WEIGHTS FOR CLASSICAL GROUPS

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Abstract. This paper proves the Alperin's weight conjecture for the finite unitary groups when the characteristic \( r \) of modular representation is odd. Moreover, this paper proves the conjecture for finite odd dimensional special orthogonal groups and gives a combinatorial way to count the number of weights, block by block, for finite symplectic and even dimensional special orthogonal groups when \( r \) and the defining characteristic of the groups are odd.

Introduction

Let \( G \) be a finite group and \( r \) a prime. A weight of \( G \) is a pair \((R, \varphi)\) of an \( r \)-subgroup \( R \) of \( G \) and an irreducible character \( \varphi \) of \( N(R) \) such that \( \varphi \) is trivial on \( R \) and in an \( r \)-block of defect 0 of \( N(R)/R \), where \( N(R) = N_G(R) \) is the normalizer of \( R \) in \( G \). A radical subgroup \( R \) of \( G \) is an \( r \)-subgroup of \( G \) such that \( R = O_r(N(R)) \), where \( O_r(N(R)) \) is the largest normal \( r \)-subgroup of \( N(R) \). If \((R, \varphi)\) is a weight of \( G \), then \( R \) is necessarily a radical subgroup of \( G \). A weight \((R, \varphi)\) is a \( B \)-weight for an \( r \)-block \( B \) of \( G \) if \( \varphi \) is contained in an \( r \)-block \( b \) of \( N(R) \) such that \( B = b^G \), that is, \( B \) corresponds to \( b \) by the Brauer homomorphism. In his paper [2], Alperin introduced the concept of weight in the modular representation theory of finite groups and conjectured that the number of weights of \( G \) should equal the number of modular irreducible representations. Moreover, this equality should hold block by block. Here a weight \((R, \varphi)\) is identified with its conjugates in \( G \). Alperin and Fong in [3] have proved this conjecture for symmetric groups and for finite general linear groups when the characteristic \( r \) of modular representation is odd. The author in [4, 5] proved the conjecture for finite general linear and unitary groups when \( r \) is even. In this paper, we prove the conjecture for the finite unitary groups when \( r \) is odd. Moreover, we prove the conjecture for odd dimensional special orthogonal groups and give a combinatorial way to count the number of weights, block by block, for both finite symplectic and even dimensional special orthogonal groups when \( r \) and the defining characteristic \( p \) of groups are odd. We may suppose \( p \) is different from \( r \) since the result is known when \( p \) is \( r \) (see [2]).

In the first two sections, we describe the local structures of radical subgroups of a finite classical group, and in §3 we count the number of weights when the center of a radical subgroup is cyclic. The conjecture has been proved for unitary groups in (4D) and for odd dimensional special orthogonal groups in
JIANBEI AN

2

(4G) and its remarks. Finally, the numbers of weights for symplectic and even dimensional special orthogonal groups have been counted in (4F) and (4H) respectively.

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1. THE GROUPS OF SYMPLECTIC TYPE

Throughout this paper we shall follow the notation of [3, 5, 7], and [12]. In particular, \( r \) is an odd prime and \( E \) is an extraspecial \( r \)-group of order \( r^{2y+1} \) with center \( Z(E) = \langle y \rangle \). Then \( E = \langle x_1, x_2, \ldots, x_{2y-1}, x_{2y} \rangle \) such that \( [x_{2i-1}, x_{2i}] = x_{2i-1}^{-1}x_{2i}^{-1}x_{2i-1}x_{2i} = y, [x_{2i}, x_{2i+1}] = 1 \), for \( 1 \leq i \leq y \), \( [x_i, x_j] = 1 \) for \( |i - j| \geq 2 \), \( x_i^r = 1 \) for \( i \neq 2 \). Thus \( E \) has exponent \( r \) or \( r^2 \) according as \( x_i^r = 1 \) or \( y \). An \( r \)-group \( R \) is of symplectic type if \( R \) is a central product of a nontrivial cyclic \( r \)-group \( Z \) and an extraspecial group \( E \), where \( Z(E) \) is identified with \( E\langle x \rangle \). If \( R > E \), then \( R \) can be rewritten as the central product of \( Z \) and an extraspecial group of exponent \( r \), so that we may suppose \( E \) has exponent \( r \) and \( E = \langle x \rangle \). Let \( \text{Aut} \, R \) be the automorphism group of \( R \), \( \text{Inn} \, R \) the group of inner automorphisms, and \( \text{Aut}^0R = \{ \sigma \in \text{Aut} \, R : [\sigma, Z] = 1 \} \). Since every \( \sigma \) in \( \text{Aut}^0 \, R \) restricts to an element \( \text{Aut}^0 \, E \) and every \( \sigma \) in \( \text{Aut}^0 \, E \) extends to an element of \( \text{Aut} \, R \), it follows that \( \text{Aut}^0 \, R = \text{Aut}^0 \, E \). Denote

\[
K = \begin{cases} 
\text{Sp}(2y, r) & \text{if } E \text{ has exponent } r, \\
\text{Sp}(2y - 2, r) \rtimes r^{2(\gamma - 1) + 1} & \text{if } E \text{ has exponent } r^2,
\end{cases}
\]

where \( r^{2(\gamma - 1) + 1} \) denotes the extraspecial group of order \( 2(\gamma - 1) + 1 \) and exponent \( r \), and \( \text{Sp}(0, r) \rtimes r^1 \) is interpreted as a group of order \( r \). By [20, Theorem 1 or 15, p. 404] \( \text{Aut}^0 \, E = K \rtimes \text{Inn} \, E \) (see also [3, p. 10]). In the following we shall consider the embeddings of \( R \) into classical groups and determine the local structures of these embeddings.

Let \( \mathbb{F}_q \) be the field of \( q \) elements and \( \eta = \pm 1 \) a sign, where \( q \) is a power of prime \( p \) distinct from \( r \). We first consider the embedding of \( E \) in the groups \( G = \text{GL}(n, \eta q) \). Here following [7], we denote \( \text{U}(n, q) \) by \( \text{GL}(n, -q) \). The proofs of the following two lemmas are similar to that of [5, (1D), (1E), and (1F)] and in the proofs such terms as orthogonal, orthonormal, and isometric will have meaning only in contexts involving \( \text{U}(n, q) \) and unitary spaces, but no meaning in contexts involving \( \text{GL}(n, q) \) and linear spaces.

(1A). Let \( E \) be an extraspecial group of order \( r^{2\gamma + 1} \) and \( G = \text{GL}(r^\gamma, \eta q) \). If \( r \) divides \( q - \eta \) (written \( r \mid q - \eta \)), then \( G \) contains a unique conjugacy class of subgroups isomorphic to \( E \). Moreover, if \( r \mid q - 1 \), then \( \mathbb{F}_q \) is a splitting field of \( E \).

Proof. Given \( 1 \leq i \leq \gamma \), let \( E_i = \langle x_{2i-1}, x_{2i} \rangle \), and \( V_i \) a linear space of dimension \( r \) over \( \mathbb{F}_q \) or a unitary space of dimension \( r \) over \( \mathbb{F}_{q^2} \) according as \( \eta = 1 \) or \( -1 \). Then \( E_i \) acts faithfully, irreducibly, and isometrically on \( V_i \). Namely, let \( w \) be an \( r \)th root of unity in \( \mathbb{F}_{q^2} \) and \( \{v_1^i, v_2^i, \ldots, v_{r^\gamma}^i\} \) an orthonormal basis of \( V_i \). If \( E \) has exponent \( r \), then define

\[
(1.2) \quad x_{2i-1}: v_j^i \mapsto w^j v_j^i, \quad x_{2i}: v_j^i \mapsto v_{j+1}^i,
\]
where $1 \leq j \leq r$. If $E$ has exponent $r^2$, then define

$$x_{2i-1}: v_j^i \mapsto w^iv_j^i, \quad x_{2i}: v_j^i \mapsto \begin{cases} \quad wv_j^i & \text{if } i = 1 \text{ and } j = r, \\ v_{j+1}^i & \text{otherwise,} \end{cases}$$

where $1 \leq j \leq r$. Here subscripts on basis vectors are naturally read modulo $r$. In particular, $y: v_j^i \mapsto uv_j^i$ for all $j$.

Since $E$ is the central product of the $E_i$'s and the element $y$ in $Z(E_i)$ is represented on $V_i$ by the scalar matrix $wI$, $E$ acts faithfully and irreducibly on $V = V_1 \otimes V_2 \otimes \cdots \otimes V_r$. To see that the actions are by isometries, we first simplify notation and write

$$v_j^1 \otimes v_j^2 \otimes \cdots \otimes v_j^r = [j_1, j_2, \ldots, j_r], \quad 1 \leq j_i \leq r.$$  

The $r^r$ elements $[j_1, j_2, \ldots, j_r]$ form an orthonormal basis for $V$. So

$$x_{2i-1}: [j_1, j_2, \ldots, j_r] \mapsto w^2[j_1, j_2, \ldots, j_r],$$

$$x_{2i}: [j_1, j_2, \ldots, j_r] \mapsto [j_1, \ldots, j_i, j_i+1, j_i+1, \ldots, j_r],$$

except when $E$ has exponent $r^2$, in which case the actions of $x_i$ for $i \neq 2$ are given by (1.4) and

$$x_2: [j_1, j_2, \ldots, j_r] \mapsto \begin{cases} [j_1 + 1, j_2, \ldots, j_r] & \text{if } j_1 \neq r, \\ w[1, j_2, \ldots, j_r] & \text{if } j_1 = r. \end{cases}$$

Since basic vectors are mapped onto orthonormal vectors by generating elements of $E$, $E$ acts on $V$ by isometries, so that $G$ contains a copy of $E$.

Suppose $r|q - 1$. Replacing $w$ by $w^k$ for $1 \leq k < r$ in the proof above, we get $r - 1$ faithful and irreducible representations of $E$. By [14, 5.5.4] $E$ has $r - 1$ nonlinear characters and all linear characters are realizable over $\mathbb{F}_q$ since $E/Z(E)$ is an elementary abelian $r$-group. Thus $\mathbb{F}_q$ is a splitting field of $E$.

To prove the uniqueness, it suffices to show that if $E$ is embedded as a subgroup of $G$, then there exists an orthonormal basis of the underlying space $V$ such that (1.4) or (1.5) holds according as $E$ has exponent $r$ or $r^2$. By Schur's lemma $y = w^kI$ for some integer $1 \leq k < r$. We may suppose $y = wI$ since $E = (x_1, x_2^k, x_3, x_4^k, \ldots, x_{2r-1}, x_{2r}^k)$ and $[x_{2i-1}, x_{2i}^k] = y^k$.

Let $W_j = \{v \in V: x_1v = w^jv\}$ for $1 \leq j \leq r$. Then $V$ is the orthogonal sum of the $W_j$, so the $W_j$ for $1 \leq j \leq r$ are nondegenerate subspaces of $V$ and they are permuted by $x_2$ cyclically

$$x_2W_1 = W_2, \quad x_2^2W_1 = W_3, \ldots, x_2^rW_1 = W_1,$$

since $x_1x_2 = wx_2x_1$. In particular, $W_j$ for $1 \leq j \leq r$ have the same dimension.

If $\gamma = 1$ and $\{v_1\}$ is an orthonormal basis of $W_1$, then $\{v_1, x_2v_1, \ldots, x_2^{r-1}v_1\}$ is an orthonormal basis of $V$ and the actions of $x_1$ and $x_2$ on the basis are given by (1.2) or (1.3) according as $E$ has exponent $r$ or $r^2$. If $\gamma \geq 2$, then $L = (x_3, x_4, \ldots, x_{2r})$ is an extraspecial group of order $r^{2\gamma - 1}$ and exponent $r$ acting faithfully on $W_1$. We may suppose by induction that $x_3, x_4, \ldots, x_{2r}$ act on $W_1$ by (1.4) relative to the orthonormal basis $\{[j_2, j_3, \ldots, j_r]\}$ of $W_1$, where $1 \leq j_i \leq r$. Thus $\{[j_2, j_3, \ldots, j_r] = x_2^{j_i-1}[j_2, \ldots, j_r]: 1 \leq j_i \leq r\}$ is an orthonormal basis of $V$ and $x_1, x_2, \ldots, x_{2r}$ act on the basis by (1.4) or (1.5). Thus any two embeddings of $E$ in $G$ are conjugate.
Remark. (1) Suppose \( r|q - \eta \) and \( E \) is embedded in \( G = GL(n, \eta q) \) as a subgroup such that \( y \) is represented by a scalar multiple of the identity matrix. Then \( n = m\tau \) for some integer \( m \geq 1 \), and there exists an orthonormal basis \( \{ [j_1, j_2, \ldots, j_r]_k \} \) of the underlying space \( V \) of \( G \), where \( 1 \leq j_i \leq r \) and \( 1 \leq k \leq m \) such that for each \( k \) the actions of \( x_{2i-1} \) and \( x_{2i} \) are given by (1.4) or (1.5) with \( [j_1, j_2, \ldots, j_r] \) replaced by \( [j_1, j_2, \ldots, j_r]_k \). In particular, by (1A) such embedding of \( E \) in \( G \) is uniquely determined up to conjugacy in \( G \). The proof of the remark is similar to that of the uniqueness of (1A) and Remark (2) of [5, (1D)].

(2) Suppose \( r|q - \eta \), \( E \) has exponent \( r \), and \( E \) is embedded in \( GL(r^n, \eta q) \) as a subgroup. In the notation of (1A), we claim that \( V \) has an orthonormal basis \( \{ [j_1, j_2, \ldots, j_r]' \} \), where \( 1 \leq j_i \leq r \) such that the actions of \( x_{2i-1} \) and \( x_{2i} \) for \( i \geq 2 \) are given by (1.4) with \( [j_1, j_2, \ldots, j_r] \) replaced by \( [j_1, j_2, \ldots, j_r]' \), and

\[
\begin{align*}
x_1 & : [j_1, j_2, \ldots, j_r]' \mapsto [j_1 + 1, j_2, \ldots, j_r]', \\
x_2 & : [j_1, j_2, \ldots, j_r]' \mapsto w^{-j_1}[j_1, j_2, \ldots, j_r]').
\end{align*}
\]

Indeed let \( V_j' = \{ v \in V : x_j v = w^{-j} v \} \) for \( 1 \leq j \leq r \). Then \( V_j' \) are non-degenerate subspaces permuted by \( x_1 \) cyclically. If \( \gamma = 1 \) and \( \{ v_1 \} \) is an orthonormal basis of \( V_1' \), then \( \{ [j_1]' = x_1^{j_1-1}v_1 \} \), where \( 1 \leq j_1 \leq r \), is a required basis. Suppose \( \gamma \geq 2 \) and \( \{ [j_2, j_3, \ldots, j_r]' \} \), where \( 1 \leq j_i \leq r \), is an orthonormal basis of \( V_1' \) such that the actions of \( x_3, \ldots, x_{2r} \) on the basis are given by (1.4) with \( [j_2, j_3, \ldots, j_r] \) replaced by \( [j_2, j_3, \ldots, j_r]' \). Let \( [j_1, j_2, \ldots, j_r]' = x_1^{j_1-1}[j_2, \ldots, j_r]' \). Then \( \{ [j_1, j_2, \ldots, j_r]' : 1 \leq j_i \leq r \} \) is a required basis.

(1B). Suppose \( r|q - \eta \). Let \( G = GL(r^n, \eta q) \) and \( R = Z \) be an \( r \)-subgroup of symplectic type of \( G \), where \( Z = Z(G) \) and \( E \) is an extraspecial subgroup of order \( r^{2r+1} \) of \( G \). Set \( C = C_G(R) \) and \( N = N_G(R) \). Then \( C = Z(G) = Z(N) \) and if \( E \) has exponent \( r \), then \( N/RC \cong Sp(2\gamma, q) \). In addition, if \( R \) is radical in \( G \), then \( E \) has exponent \( r \). Moreover, each linear character of \( Z(N) \) acting trivially on \( O_r(Z(N)) \) has an extension to \( N \) trivial on \( R \).

Proof. By (1A) \( \mathbb{F}_q^2 \) is a splitting field, so that \( C = Z(G) = Z(N) \). The proof of the last assertion is the same as that of [5, (1E)] with 2 replaced by \( r \). If \( R > E \), then \( E \) may be assumed to have exponent \( r \). The elements of \( N \) induce automorphisms in \( Aut^0 E = Aut^0 R \). Suppose \( E \) has exponent \( r \) and acts on the underlying space \( V \) of \( G \) by (1.4). We shall exhibit elements in \( N \) which together with \( R \) generate \( Aut^0 E \).

(1) Let \( g \) be the element in \( G \) such that

\[
g : [j_1, j_2, \ldots, j_i, \ldots, j_r] \mapsto [j_1, j_2, \ldots, j_i, \ldots, j_r] \cdot
\]

Then \( g^{-1}x_1g = x_{2i-1} \), \( g^{-1}x_2g = x_1 \), \( g^{-1}x_2g = x_{2i} \), \( g^{-1}x_2g = x_2 \), and \( g^{-1}x_kg = x_k \) for all other indices. Thus \( N \) contains a subgroup inducing the symmetric group \( S(\gamma) \) on the set \( \{ E_1, E_2, \ldots, E_r \} \).

