 - s_1|.
\end{aligned} \tag{3.3}$$

This shows that $v\psi$ is Lipschitz continuous on $\mathbb{R}^-$, and hence the same is true for $v\varphi$ in view of (v2*).

Step 3. In order to prove the reverse inclusion, take $\varphi \in \operatorname{cl} D(A)$ such that $v\varphi$ is Lipschitz continuous on $\mathbb{R}^-$ with Lipschitz constant C_φ and $\varphi(0) \in \widehat{D}(B)$. For $\lambda > 0$ with $\lambda\gamma < 1$, choose $\psi = \varphi_\lambda \in D(A)$ as in Step 1. Then, for $s \le 0$,

$$\begin{aligned}(A_\lambda\varphi)(s) &= \frac{1}{\lambda}(\varphi - J_\lambda\varphi)(s)\\ &= \frac{1}{\lambda}\left\{\varphi(s) - e^{s/\lambda}\psi(0) - \frac{e^{s/\lambda}}{\lambda}\int_s^0 e^{-\xi/\lambda}\varphi(\xi)\,d\xi\right\}\\ &= \frac{e^{s/\lambda}}{\lambda}\left\{(\varphi(s)-\varphi(0)) + \left(\varphi(0) - \left(I + \frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\varphi(0)\right)\right.\\ &\qquad + \left(\left(I + \frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\varphi(0)\right.\\ &\qquad\qquad \left. - \left(I + \frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\left[\frac{1}{1+\lambda\alpha}(\varphi(0) + \lambda F(\psi))\right]\right)\\ &\qquad\qquad \left. - \frac{1}{\lambda}\int_s^0 e^{-\xi/\lambda}(\varphi(\xi) - \varphi(s))\,d\xi\right\}.\end{aligned}$$

Using (v1) and (v2*), some elementary calculation leads to

$$\begin{aligned} v(s)\|(A_\lambda\varphi)(s)\| &\le \frac{e^{s/\lambda}}{\lambda}\{C_\varphi + M_v\|\varphi(0)\|\}|s|\\ &\quad + e^{s/\lambda}v(s)\frac{1}{1+\lambda\alpha}\left\|\frac{1+\lambda\alpha}{\lambda}\left[\varphi(0) - \left(I + \frac{\lambda}{1+\lambda\alpha}B\right)^{-1}\varphi(0)\right]\right\|\\ &\quad + e^{s/\lambda}v(s)\left\|\frac{\alpha}{1+\lambda\alpha}\varphi(0) - \frac{1}{1+\lambda\alpha}F(\psi)\right\|\\ &\quad + \frac{e^{s/\lambda}}{\lambda^2}\left\{\int_s^0 e^{-\xi/\lambda}C_\varphi(\xi - s)\,d\xi + \int_s^0 e^{-\xi/\lambda}\left|\frac{v(s)}{v(\xi)} - 1\right| v(\xi)\|\varphi(\xi)\|\,d\xi\right\}\\ &\le \frac{1}{e}\{C_\varphi + M_v\|\varphi(0)\|\} + \frac{1}{1+\lambda\alpha}\|B_\rho(\varphi(0))\|\\ &\quad + \left\|\frac{\alpha}{1+\lambda\alpha}\varphi(0) - \frac{1}{1+\lambda\alpha}F(\psi)\right\| + (C_\varphi + M_v\|\varphi\|)\\ &\le \left(1 + \frac{1}{e}\right)(C_\varphi + M_v\|\varphi\|) + \frac{|\alpha|}{1+\lambda\alpha}\|\varphi(0)\| + \frac{M\lambda}{1+\lambda\alpha}\|A_\lambda\varphi\|\\ &\quad + \frac{1}{1+\lambda\alpha}\|F(\varphi)\| + \frac{1}{1+\lambda\alpha}\|B_\rho(\varphi(0))\|,\end{aligned}$$

where $\rho = \lambda/(1+\lambda\alpha)$. We conclude from this estimate that

$$|A\varphi| = \lim_{\lambda\to 0^+}\|A_\lambda\varphi\| < \infty,$$

and the proof of Proposition 3.3 is complete.