(2) Let \( \{ [j_1, j_2, j_3, \ldots, j_r]' \} \) be the orthonormal basis of \( V \) given by Remark (2), and \( g \) the element in \( G \) such that

\[
g : [j_1, j_2, \ldots, j_r]' \mapsto [j_1, j_2, \ldots, j_r] \cdot
\]

Then \( g^{-1}x_1g = x_2^{-1} \), \( g^{-1}x_2g = x_1 \), and \( g^{-1}x_kg = x_k \) for \( k \geq 3 \). By (1) for each \( 1 \leq i \leq \gamma \), there exists \( h \in G \) such that \( h^{-1}x_{2i-1}h = x_{2i}^{-1} \),
Let $g$ be the element in $G$ such that

$$g: [j_1, j_2, j_3, \ldots, j_\gamma] \mapsto [\lambda j_1, j_2, j_3, \ldots, j_\gamma],$$

where $\lambda$ is a nonzero element of $\mathbb{Z}/\mathbb{Z}$. Then $g^{-1}x_1g = x_1^\lambda$, $g^{-1}x_2g = x_2^{\lambda^{-1}}$, and $g^{-1}x_kg = x_k$ for $k > 2$. In addition, let $g$ be the element in $G$ such that

$$(1.6) \quad g: [j_1, j_2, j_3, \ldots, j_\gamma] \mapsto [j_1 + j_2, j_2, j_3, \ldots, j_\gamma].$$

Then $g^{-1}x_1g = x_1x_3$, $g^{-1}x_4g = x_4x_2^{-1}$, and $g^{-1}x_kg = x_k$ for all other indices. Since $(x_1, x_3, \ldots, x_{2\gamma-1})$ and $(x_2, x_4, \ldots, x_{2\gamma})$ give a hyperbolic decomposition of $R/\mathbb{Z}(R)$, the element $g$ of (1.6) induces

$$\left(\begin{array}{cc}
1 & 1 \\
0 & 1
\end{array}\right)$$

$$\left(\begin{array}{cc}
1 & 0 \\
-1 & 1
\end{array}\right)$$

relative to this decomposition of $R/\mathbb{Z}(R)$. By (1) we may replace $E_1$ and $E_2$ by $E_1$ and $E_\gamma$ for $1 \leq i \neq \gamma \leq \gamma$. Thus $N$ contains a subgroup inducing

$$\left\langle \left(\begin{array}{c}
A \\
(A^{-1})^t
\end{array}\right) : A \in \text{GL}(\gamma, r) \right\rangle$$
on $R/\mathbb{Z}(R)$.

(4) We claim there are elements in $N$ inducing

$$\left(\begin{array}{cc}
I & X \\
0 & I
\end{array}\right)$$
on $R/\mathbb{Z}(R)$ for any $X$ such that $X^t = X$. By (3) it suffices to show this when $X = \text{diag}(1, 0, 0, \ldots, 0)$. Indeed, let $g$ be the element in $G$ such that

$$(1.7) \quad g: [j_1, j_2, \ldots, j_\gamma] \mapsto w^{(j_1+1)/2} [j_1, j_2, \ldots, j_\gamma],$$

where $w$ is the $r$th root of unity in $\mathbb{F}_{q^2}$ given by (1.4). Then $g^{-1}x_2g = x_1x_2$, and $g^{-1}x_kg = x_k$ for all other indices. Thus the claim holds.

By (3) and (4) $N$ contains a subgroup inducing a Borel subgroup of $\text{Sp}(2\gamma, r)$ on $R/\mathbb{Z}(R)$. Thus $N$ induces $\text{Sp}(2\gamma, r)$ on $R/\mathbb{Z}(R)$. Suppose $R$ is radical in $G$. If $E$ has exponent $r^2$, then $R = E$ and the element $g$ defined by (1.7) lies in $N \setminus R$. Moreover, as shown in the proof of [20, p. 166], $g$ induces an element of $Z(K)$, where $K \simeq \text{Aut}^0 E/\text{Inn} E$ is given by (1.1). Let $Q = (g, E)$, so that $Q \leq N$. We claim that $Q \leq O_r(N)$. Indeed for any $h \in N$, $h$ induces an element of $\text{Aut}^0 E$. Replacing $h$ by $hx$ for some $x \in E$, we may suppose $h$ induces an element of $K$. Thus $[h, g]$ induces a trivial action on $E$ and then $[h, g] \in C = Z(G)$, so that $hgh^{-1} = zg$ for some $z \in C$ and $z \in O_r(C) = Z(R)$ since $zg$ and $g$ are $r$-elements. So $h$ normalizes $Q$ and
the claim holds. It follows that \( R \) is nonradical in \( G \) and we may suppose \( E \) has exponent \( r \). This proves (1B).

We now consider the embedding of \( R \) into finite classical groups. Let \( G = \text{U}(n, q), \text{Sp}(2n, q), \text{O}(2n+1, q), \) or \( \text{O}^*(2n, q) \), and let \( V \) be the underlying space of \( G \), where \( \eta = \pm 1 \). If \( V \) is a symplectic or orthogonal space, we always suppose the characteristic \( p \) of \( F^q \) is odd. Moreover, we denote by \( I(V) \) the group of isometries of \( V \), \( I_0(V) \) the subgroups of \( I(V) \) of determinant 1, and \( \eta(V) \) the type of \( V \) if \( V \) is orthogonal. For simplicity, we set \( \eta(V) = 1 \) if \( V \) is symplectic.

We define the integers \( e, a, \) and \( \text{sign } e = \pm 1 \) as follows: In the case \( G = \text{U}(n, q) \), let \( e \) be the order of \(-q\) modulo \( r \) and \( e = 1 \) or \(-1\) according as \( e \) is even or odd; in the remaining cases, let \( e \) be the order of \( q^2 \) modulo \( r \) and \( e \) the sign chosen so that \( r^a \) divides \( q^e - 1 \). In all cases, let \( r^a \) be the exact power of \( r \) dividing \( q^{2e} - 1 \). In the case \( G = \text{U}(n, q) \), our definition of \( e \) above is different from that of \cite[p. 125]{11}. In fact, if \( r | q^e + 1 \), then our \( e \) is the same as that of \[11]. If \( r | q^e - 1 \), then our \( e \) is the double of that of \[11].

We recall that there exists a set \( \mathcal{F} \) of polynomials serving as elementary divisors for all semisimple elements of each of these groups. First suppose \( G = \text{U}(n, q) \). For each monic polynomial \( A(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_1X + a_0 \) of \( \mathbb{F}_q[X] \) with nonzero roots, let \( \tilde{A}(X) = (a_0^{-1})^q X^m \Delta^q(X) \). Then define

\[
\mathcal{F} = \{ A: A \text{ is monic, irreducible, } \Delta \neq X, \ \Delta = \tilde{\Delta} \},
\]

and \( \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \). Suppose \( G \) is a symplectic or orthogonal group. For each monic polynomial \( A(X) \) in \( \mathbb{F}_q[X] \) with nonzero roots, let \( \tilde{A}(X) \) be the monic polynomial in \( \mathbb{F}_q[X] \) whose roots are the inverses of the roots of \( A(X) \). Define

\[
\tilde{A}(X) = (a_0^{-1})^q X^m \Delta^q(X) - 1.
\]

In all cases, let \( a \) be the degree of \( \Delta(X) \) and \( \Delta(X) \) be the monic polynomial in \( \mathbb{F}_q[X] \) whose roots are the inverses of the roots of \( \Delta(X) \). Define

\[
\mathcal{F} = \{ X - 1, X + 1 \},
\]

\[
\mathcal{F}_1 = \{ \Delta: \Delta \text{ is monic, irreducible, } \Delta \neq X, \ \Delta \neq X \pm 1, \ \text{and } \Delta = \Delta^* \},
\]

and \( \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \). Given \( \Gamma \in \mathcal{F} \), denote \( d_\Gamma \) its degree and \( \delta_\Gamma \) its reduced degree defined by

\[
d_\Gamma = \begin{cases} d_\Gamma & \text{if } G = \text{U}(n, q) \text{ and } \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2, \\ d_\Gamma & \text{if } G \neq \text{U}(n, q) \text{ and } \Gamma \in \mathcal{F}_0, \\ \frac{1}{2}d_\Gamma & \text{if } G \neq \text{U}(n, q) \text{ and } \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2. \end{cases}
\]

Thus \( \delta_\Gamma \) is an integer. We define a sign \( \varepsilon_\Gamma \) for \( \Gamma \in \mathcal{F} \) by

\[
\varepsilon_\Gamma = \begin{cases} \varepsilon & \text{if } \Gamma \in \mathcal{F}_0, \\ -1 & \text{if } \Gamma \in \mathcal{F}_1, \\ 1 & \text{if } \Gamma \in \mathcal{F}_2. \end{cases}
\]

Given a semisimple element \( s \in G \), there exists a unique orthogonal decomposition

\[
(1.8) \quad V = \sum_\Gamma V_\Gamma(s), \quad s = \prod_\Gamma s(\Gamma),
\]
where the $V_{r}(s)$ are nondegenerate subspaces of $V$, $s(\Gamma) \in U(V_{r}(s))$ or $I(V_{r}(s))$ according as $V$ is or is not a unitary space, and $s(\Gamma)$ has minimal polynomial $\Gamma \in \mathcal{F}$. The decomposition (1.8) will be called the primary decomposition of $s$ in $G$. Let $m_{r}(s)$ be the multiplicity of $\Gamma$ in $s(\Gamma)$. Then

\begin{equation}
C_{G}(s) = \prod_{r} C_{r}(s) ,
\end{equation}

where $C_{r}(s) = C_{U(V_{r}(s))(s(\Gamma))}$ or $C_{I(V_{r}(s))(s(\Gamma))}$. Moreover, by [11, (1A)] or [12, (1.13)]

\begin{equation}
C_{r}(s) = \begin{cases} 
I(V_{r}(s)) & \text{if } \Gamma \in \mathcal{F}_{0}, \\
\text{GL}(m_{r}(s), \varepsilon_{r}q^{\delta_{r}}) & \text{if } \Gamma \in \mathcal{F}_{1} \cup \mathcal{F}_{2}.
\end{cases}
\end{equation}

A semisimple element $s \in G$ is primary if $s = s(\Gamma)$.

Suppose $V$ is a symplectic or orthogonal space and $s$ decomposes as (1.8). Let $\eta_{r}(s)$ be the type of $V_{r}(s)$, where $\eta_{r}(s) = 1$ for all $\Gamma \in \mathcal{F}$ if $V$ is symplectic. So $s$ lies in $I_{0}(V)$ if and only if $m_{x+1}(s)$ is even. By [12, (1.12)], the multiplicity and type functions $\Gamma \mapsto m_{r}(s)$, $\Gamma \mapsto \eta_{r}(s)$ satisfy the following relations

\begin{equation}
\dim V = \sum_{r} d_{r} m_{r}(s) ,
\end{equation}

\begin{equation}
\eta(V) = (-1)^{(q-1)/2m_{x-1}(s)m_{x+1}(s)} \prod_{r} \eta_{r}(s) ,
\end{equation}

\begin{equation}
\eta(V_{r}(s)) = \varepsilon_{r}^{m_{r}(s)} \text{ for } \Gamma \in \mathcal{F}_{1} \cup \mathcal{F}_{2} , \text{ and } V \text{ orthogonal}.
\end{equation}

Conversely, if $\Gamma \mapsto \eta_{r}$, $\Gamma \mapsto \eta_{r}$ are functions from $\mathcal{F}$ to $\mathbb{N}$, $\{\pm 1\}$ respectively satisfying (1.11) with $m_{r}(s)$ and $\eta_{r}(s)$ replaced by $\eta_{r}$ and $\eta_{r}$, then there exists a semisimple element $s$ of $I(V)$ with those functions as multiplicity and type functions. Moreover, two semisimple elements $s$ and $s'$ of $I(V)$ are conjugate in $I(V)$ if and only if $m_{r}(s) = m_{r}(s')$ and $\eta_{r}(s) = \eta_{r}(s')$.

Let $Z = \langle z \rangle$ be a cyclic $r$-group of order $r^{a+\alpha}$ with $\alpha \geq 0$, $E$ an extraspecial $r$-group of order $r^{2\gamma+1}$, and $R = ZE$ a group of symplectic type with $Z(R) = Z$. Moreover, we may suppose $E$ has exponent $r$ if $R > E$.

(1C). Let $G = U(n,q)$, $Sp(2n,q)$, $O(2n+1,q)$, or $O^{0}(2n,q)$, where $q = \pm 1$. Suppose $F$ and $F'$ are two embeddings of $R$ in $G$ such that $F(z)$ and $F'(z)$ are primary elements of $G$. Then $n = m r^{a+\gamma}$ for some $m \geq 1$, $F(R)$ and $F'(R)$ are conjugate in $G$, and $\eta = \varepsilon^{m}$ if $G = O^{0}(2n,q)$. Identify $R$ with $F(R)$ and let $C = C_{G}(R)$, $N = N_{G}(R)$, and $N_{0} = \{ g \in N : [g, Z] = 1 \}$. Then $C \simeq \text{GL}(m, \varepsilon q^{\sigma_{s}})$. Furthermore, suppose $R$ is a radical subgroup of $G$.

(1) $E$ has exponent $r$ and $N_{0} = LC$, where $R \subseteq L$, $L \cap C = Z(C) = Z(C_{G}(z)) = Z(L)$, $L/RZ(L) \simeq \text{Sp}(2\gamma, r)$, and $[C, L] = 1$. Moreover, each linear character of $Z(L)$ acting trivially on $O_{r}(Z(L))$ can be extended as a character of $L$ acting trivially on $R$.

(2) $N/N_{0} \simeq N_{G}(Z)/C_{G}(Z)$ is cyclic of order $er^{a}$ or $2er^{a}$ according as $G = U(n, q) \text{ or } G \neq U(n, q)$.

Proof. Since both $Z(F(R))$ and $Z(F'(R))$ are cyclic groups of order $r^{a+\alpha}$ generated by primary elements $F(z)$ and $F'(z)$ respectively, they are conjugate in $G$, so that we may suppose $Z(F(R)) = Z(F'(R))$. Thus $F(E)$ and $F'(E)$
are subgroups of $C_G(F(z))$. Let $H = C_G(F(z))$ and $\Gamma$ be the unique elementary divisor of $F(z)$. Then $H \cong \text{GL}(m\Gamma(F(z)), \text{eq} e^{re})$ and the two embeddings $F(E)$ and $F'(E)$ of $E$ in $H$ can be viewed as embeddings of $E$ in $\text{GL}(m\Gamma(F(z)), \text{eq} e^{re})$ in which a generator $y$ of $Z(E)$ is represented by scalar multiples of the identity matrix. It then follows by Remark (1) of (1A) that $F(E)$ and $F'(E)$ are conjugate in $H$ and $m\Gamma(F(z)) = mr^r$ for some $m \geq 1$. So $F(R)$ and $F'(R)$ are conjugate in $G$, and $\eta = e^{m'r^r} = e^m$ if $G = O^n(2n, q)$.

Identify $H$ with $\text{GL}(mr^r, \text{eq} e^{re})$. Let $W$ be the faithful and irreducible representation of $E$ in $\text{GL}(r^r, \text{eq} e^{re})$ given by (1A), and let $H\gamma$ be the normalizer of $W(E)$ in $\text{GL}(r^r, \text{eq} e^{re})$. Then the commuting algebras of $H\gamma$ and $E$ on the underlying space of $\text{GL}(r^r, \text{eq} e^{re})$ are $\text{F}_q e^{re}$ or $\text{F}_q e^{re}$ according as $r = 1$ or $-1$. Moreover, if $E$ has exponent $r$, then $H\gamma/Z(H\gamma) \cong \text{Aut}^0 E$. By Remark (1) of (1A) $F(E)$ in $H$ can be viewed as an $m$-fold diagonal embedding of $E$ into $\text{GL}(mr^r, \text{eq} e^{re})$ given by

$$(1.12) \quad \begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix}, \quad g \in W(E).$$

In particular, $C = C_H(F(R)) \cong \text{GL}(m, e^{re})$. Let $L$ be the image of $L\gamma$ under (1.12), so that $F(R) \leq L$, $L \leq N^0 = N_H(F(R))$, $C_H(L) = C_H(E) = C$, and $[L, C] = 1$. Suppose $F(R)$ is radical in $G$ and $E$ has exponent $r^2$, so that $R = E$. As shown in the proof of (4) of (1B), there exists an $r$-element $x$ of $L\gamma$ such that $x \notin W(E)$ and $x$ induces an element of $Z(\text{Aut}^0 E/\text{Inn} E)$, so that the image $w$ of $x$ under (1.12) is an $r$-element of $L \setminus F(E)$. If $Q = \langle w, F(E) \rangle$, then $C_H(F(E)) = C_H(Q) = C$. Since $N^0 \leq N$ and $F(E)$ is radical in $G$, it follows that $F(E) = O_r(N^0)$ and each element of $N^0$ induces an element of $\text{Aut}^0 E$, so that $w$ induces an element of $Z(\text{Aut}^0 E/\text{Inn} E)$. We claim $Q \leq O_r(N^0)$. Indeed for each $h \in N^0$, we may suppose $h$ induces an element of $\text{Aut}^0 E/\text{Inn} E$ and then $[h, w]$ acts trivially on $E$, so that $[h, w] \in C$. Since $h$ normalizes $C$ and $w$ commutes with $C$, $[h, w]$ commutes with $C$ and $hwh^{-1} = gw$ for some $g \in Z(C) = Z(H)$. Since $gw$ and $w$ are commutative $r$-elements, $g$ is an $r$-element of $Z(H)$, so that $g \in O_r(H) \leq F(E)$. Thus $h$ normalizes $Q$ and $Q \leq O_r(N^0)$. This is a contradiction and $E$ has exponent $r$.

Identify $R$ with $F(R)$. Since $L/Z(L) \cong \text{Aut}^0 R$ and $N^0$ induces a subgroup of $\text{Aut}^0 R$, it follows that $N^0 = LC$. Thus $Z(H) \leq Z(N^0) \leq Z(L)Z(C)$, $Z(L) \leq Z(C) = Z(H)$, and $L \cap C \leq Z(C)$, so that $Z(L) = Z(H) = Z(C) = L \cap C$. The last assertion of (1) follows by (1B) since $L \cong L\gamma$. Finally, $N_G(Z)/C_G(Z)$ is cyclic of order $er^a$ or $2er^a$ according as $G = U(n, q)$ or $G \neq U(n, q)$ by [11, (3D)] or [12, (5B)]. Suppose $g$ generates $N_G(Z)$ modulo $C_G(Z)$. Then $E$ and $g^{-1}Eg$ are extraspecial subgroups of $H = C_G(Z)$, and they are conjugate in $H$ by Remark (1) of (1A), so that $h^{-1}g^{-1}Egh = E$ for some $h \in H$ and $gh \in N$. On the other hand, $N \leq N_G(Z)$ and $N^0 = N \cap C_G(Z)$, so that $N/N^0 \cong N_G(Z)/C_G(Z)$ and (1C) holds.

Remark. In the notation of (1C), let $E = \langle x_1, x_2, \ldots, x_{2\gamma} \rangle$, $R' = \langle x_1, x_3, \ldots, x_{2\gamma-1}, Z \rangle$. Identify $R$ with $F(R)$ and $R'$ with $F(R')$. Then $R' \leq R$ and
$C_G(R') = C_1 \times C_2 \times \cdots \times C_r$ is a regular subgroup $G$, where $C_i \simeq \text{GL}(m, \mathbb{F}_q^{er})$ for all $i$. Indeed by Remark (1) of (1A) we may suppose the underlying space of $H = C_G(Z)$ has an orthonormal basis $\{[j_1, j_2, \ldots, j_y]\}$, where $1 \leq j_i \leq r$ and $1 \leq k \leq m$, such that the actions of $x_1, x_2, \ldots, x_2y$ on the basis are given by (1.4) or (1.5) with $[j_1, j_2, \ldots, j_y]$ replaced by $[j_1, j_2, \ldots, j_y]k$. Thus each $x_{i-1}$ is a diagonal matrix with respect to the basis for $1 \leq i \leq y$, so $C_H(R') = C_G(R') = C_1 \times C_2 \times \cdots \times C_r$, where $C_i \simeq \text{GL}(m, \mathbb{F}_q^{er})$ for all $i$.

2. THE RADICAL SUBGROUPS

In this section we shall give a description of the radical subgroups of classical groups. We first consider the unitary group $G = U(n, q)$.

For integers $a \geq 0$ and $\gamma \geq 0$, let $Z_a$ be a cyclic group of order $r^{a+\gamma}$, $E_\gamma$ an extraspecial group of order $r^{2\gamma+1}$, and $Z_aE_\gamma$ a central product over $\Omega_1(Z_a) = Z(E_\gamma)$. By (1A) $Z_aE_\gamma$ can be embedded as a subgroup of $\text{GL}(r^{\gamma}, \mathbb{F}_q^{er})$ such that $Z_a$ is identified with $O_r(Z(\text{GL}(r^{\gamma}, \mathbb{F}_q^{er})))$. Let $\Lambda_a$ be a polynomial in $\mathcal{S}$ having a primitive $r^{a+\gamma}$th root of unity as a root. The degree of $\Lambda_a$ is $r^{a+\gamma}$ (cf. [11, p. 126]), so that $U(\mathbb{F}_q^{er^{a+\gamma}}, q)$ has a primary element $g$ with $\Lambda_a$ as a unique elementary divisor of multiplicity $r^\gamma$. By (1.10)

$$C(g) \simeq \text{GL}(r^{\gamma}, \mathbb{F}_q^{er^{\gamma}}).$$

We may identify $\text{GL}(r^{\gamma}, \mathbb{F}_q^{er^{\gamma}})$ with $C(g)$, so that $\text{GL}(r^{\gamma}, \mathbb{F}_q^{er^{\gamma}})$ is embedded as a subgroup of $U(\mathbb{F}_q^{er^{a+\gamma}}, q)$ and $Z_a = \langle g \rangle$. Let $R_{a, \gamma}$ be the image of $Z_aE_\gamma$ under the composition

$$Z_aE_\gamma \hookrightarrow \text{GL}(r^{\gamma}, \mathbb{F}_q^{er^{\gamma}}) \hookrightarrow U(\mathbb{F}_q^{er^{a+\gamma}}, q).$$

Since $Z_a = \langle g \rangle$, a generator of $Z(R_{a, \gamma})$ is primary, so that by (1C) $R_{a, \gamma}$ is uniquely determined by $Z_aE_\gamma$ up to conjugacy. For integer $m \geq 1$, let $R_{m, a, \gamma}$ be the image of the $m$-fold diagonal mapping of $R_{a, \gamma}$ in $U(m\mathbb{F}_q^{er^{a+\gamma}}, q)$ given by

$$g \mapsto \begin{pmatrix} g & & & g \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix}, \quad g \in R_{a, \gamma}.$$

Then a generator of $Z(R_{m, a, \gamma})$ is the image of a generator of $Z(R_{a, \gamma})$ under the embedding above, so that it is primary in $U(m\mathbb{F}_q^{er^{a+\gamma}}, q)$ and then $R_{m, a, \gamma}$ is uniquely determined by $m$ and $Z_aE_\gamma$ up to conjugacy. Let $C_{m, a, \gamma}$ and $N_{m, a, \gamma}$ be the centralizer and normalizer of $R_{m, a, \gamma}$ in $U(m\mathbb{F}_q^{er^{a+\gamma}}, q)$, and let $N_{0, m, a, \gamma} = \{g \in N_{m, a, \gamma}: [g, Z(R_{m, a, \gamma})] = 1\}$. By (1C) $C_{m, a, \gamma} \simeq \text{GL}(m, \mathbb{F}_q^{er^{\gamma}}) \otimes I_\gamma$, where $I_\gamma$ is the identity matrix of order $r^\gamma$ and $\text{GL}(m, \mathbb{F}_q^{er^{\gamma}}) \otimes I_\gamma$ is the group $\{g \otimes I_\gamma: g \in \text{GL}(m, \mathbb{F}_q^{er^{\gamma}})\}$. If $R_{m, a, \gamma}$ is radical, then $E_\gamma$ has exponent $r$, $N_{0, m, a, \gamma} = L_{m, a, \gamma}C_{m, a, \gamma}$, and $N_{m, a, \gamma}/N_{0, m, a, \gamma}$ is cyclic of order $er^\gamma$, where $L_{m, a, \gamma}$ is a subgroup of $N_{0, m, a, \gamma}$ containing $R_{m, a, \gamma}$ such that $L_{m, a, \gamma} \cap C_{m, a, \gamma} = Z(L_{m, a, \gamma}) = Z(C_{m, a, \gamma})$, $[L_{m, a, \gamma}, C_{m, a, \gamma}] = 1$, and $L_{m, a, \gamma}/Z(L_{m, a, \gamma})R_{m, a, \gamma} \simeq \text{Sp}(2\gamma, r)$. In particular, $R_{m, a, \gamma}$ is uniquely determined by $m$, $a$, and $\gamma$ up to conjugacy. Moreover, each linear character of $Z(L_{m, a, \gamma})$ acting trivially on $O_r(Z(L_{m, a, \gamma}))$ can be extended as a character of $L_{m, a, \gamma}$ trivial on $R_{m, a, \gamma}$. 
For integer $c \geq 1$, let $A_c$ denote the elementary abelian $r$-subgroup of order $r^c$ represented by its regular permutation representation. For any sequence $c = (c_1, c_2, \ldots, c_l)$ of nonnegative integers, let $A_c = A_{c_1} \wr A_{c_2} \cdots \wr A_{c_l}$, and let

$$R_{m, \alpha, \gamma, c} = R_{m, \alpha, \gamma} \wr A_c$$

be the wreath product in $U(d, q)$, where $d = m \alpha \gamma + c_1 + \cdots + c_l$. Then $R_{m, \alpha, \gamma, c}$ is determined up to conjugacy in $U(d, q)$. By [3, (1.4)], which applies to $U(d, q)$ with some obvious modifications,

$$C_{U(d, q)}(R_{m, \alpha, \gamma, c}) = C_{m, \alpha, \gamma} \otimes I_c,$$

where $I_c$ is the identity matrix of order $u = r^{c_1 + c_2 + \cdots + c_l}$ and $C_{m, \alpha, \gamma} \otimes I_c$ is defined as before. Moreover,

$$N_{U(d, q)}(R_{m, \alpha, \gamma, c}) = (N_{m, \alpha, \gamma} / R_{m, \alpha, \gamma}) \otimes N_{S(U)}(A_c),$$

$$N_{U(d, q)}(R_{m, \alpha, \gamma, c}) / R_{m, \alpha, \gamma, c} \simeq (N_{m, \alpha, \gamma} / R_{m, \alpha, \gamma}) \times GL(c_1, r) \times \cdots \times GL(c_l, r),$$

where $(N_{m, \alpha, \gamma} / R_{m, \alpha, \gamma}) \otimes N_{S(U)}(A_c)$ is defined as [3, (1.5)]. The proof of (2.2) is the same as that of [3, (4.1)] with GL replaced by U and some obvious modifications. We shall call $R_{m, \alpha, \gamma, c}$ a basic subgroup of $U(d, q)$, $d$ the degree $d(R_{m, \alpha, \gamma, c})$ of $R_{m, \alpha, \gamma, c}$, and $l$ the length $l(R_{m, \alpha, \gamma, c})$ of $R_{m, \alpha, \gamma, c}$.

Let $V$ be a unitary space over $F_{q^2}$, or a symplectic or orthogonal space over $F_q$ with type $\eta = \pm 1$ if $V$ is orthogonal. Let $G = U(V)$ or $I(V)$, and let $R$ be an $r$-subgroup of $G$. We shall say that an $R$-submodule $W$ of $V$ is nondegenerate or totally isotropic if $W$ is respectively a nondegenerate or a totally isotropic subspace of $V$.

(2A). Let $R$ be an $r$-subgroup of $G$. Then $V$ has an $R$-module decomposition

$$V = V_1 \perp V_2 \perp \cdots \perp V_v \perp (U_{v+1} \oplus U'_{v+1}) \perp \cdots \perp (U_w \oplus U'_w),$$

where the $V_i$ for $1 \leq i \leq v$ are nondegenerate simple $R$-submodules, the $U_j$ and $U'_j$ for $v + 1 \leq j \leq w$ are totally isotropic simple $R$-submodules such that $U_j \oplus U'_j$ is nondegenerate and has no proper nondegenerate $R$-submodule. Moreover, if $R$ is abelian and the set of vectors $\{V, R\}$ moved by $R$ is $V$, then $v = 0$ or $v = w$ according as $\varepsilon = 1$ or $-1$.

Proof. The first half of (2A) follows by the proof of [5, (1B)]. Suppose $R$ is abelian and $[V, R] = V$. Let $F_i$ be the representation of $R$ on $V_i$ or $U_i \oplus U'_i$ according as $i \leq v$ or $i \geq v + 1$. If $i \leq v$, then $V_i$ is a simple $R$-module and the commuting algebra $D$ of $R$ on $V_i$ contains $F_i(R)$. If $i \geq v + 1$, then $U_i$ is a simple $R$-module and the representation of $R$ on $U'_i$ is the contragredient of the representation $W$ of $R$ on $U_i$ composed with a field automorphism. Thus the commuting algebra $D$ of $R$ on $U_i$ contains $W(R)$. Since $D$ is a field and $D^x = D \setminus \{0\}$ is a cyclic group, $F_i(R)$ is cyclic generated by $g_i$ for some $g_i \in I(V_i)$ or $I(U_i \oplus U'_i)$ according as $i \leq v$ or $i \geq v + 1$, so that $V_i$ or $U_i$ is a simple $(g_i)$-module. By (1.8) $g_i$ is primary with a unique elementary divisor $\Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2$ of multiplicity 1. Since $g_i$ is an $r$-element, it follows that $\Gamma \in \mathcal{F}_1$ or $\mathcal{F}_2$ according as $\varepsilon = -1$ or 1. Thus the underlying space of $F_i(R)$ has the form $V_i$ or $U_i \oplus U'_i$ according as $\varepsilon = -1$ or 1. This proves (2A).
Let $G = U(V)$, $R$ a radical $r$-subgroup of $G$, and $N = N_G(R)$. Then there exists a corresponding decomposition
\[ V = V_0 \perp V_1 \perp \cdots \perp V_i, \quad R = R_0 \times R_1 \times \cdots \times R_i \]
such that $R_0$ is the trivial subgroup of $U(V_0)$ and $R_i$ is a basic subgroup of $U(V_i)$ for $i \geq 1$. Moreover, the extraspecial components of $R_i$ for $i \geq 1$ have exponent $r$.

**Proof.** Let $V_0 = C(V(R))$ be the set of vectors in $V$ fixed by each element of $R$ and $V_+ = [V, R]$. Then $V = V_0 \perp V_+$ and $R = R_0 \times R_+$, where $R_0 = \{1\}$ and $R_+ \leq U(V_+)$. So $N = U(V_0) \times N_{U(V_+)}(R_+)$ and $R_+$ is necessarily radical in $U(V_+)$. We may suppose $V = V_+$ by induction. Let $F$ be the natural representation of $R$ in $G$. The same proof with some obvious modifications as that of [5, (2B)] shows that $R$ can be reduced to the following case: Every characteristic abelian subgroup of $R$ is cyclic and $V = wV_1$ for some $w \geq 1$ such that either $V_i$ is a nondegenerate simple $R$-module or $V_i$ decomposes as $U_l \otimes U_l'$, where $U_l$ and $U_l'$ are totally isotropic simple $R$-modules and $V_1$ has no proper nondegenerate $R$-submodule. In particular, $Z(F(R))$ is cyclic.

By a result of Hall, [14, 5.4.9], $R$ is a group $ZE$ of symplectic type, where $Z$ is a cyclic $r$-group and $E$ is an extraspecial $r$-group of order $r^{2y+1}$. Thus $Z(F(R)) = F(Z)$ and we may suppose $F(Z) = \langle z \rangle$. Let $H = C_G(F(Z))$ and $C = C_G(F(R))$. Then $F(R) \leq H$ and $C \leq H$, so $Z(H) \leq C$. Since $F(R)$ is radical in $G$ and $C \leq N$, it follows $O_r(C) \leq Z(F(R))$, so that $O_r(Z(H)) \leq O_r(C) \leq Z(F(R))$ and $O_r(Z(C_G(z))) \leq F(Z)$. Thus $O_r(Z(C_G(z)))$ is cyclic and by (1.9) and (1.10) $z$ is primary with a unique elementary divisor $\gamma$. So

\[ H \simeq GL(m_r(z), e \gamma^r). \]

Identify $H$ with $GL(m_r(z), e \gamma^r)$. Then a generator of $F(Z(E))$ is represented by a scalar multiple of the identity matrix, so that $m_r(z) = m_r^r$ for some integer $m \geq 1$ by Remark (1) of (1A). Since $O_r(Z(H)) \leq F(Z)$ and $z \in O_r(Z(H))$, $F(Z) = O_r(Z(H))$, so that $|Z| = r^{2y+1}$ for some integer $\alpha \geq 0$. By (1C) $R = R_{m, \alpha, \gamma}$ and $E$ has exponent $r$. This proves (2B).

Let $(R, \varphi)$ be a weight of $G = U(V)$ and let

\[ V = V_0 \perp V_1 \perp \cdots \perp V_i, \quad R = R_0 \times R_1 \times \cdots \times R_i \]

be the corresponding decomposition of (2B). We define

\[ V(m, \alpha, \gamma, c) = \sum_i V_i, \quad R(m, \alpha, \gamma, c) = \prod_i R_i, \]

where $i$ runs over all indices such that $R_i = R_{m, \alpha, \gamma, c}$.

(2C). With the preceding notation

\[ N(R) = U(V_0) \times \prod_{m, \alpha, \gamma, c} N_{U(V(m, \alpha, \gamma, c))}(R(m, \alpha, \gamma, c)), \]

\[ N(R)/R = U(V_0) \times \prod_{m, \alpha, \gamma, c} N_{U(V(m, \alpha, \gamma, c))}(R(m, \alpha, \gamma, c))/R(m, \alpha, \gamma, c). \]

Moreover,

\[ N_{U(V(m, \alpha, \gamma, c))}(R(m, \alpha, \gamma, c)) = N_{U(V_{m, \alpha, \gamma, c})}(R_{m, \alpha, \gamma, c}) \otimes S(u), \]

\[ N_{U(V(m, \alpha, \gamma, c))}(R(m, \alpha, \gamma, c))/R(m, \alpha, \gamma, c) \]

\[ = (N_{U(V_{m, \alpha, \gamma, c})}(R_{m, \alpha, \gamma, c})/R_{m, \alpha, \gamma, c}) \otimes S(u), \]
where \( V_{m,a,y,c} \) is the underlying space of \( R_{m,a,y,c} \) and \( u \) is the number of basic components \( R_{m,a,y,c} \) in \( R(m, \alpha, \gamma, c) \).

**Proof.** The proof of [3, (4B)] can be applied here with \( GL \) replaced by \( U \) and some obvious modifications.

We now consider radical subgroups of classical groups and as before, we suppose \( q \) is odd. For integers \( \alpha \geq 0 \) and \( \gamma \geq 0 \), let \( \Lambda_\alpha \) be a polynomial in \( S \) having a primitive \( r^{\alpha+\gamma} \)-th root of unity as a root. Then the degree of \( \Lambda_\alpha \) is \( 2er^\alpha \) and \( \Lambda_\alpha \in S_1 \) or \( S_2 \) according as \( \varepsilon = -1 \) or \( 1 \) (see [12, (5.1)]). Let \( V_{a,y} \) be a symplectic or orthogonal space over \( \mathbb{F}_q \) of dimension \( 2er^{a+y} \) and \( \eta(V_{a,y}) = \varepsilon \) if \( V_{a,y} \) is orthogonal. Then by (1.11) \( I(V_{a,y}) \) has a primary element \( g \) with a unique elementary divisor \( \Lambda_\alpha \) of multiplicity \( r^\gamma \).

By (1.10) \( C_{I(V_{a,y})}(g) \cong GL(r^\gamma, \varepsilon q^{er^\alpha}) \) and we may identify these two groups. By (1A) \( Z(\mathbb{F}_q) \) can be embedded as a subgroup of \( GL(r^\gamma, \varepsilon q^{er^\alpha}) \) such that \( Z_\alpha = O_\varepsilon(\mathbb{Z}(GL(r^\gamma, \varepsilon q^{er^\alpha}))) \), where \( Z(\mathbb{F}_q) \) is defined as before. The image \( R_{a,y} \) of \( Z(\mathbb{F}_q) \) under the composition

\[
Z(\mathbb{F}_q) \hookrightarrow GL(r^\gamma, \varepsilon q^{er^\alpha}) \hookrightarrow I(V_{a,y})
\]

is then determined up to conjugacy. A generator of \( Z(R_{a,y}) \) is primary, so by (1C) \( R_{a,y} \) is uniquely determined by \( Z_\alpha \) up to conjugacy.

For integer \( m \geq 1 \), let \( V_{m,a,y} = V_{a,y} \perp V_{a,y} \perp \cdots \perp V_{a,y} \) (\( m \) terms), and let \( R_{m,a,y} \) be the image of the \( m \)-fold diagonal mapping of \( R_{a,y} \) in \( I(V_{m,a,y}) \) given by

\[
g \mapsto \begin{pmatrix} g & \cdots & g \\ g & \cdots & g \end{pmatrix}, \quad g \in R_{a,y}.
\]

The same proof as the unitary case shows that \( R_{m,a,y} \) is also uniquely determined by \( m \) and \( Z_\alpha \) up to conjugacy. In addition, \( \eta(V_{m,a,y}) = \varepsilon^m \) if \( V_{m,a,y} \) is orthogonal.