3.4 *Remarks.* 1. The requirement that $v\varphi$ be uniformly continuous on $\mathbb{R}^-$ is essential for (E3)(a), and thus for Lemma 2.2(a) to hold. In the transition from finite to infinite delays, this has sometimes been overlooked in the literature (concerning the global case) with claims to the effect that the conclusion of Theorem 2.1 would hold for $E = C_b(\mathbb{R}^-, X)$.

2. For $\mu \geq 0$, the weight functions $v_1(s) = e^{\mu s}$ and $v_2(s) = (1+|s|)^{-\mu}$, $s \in \mathbb{R}^-$, both satisfy (v1) and (v2*).

4. The examples revisited

Before returning to the applications presented in §1, we briefly consider an instance where the automatic criteria of Proposition 2.7(b) apply. The following example also serves to show that our approach to (FDE) properly extends Brewer's result [7, Theorem 2, p. 375] for the locally Lipschitz continuous case.

4.1 **Example.** Consider the scalar $(X = \mathbb{R})$ functional differential equation

$$(1) \qquad \begin{cases} x'(t) + x(t) = x(t)\|x_t\|_\infty, & t \geq 0 \\ x(t) = \varphi(t), & t \leq 0, \end{cases}$$

where $\varphi \in E = (BUC(\mathbb{R}^-, \mathbb{R}), \|\cdot\|_\infty)$, and put $\widehat{E} = \{\varphi \in E: \|\varphi\|_\infty \leq 1\}$. Then

(i) 4.1(1) has a unique solution $x_\varphi: \mathbb{R} \to \mathbb{R}$ corresponding to each $\varphi \in \widehat{E}$;
(ii) there exists a strongly continuous semigroup of operators $(S(t))_{t\geq 0}$ on $\widehat{E}$ such that, given any $\varphi \in \widehat{E}$, $x_\varphi(t) = (S(t)\varphi)(0) \in [-1, 1]$, $t \geq 0$.

Proof. As noted in the preceding section, since $E = E_v$ where $v(s) = 1$, $s \leq 0$, E satisfies (E1)–(E3). Now, choosing $\widehat{X} = \{x \in \mathbb{R}: |x| \leq 1\}$ as specified in Remark 2.8, since $\varphi \in \widehat{E}$ implies that $|F(\varphi)| = |\varphi(0)|\,\|\varphi\|_\infty \leq 1$, it is immediate from (ii) of Proposition 2.7(b) that Theorem 2.1 applies. Consequently, $\operatorname{cl} D(A) = \widehat{E}$ by Proposition 2.6, whereby the desired conclusions now follow from Theorem 2.5(c).

Remarks. 1. Using the notation of (FDE) in the context of Example 4.1, since $F|_{\widehat{E}}$ has Lipschitz constant $M = 2$ while $\alpha = 1$, [7, Theorem 2, p. 375] does not apply.

2. The semigroup $(S(t))_{t\geq 0}$ on $\widehat{E}$ (specified by 4.1(ii)) has exactly three equilibria; namely, the motions through the constant functions $\varphi_1, \varphi_0, \varphi_{-1} \in \widehat{E}$ which are identically 1, 0, and -1, respectively. On the other hand, given any $c \in [-1, 1]$, it is easy to see that there exists $\varphi_c \in \widehat{E}$ such that $x_{\varphi_c}(t) = c$, $t \geq 0$.

3. Even so, the (zero) solution x_{φ_0} is asymptotically stable in the sense that, given $\psi \in E$ with $\|\psi\|_\infty = \beta < 1$, $\lim_{t\to\infty} x_\psi(t) = 0$. Indeed, when applied to $E_\beta = \{\varphi \in E: \|\varphi\|_\infty \leq \beta\}$ in place of $\widehat{E}$, the argument used to establish 4.1 shows that $\|x_\psi\|_\infty \leq \beta$ whence $x_\psi(t) = \psi(0)\exp(\beta - 1)t$, $t \geq 0$.