Let \( C_{m,a,y} \) and \( N_{m,a,y} \) be the centralizer and normalizer of \( R_{m,a,y} \) in \( I(V_{m,a,y}) \) respectively, and let \( N_{m,a,y}^0 = \{ g \in N_{m,a,y} : [g, Z(R_{m,a,y})] = 1 \} \). Then \( N_{m,a,y}^0 \leq N_{m,a,y} \) and by (1C) \( C_{m,a,y} \cong GL(m, \varepsilon q^{er^\alpha}) \otimes I_\gamma \), where \( I_\gamma \) is the identity matrix of degree \( r^\gamma \) and \( GL(m, \varepsilon q^{er^\alpha}) \otimes I_\gamma \) is defined as in the unitary case. In particular, if \( R_{m,a,y} \) is radical in \( I(V_{m,a,y}) \), then \( R_{m,a,y} \) has exponent \( r \), \( N_{m,a,y}^0 = L_{m,a,y} C_{m,a,y} \), and \( N_{m,a,y}/N_{m,a,y}^0 \) is cyclic of order \( 2er^\alpha \), where \( L_{m,a,y} \cap C_{m,a,y} = Z(L_{m,a,y}) = Z(C_{m,a,y}) \), \( [L_{m,a,y}, C_{m,a,y}] = 1 \), \( R_{m,a,y} \leq L_{m,a,y} \), and \( L_{m,a,y}/Z(L_{m,a,y})R_{m,a,y} \cong Sp(2\gamma, r) \). So \( R_{m,a,y} \) is uniquely determined by \( m, \alpha, \) and \( \gamma \) up to conjugacy in \( I(V_{m,a,y}) \). Moreover, by (1C) each linear character of \( Z(L_{m,a,y}) \) acting trivially on \( O_\varepsilon(Z(L_{m,a,y})) \) can be extended as a character of \( L_{m,a,y} \) acting trivially on \( R_{m,a,y} \).

For each sequence \( c = (c_1, c_2, \ldots, c_l) \) of nonnegative integers, let

\[
(2.4) \quad V_{m,a,y,c} = V_{a,y} \perp V_{a,y} \perp \cdots \perp V_{a,y} \quad (u \text{ terms}),
\]

\[
A_c = A_{c_1} \cup A_{c_2} \cup \cdots \cup A_{c_l}, \quad R_{m,a,y,c} = R_{m,a,y} \cup A_c,
\]

where \( u = r^{c_1+c_2+\cdots+c_l} \) and each \( A_{c_i} \) is defined as before. Then \( R_{m,a,y,c} \) is determined up to conjugacy in \( I(V_{m,a,y,c}) \) and \( \eta(V_{m,a,y,c}) = \varepsilon^m \) if \( V_{m,a,y,c} \)
is orthogonal. By [3, (1.4)] with some obvious modifications
\[ C_{I(V_{m,a,y,c})}(R_{m,a,y,c}) = C_{m,a,y} \otimes I_c, \]
where \( I_c \) is the identity matrix of order \( u \) and the right-hand sides is defined as before. Moreover, the same proof as that of [3, (4.1)] with \( \text{GL} \) replaced by \( I \) shows that
\[ N_{I(V_{m,a,y,c})}(R_{m,a,y,c}) = (N_{m,a,y}/R_{m,a,y}) \otimes N_{S(u)}(A_c), \]

(2.5) \[ N_{I(V_{m,a,y,c})}(R_{m,a,y,c})/R_{m,a,y,c} = (N_{m,a,y}/R_{m,a,y}) \times \text{GL}(c_1, r) \times \cdots \times \text{GL}(c_l, r), \]
where \((N_{m,a,y}/R_{m,a,y}) \otimes N_{S(u)}(A_c)\) is defined as [3, (1.5)]. We shall call \( R_{m,a,y,c} \) a basic subgroup of \( I(V_{m,a,y,c}) \), \( \dim V_{m,a,y,c} \) the degree \( d(R_{m,a,y,c}) \) of \( R_{m,a,y,c} \), and \( l \) the length \( l(R_{m,a,y,c}) \) of \( R_{m,a,y,c} \).

(2D). Let \( V \) be a symplectic or orthogonal space over \( F_q \), \( G = I(V) \) the group of all isometries of \( V \), and \( R \) a radical subgroup of \( G \). Then there exists a corresponding decomposition
\[ V = V_0 \perp V_1 \perp \cdots \perp V_i, \quad R = R_0 \times R_1 \times \cdots \times R_l, \]
such that \( R_0 \) is the trivial subgroup of \( I(V_0) \) and \( R_i \) is a basic subgroup of \( I(V_i) \) for \( i \geq 1 \). Moreover, the extraspecial components of \( R_i \) for \( i \geq 1 \) have exponent \( r \).

Proof. Let \( V_0 = C_V(R) \) and \( V_+ = [V, R] \). Then \( V = V_0 \perp V_+ \) and \( R = R_0 \times R_+ \), where \( R_0 = \langle V_0 \rangle \) and \( R_+ \leq I(V_+) \). In particular, \( N(R) = I(V_0) \times N_{I(V_+)}(R_+) \) and \( R_+ \) is necessarily a radical subgroup of \( I(V_+) \). By induction we may suppose \( V = V_+ \). Thus \( Z(R) \) is abelian and \([V, Z(R)] = V\). By (2A) we may write the \( Z(R) \)-module \( V \) as
\[ V = m_1V_1 \perp m_2V_2 \perp \cdots \perp m_wV_w, \]
where each \( V_i \) is either a nondegenerate simple \( Z(R) \)-submodule or a sum \( U_i \oplus U'_i \) of totally isotropic simple \( Z(R) \)-submodules \( U_i, U'_i \) according as \( e = -1 \) or \( 1 \), and \( m_i \) is the multiplicity of \( V_i \) in \( V \) for all \( i \geq 1 \). If \( e = -1 \), then \( r|q^e - 1 \) and \( F_{q^{m_i}} \) is the commuting algebra of \( Z(R) \) on \( V_i \) for some \( \alpha_i \geq 0 \) since \([V_i, Z(R)] = V_i \) and \( Z(R) \) is an \( r \)-group. Similarly, if \( e = 1 \), then \( r\alpha_i - 1 \), \( V_i = U_i \oplus U'_i \), and \( F_{q^{m_i}} \) is the commuting algebra of \( Z(R) \) on \( U_i \) for some integer \( \alpha_i \geq 0 \). In all cases \( \dim V_i = 2\alpha_i \). Let \( N^0 = \{g \in N(R) : [g, Z(R)] = 1\} \), and let \( H = C_G(Z(R)) \). Then \( h(m_iV_i) = m_iV_i \) for \( h \in H \) and all \( i \geq 1 \). Thus there exists a corresponding decomposition
\[ H = H_1 \times H_2 \times \cdots \times H_w \]
such that \( H_i \simeq \text{GL}(m_i, q^{e\alpha_i}) \leq I(m_iV_i) \) for all \( i \geq 1 \). Since \( R \) is radical and \( N^0 \leq N \), it follows \( O_r(N^0) \leq O_r(N) = R \). On the other hand, \( R \leq N^0 \) and \( N^0 = N_H(R_i) \), so \( R = O_r(N^0) \) and \( R \) is radical in \( H \).

Let \( R_i \) be the group of linear operators which agree with an element of \( R \) on \( m_iV_i \) and are the identity on \( m_jV_j \) for \( j \neq i \). Then \( N^0 \) permutes the pairs \((m_iV_i, R_i)\) for \( 1 \leq i \leq w \), so that \( R \leq N^0 \cap R_1 \times R_2 \times \cdots \times R_w \leq N^0 \). It follows that \( R = R_1 \times R_2 \times \cdots \times R_w \) and \( R_i = O_r(N_i) \), where \( N_i = N_H(R_i) \). Thus \( R_i \) is radical in \( H_i \) for all \( i \). By induction on \( \dim V \), we may suppose \( w = 1 \), so that \( V = m_1V_1 \), \( R = R_1 \), \( H = H_1 \), and \( Z(R) = Z(R_1) \) is cyclic generated by some
\[ x \in I(V). \] But \( H = C_G(x) \) and \( O_r(Z(H)) \leq O_r(H) \), so \( O_r(Z(H)) \leq Z(R) \). By (1.9) and (1.10) \( x \) is primary in \( G \). Apply [3, (4A)] or (2B) to \( H \cong GL(m_1, eqr^m) \). So \( R \) is a basic subgroup \( R_{m,a,y,c} \) of \( H \), where \( m, y, \alpha \) are integers, and \( c = (c_1, \ldots, c_t) \) is a sequence of nonnegative integers such that \( \alpha > \alpha_1 \), and \( mer^{a+y+c_1+\ldots+c_t} = m_1e^{ra_1} \). Moreover, the extraspecial components of \( R_{m,a,y,c} \) have exponent \( r \). In particular, \( \dim V = 2mer^{a+y+c_1+\ldots+c_t} \) and \( \eta(V) = e^m = e^{m_1} \) if \( V \) is orthogonal. Thus \( I(V) \) has a basic subgroup \( R' \) of the form \( R_{m,a,y,c} \) defined by (2.4), where the extraspecial components of \( R' \) have exponent \( r \). So \( Z(R) \) and \( Z(R') \) are cyclic generated by primary elements of order \( r^{a+\alpha} \) in \( I(V) \), and they are conjugate in \( I(V) \). Thus we may suppose \( Z(R) = Z(R') \), so that \( R' \leq H \). By definition \( R' \) still has the type \( R_{m,a,y,c} \) as a subgroup of \( H \), so that \( R' \) and \( R \) are conjugate in \( H \). Thus \( R = R_{m,a,y,c} \) is a basic subgroup of \( I(V) \) and (2D) follows.

**Remark.** In the notation of (2B) or (2D), suppose \( t \neq 0 \). Then there exists an element \( \rho \) of \( Z(R) \) such that (1) \( |\rho| = r^a \); (2) \( [V, \rho] = \sum_{i=1}^t V_i \); (3) the restriction of \( \rho \) on \( [V, \rho] \) is primary. Such an element exists by (2B) or (2D) and will be called a primary element of \( R \). If \( \rho \) is a primary element of \( R \), then \( \langle \rho \rangle \) is uniquely determined by \( R \) up to conjugacy and \( C_{G}(\rho) \cong U(V_0)^{\times}GL(m, eq^e) \) or \( C_{G}(\rho) \cong I(V_0)^{\times}GL(m, eq^e) \) for some \( m \geq 1 \) according as \( G = U(V) \) or \( I(V) \).

Let \((R, \varphi)\) be a weight of \( G = I(V) \) and let
\[
V = V_0 \perp V_1 \perp \cdots \perp V_t, \quad R = R_0 \times R_1 \times \cdots \times R_t,
\]
be the corresponding decomposition of (2D). We define
\[
V(m, \alpha, \gamma, c) = \sum_i V_i, \quad R(m, \alpha, \gamma, c) = \prod_i R_i,
\]
where \( i \) runs over all indices such that \( R_i = R_{m,a,y,c} \).

(2E). With the preceding notation
\[
N(R) = I(V_0) \times \prod_{m,a,y,c} N_{I(V(m,\alpha,\gamma,c))}(R(m, \alpha, \gamma, c)),
\]
\[
N(R)/R = I(V_0) \times \prod_{m,a,y,c} N_{I(V(m,\alpha,\gamma,c))}(R(m, \alpha, \gamma, c))/R(m, \alpha, \gamma, c).
\]

Moreover,
\[
N_{I(V(m,\alpha,\gamma,c))}(R(m, \alpha, \gamma, c)) = N_{I(V_{m,\alpha,\gamma,c})}(R_{m,\alpha,\gamma,c}) \cdot S(u),
\]
\[
N_{I(V(m,\alpha,\gamma,c))}(R(m, \alpha, \gamma, c))/R(m, \alpha, \gamma, c) = (N_{I(V_{m,\alpha,\gamma,c})}(R_{m,\alpha,\gamma,c})/R_{m,\alpha,\gamma,c}) \cdot S(u),
\]
where \( V_{m,\alpha,\gamma,c} \) is the underlying space of \( R_{m,\alpha,\gamma,c} \) and \( u \) is the number of basic components \( R_{m,\alpha,\gamma,c} \) in \( R(m, \alpha, \gamma, c) \).

**Proof.** The proof is essentially the same as that of [3, (4B)] with \( GL \) replaced by \( I \) and some obvious modifications, except the minimal elements of \( \mathcal{E}_i \) have dimension \( 2mer^{a+\gamma} \) when \( R_i = R_{m,a,y,c} \), where \( \mathcal{E}_i \) is defined there.

### 3. More on basic subgroups

Let \( R \) be a radical subgroup of a finite group \( G \), \( N = N(R) \), \( C = C(R) \). The stabilizer in \( N \) of an irreducible character \( \theta \) of \( CR \) will be denoted by
WEIGHTS FOR CLASSICAL GROUPS

We denote the sets of irreducible characters of $N(\theta)$ and $N$ which cover $\theta$ and which have defect 0 as characters of $N(\theta)/R$ and $N/R$ respectively by $\text{Irr}^0(N(\theta), \theta)$ and $\text{Irr}^0(N, \theta)$. By Clifford theory the induction mapping $\psi \mapsto I(\psi) = \text{Ind}_{N(\theta)}^N(\psi)$ induces a bijection from $\text{Irr}^0(N(\theta), \theta)$ to $\text{Irr}^0(N, \theta)$. Since $\psi(1) = d(\psi)\theta(1)$ for some integral divisor $d(\psi)$ of $(N(\theta): CR)$, it follows that $(R, I(\psi))$ is a weight of $G$ if and only if

$$d(\psi)_r = (N(\theta): CR)_r, \quad \theta(1)_r = (CR: R)_r,$$

and in particular, $\theta$ then has defect 0 as a character of $CR/R$. In this case the block $b$ of $CR$ containing $\theta$ has a defect group $R$ and the canonical character $\theta$. Moreover, for any $\psi$ of $\text{Irr}^0(N(\theta), \theta)$, $I(\psi)$ is a character of $b^N$ and $(R, I(\psi))$ is a $b^G$-weight of $G$. Following [3, p. 3] all $B$-weights for a block $B$ of $G$ have the form $(R, I(\psi))$, where $R$ runs over representatives for the conjugacy $G$-classes of radical subgroups, $b$ runs over representatives for the conjugacy $N(R)$-classes of blocks of $C(R)R$ such that $b$ has defect group $R$ and $b^G = B$, and $\psi$ runs over $\text{Irr}^0(N(\theta), \theta)$. Here $\theta$ is the canonical character of $b$. A pair $(R, b)$ of an $r$-subgroup $R$ of $G$ and a block $b$ of $C$ is called a Brauer pair of $G$. In particular, pairs $(1, B)$ correspond to blocks $B$ of $G$.

Now we consider the unitary group $G = U(n, q)$. Given $\Gamma \in \mathcal{S}$, let $e_\Gamma$, $\alpha_\Gamma$, $m_\Gamma$ be integers defined as follows: $e_\Gamma$ is the multiplicative order of $e_\Gamma q^{d_\Gamma}$ modulo $r$, $r^{e_\Gamma} = (d_\Gamma)_r$, and $m_\Gamma r^{e_\Gamma} = d_\Gamma e_\Gamma$. By [7, (3.2)] the Brauer pairs $(R, b)$ of $G$ are labeled by ordered triples $(R, s, \kappa)$, where $s$ is a semisimple $r'$-element of a dual group $G^*$ of $G$, and $\kappa = \prod_{\Gamma \in \mathcal{S}} \kappa_\Gamma$ is a product of partitions $\kappa_\Gamma$ such that each $\kappa_\Gamma$ is an $e_\Gamma$-core of a partition of the multiplicity $m_\Gamma(s)$ of $s$. This labeling extends the labeling [11, (5D)] by Fong and Srinivasan for blocks $B$ of $G$ by ordered pairs $(s, \kappa)$. Since $G^* \simeq G$, we may identify $C^* \simeq G$.

Let $\mathcal{S}'$ be the subset of $\mathcal{S}$ consisting of polynomials whose roots have $r'$-order. In [11, (5A)] each $\Gamma$ in $\mathcal{S}'$ determines a block $\Gamma_\Gamma$ of $G = U(e_\Gamma q^{d_\Gamma}, q)$ with defect group $R_\Gamma = R_{e_\Gamma q^{d_\Gamma}, q}$ as follows: Let $C_\Gamma = C_{G_\Gamma}(R_\Gamma)$, $N_\Gamma = N_{G_\Gamma}(R_\Gamma)$, so that $C_\Gamma \simeq \text{GL}(m_{\Gamma r^{e_\Gamma}}, e_\Gamma r^{e_\Gamma})$ and $N_\Gamma/C_\Gamma$ is cyclic of order $e_\Gamma r^{e_\Gamma}$. Then $C_\Gamma$ has a block $b_\Gamma$ with defect group $R_\Gamma$ and label $(s_\Gamma, -)$ in $C^*_\Gamma$ such that as an element of $G^*_\Gamma$, $s_\Gamma$ is primary with a unique elementary divisor $\Gamma$ of multiplicity $e_\Gamma$. If $\theta_\Gamma$ is the canonical character of $b_\Gamma$ and $N(\theta_\Gamma)$ is its stabilizer in $N_\Gamma$, then $(N(\theta_\Gamma): C_\Gamma) = e_\Gamma$. The block $b_\Gamma$ induces a block $b_{\Gamma r^{e_\Gamma}}$ of $G$ which will be denoted by $B_\Gamma$. Since $(e_\Gamma, r) = 1$, $B_\Gamma$ has a defect group $R_\Gamma$ and the label $(s_\Gamma, -)$ (see [7, 3.2]). We shall also write $s_\Gamma$ as $e_\Gamma \Gamma$. Conversely, let $G = U(m_{\Gamma r^{e_\Gamma}}, q)$, and $B$ a block of $G$ with defect group $R = R_{m_{\Gamma r^{e_\Gamma}}, q}$. By [11, (4B) and (5A)] $(m, r) = 1$ and there exists a unique $\Gamma \in \mathcal{S}'$ such that $\Gamma$ and $B$ correspond in the preceding manner. In particular, $m = m_\Gamma$ and $\alpha = \alpha_\Gamma$.