We now return to the single species population model 1.1(2).

4.2 **Example.** Again consider the initial value problem (1.1(2))

$$\begin{cases} x'(t) = x(t)(a - bx(t) - H(x_t)), & t \geq 0 \\ x(t) = \varphi(t), & t \leq 0, \end{cases}$$

where $a, b > 0$, $E = (BUC(\mathbb{R}^-, \mathbb{R}), \|\cdot\|_\infty)$, $\varphi \in E$, and $H: E \to \mathbb{R}$ satisfies conditions (H1) and (H2) of 1.1. Further, assume that there exists $\beta \geq a/b$

such that (i) $l(\beta) \leq a$, and (ii) $a + l(\beta) \leq b\beta$. Then the following assertions hold:

1. Corresponding to each $\varphi \in \widehat{E} = \{\psi \in E : 0 \leq \psi \leq \beta\}$, equation 1.1(2) has a unique solution $x_\varphi : \mathbb{R} \to \mathbb{R}$, $x_\varphi(t) \in [0, \beta]$ for each $t \geq 0$, and $x_\varphi(t) > 0$ for all $t \geq 0$ in case $\varphi(0) > 0$.

2. For each $\alpha \in [0, 2(a - l(\beta))]$, there exists a strongly continuous operator semigroup $(S_\alpha(t))_{t \geq 0}$ on $\widehat{E}_\alpha = \{\psi \in \widehat{E} : \alpha/2b \leq \psi(0) \leq \beta\}$ such that, given any $\varphi \in \widehat{E}_\alpha$, $x_\varphi(t) = (S_\alpha(t)\varphi)(0) \in [\alpha/2b, \beta]$, $t \geq 0$.

3. If $l(\beta) < a$ and $\alpha = 2(a - l(\beta))$, then there exists (a constant function) $\varphi \in \widehat{E}_\alpha$ such that $S_\alpha(t)\varphi = \varphi$, $t \geq 0$, whereby 1.1(2) has a constant solution x_φ satisfying $x_\varphi(t) = \varphi(0) \in [\alpha/2b, \beta]$, $t \geq 0$.

4. Setting $M(\beta) = a + l(\beta) + \beta m(\beta)$, if there exists $\alpha \in [0, 2(a - l(\beta))]$ such that $\alpha \geq M(\beta)$, then $\|S_\alpha(t)\varphi - S_\alpha(t)\psi\|_\infty \leq \|\varphi - \psi\|_\infty$ for all φ, $\psi \in \widehat{E}_\alpha$ and all $t \geq 0$.

Proof. Fixing $\alpha \in [0, 2(a - l(\beta))]$, equation 1.1(2) can be written as

$$x'(t) + \alpha x(t) + B_\alpha x(t) = F(x_t), \qquad t \geq 0,$$

where $B_\alpha x = bx^2 - \alpha x$ for $x \in \mathbb{R}$ and $F(\varphi) = \varphi(0)(a - H(\varphi))$ for $\varphi \in E$. Now, recall that E satisfies (E1)–(E3), put $\widehat{X}_\alpha = [\alpha/2b, \beta]$, and note that $B_\alpha|_{[\alpha/2b, \infty)}$ is accretive. Since $F|_{\widehat{E}_\alpha}$ is Lipschitz continuous (with Lipschitz constant $M_\alpha \leq M(\beta) = a + l(\beta) + \beta m(\beta)$) in view of (H1) and (H2), we thus have that (A1) holds, while it is obvious that (A2) is also satisfied. Turning to (A3), let $\psi \in \widehat{E}_\alpha$, $x \in \widehat{X}_\alpha$, and $\lambda > 0$. As a consequence of conditions (i) and (ii), (H2), and the choice of α, we have the following estimates:

$$\psi(0) + \lambda F(\varphi_x) \leq \beta + \lambda\beta(a + l(\beta)) \leq \beta + \lambda b\beta^2; \tag{1}$$

$$\psi(0) + \lambda F(\varphi_x) \geq \frac{\alpha}{2b} + \lambda(a - l(\beta))\frac{\alpha}{2b} \geq \frac{\alpha}{2b} + \lambda\frac{\alpha^2}{4b}. \tag{2}$$