The proofs of the following two lemmas are similar to that of [4, (3A) and (3B)].

(3A). Given $\Gamma \in \mathcal{S}'$, let $G = U(r^e e_\Gamma q^{d_\Gamma}, q)$, $R = R_{m_{\Gamma r^{e_\Gamma}}, \alpha_{\Gamma r^{e_\Gamma}}, q}$ a basic subgroup of $G$, and $C = C_{G}(R)$. Then $C = C_{\Gamma} \otimes I_\Gamma$, where $I_\Gamma$ is the identity matrix of order $r^e$. The irreducible character $\theta = \theta_\Gamma \otimes I_\Gamma$ of $C$ defined by $\theta(c \otimes I_\Gamma) = \theta_\Gamma(c)$ for $c \in C_{\Gamma}$ is then a character of defect 0 of $CR/R$, and $|\text{Irr}^0(N(\theta), \theta)| = e_\Gamma$. 


Proof. All statements but the last are clear. Let \( N = N_G(R) \), and \( N^0 \) the subgroup \( \{ g \in N : [g, Z(R)] = 1 \} \) of \( N \). By (1C) \( N^0 = LC \) and \( N/N^0 \) is cyclic of order \( e_r \), where \( R \leq L \), \( L \cap C = Z(L) = Z(C) \), \( [L, C] = 1 \), and \( L/Z(L)R \simeq \text{Sp}(2\gamma, r) \). Moreover, each linear character of \( Z(L) \) acting trivially on \( O_c(Z(L)) \) can be extended as a character of \( L \) trivial on \( R \). Thus \( N^0 \leq N(\theta) \), and \( N(\theta)/N^0 \) is cyclic. An irreducible constituent of the restriction of \( \theta \) to \( Z(C) \) is a linear character trivial on \( O_c(Z(C)) \) and so has an extension \( \xi \) to \( L \) trivial on \( R \). Thus \( \xi \theta \) is an extension of \( \theta \) to \( N^0 \). Since \( N^0/RC \simeq L/Z(L)R \simeq \text{Sp}(2\gamma, r) \), the Steinberg character \( St \) of \( N_0/RC \) can be regarded as a character of \( N^0 \) trivial on \( CR \). Thus \( \theta = St\xi \theta \) is irreducible since its restriction to \( C \) is irreducible. By (3.1) \( \theta \in \text{Irr}^0(N^0, \theta) \). Suppose \( \psi \) is a character of \( \text{Irr}^0(N^0, \theta) \). Then by Clifford theory \( \psi = \chi\xi \theta \) for some irreducible character \( \chi \) of \( N^0 \) trivial on \( C \). Since \( \psi \) and \( \xi \theta \) act trivially on \( R \), \( \chi \) acts trivially on \( R \), so that \( \chi \) is an irreducible character of \( N^0/CR \). Since \( \psi \) has defect 0 as a character of \( N^0/R \), \( \chi \) has defect 0 as a character of \( N^0/RC \simeq \text{Sp}(2\gamma, r) \). Thus \( \chi = St \) and \( \text{Irr}^0(N^0, \theta) = \{ \theta \} \). If \( N(\theta) \) is the stabilizer of \( \theta \) in \( N \), then \( N(\theta) = N(\theta) \) and \( \text{Irr}^0(N(\theta), \theta) = \text{Irr}^0(N(\theta), \theta) \).

By (1C) a generator \( \sigma \) of \( N/N^0 \) induces a field automorphism of order \( e_r \) on \( C(Z(R)) \). Since \( C = C_\Gamma \otimes I_\gamma \) is a subgroup of \( C(Z(R)) \) invariant under \( \sigma \), \( \sigma \) also induces a field automorphism of order \( e_r \) on \( C \). But a generator \( \sigma_1 \) of \( N Rafael/C_\Gamma \) also induces a field automorphism of order \( e_r \) on \( C_\Gamma \simeq GL(m_\Gamma, r) \). By replacing generators, we may suppose \( \sigma \) induces \( \sigma_1 \) on \( C_\Gamma \). It follows that \( N(\theta)/N^0 \simeq N(\theta)/C_\Gamma \) and \( N(\theta)/N^0 \) is cyclic, \( \theta \) extends in \( e_r \) ways to irreducible characters of \( N(\theta) \) which cover \( \theta \), and since \( e_r \) is prime to \( r \), these extensions are in \( \text{Irr}^0(N(\theta), \theta) \).

Remark. The weights \( (R, I(\psi)) \) of \( G \) for \( \psi \in \text{Irr}^0(N(\theta), \theta) \) are \( B \)-weights, where \( B \) is the block of \( G \) labeled by \( (r^2e_1\Gamma, -) \), \( I \) is the induction operator from \( N(\theta) \) to \( N \), and \( r^2e_1\Gamma \) represents an element of \( U(r^2e_1\Gamma, -) \) with a unique elementary divisor \( \Gamma \) of multiplicity \( r^2e_1 \). Indeed, if \( b \) is the block of \( C \) containing \( \theta \), then \( (R, b) \) is labeled by \( (R, r^2e_1\Gamma, -) \) and the weights are \( h^G \)-weights. Moreover, by (7, 3.2) \( h^G \) is labeled by \( (r^2e_1\Gamma, -) \).

Given \( \Gamma \in \mathcal{F} \), let \( G = U(\Gamma, q) \) and \( R = R_m, r, \gamma, e \) a basic subgroup of \( G \), where \( d \) and \( \gamma \) are nonnegative integers, \( e = (c_1, c_2, \ldots, c_l) \) such that \( \gamma + c_1 + c_2 + \cdots + c_l = d \). Then \( C = C_G(R) = C_\Gamma \otimes I_\gamma \otimes I_e \), where \( I_\gamma, I_e \) are identity matrices of orders \( r^2 \) and \( r^{c_1+c_2+\cdots+c_l} \), respectively. The irreducible character of \( C \) defined by

\[
\theta(c \otimes I_\gamma \otimes I_e) = \theta_\Gamma(c)
\]

for \( c \in C_\Gamma \) is then a character of defect 0 of \( CR/R \). We shall say the pair \( (R, \theta) \) is of type \( \Gamma \). If \( b \) is the block of \( C \) containing \( \theta \), then \( (R, b) \) has label \( (R, r^2e_1\Gamma, -) \).

(3B). Let \( G = U(n, q), R \) a basic subgroup of \( G \), \( b \) a block of \( C(R)R \) with defect group \( R \), and \( \theta \) the canonical character of \( b \). Then \( (R, \theta) \) has type \( \Gamma \) for some \( \Gamma \in \mathcal{F} \).

Proof. Suppose \( R = R_m, \alpha, \gamma, e \). Set \( G_1 = U(m, q) \), \( R_1 = R_m, \alpha, 0 \), \( C_1 = C_{G_1}(R_1) \). So \( C_1 \simeq GL(m, eqe^r) \) and \( C = C_1 \otimes I_\gamma \otimes I_e \). Then \( \theta \) has the form
$\theta_1 \otimes I_r \otimes I_c$, where $\theta_1$ is a character of $C_1$. Since $\theta$ has defect $0$ as a character of $CR/R$ and $CR/R \simeq C_1/R_1$, $\theta_1$ also has defect $0$ as a character of $C_1/R_1$. The block $b_1$ of $C_1$ containing $\theta_1$ then has defect group $R_1$. By [11, (5A)] there is a unique $\Gamma \in \mathcal{F}$ such that $R_1 = R_\Gamma$ and $\theta_1 = \theta_\Gamma$. Thus $m = m_\Gamma$, $\alpha = \alpha_\Gamma$, and $(R, \theta)$ has type $\Gamma$.

Following the notation of [12], we denote $V$ and $V^*$ finite-dimensional symplectic or orthogonal spaces over $\mathbb{F}_q$ related as follows:

<table>
<thead>
<tr>
<th>Type</th>
<th>$\dim V$</th>
<th>$\dim V^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>symplectic</td>
<td>$2n$</td>
<td>$2n + 1$</td>
</tr>
<tr>
<td>orthogonal</td>
<td>$2n + 1$</td>
<td>$2n$</td>
</tr>
</tbody>
</table>

where $\eta(V) = \eta(V^*) = 1$ in the first two cases and $\eta(V) = \eta(V^*)$ in the third case. Here $\eta(V) = 1$ for a symplectic space as before. Moreover, $I(V)$ and $I(V^*)$ are the groups of isometries of $V$ and $V^*$, $I_0(V)$ and $I_0(V^*)$ the subgroups of $I(V)$ and $I(V^*)$ of determinant $1$. We shall call $I_0(V^*)$ the dual group of $I_0(V)$. Let $G = I_0(V)$ and $G^* = I_0(V^*)$. Given a semisimple element $s$ of $G^*$, let $(s)$ be the conjugacy class of $s$ in $G^*$, and let $\mathcal{E}(G, (s))$ be defined by [8, p. 57]. Namely $\mathcal{E}(G, (s))$ is the set of the irreducible constituents of Deligne-Lusztig generalized characters associated with $(s)$. Given a semisimple $r'$-element $s$ of $G^*$, let

$$\mathcal{E}_r(G, (s)) = \bigcup_u \mathcal{E}(G, (su)),$$

where $u$ runs over all the $r$-elements of $C_{G^*}(s)$. By [8, 2.2], $\mathcal{E}_r(G, (s))$ is a union of $r$-blocks.

The following lemma is due to Fong and Olsson.

(3C). Let $\rho$ be an $r$-element of $G$, $b$ a block of $H = C_G(\rho)$, and $B$ a block of $G$. Suppose $H$ is regular subgroup of $G$, $B \subseteq \mathcal{E}_r(G, (s))$, and $b \subseteq \mathcal{E}_r(H, (t))$. If $b^G = B$, then $s$ and $t$ are conjugate in $G^*$.

Proof. By Brauer's Second Main Theorem there exists a nonzero generalized decomposition number $d_{\chi, \varphi}^b$ for some irreducible character $\chi \in B$ and irreducible modular character $\varphi \in b$. Let $\chi^{(b)}(\rho \tau) = \sum_{\varphi' \in b'} d_{\chi, \varphi'}^b(\tau')$, where $b'$ is a block of $H$, $\tau$ runs over the $r'$-elements in $H$, and $\varphi'$ runs over the irreducible modular characters in $b'$. Then $\chi(\rho \tau) = \sum_{b'} \chi^{(b')}(\rho \tau)$. On the other hand, by the theorem of Curtis type [9, (3.7)],

$$\chi(\rho \tau) = \sum_{b'} \sum_{\zeta \in b'} (\chi, R_H^G(\zeta)) \zeta(\rho \tau),$$

where $R_H^G(\zeta)$ is the generalized Deligne-Lusztig character, $b'$ runs over blocks of $H$, and $\zeta$ runs over the irreducible characters of $b'$. Since the $\zeta(\rho \tau)$ for $\zeta \in b'$ are linear combinations of the Brauer characters $\varphi(\tau)$ for $\varphi \in b'$ and the $\varphi$ are linear independent, it follows that

$$\chi^{(b)}(\rho \tau) = \sum_{\zeta \in b} (\chi, R_H^G(\zeta)) \zeta(\rho \tau),$$

and $\chi^{(b)}(\rho \tau) \neq 0$ for some $r'$-element $\tau$. So $(\chi, R_H^G(\zeta)) \neq 0$ for some $\zeta \in b$. Suppose $\chi \in \mathcal{E}(G, (su))$ and $\zeta \in \mathcal{E}(H, (tv))$, where $u$ is an $r$-element in
18

Let \( R \) be a radical \( r \)-subgroup of \( G \), \( b \) a block of \( C_G(R)R \) with defect group \( R \), \( V_0 = C_V(R) \), and \( V_+ = [V, R] \). Then \( b^G \) is well defined and \( b^G \subseteq \mathcal{E}_r (G, (s)) \) for some \( s \in G^* \). We shall give a decomposition of \( s \) corresponding to the decomposition \( V_0 \perp V_+ \) of \( V \) and give a label to the Brauer pair \( (R, b) \) when \( V = V_+ \), where \( b \) is regarded as a block of \( C_G(R) \). Let \( \rho \) be a primary element of \( R \) given by the remark of (2D), and let \( K = C_G(\rho) \). Then \( K = K_0 \times K_+ \), where \( K_0 = I_0(V_0) \) and \( K_+ \simeq \GL(m, \mathbb{F}_e) \) for some \( m \geq 0 \). Since \( \langle \rho \rangle \leq R \), there exists a unique block \( B_\rho \) of \( K \) such that

\[
(1, b^G) \leq (\langle \rho \rangle, B_\rho) \leq (R, b).
\]

Let \( B_\rho = B_{\rho, 0} \times B_{\rho, +} \), where \( B_{\rho, 0} \), \( B_{\rho, +} \) are blocks of \( K_0 \), \( K_+ \) respectively. Then \( B_{\rho, 0} \subseteq \mathcal{E}_r (K_0, (s_0)) \) and \( B_{\rho, +} \subseteq \mathcal{E}_r (K_+, (s_+)) \) for some \( s_0 \in K_0^* \) and \( s_+ \in K_+^* \). By (3C) \( s_0 \times s_+ \) and \( s \) are conjugate in \( G^* \) and we may suppose \( s = s_0 \times s_+ \), so that this gives a decomposition of \( s \). Moreover, the decomposition depends only on \( b^G \) not on the choice of \( R \). Indeed there exists a defect group \( D \) of \( b^G \) such that \( Z(D) \leq Z(R) \leq R \leq D \), so that \( V_0 = C_V(D) \) and \( V_+ = [V, D] \) and a primary element of \( D \) is a primary element of \( R \). Thus we may suppose \( \rho \in Z(D) \) is a primary element of \( D \) and then the decomposition \( s = s_0 \times s_+ \) is determined by \( b^G \). Suppose now \( V = V_+ \). Then \( B_\rho = B_{\rho, +} \) and \( B_\rho \subseteq \mathcal{E}_r (K_+, (s)) \). Since \( C_G(R) = C_K(R) \), we may view \((R, b)\) as a Brauer pair of \( K \) and then \((R, b)\) has a Broué labeling \((R, t, -)\), where \( t \in K^* \). Here, the third component of the label is empty since \( K \simeq \GL(m, \mathbb{F}_e) \) and \( R \) acts fixed-point freely on the underlying space of \( K \). By definition of normal inclusion of Brauer pairs, \((1, B_\rho) \leq (R, b) \) holds in \( K \) and by [7, (3.2)], \( t \) and \( s \) are conjugate in \( K^* \). In particular, \( t \) determines a unique conjugacy class of \( G^* \). We then give \((R, b)\) the label \((R, t, -)\).

Given \( \Gamma \) in \( \mathcal{S} \), let \( e_\Gamma \), \( o_\Gamma \), and \( m_\Gamma \) be the following integers: \( e_\Gamma \) is the multiplicative order of \( q^{2\delta_\Gamma} \) or \( q^{\delta_\Gamma} \) modulo \( r \) according as \( \Gamma \in \mathcal{S}_1 \) or \( \Gamma \in \mathcal{S}_2 \), \( r^{en_\Gamma} (\delta_\Gamma) \), and \( m_\Gamma e^{en_\Gamma} = \delta_\Gamma e_\Gamma \). In addition, let \( \beta_\Gamma = 1 \) or \( 2 \) according as \( \Gamma \in \mathcal{S}_1 \cup \mathcal{S}_2 \) or \( \Gamma \in \mathcal{S}_0 \).

Suppose \( \dim V \) is even and \( s \) is a semisimple element of \( I_0(V^*) \) with primary decomposition

\[
V^* = \sum_{\Gamma} V^{\ast}_\Gamma(s), \quad s = \prod_{\Gamma} s(\Gamma).
\]

We define a semisimple element \( s^* \) of \( I_0(V) \), which is determined uniquely up to conjugacy in \( I(V) \), as follows: If \( V \) is orthogonal, then \( V \) and \( V^* \) have the same dimension and type, so that \( m_\Gamma(s) \) and \( \eta_\Gamma(s) \) satisfy the relations (1.11). Thus a semisimple element, denoted by \( s^* \), exists in \( I(V) \) such that \( m_\Gamma(s^*) = m_\Gamma(s) \) and \( \eta_\Gamma(s^*) = \eta_\Gamma(s) \). Since \( s \in G^* \), it follows that \( s^* \in G \). If \( V \) is symplectic, then \( V^* \) is an odd dimensional orthogonal space. Let \( \eta_\Gamma = 1 \) for all \( \Gamma \in \mathcal{S} \), and \( n_\Gamma = m_\Gamma(s) \) except when \( \Gamma = X - 1 \), in which case, \( n_\Gamma = m_\Gamma(s) - 1 \). Then \( n_\Gamma \) and \( \eta_\Gamma \) satisfy the relations (1.11) with \( \eta_\Gamma(s) \) and \( \eta_\Gamma(s) \) replaced by \( n_\Gamma \) and \( \eta_\Gamma \) respectively. So a semisimple element, denote by \( s^* \), exists in \( G \) such that \( m_\Gamma(s^*) = n_\Gamma \) and \( \eta_\Gamma(s^*) = \eta_\Gamma = 1 \). Thus \( s^* \) is
uniquely determined up to conjugacy in \( I(V) \) and \( \det s^* = 1 \). We shall call \( s^* \) a dual of \( s \).