From (1) and (2), since $f = (1 + \lambda\alpha)I + \lambda B_\alpha$ is continuous and nondecreasing on $[\alpha/2b, \beta]$, it is then immediate that there exists $y \in \widehat{X}_\alpha$ such that $f(y) = \psi(0) + \lambda F(\varphi_x)$, whereby (A3) holds as well. Since any solution x_φ of 1.1(2) necessarily satisfies

$$x_\varphi(t) = \varphi(0)\exp\left(\int_0^t (a - bx(s) - H(x_s))ds\right), \qquad t \geq 0,$$

assertions 1, 2, and 4 can now be read from Theorem 2.5(c), Theorem 2.1, and Proposition 2.6. As for the remaining assertion, given $x \in \widehat{X}_\alpha$, if we put $\tilde{x}(r) = x$ for $r \leq 0$, then $\tilde{x} \in \widehat{E}_\alpha$. Furthermore, an argument similar to the one used to establish (A3) shows that $F(\tilde{x}) = \alpha y + B_\alpha y$ for some $y \in \widehat{X}_\alpha$. Since $\|\tilde{x}\|_\infty = \|x\|$ in this setting, Proposition 2.9 applies to complete the proof.

Remark. In Example 4.2, assertion 4.2.2 can be used to strengthen the conclusion of 4.2.1 whenever $l(\beta) < a$. In this case, if $\varphi \in \widehat{E}$ with $\varphi(0) > 0$, it even follows that

$$x_\varphi(t) \geq \min\left\{\varphi(0), \frac{a - l(\beta)}{b}\right\} \quad \text{for all } t \geq 0.$$

Before leaving Example 4.2, we note that, roughly speaking, conditions (i) and (ii) relating the parameters a and b with the function $l: \mathbb{R}^+ \to \mathbb{R}^+$ specified by (H2) can be viewed as a limitation on how rapidly the "crowding effect" induced by H can (initially) increase relative to historical population sizes. As illustration, consider the special case of the delay logistic equation 1.1(1). Then $H: E \to \mathbb{R}$ is defined for $\varphi \in E$ by $H(\varphi) = \int_{-\infty}^{0} k(-s)\varphi(s)\,ds$ where $k \in L^1(\mathbb{R}^+)$, and hence (H1) and (H2) hold with $m(\beta) = \|k\|_1$ and $l(\beta) = \|k\|_1\beta$, $\beta \geq 0$. Thus, (i) and (ii) will also be satisfied if (and only if) $2\|k\|_1 \leq b$ in which case

$$\frac{a}{b - \|k\|_1} \leq \beta \leq \frac{a}{\|k\|_1}.$$

Of course, $l(\beta) < a$ if (and only if)

$$2\|k\|_1 < b \quad \text{and} \quad \frac{a}{b - \|k\|_1} \leq \beta < \frac{a}{\|k\|_1},$$

while the condition $2(a - l(\beta)) \geq M(\beta)$ holds if (and only if) $5\|k\|_1 \leq b$ and

$$\frac{a}{b - \|k\|_1} \leq \beta \leq \frac{a}{4\|k\|_1}.$$

The next example treats the model 1.2(1) for the classical Goodwin oscillator with infinite delay (in the case $n = 2$).

4.3 **Example.** Recalling 1.2(1), let $a_i, b_i > 0$ for $i = 1, 2$, $k \in L^1(\mathbb{R}^+) \cap C(\mathbb{R}^+)$ with $k \geq 0$, and $m \in \mathbb{N}$. Further, taking $X = \mathbb{R}^2$ and

$$E = (BUC(\mathbb{R}^-, X), \|\cdot\|_\infty),$$

let E^+ denote the cone of all

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in E$$

such that $\varphi_i \geq 0$, $i = 1, 2$, and define $F: E^+ \to X$ by

$$F(\varphi) = \begin{pmatrix} f_1(\varphi_2) \\ f_2(\varphi_1) \end{pmatrix},$$

where

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in E^+, \qquad f_1(\varphi_2) = b_1 \left[1 + \left(\int_{-\infty}^{0} k(-s)\varphi_2(s)\,ds\right)^m\right]^{-1},$$

and

$$f_2(\varphi_1) = b_2 \int_{-\infty}^{0} k(-s)\varphi_1(s)\,ds.$$

If we now put $\alpha = \min\{a_1, a_2\}$ and

$$B = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} - \alpha I,$$

then 1.2(1) (in the case $n = 2$) is given by

$$(1) \qquad \begin{cases} x'(t) + \alpha x(t) + Bx(t) = F(x_t), & t \geq 0 \\ x|_{\mathbb{R}^-} = \varphi \in E^+, \end{cases}$$

and the following assertions hold:

1. Corresponding to each $\varphi \in E^+$, 4.3(1) has a unique solution $x_\varphi: \mathbb{R} \to \mathbb{R}^2$;

2. (i) There exists a strongly continuous semigroup $(S(t))_{t\geq 0}$ of operators on E^+ such that, given $\varphi \in E^+$, $x_\varphi(t) = S(t)\varphi(0)$, $t \geq 0$, whereby

$$x_\varphi(t) \in X^+ = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1, x_2 \geq 0 \right\} \quad \text{for all } t \geq 0,$$

and (ii) $(S(t))_{t\geq 0}$ has a unique fixed point $\varphi_e \in E^+$;

3. If $b_2\|k\|_1 < \alpha$, then (i) $\|\varphi_e\|_\infty \leq b_1[\alpha^2 - (b_2\|k\|_1)^2]^{-1/2}$, and (ii)

$$\sup\{\|x_\varphi(t)\|: t \geq 0\} \leq \max\{\|\varphi\|_\infty, b_1[\alpha^2 - (b_2\|k\|_1)^2]^{-1/2}\}$$

for each $\varphi \in E^+$;

4. If $\|k\|_1^2[(b_1c(m))^2 + b_2^2] \leq \alpha^2$, where $c(1) = 1$ and

$$c(m) = \frac{(m+1)^2}{4m}\left(\frac{m-1}{m+1}\right)^{\frac{m-1}{m}}, \qquad m > 1,$$

then $(S(t))_{t\geq 0}$ is a contraction semigroup on E^+.

Proof. For the function $f(t) = (1 + t^m)^{-1}$, $t \geq 0$, it is an elementary exercise to see that $|f(s) - f(t)| \leq c(m)|s - t|$ for all $s, t \in \mathbb{R}^+$. With this observation in hand, the obvious estimate then shows that F is Lipschitz continuous (on E^+) with Lipschitz constant no greater than $M = \|k\|_1[(b_1c(m))^2 + b_2^2]^{1/2}$. Thus, since B is (even) m-accretive, (A1) holds for $\widehat{E} = E^+$ and $\widehat{X} = X^+$. Clearly, both (A2) and (A3) also hold in this setting, while E satisfies (E1)–(E3), and hence assertions 1, 2(i), and 4 directly follow from Theorems 2.1 and 2.5(c) taken together with Proposition 2.6. Using the notation of Proposition 2.9, moreover, a straightforward calculation shows that there exists exactly one $x \in X^+$ such that $\tilde{x} \in E^+$ and $F(\tilde{x}) = (\alpha I + B)x$, whereby 2(ii) holds with $\varphi_e = \tilde{x}$ (in view of Proposition 2.9, for example).