The following proposition is due to Fong and Olsson.

\[
(3D). \text{The dual mapping } s \mapsto s^* \text{ induces a bijection } f: (s) \mapsto (s^*) \text{ from the conjugacy classes of r-elements of } I_0(V^*) \text{ onto the conjugacy classes of r-elements of } I_0(V) \text{ such that}
\]

\[
(3.4) \quad C_{I_0(V)}(s^*) \simeq C_{I_0(V^*)}(s^*).
\]

**Proof.** Suppose \( s \) is an \( r \)-element and decomposes as \((3.3)\). Then \(-1\) is not an eigenvalue of \( s \), so that \( \dim V^*_r(s) = m_r(s) d_r \) and \( \eta_r(s) = e^{m_r(s)} \) for \( \Gamma \neq X - 1 \). Thus

\[
m_{X-1}(s) = \dim V^* - \sum_{\Gamma \neq X-1} \dim V^*_r(s)
\]

and

\[
\eta_{X-1}(s) = (-1)^{(q-1)/2} \eta_{X-1}(s) \prod_{\Gamma \neq X-1} \eta_r(s),
\]

so that \( s \) is determined uniquely up to conjugacy in \( I(V^*) \) by its multiplicity function \( m_r(s) \). Moreover, \( s \in I_0(V^*) \) and the \( I(V^*) \)-class of \( s \) decomposes into one or two conjugacy classes of \( I_0(V^*) \) according as \( 1 \) is or is not an eigenvalue of \( s \). Similar statements hold for \( r \)-elements of \( I(V) \). If \( V \) is symplectic, then the dual mapping induces a bijection of the conjugacy classes of \( r \)-elements of \( I_0(V^*) \) onto the conjugacy classes of \( r \)-elements of \( I_0(V) \). If \( V \) and \( V^* \) are even dimensional orthogonal spaces, then the dual mapping induces a bijection of the conjugacy classes of \( r \)-elements of \( I_0(V^*) \) onto the conjugacy classes of \( r \)-elements of \( I(V) \). Moreover, the \( I(V^*) \)-class of \( s \) is a single \( I_0(V^*) \)-class if and only if the \( I(V) \)-class of \( s^* \) is a single \( I_0(V) \)-class. So the dual mapping induces a bijection of the conjugacy classes of \( r \)-elements of \( I_0(V^*) \) and \( I_0(V) \). The isomorphism \((3.4)\) follows by [12, (3A)].

Given \( m \geq 1 \), let \( V \) be a symplectic or orthogonal space of dimension \( 2em \) and type \( e^m \) if \( V \) is orthogonal. Let \( G = I_0(V) \) and \( G^* = I_0(V^*) \). By [12, (5.2)] \( G \) has a basic subgroup \( R \) of the form \( R_{m,0,0} \), and we denote by \( u^* \) a primary element of \( R \) and \( u \) a dual of \( u^* \) given by \((3D)\), so that \( |u^*| = r^2 \), \( u^* = u^*(\Gamma) \) for a unique \( \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2 \), and \( C_G(u^*) = C_{I(V)}(u^*) \simeq GL(m, eq^e) \). Moreover, the subgroup \( \langle u^* \rangle \) is uniquely determined up to conjugacy in \( I(V) \). Namely, if \( v^* \in G \) is an element of order \( r^2 \) and \( v^* = v^*(\Gamma) \) for a unique \( \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2 \), then \( \langle v^* \rangle \) and \( \langle u^* \rangle \) are conjugate in \( I(V) \). Let \( \mathcal{S} \) and \( \mathcal{S}^* \) be the sets of conjugacy classes of \( G \) and \( G^* \) of semisimple elements in the \( r \)-sections containing \( u^* \) and \( u \) respectively. Here the \( r \)-section containing \( u^* \) in \( G \), by definition, is the set of all elements in \( G \) whose \( r \)-part is conjugate with \( u^* \) in \( G \). Thus each class of \( \mathcal{S} \) has the form \( \langle h^* u^* \rangle \) for some semisimple \( r' \)-element \( h^* \in C_G(u^*) \). Define

\[
\mathcal{S}' = \{ [h^*]; (h^* u^*) \in \mathcal{S} \}, \quad \mathcal{S}'^* = \{ [s]; (su) \in \mathcal{S}^* \},
\]

where \([h^*]\) and \([s]\) are conjugacy classes of \( h^* \) and \( s \) in \( I(V) \) and \( I(V^*) \) respectively.
The dual mapping \( s \mapsto s^* \) from the semisimple elements of \( I_0(V^*) \) to the semisimple elements of \( I_0(V) \) induces a bijection \( f: [s] \mapsto [s^*] \) from \( \mathcal{S}^\ast \) onto \( \mathcal{S}^\ast \) such that

\[
C_{I_0(V^*)}(u^*, s^*) \approx C_{I_0(V)}(u, s).
\]

**Proof.** Let \([s] \in \mathcal{S}^\ast\), \( s^* \) a dual of \( s \) in \( G \), \( K = C_G(u^*) \), and \( K^* = C_G^\ast(u) \), so that \( K^* \) is a dual of \( K \). Then \( s \) and \( s^* \) have primary decompositions

\[
V = \sum_{\Gamma} V_{\Gamma}(s^*), \quad s^* = \prod_{\Gamma} s^*(\Gamma), \quad V^* = \sum_{\Gamma} V_{\Gamma}^*(s), \quad s = \prod_{\Gamma} s(\Gamma).
\]

Thus \( C_{I(V^*)}(s) = \prod_{\Gamma} C_{\Gamma}(s) \), where \( C_{\Gamma}(s) = C_{I(V^*)}(s)(\Gamma) \). Moreover, by (1.10)

\[
C_{\Gamma}(s) \simeq \begin{cases} I(V^*) & \text{if } \Gamma \in \mathcal{S}_0, \\ GL(m_{\Gamma}(s), e_{\Gamma} q^{\delta_{\Gamma}}) & \text{if } \Gamma \in \mathcal{S}_1 \cup \mathcal{S}_2. \end{cases}
\]

Let \( u_\Gamma \) be the restriction of \( u \) to \( V_{\Gamma}(s) \). Then \([V_{\Gamma}(s), u_\Gamma] = V_{\Gamma}(s) \) for \( \Gamma \neq X - 1 \) and \( u_\Gamma \in C_{\Gamma}(s) \). Thus

\[
m_{\Gamma}(s) = \begin{cases} e_{\Gamma} w_{\Gamma}(s) & \text{if } \Gamma \in \mathcal{S}_1 \cup \mathcal{S}_2, \\ 2e_{\Gamma} w_{\Gamma}(s) & \text{if } \Gamma \in \mathcal{S}_0 \text{ and } \dim V^* \text{ is even}, \\ 2e_{\Gamma} w_{\Gamma}(s) & \text{if } \Gamma = X + 1 \text{ and } \dim V^* \text{ is odd}, \\ 2e_{\Gamma} w_{\Gamma}(s) + 1 & \text{if } \Gamma = X - 1 \text{ and } \dim V^* \text{ is odd}, \end{cases}
\]

for some integer \( w_{\Gamma}(s) \), and \( \eta_{X+1}(s) = e_{w_{X+1}(s)} \), \( \eta_{\Gamma}(s) = e_{w_{\Gamma}(s)}^{m_{\Gamma}(s)} \) for \( \Gamma \in \mathcal{S}_1 \cup \mathcal{S}_2 \). Moreover, \( \eta_{X-1}(s) \) is determined by the equation

\[
\eta(V^*) = (-1)^{(q-1)/2} m_{X-1}(s) \prod_{\Gamma} \eta_{\Gamma}(s).
\]

Thus the type function \( \eta_{\Gamma}(s) \) is uniquely determined by the multiplicity function \( m_{\Gamma}(s) \), so that \([s] = [s']\) for \([s], [s'] \in \mathcal{S}^\ast \) if and only if \( m_{\Gamma}(s) = m_{\Gamma}(s') \) for all \( \Gamma \in \mathcal{S} \). It is clear that \( C_{K^*}(s) = C_{G^*}(u, s) = C_{G^*}(s)(u) \) and \( C_{K^*}(s) = \prod_{\Gamma} C_{G^*}(u, s) \), where \( C_{G^*}(u, s) = C_{G^*}(u_\Gamma) \) for \( \Gamma \in \mathcal{S}_1 \cup \mathcal{S}_2 \) and \( C_{G^*}(u, s) = C_{I_0(V^*)}(u_\Gamma) \) for \( \Gamma \in \mathcal{S}_0 \). By (3.7) and (3.8)

\[
C_{\Gamma}(u, s) \simeq GL(w_{\Gamma}(s), e_{\Gamma} q^{\delta_{\Gamma}})
\]

for all \( \Gamma \in \mathcal{S} \). Similarly, \( C_{I(V)}(s^*) = \prod_{\Gamma} C_{\Gamma}(s^*) \), where \( C_{\Gamma}(s^*) = C_{I(V^*)}(s^*)(\Gamma) \). Moreover

\[
C_{\Gamma}(s^*) = \begin{cases} I(V_{\Gamma}(s^*)) & \text{if } \Gamma \in \mathcal{S}_0, \\ GL(m_{\Gamma}(s^*), e_{\Gamma} q^{\delta_{\Gamma}}) & \text{if } \Gamma \in \mathcal{S}_1 \cup \mathcal{S}_2. \end{cases}
\]

By definition of \( s^* \), \( m_{\Gamma}(s^*) = m_{\Gamma}(s) \) except when \( \Gamma = X - 1 \) and \( V \) is symplectic, in which case, \( m_{\Gamma}(s^*) = m_{\Gamma}(s) - 1 \). Thus \( m_{\Gamma}(s^*) = \beta_{\Gamma} r_{\Gamma} w_{\Gamma}(s) \), where \( \beta_{\Gamma} = 1 \) or 2 according as \( \Gamma \in \mathcal{S}_1 \cup \mathcal{S}_2 \) or \( \Gamma \in \mathcal{S}_0 \). Let \( w_{\Gamma}(s) = \sum_{\beta} n_{\beta} r_{\beta} \) be the \( \beta \)-adic expansion of \( w_{\Gamma}(s) \), and \( c_{\beta} = (1, 1, \ldots, 1) \) (\( \beta \)-terms). Then a Sylow \( r \)-subgroup \( D(\Gamma) \) of \( C_{\Gamma}(s^*) \) is of the form \( \prod_{\beta} (R_{m_{\beta}, \alpha_{\beta}, \ldots, \alpha_{\beta}})^{n_{\beta}} \). Thus a Sylow \( r \)-subgroup \( P \) of \( C_{I(V)}(s^*) \) is of the form \( \prod_{\Gamma} D(\Gamma) \) as a subgroup of \( I(V) \), so that \( P \) has a primary element \( v^* \) and \( \langle v^* \rangle \) is conjugate with \( \langle u^* \rangle \) in \( I(V) \). Thus a conjugate of \( s^* \) in \( I(V) \) lies in \( K \). Replacing \( s^* \) by its conjugate, we may suppose \( s^* \in K \). So \( C_K(s^*) = C_G(u^*, s^*) = C_{G^*}(s^*)(u^*) \) and
if $u^*_r$ is the restriction of $u^*$ to $V_\Gamma(s^*)$, then $C_K(s^*) = \prod_{\Gamma} C_{\Gamma}(u^*, s^*)$, where $C_{\Gamma}(u^*, s^*) = C_{G}(s^*)(u^*_{\Gamma})$. Moreover,

\[(3.11) \quad C_{\Gamma}(u^*, s^*) \simeq GL(w_{\Gamma}(s), \varepsilon_{\Gamma}q^{m(\Gamma)}) ,\]

for all $\Gamma \in \mathcal{F}$. Since $s^*$ is an $r'$-element and $s^* \in K$, it follows $(s^* u^*) \in \mathcal{F}$ and $[s^*] \in \mathcal{F}'$.

Conversely, given $[s^*] \in \mathcal{F}'$, suppose $s^*$ decomposes as $(3.6)$. Since $u^* \in C_G(s^*)$ and the restriction $u^*_{\Gamma}$ of $u^*$ to $V_\Gamma(s^*)$ lies in $C_{\Gamma}(s^*)$, it follows $m_{\Gamma}(s^*) = \beta_{\Gamma} \varepsilon_{\Gamma} w_{\Gamma}(s^*)$. Define $n_{\Gamma} = m_{\Gamma}(s^*)$ except when $\Gamma = X - 1$ and $V$ is symplectic, in which case, $n_{\Gamma} = m_{\Gamma}(s^*) + 1$. In addition, define $\eta_{\Gamma} = \varepsilon_{\Gamma} m_{\Gamma}(s^*)$ for $\Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2$, $\eta_{X+1} = \varepsilon_{\Gamma} w_{X+1}(s^*)$, and $\eta_{X-1}$ is chosen so that $(3.9)$ holds with $\eta_{\Gamma}(s)$ and $m_{\Gamma}(s)$ replaced by $\eta_{\Gamma}$ and $n_{\Gamma}$ respectively. Thus $n_{\Gamma}$ and $\eta_{\Gamma}$ satisfy the relation $(1.11)$ for $V^*$ with $m_{\Gamma}(s)$ and $\eta_{\Gamma}(s)$ replaced by $n_{\Gamma}$ and $\eta_{\Gamma}$, so that a semisimple element, denote by $s^*$, exists in $I_0(V^*)$ such that $m_{\Gamma}(s) = n_{\Gamma}$ and $\eta_{\Gamma}(s) = \eta_{\Gamma}$. Such an element is determined uniquely up to conjugacy in $I(V^*)$. Thus $m_{\Gamma}(s)$ satisfy equation $(3.8)$ with $w_{\Gamma}(s)$ replaced by $w_{\Gamma}(s^*)$. A similar proof to above shows that a Sylow $r$-subgroup of $C_{I(V^*)}(s)$ has a primary element conjugate with $u$ in $I(V^*)$. We may suppose $u \in C_{I(V^*)}(s)$ and $(su) \in \mathcal{F}^*$, so that $[s] \in \mathcal{F}^*$. But $[s] = [s']$ for $[s], [s'] \in \mathcal{F}^*$ if and only if $m_{\Gamma}(s) = m_{\Gamma}(s')$ for all $\Gamma \in \mathcal{F}$, so the two maps induced by $s \mapsto s^*$ and $s^* \mapsto s$ are inverse each other and both are bijections. The isomorphism $(3.5)$ follows by $(3.10)$ and $(3.11)$.

Remark. As shown in the proof of $(3E)$, if $s^*$ is a semisimple $r'$-element of $I_0(V)$ such that a Sylow $r$-subgroup of $C_{I(V)}(s^*)$ acts fixed-point freely on $V$, then $m_{\Gamma}(s^*) = \beta_{\Gamma} \varepsilon_{\Gamma} w_{\Gamma}(s^*)$, so that a dual $s$ of $s^*$ is a well-defined semisimple $r'$-element of $I_0(V^*)$. Moreover, if $u^*$ is a primary element of a Sylow $r$-subgroup of $C_{I(V)}(s^*)$ and $u$ is its dual, then we may suppose $u$ is a primary element of a Sylow $r$-subgroup of $C_{I(V^*)}(s)$ and $C_{I_0(V^*)}(u^*, s^*) \simeq C_{I_0(V^*)}(u, s)$.

$(3F)$. Given integer $m \geq 1$, let $V$ be a symplectic or orthogonal space over $\mathbb{F}_q$ of dimension $2em$ and $\eta(V) = e^m$ if $V$ is orthogonal. Let $G = I_0(V)$, and $B$ a block of $G$ contained in $\mathcal{F}_r(G, (s))$ for some semisimple $r'$-element $s$ of $G^*$. If a defect group $R$ of $B$ acts fixed-point freely on $V$, then $R$ is conjugate in $I(V)$ with a Sylow $r$-subgroup of $C_G(s^*)$, where $s^*$ is a dual of $s$ in $G$.

Proof. Since $R$ is radical in $I(V)$, it has a primary element $z^*$. Let $K = C_G(z^*)$ and $K^*$ its dual. Then $z^* = z^*(\Gamma)$ for a unique $\Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2$, $K = C_{I(V)}(z^*) \simeq GL(m, \varepsilon q^s)$, and $K^*$ is embedded as a regular subgroup in $G^*$. Suppose $(z^*, B_z^*)$ is a major subsection associated with $B_z$, in the sense of [6], and $B_z^* \subseteq \mathcal{F}_r(K, (t))$. Then $s$ and $t$ are conjugate in $G^*$ by $(3C)$ and $R$ is a defect group of $B_z^*$. Replace $s$ by a conjugate we may suppose $s = t$, so that $R$ is conjugate with a Sylow $r$-subgroup of $C_{K^*}(s)^*$ in $K$ by a result of [11, §5]. Let $s^*$ be a dual of $s$ and $\rho$ an element of order $r^s$ in $Z(K^*)$. Such an element $\rho$ exists since $K \simeq K^*$. Thus $K^* \leq C_{G^*}(\rho)$ and $\delta_{\Gamma} = e$ for all $\Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2$ with $m_{\Gamma}(\rho) \neq 0$. By $(1.9)$ and $(1.10)$ $C_{G^*}(\rho) = K^*$, so that $\rho$ is a primary element of $O_r(Z(K^*))$. Thus $(\rho)$ is conjugate in $I(V^*)$ with the subgroup generated by a dual of $z^*$ given by $(3D)$. Replacing $\rho$ by $\rho^k$ for some integer $k$, we may suppose $\rho$ is a dual of $z^*$. Since $s$ lies in the $r$-section containing $\rho$, we may suppose $s^*$ lies in the $r$-section containing $z^*$.
and \( C_K(s^*) \cong C_K(s) \) by \((3E)\). By \((3.10)\) and \((3.11)\) \( C_K(s^*) \) and \( C_K\cdot(s)* \) are conjugate in \( K \). Thus \( R \) is conjugate with a Sylow \( r \)-subgroup of \( C_K(s^*) \).

We may suppose \( R \) is a Sylow \( r \)-subgroup of \( C_K(s^*) \). Let \( P \) be a Sylow \( r \)-subgroup of \( C_{(V)}(s^*) \) containing \( R \) and \( u^* \) a primary element of \( P \). So \( u^* \in Z(P) \), \( R \leq C_P(z^*) \leq C_K(s^*) \), and \( R = C_P(z^*) \) since \( R \) is Sylow in \( C_K(s^*) \). Thus \( u^* \in Z(R) \) and \( u^* \) is a primary element of \( R \). So \( \langle z^* \rangle = \langle u^* \rangle \leq Z(P), P = C_P(z^*) = R \), and \((3F)\) holds.

Let \( \mathcal{S}' \) be the subsets of polynomials in \( \mathcal{S} \) whose roots have \( r' \)-order. Given \( \Gamma \in \mathcal{S}' \), we shall define \( G_\Gamma, R_\Gamma, C_\Gamma, \theta_\Gamma, \) and \( s_\Gamma \) as follows: Let \( V_\Gamma \) denote a symplectic or orthogonal space of dimension \( 2e_\Gamma \delta_\Gamma \) over \( \mathbb{F}_q \) and of type \( e_\Gamma^\xi \) or \( \varepsilon \) according as \( \Gamma \in \mathcal{S}_1 \cup \mathcal{S}_2 \) or \( \Gamma \in \mathcal{S}_0 \) if \( V_\Gamma \) is orthogonal. Thus \( I(V_\Gamma) \) has a primary element \( s_\Gamma \) with a unique elementary divisor \( \Gamma \) of multiplicity of \( \beta_\Gamma e_\Gamma \) and \( I(V_\Gamma) \) has a basic subgroup \( R_\Gamma \) of form \( R_{m_\Gamma, \alpha_\Gamma, 0} \) by \([12, (1.12) \) and \((5.2)\)\]. Let \( G_\Gamma = I(V_\Gamma), G_\Gamma^0 = I_0(V_\Gamma), \) and \( C_\Gamma = C_G(\Gamma) \). Then \( s_\Gamma^* \in G_\Gamma^0 \) and \( C_\Gamma \cong \text{GL}(m_\Gamma, q^\delta_\Gamma) \), so that a Coxeter torus \( T_\Gamma \) of \( C_\Gamma \) has order \( q^{m_\Gamma \delta_\Gamma} - 1 \). The dual \( T_\Gamma^* \) is embedded as a regular subgroup of \( C_\Gamma^* \), and in turn, \( C_\Gamma^* \) is embedded as a regular subgroup of \( G_\Gamma^0 \). We claim that there exists an element \( s_\Gamma \) in \( T_\Gamma^* \) such that \( C_{C_\Gamma}(s_\Gamma) = T_\Gamma^* \) and as an element of \( G_\Gamma^0 \), \( s_\Gamma \) and \( s_\Gamma^* \) are dual each other in the sense of \((3E)\). Indeed

\[
C_{G_\Gamma}(s_\Gamma) = \begin{cases} I(V_\Gamma) & \text{if } \Gamma = X \pm 1, \\
\text{GL}(\varepsilon_\Gamma, e_\Gamma q^{\delta_\Gamma}) & \text{if } \Gamma \neq X \pm 1,
\end{cases}
\]

so that a Sylow \( r \)-subgroup of \( C_{G_\Gamma}(s_\Gamma) \) acts fixed-point freely on \( V_\Gamma \). By the remark of \((3E)\) a dual \( s_\Gamma \) of \( s_\Gamma^* \) exists in \( G_\Gamma^0 \) and

\[
C_{G_\Gamma^0}(s_\Gamma) = \begin{cases} \\
\text{GL}(\varepsilon_\Gamma, e_\Gamma q^{\delta_\Gamma}) & \text{if } \Gamma \neq X \pm 1, \\
\text{SO}^e(2e, q) & \text{if } \Gamma = X \pm 1 \text{ and } V_\Gamma \text{ is orthogonal}, \\
\text{SO}(2e + 1, q) & \text{if } \Gamma = X - 1 \text{ and } V_\Gamma \text{ is symplectic}, \\
\langle w, 1 \times \text{SO}^e(2e, q) \rangle & \text{if } \Gamma = X + 1 \text{ and } V_\Gamma \text{ is symplectic},
\end{cases}
\]

where \( w \) is an element in \( \text{SO}(V_\Gamma) \) such that \( w^2 = 1 \times \text{SO}^e(2e, q) \), and \( 1 \) is the identity matrix of size \( 1 \). Let \( R_\Gamma^* \) be a Sylow \( r \)-subgroup of \( C_{G_\Gamma^0}(s_\Gamma) \), \( C_\Gamma^* = C_{G_\Gamma^0}(s_\Gamma) \), and \( T_\Gamma^* = C_{C_\Gamma^*}(s_\Gamma) \). Then \( s_\Gamma \in T_\Gamma^* \) and \( T_\Gamma^* = C_{C_{G_\Gamma^0}(s_\Gamma)}(R_\Gamma^*) \). Thus \( T_\Gamma^* \) has order \( q^{e_\Gamma \delta_\Gamma} - 1 \). But \( e_\Gamma \delta_\Gamma = m_\Gamma e_\Gamma r^{ar} \), \( r \) divides both \( q^{m_\Gamma e_\Gamma r^{ar} - 1} \) and \( q^{e_\Gamma \delta_\Gamma} - 1 \), so \( q^{e_\Gamma \delta_\Gamma} = q^{m_\Gamma e_\Gamma r^{ar}} \) and \( R_\Gamma^* \) is a Sylow \( r \)-subgroup of \( G_\Gamma^0 \). In particular, \( R_\Gamma^* \) is cyclic of order \( r^{a+ar} \) and has type \( R_{m_\Gamma, \alpha_\Gamma, 0} \) as a subgroup of \( I(V_\Gamma^*) \). Let \( R_\Gamma^* \) be the Sylow \( r \)-subgroup of \( T_\Gamma^* \). Then \( R_\Gamma^* \) is cyclic of order \( r^{a+ar} \) and there exists \( g \in I(V_\Gamma^*) \) such that \( (R_\Gamma^*)^g = R_\Gamma^* \), so that \( (C_\Gamma^*)^g = C_\Gamma^* \). Thus \( (T_\Gamma^*)^hT_\Gamma^* \), and \( h^{-1}g^{-1} \in T_\Gamma^* \) for some \( h \in C_\Gamma^* \). Thus \( s_\Gamma^* \) is a dual of \( s_\Gamma^* \) in \( G_\Gamma^0 \) and \( C_{C_\Gamma^*}(s_\Gamma^* \) = \( T_\Gamma^* \). We may denote \( s_\Gamma^h \) by \( s_\Gamma \) and then the claim holds. By \((3E)\) \( s_\Gamma \) is uniquely determined by \( \Gamma \) up to conjugacy in \( I(V_\Gamma^*) \).

Let \( \phi_\Gamma \) be the character of \( T_\Gamma \) corresponding to \( s_\Gamma \), and let

\[
\theta_\Gamma = \pm R_{G_\Gamma^0}(\phi_\Gamma) = \pm R_{G_\Gamma^0}(s_\Gamma),
\]

where the sign is chosen so that \( \theta_\Gamma \) is an irreducible character of \( C_\Gamma \). The
block \( b_T \) of \( C_T \) containing \( \theta_T \) then has defect group \( R_T \) by [11, (4C)] and the Brauer pair \( (R_T, b_T) \) of \( G_T^0 \) has the label \( (R_T, s_r, -) \).

(3G). Let \( N_T = N_{G_T}(R_T) \), and \( N(\theta_T) \) the stabilizer of \( \theta_T \) in \( N_T \).

(a) \( (N(\theta_T): C_T) = \beta T \). In particular, \( |\text{Irr}^0(N(\theta_T), \theta_T)| = \beta T \) and \( R_T \) is a defect group of \( b_T^{G_T} \).

(b) Let \( \Gamma, \Gamma' \in \mathcal{F} \) such that \( G_T = G_{\Gamma'} \) and \( R_T = R_{\Gamma'} \), so that \( C_T = C_{\Gamma'} \) and \( N_T = N_{\Gamma'} \). Let \( \theta_T \) and \( \theta_{\Gamma'} \) be the canonical characters of \( b_T \) and \( b_{\Gamma'} \), respectively. Then \( b_T = b_{\Gamma'} \) for some \( \tau \in N_T \) if and only if \( s_T \) and \( s_{\Gamma'} \) are conjugate in \( I(V_T^\ast) \), where \( V_T^\ast \) is the underlying space of \( G_T^{b_T} \).

Proof. (a) It suffices to show \( (N(\theta_T): C_T) = \beta T \) since \( N_T/C_T \) is cyclic of order \( 2\epsilon T \). If \( \Gamma \in \mathcal{F}_2 \), then \( C_T = T_T \), \( \theta_T = \phi_T \), and \( \theta_T \) is either the identity character or the character of order 2 of \( T_T \). Thus \( N(\theta_T) = N_T \) and \( (N(\theta_T): C_T) = 2\epsilon T \).

Suppose \( \Gamma \in \mathcal{F}_2 \cup \mathcal{F}_3 \), so that \( T_T = C_{G_T}(r) \) for some \( r \in T_T \) and \( T_T = C_{G_T}(r) \) for any generator \( r \) of \( R_T \). Let \( \Delta \) be the unique elementary divisor of \( \mu r \) and \( N(T_T) = N_{G_T}(r) \). Following [12, p. 149], if \( \Delta \in \mathcal{F}_3 \), we have \( N(T_T) = \langle x, t \rangle \), where \( x : t \mapsto t^a \) for \( t \in T_T \). Here \( x \) has order \( 2m_{T_T}T_T \) in \( N(T_T)/T_T \) and \( \sigma_{m_{T_T}}T_T \) inverts \( T_T \). If \( \Delta \in \mathcal{F}_2 \), we have \( N(T_T) = \langle \beta, \gamma, T_T \rangle \), where \( \beta : t \mapsto t^a \), \( \gamma : t \mapsto t^{-1} \) for \( t \in T_T \). Here \( \beta \) and \( \gamma \) have order \( m_{T_T}T_T \) and \( 2 \) respectively in \( R_T \). Moreover, \( N_T = N(T_T) \).

Let \( N_T \) act on the pairs \( (T, \phi) \) by conjugation and let \( [T, \phi] \) be the \( C_T \)-orbit of the pair \( (T, \phi) \), where \( T \) is a Coxeter torus of \( C_T \) and \( \phi \) is an irreducible character of \( T \). Then \( N_T \) induces an action on the \( C_T \)-orbits and the \( N_T \)-orbit of \( [T_T, \phi_T] \) consists of \( \{[T_T, \phi_T^{t\epsilon T}] | 1 \leq t \leq m_{T_T}T_T \} \). Moreover, we claim that for \( \tau \in N_T \), \( \phi \in N(T_T) \), \( [T_T, \phi_T] = [T_T, \phi_T^\tau] \) if and only if \( (R_T^G(\phi_T^\tau))^{t\epsilon T} = R_T^G(\phi_T^\tau) \). Indeed given \( \tau \in N_T \), then \( T_T^{\epsilon T} = T_T^{t\epsilon T} \) for some \( \epsilon T \in C_T \) and \( (R_T^G(\phi_T^{t\epsilon T}))^{t\epsilon T} = (R_T^G(\phi_T^{t\epsilon T}))^{t\epsilon T} = R_T^G(\phi_T^{t\epsilon T}) \). Thus \( [T_T, \phi_T] = [T_T, \phi_T^\tau] \) if and only if \( (R_T^G(\phi_T)^{t\epsilon T} = R_T^G(\phi_T^\tau) \). Thus the claim holds. In particular, \( N(\theta_T) \) is the stabilizer of \( [T_T, \phi_T] \) in \( N_T \).

The group \( C_T \) acts on the pairs \( (T, \phi) \) by conjugation. Let \( [T^*, s] \) be the conjugacy \( C_T \)-class of \( (T, \phi) \). By [18, (7.5)] the \( C_T \)-classes \( [T, \phi] \) are in bijection with the \( C_T \)-classes \( [T^*, s] \) and if \( [T, \phi] \) corresponds to \( [T^*, s] \), then \( R_T^G(\phi) = R_T^G(s) \) and \( [T, \phi^k] \) corresponds to \( [T^*, s^k] \) for any integer \( k \). Let \( R_T^* \) be the Sylow \( r \)-subgroup of \( C_T \) and \( N_T^* = N_I(V_T^*) \) be its normalizer in \( I(V_T^*) \). Then \( |R_T^*| = r^{\ast + \epsilon T} \), \( R_T^* \leq Z(C_T) \), and \( R_T^* \) has form \( R_{m_{T_T}T_T, 0} \) as a subgroup of \( I(V_T^*) \). So \( C_T = C_{I(V_T^*)}(R_T^*) \). Let \( N(T_T^*) = N_I(V_T^*)(T_T^*) \). Then \( N_T^* = N(T_T^*)C_T^* \) and \( N_T^* \) acts on the pairs \( (T^*, s) \) by conjugation, so that \( N_T^* \) induces an action on classes \( [T^*, s] \). If \( G_T \) is a symplectic group, then \( I(V_T^*) \simeq 0(\phi, T_T, T_T, q) \), and the action of \( N(T_T^*) \) on \( T_T^* \) is similar to that of \( N(T_T) \) on \( T_T \), namely for \( g \in N(T_T^*) \), \( g \) acts on \( T_T^* \) by \( g : t \mapsto t^{q^k} \) where \( t \in T_T^* \) and \( 1 \leq l \leq m_{T_T}T_T \). If \( G_T \) is an orthogonal group, then \( I(V_T^*) \simeq 0(\phi, T_T, T_T, q) \), and the action of \( N(T_T^*) \) on \( T_T^* \) is similar to that of \( N(T_T) \) on \( T_T \). Thus the \( N_T^* \)-orbit of \( [T_T^*, s_T] \) consists of \( \{[T_T^*, s_T^{q^k}] \} \), where \( 1 \leq k \leq m_{T_T}T_T \) and the elements in this orbit are in bijection with that in the \( N_T \)-orbit of \( [T_T, \phi_T] \). So \( (N_T : N(\theta_T)) = (N_T^* : N([T_T^*, s_T])) \), where \( N([T_T^*, s_T]) \)
is the stabilizer of \([T^*_r, s_r]\) in \(N^*_r\). Let \(H^* = N([T^*_r, s_r])\) or \(N([T^*_r, s_r]) \cap I_0(V^*_r)\) according as \(V_r\) is orthogonal or symplectic. Then \(H^* \geq C^*_r\) and 
\(|N(\theta^*)| = |H^*|\) since \(|N_r| = |N^*_r|\) or \(\frac{1}{2}|N^*_r|\) according as \(V_r\) is orthogonal or symplectic. Moreover, \((N(\theta^*_r) : C^*_r) = (H^*_r : C^*_r)\).

Now fix the \(C^*_r\)-classes \([T^*_r, s_r]\). Then it is clear that \(C^*_r\) and \(H^*\) act transitively on the class and so 
\((H^* : N_{H^*}(T^*_r, s_r)) = (C^*_r : N_{C^*_r}(T^*_r, s_r))\), where \(N_{H^*}(T^*_r, s_r)\) and \(N_{C^*_r}(T^*_r, s_r)\) are the stabilizers of the pair \((T^*_r, s_r)\) in \(H^*\) and \(C^*_r\) respectively. But \(H^* \geq C^*_r\), \(N_{C^*_r}(T^*_r, s_r) = T^*_r\), and 
\((H^*_r : T^*_r) = (H^*_r : C^*_r)(C^*_r : T^*_r) = (H^* : N_{H^*}(T^*_r, s_r))(N_{H^*}(T^*_r, s_r) : T^*_r)\),
so 
\((H^* : C^*_r) = (N_{H^*}(T^*_r, s_r) : T^*_r)\). If \(V^*_r\) is orthogonal, then \(C_{I(V^*_r)}(s_r) = C_{I(V^*_r)}(s_r)\) by \(\Gamma \in F_1 \cup F_2\). Thus in any case \(N_{H^*}(T^*_r, s_r) \leq I_0(V^*_r)\). Let \(K^* = C_{I(V^*_r)}(s_r)\). Then \(K^* \simeq GL(e_T, e_T^d)\) and \(N_{H^*}(T^*_r, s_r) = N_{K^*}(T^*_r)\). Since \(T^*_r\) is a Coxeter torus of \(K^*\), \((N_{K^*}(T^*_r) : T^*_r^c) = e_T\) and then \((N(\theta^*_r) : C^*_r) = e_T\).

(b) Let \(\theta_\tau = r\mathbf{R}T^{r}_\tau(s_r)\). Suppose \(\theta_\tau = \theta_T\) for some \(\tau \in N^*_r\). Then \([T^*_r, \theta_T]\) corresponds to \([T^*_r, s_{_\tau^n}]\) for some \(n \in N(T^*_r)\) since the elements in the \(N^*_r\)-orbit of \([T^*_r, s_r]\) and \(N^*_r = N(T^*_r)C^*_r\). Thus \(\theta_\tau = r\mathbf{R}T^{r}_\tau(s_{_\tau^n})\) and \([T^*_r, s_{_\tau^n}] = [T^*_r, s_r]\). So \(s_r\) is conjugate with \(s_\tau^n\) in \(I(V^*_r)\). Conversely, suppose \(s_r\) and \(s_\tau^n\) are conjugate in \(I(V^*_r)\). Since \(T^*_r\) and \(T^*_r\) are Coxeter tori of \(C^*_r\), \(T^*_r \cap s_r = T^*_r \cap s_\tau^n\) and \(s_r = s_\tau^n\) for some \(c \in C^*_r\) and \(w \in I(V^*_r)\). If \(\Gamma \in F_0\), then \(C^*_r = T^*_r = T^*_r\) and \(s_r = s_\tau^n\) so that both \(s_r\) and \(s_\tau^n\) are elements of \(T^*_r\) of order 1 or 2 according as \(\Gamma = X - 1\) or \(\Gamma = X + 1\). Thus \(s_r = s_\tau^n\) and \(\theta_\tau = \theta_T\). Suppose \(\Gamma \in F_1 \cup F_2\), so that \(K^* = C_{I(V^*_r)}(s_r) = e_T^{c-1}\) and hence \(T^*_r, T^*_r^{cw-1}\) are Coxeter tori of \(K^*\). So \(T^*_r^{gw} = T^*_r^{cw-1}\) for some \(g \in K^*, T^*_r^{gw} = T^*_r^{cw-1}\), and \(gw \in N^*_r\). It follows that 
\([T^*_r, s_r]^{gw} = [T^*_r, s_r]^{gw} = [T^*_r, s_r]^{cw-1}, s_r^{c-1}] = [T^*_r, s_r]\). Since \(gw \in N^*_r\), \([T^*_r, s_r]^{gw}\) corresponds to \([T^*_r, \theta_T]\) for some \(\tau \in N^*_r\) and then \(\theta_\tau = \theta_T\). This completes the proof.

**Remark.** Let \(G_T\) be an orthogonal group, and \(N_0(\theta_T) = N(\theta_T) \cap G_0^0\). By \([12, (6B)](N(\theta_T) : N_0(\theta_T) = \beta_T\).

For each \(\alpha \geq 0\) and \(m \geq 0\), let \(V^*_m,0,0\) denote a symplectic or orthogonal space over \(F_q\) of dimension \(2mer^\alpha\) and type \(e^m\) if \(V^*_m,0,0\) is orthogonal. Thus \(I(V^*_m,0,0)\) has a basic subgroup of form \(R_{m,0,0}\) (see §2).

(3H). Let \(G = I(V^*_m,0,0), R = R_{m,0,0}\) a basic subgroup of \(G\), \(b\) a block of \(C_R(G)\) with defect group \(R\), and \(\theta\) the canonical character of \(b\). If \(N(\theta)\) is the stabilizer of \(\theta\) in \(N\), and \((N(\theta) : C_R(G))\) is cyclic of order \(2er^\alpha\).