Regarding assertion 3, assume that $b_2\|k\|_1 < \alpha$ and put

$$\beta = b_1[\alpha^2 - (b_2\|k\|_1)^2]^{-1/2}.$$

Since conditions (i)–(iii) of Proposition 2.9.2 are clearly satisfied in the present setting, consider the map $T: X^+ \to X^+$ specified by 2.9.2. Then, given $x \in X_\beta = \{y \in X^+: \|y\| \leq \beta\}$, we have that

$$\|Tx\| \leq \frac{1}{\alpha}\|F(\tilde{x})\| \leq \frac{1}{\alpha}[b_1^2 + (b_2\|k\|_1\beta)^2]^{1/2} = \beta;$$

i.e., $T(X_\beta) \subseteq X_\beta$. In view of the Brouwer fixed point theorem, Proposition 2.9.2 and 2(ii) now combine to establish 3(i). On the other hand, given $\varphi \in E^+$, put $\hat{\beta} = \max\{\|\varphi\|_\infty, \beta\}$, and note that (A1) and (A2) also hold for $\widehat{E} = \{\psi \in E^+: \|\psi\|_\infty \leq \hat{\beta}\}$ and $\widehat{X} = \{x \in X^+: \|x\| \leq \hat{\beta}\}$. If (A3) were to hold as well, then another application of Theorems 2.1 and 2.5(c) combined with Proposition 2.6 in this new context would certainly serve to establish 3(ii). Given $\psi \in \widehat{E}$, $x \in \widehat{X}$, and $\lambda > 0$, however, since B is m-accretive with $B(0) = 0$ and

$$\begin{aligned}\frac{1}{1+\lambda\alpha}\|\psi(0) + \lambda F(\varphi_x)\| &\leq \frac{1}{1+\lambda\alpha}(\hat{\beta} + \lambda[b_1^2 + (b_2\|k\|_1\hat{\beta})^2]^{1/2} \\ &\leq \frac{1}{1+\lambda\alpha}(\hat{\beta} + \lambda\alpha\hat{\beta}) = \hat{\beta},\end{aligned}$$

we indeed have that (A3) is satisfied, and the proof is thereby complete.

As our concluding illustration, we take up the modified Goodwin oscillator mentioned under 1.2.

4.4 **Example.** Again, consider the initial value problem

$$(1)\qquad \begin{cases} x'(t) + \alpha x(t) + Bx(t) = F(x_t), & t \geq 0 \\ x|_{\mathbb{R}^-} = \varphi \in E^+ \end{cases}$$

specified in Example 4.3, except now take $F\colon E^+ \to \mathbb{R}^2$ to be defined by

$$F(\varphi) = \begin{pmatrix} f_1(\varphi_2) \\ \varphi_2(0) f_2(\varphi_1) \end{pmatrix} \quad \text{for } \varphi = \begin{pmatrix} \varphi_2 \\ \varphi_2 \end{pmatrix} \in E^+.$$

In this case, assume that $0 < 2b_1 b_2 \|k\|_1 \leq \alpha^2$. Then, setting $l = 2b_1 b_2 \|k\|_1$,

$$\mathscr{B} = \{\beta \geq 0\colon \alpha^2 - (\alpha^4 - l^2)^{1/2} \leq 2(b_2\|k\|_1)^2 \beta^2 \leq \alpha^2 + (\alpha^4 - l^2)^{1/2}\},$$

$\beta_0 = \inf \mathscr{B}$, $\beta_1 = \sup \mathscr{B}$, and $E_\beta = \{\varphi \in E^+\colon \|\varphi\|_\infty \leq \beta\}$ for $\beta \in \mathscr{B}$, the following assertions hold:

1. Corresponding to each $\varphi \in E_{\beta_1}$, 4.4(1) has a unique solution $x_\varphi\colon \mathbb{R} \to \mathbb{R}^2$;
2. For each $\beta \in \mathscr{B}$, there exists a strongly continuous semigroup $(S_\beta(t))_{t\geq 0}$ of operators on E_β such that, given $\varphi \in E_\beta$, $x_\varphi(t) = (S_\beta(t)\varphi)(0)$, $t \geq 0$, whereby $x_\varphi(t) \in X_\beta = \{x \in X^+\colon \|x\| \leq \beta\}$ for all $t \geq 0$;
3. Setting
$$\varphi_e(r) = \begin{pmatrix} b_1/a_1 \\ 0 \end{pmatrix} \quad \text{for } r \leq 0,$$
   (i) $\varphi_e \in E_{\beta_0}$ and (ii) φ_e is the unique fixed point of $(S_\beta(t))_{t\geq 0}$ for each $\beta \in \mathscr{B}$;
4. For $p = b_1 \|k\|_1 c(m)$, if $p^2 + 2(p^4 + 3l^2)^{1/2} \leq 3\alpha^2$, then
   (i) $\alpha^2 - (\alpha^4 - l^2)^{1/2} \leq \frac{1}{2}(\alpha^2 - p^2) \leq \alpha^2 + (\alpha^4 - l^2)^{1/2}$, and
   (ii) $(S_\beta(t))_{t\geq 0}$ is a contraction semigroup on E_β for each $\beta \in \mathscr{B}$ such that $2(b_2\|k\|_1)^2 \beta^2 \leq \frac{1}{2}(\alpha^2 - p^2)$.

Proof. For any $\beta > 0$, put $E_\beta = \{\varphi \in E^+\colon \|\varphi\|_\infty \leq \beta\}$ and $X_\beta = \{x \in X^+\colon \|x\| \leq \beta\}$. Then, as can be readily verified, $F|_{E_\beta}$ is Lipschitz continuous with Lipschitz constant no greater than $M(\beta) = \|k\|_1[(b_1 c(m))^2 + (2b_2\beta)^2]^{1/2}$, and hence (A1) and (A2) are clearly satisfied for $\widehat{E} = E_\beta$ and $\widehat{X} = X_\beta$. Given $\psi \in E_\beta$, $x \in X_\beta$, and $\lambda > 0$, moreover, if $\beta \in \mathscr{B}$, then

$$\begin{aligned} &(2[b_2\|k\|_1\beta]^2 - [\alpha^2 - (\alpha^4 - l^2)^{1/2}])(2[b_2\|k\|_1\beta]^2 - [\alpha^2 + (\alpha^4 - l^2)^{1/2}]) \\ &\quad = (2[b_2\|k\|_1\beta]^2 - \alpha^2)^2 - (\alpha^4 - l^2) \leq 0. \end{aligned}$$

Since this implies that $b_1^2 + (b_2\|k\|_1\beta^2)^2 \leq (\alpha\beta)^2$, we conclude that

$$\frac{1}{1+\lambda\alpha}\|\psi(0) + \lambda F(\varphi_x)\| \leq \frac{1}{1+\lambda\alpha}(\beta + \lambda[b_1^2 + (b_2\|k\|_1\beta^2)^2]^{1/2}) \leq \beta,$$

which suffices to show that (A3) also holds in case $\beta \in \mathscr{B}$. Assuming that $p^2 + 2(p^4 + 3l^2)^{1/2} \leq 3\alpha^2$, a routine calculation yields that

$$\alpha^2 - (\alpha^4 - l^2)^{1/2} \leq \frac{1}{2}(\alpha^2 - p^2),$$

whereby 4(i) holds, and it is easy to see that $M(\beta) \leq \alpha$ if $2(b_2\|k\|_1)^2\beta^2 \leq \frac{1}{2}(\alpha^2 - p^2)$. Assertions 1, 2, and 4(ii) now follow by applying Theorems 2.1 and 2.5(c) together with Proposition 2.6. As for assertion 3, 3(i) can be checked directly. In view of Proposition 2.9, the same is true of 3(ii), and the proof is complete.

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Fachbereich Mathematik, Universität Essen, 45117 Essen, Germany
E-mail address: `mate00@vm.hrz.uni-essen.de`

Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701
E-mail address: `ws24788@uafsysb.uark.edu`