Proof. Let \(C = C_R(G), N = N_G(G), \) and \(G_0 = I_0(V^*_m,0,0)\). Then \(C = C_R(G)\) and \(N/C\) is cyclic of order \(2er^\alpha\).

Since \(C \simeq GL(m, e^\alpha r^\alpha)\), it follows by \([11, (4B)\) and \( (4C)\) that \(\theta = e_T R_T^\alpha(\phi)\),
where \(e_T = \pm 1\), \(T\) is a Coxeter torus of \(C\) and \(\phi\) is an \(r\)-rational irreducible character of \(T\). Moreover, the dual \(T^*\) is embedded as a regular subgroup of
C*, and C* is embedded as a regular subgroup of G*. There is an element s of T* such that s corresponds to \( \phi \) and \( T^* = C_{C*}(s) \). In particular, if \( \phi^2 = 1 \), then \( s^2 = 1 \), \( T^* = C^* \), \( m = 1 \), and \( \theta = \phi \). Thus \( N = N(\theta) \) and \( (N(\theta) : C)_r = (N : C)_r = 1 \), so that \( \alpha = 0 \). In this case \( R = R_{x\pm 1} \), and \( \theta = \theta_{x\pm 1} \) (see [12, p. 148]).

Suppose \( \phi^2 \neq 1 \). Then as an element of \( C^* \), s has a unique elementary divisor \( \Delta \) with multiplicity 1 since \( T^* = C_{C^*}(s) \) is the Coxeter torus of \( C^* \). Regard s as an element of \( G_0^* \). By [12, (9A) and (9.2)] there is a unique \( \Gamma \in F_1 \cup F_2 \) such that the multiplicity of \( \Gamma \) in s is \( e_\Gamma r^l \) and \( e_\Gamma r^ld_\Gamma = 2mer^a \) for some \( l \geq 0 \). So \( C_{G_0^*}(s) \cong GL(e_\Gamma r^l, e_\Gamma q^{d_\Gamma}) \). A similar proof to that of (3G)(a) shows that \( (N(\theta) : C) = (N_{C_{G_0^*}(T^*)}(T^*) : T^*) = e_\Gamma r^l \). Thus \( l = 0 \) and \( e_\Gamma d_\Gamma = 2mer^a \) since \( (N(\theta) : C)_r = 1 \). But \( (m, r) = 1 \) by [11, (4B)]. It follows that \( m = m_\Gamma, \alpha = \alpha_\Gamma \), and \( G = G_\Gamma, R = R_\Gamma, \theta = \theta_\Gamma \). This completes the proof.

Given \( \Gamma \in F' \) and \( \gamma \geq 0 \). Let

\[
V_{\Gamma, \gamma} = V_\Gamma \perp V_\Gamma \perp \cdots \perp V_\Gamma,
\]

where there are \( r^\gamma \) terms \( V_\Gamma \) on the right-hand side. Then if \( V_\Gamma \) is orthogonal, \( V_{\Gamma, \gamma} \) has type \( (e_\Gamma)^{\gamma r^l} = e_\Gamma^{\gamma l} \) or \( r^l = e \) according as \( \Gamma \in F_1 \cup F_2 \) or \( \Gamma \in F_0 \).

(31). Let \( G = I(V_{\Gamma, \gamma}) \), \( R = R_{m_\Gamma, \alpha_\Gamma, \gamma} \) a basic subgroup of \( G \), and \( C = C_G(R) \). Then \( C = C_\Gamma \otimes I_\gamma \), where \( I_\gamma \) is the identity matrix of order \( r^l \). The irreducible character \( \theta = \theta_\Gamma \otimes I_\gamma \), of \( C \) defined by \( \theta(c \otimes I_\gamma) = \theta_\Gamma(c) \) for \( c \in C_\Gamma \) is then a character of defect 0 of \( CR/R \), and \( |\text{Irr}^0(N(\theta), \theta)| = \beta_{\Gamma r^l} \Gamma \). Proof. The proof is essentially the same as that of (3A), except that the automorphisms on \( C = C_\Gamma \otimes I_\gamma \) induced by \( N(R) \) have order \( 2er^a r^l \), and their actions are the same as the automorphisms on \( C_\Gamma \) induced by \( N_\Gamma/C_\Gamma \).

Remark. Suppose \( G = I(V_{\Gamma, \gamma}) \) is an orthogonal group. Let \( G_0 = I_0(V_{\Gamma, \gamma}) \) and \( N_0(\theta) = N(\theta) \cap G_0 \). Then \( |N(\theta) : N_0(\theta)| = \beta_{\Gamma} \) and for each \( \psi \in \text{Irr}^0(N(\theta), \theta) \) the restriction \( \psi|_{N_0(\theta)} \) of \( \psi \) to \( N_0(\theta) \) is irreducible. Indeed let \( N^0 = \{g \in N : [g, Z(R)] = 1 \} \). Then \( N^0 \leq N_0(\theta) \) and in the notation of (3A), \( N(\theta) = N(\theta) \) and \( N(\theta)/N^0 \cong N(\theta)/C_\Gamma \), where \( \theta \) is the unique irreducible character of \( N^0 \) covering \( \theta \) and having defect 0 as a character of \( N^0/R \). The remark of (3G) implies \( |N(\theta) : N_0(\theta)| = \beta_{\Gamma} \). Since \( \psi \) covers \( \theta \) and \( N(\theta)/N^0 \) is cyclic, \( \psi|_{N^0} = \theta \) is irreducible, so that \( \psi|_{N_0(\theta)} \) is irreducible. This completes the proof.

Given \( \Gamma \in F' \), and \( d \geq 0 \). Let \( G = I(V_{\Gamma, d}) \), and \( R = R_{m_\Gamma, c_\Gamma, \gamma} \) a basic subgroup of \( G \), where \( e = (c_1, c_2, \ldots, c_\ell) \), and \( \gamma + c_1 + c_2 + \cdots + c_\ell = d \). Then \( C = C_G(R) = C_\Gamma \otimes I_\gamma \otimes I_e \), where \( I_\gamma \) and \( I_e \) are the identity matrices of order \( r^l \) and \( r^{c_1+c_2+\cdots+c_\ell} \) respectively. The irreducible character of \( C \) defined by

\[
\theta(c \otimes I_\gamma \otimes I_e) = \theta_\Gamma(c)
\]

for \( c \in C_\Gamma \) is a character of defect 0 of \( CR/R \). We shall say that the pair \((R, \theta)\) is of type \( \Gamma \). If \((R, \theta)\) is of type \( \Gamma \), then \( \theta \) is a canonical character of a block \( b \) of \( C \) with defect group \( Z(R) \), and the Brauer pair \((R, b)\) of \( G \) is also a Brauer pair of \( G_0 = I_0(V_{\Gamma, d}) \) since \( C = C_{G_0}(R) \). Let \( D \) be
the base subgroup of \( R = R_{\text{mr}, \text{ar}, \gamma} \). Then each component \( Q \) of \( D \) is of the form \( R_{\text{mr}, \text{ar}, \gamma} \), so that by the remark of (1C) \( Q \) contains a normal subgroup \( Q' \) such that \( C_{\text{mr}}(V_{\text{mr}, \text{ar}, \gamma})(Q') = C_{\text{mr}}(V_{\text{mr}, \text{ar}, \gamma})(Q') = \prod_{i=1}^{r'} C_i \) is a regular subgroup of \( I_0(V_{\text{mr}, \text{ar}, \gamma}) \), where \( V_{\text{mr}, \text{ar}, \gamma} \) is the underlying space of \( Q \) and \( C_i \simeq \text{GL}(m_i, \varepsilon_{q^{r'}}) \) for all \( i \). Let \( R' \) be the subgroup of \( D \) with each component \( Q \) of \( D \) replaced by \( Q' \). Then \( R' \) is a normal subgroup of \( R \) and \( C' = C_{\text{mr}}(R') = \prod_{i=1}^{r'} C_i \), where \( C_i \simeq \text{GL}(m_i, \varepsilon_{q^{r'}}) \) for all \( 1 \leq i \leq r' \). Thus \( C' \) is a regular subgroup of \( I_0(V_{\gamma, d}) \) and \( C \leq C' \), so that \( C'* \) is embedded as a regular subgroup of \( I_0(V_{\gamma, d})^* \). Now we may suppose \( C_i = C^*_i \) and \( s_i \in C^*_i \) for all \( i \). Let

\[
(3.14) \quad x_\Gamma = s_\Gamma \times s_\Gamma \times \cdots \times s_\Gamma \quad \text{(r times)}
\]

be an element of \( C'* \) and \( x_\Gamma^* \) a dual of \( x_\Gamma \) in \( G \). Then as an element of \( G \), \( x_\Gamma^* \) has a unique elementary divisor \( \Gamma \) of multiplicity \( \beta_{q^r} r^d \) and type \( \eta_\Gamma(x_\Gamma^*) = \eta(V_{\gamma, d}) \). The subgroup \( C^*_i \otimes I_\gamma \otimes I_\epsilon \) can be regarded as a diagonal subgroup of \( C'* \), so that \( s_i \otimes I_\gamma \otimes I_\epsilon \in C'* \) and \( x_\Gamma \) is conjugate with \( s_i \otimes I_\gamma \otimes I_\epsilon \) in \( I(V^*) \). Thus \( (R, b) \) is labeled by \( (R, x_\Gamma, -) \). The Brauer pair \( (R, b) \) of \( G \) will also be denoted by \( (R, \theta) \).

(3J). (a) Let \( G = I(V) \), \( R \) a basic subgroup of \( G \), \( (R, \varphi) \) a weight of \( G \), and \( \theta \) an irreducible character of \( C_{\text{mr}}(R) \) covered by \( \varphi \). Then \( (R, \theta) \) is of type \( \Gamma \) for some \( \Gamma \in \mathcal{F}' \).

(b) The pair \( (R, \theta) \) of \( G \) with type \( \Gamma \) is uniquely determined by \( \Gamma \) up to conjugacy in \( N = N_G(R) \), that is, if \( (R, \theta') \) is another pair with type \( \Gamma \), then \( \theta' = \theta^n \) for some \( n \in N \).

Proof. (a) Suppose \( V = V_{m, \alpha, \gamma, \epsilon} \) and \( R = R_{m, \alpha, \gamma, \epsilon} \), where \( \epsilon = (c_1, \ldots, c_l) \). Let \( G_1 = I(V_{m, \alpha, 0}) \), \( R_1 = R_{m, \alpha, 0} \) a basic subgroup of \( G_1 \), \( C_1 = C_{G_1}(R_1) \), and \( N_1 = N_{G_1}(R_1) \). Then \( C_1 \simeq \text{GL}(m, \varepsilon_{q^{r'}}) \) and \( C = C_{G}(R) = C_1 \otimes I_\gamma \otimes I_\epsilon \). Thus \( \theta \) has the form \( \theta_1 \otimes I_\gamma \otimes I_\epsilon \), where \( \theta_1 \) is a character of \( C_1 \). Since \( \theta \) has defect 0 as a character of \( C/Z(R) \), \( \theta_1 \) has defect 0 as a character on \( C_1/R_1 \). The block of \( C_1 \) containing \( \theta_1 \) has defect group \( R_1 \).

Let \( R_{m, \alpha, \gamma} \) a basic subgroup of \( I(V_{m, \alpha, \gamma}) \), \( N_{m, \alpha, \gamma} \) and \( C_{m, \alpha, \gamma} \) the normalizer and centralizer of \( R_{m, \alpha, \gamma} \) in \( I(V_{m, \alpha, \gamma}) \). Then \( C_{m, \alpha, \gamma} = C_1 \otimes I_\gamma \) and \( (\theta_1 \otimes I_\gamma)(c \otimes I_\gamma) = \theta_1(c) \) for \( c \in C_1 \) is an irreducible character of \( C_{m, \alpha, \gamma} \). By (2.5)

\[
N/R \simeq (N_{m, \alpha, \gamma}/R_{m, \alpha, \gamma}) \otimes N_{S_{\mu}}(A_\epsilon),
\]

where \( u = r^{c_1 + \cdots + c_l} \). If \( N_{m, \alpha, \gamma} = \{ g \in N_{m, \alpha, \gamma} : [g, Z(R_{m, \alpha, \gamma})] = 1 \} \), then \( N_{m, \alpha, \gamma}/N_{m, \alpha, \gamma} \simeq N_1/C_1 \). Let \( \varphi = I(\psi) \) for some \( \psi \in \text{Irr}^\theta(N(\theta); \theta) \), and \( N(\theta_1 \otimes I_\gamma) \) be the stabilizer of \( \theta_1 \otimes I_\gamma \) in \( N_{m, \alpha, \gamma} \). Then

\[
N(\theta)/R \simeq (N(\theta_1 \otimes I_\gamma)/R_{m, \alpha, \gamma}) \times \text{GL}(c_1, r) \times \cdots \times \text{GL}(c_l, r).
\]

But \( \psi \) is a character of defect 0 of \( N(\theta)/R \), so it covers an irreducible character \( \psi_0 \) in \( \text{Irr}^\theta(N(\theta_1 \otimes I_\gamma), \theta_1 \otimes I_\gamma) \). Same proof as that of (3A) shows that \( N^0_{m, \alpha, \gamma} \leq N(\theta_1 \otimes I_\gamma) \) and \( N^0_{m, \alpha, \gamma} \) has a unique irreducible character \( \theta \) covering \( \theta_1 \otimes I_\gamma \) and having defect 0 as a character of \( N^0_{m, \alpha, \gamma}/R_{m, \alpha, \gamma} \). Moreover, \( N(\theta_1 \otimes I_\gamma) = \cdot \)
$N(\theta)$ and $N(\theta)/N_{m,a,\gamma}^{0} \simeq N(\theta)/C_1$, where $N(\theta)$ is the stabilizer of $\theta$ in $N_1$. Thus $\psi_0 \in \text{Irr}^0(N(\theta), \theta)$ and $\psi_0(1) = \psi(1)$ since $N_{m,a,\gamma}/N_{m,a,\gamma}^{0}$ is cyclic. By (3.1) $(N(\theta): N_{m,a,\gamma}^{0}) \Gamma_{r} = 1$ and hence $(N(\theta): C_1)_{r} = 1$. It follows by (3H) that $G_1 = G_{\Gamma}, R_1 = R_{\Gamma}$, and $\theta_1 = \theta_{\Gamma}$ for some $\Gamma \in \mathcal{T}$. Thus $(R_1, \theta_1)$ is labeled by $(R_1, s_\Gamma, -)$ and $(R, \theta)$ has type $\Gamma$, so (a) holds.

(b) Let $G = I(V_\Gamma, d), R = R_{m,\alpha, r, \gamma}, c$ a basic subgroup of $G, C = C_G(R), N = N_G(R), \theta = \theta_{\Gamma} \otimes I_{r} \otimes I_{c}$, and $\theta' = \theta_{\Gamma} \otimes I_{r} \otimes I_{c}$, where $\theta, \theta'$ are irreducible characters of $C_G(R)$, and $\theta$, $\theta'$ are defined as (3.13). If $(R_{\Gamma}, t_{\Gamma}, -)$ and $(R_{\Gamma}, t_{\Gamma}, -)$ are the labels of $(R_{\Gamma}, \theta_{\Gamma})$ and $(R_{\Gamma}, \theta_{\Gamma})$ respectively, then $t_{\Gamma}, t_{\Gamma}'$ are conjugate in $G_{\Gamma}$ since both Brauer pairs $(R, \theta)$ and $(R, \theta')$ are labeled by $(R, x_{\Gamma}, -)$. It follows by (3G)(b) that $\theta_{\Gamma}^w = \theta_{\Gamma}'$ for some $w \in N_{\Gamma}$.

Let $C_{m,\alpha, r, \gamma} = C_{I(V_\gamma, c)}(R_{m,\alpha, r, \gamma})$, so that $C_{m,\alpha, r, \gamma} = C_{\Gamma} \otimes I_{r}$. Let $\theta_{\Gamma} \otimes I_{r}$ and $\theta_{\Gamma}' \otimes I_{r}$ be irreducible characters of $C_{m,\alpha, r, \gamma}$ defined as (3.1). Since $N_{m,\alpha, r, \gamma}/N_{m,\alpha, r, \gamma}^{0} \simeq N_{\Gamma}/C_{\Gamma}$, it follows $(\otimes I_{r})^h = \theta_{\Gamma}^r \otimes I_{r}$ for some $h \in N_{m,\alpha, r, \gamma}$ and so $\theta^w = \theta'$ for some $n \in N$, where the structure of $N$ is given above with $m$ and $\alpha$ replaced by $m_{\Gamma}$ and $\alpha_{\Gamma}$ respectively.

Remark. Suppose $R$ is a basic subgroup of $G = I(V), b$ a block of $C_{G}(R)R$ with defect group $R$, and $\theta$ the canonical character of $b$. If $(N(\theta): C_{G}(R)R)_{n} = 1$, then $(R, \theta)$ is of type $\Gamma$ for some $\Gamma \in \mathcal{T}$. In particular, this occurs when $b$ is a root block of a block $B$ and $R$ is a defect group of $B$. Here a root block $b$ of a block $B$, in the sense of Brauer, is a block of $C_{G}(R)$ with defect group $R$ such that $b^{G} = B$, where $R$ is a defect group of $B$. Thus if $b$ is a root block of $B$ and $\theta$ is the canonical character of $b$, then $(R, \theta)$ is a maximal Brauer pair containing $(1, B)$ and $(N(\theta): C_{G}(R)R)_{r} = 1$, where $b$ is regarded as a block of $C_{G}(R)$. The proof of the remark is similar to that of (3J)(a). Indeed in the notation of (3J)(a) $N(\theta_{1} \otimes I_{r}))/N_{m,\alpha, \gamma}^{0} \simeq N(\theta_{1})/C_{1}$ and

$$N(\theta)/C_{G}(R)R \simeq (N(\theta_{1} \otimes I_{r}))/C_{m,\alpha, \gamma}R_{m,\alpha, \gamma} \otimes \text{GL}(c_{1}, r) \times \cdots \times \text{GL}(c_{1}, r).$$

Thus $(N(\theta_{1} \otimes I_{r})): C_{m,\alpha, \gamma}R_{m,\alpha, \gamma})_{r} = 1$ and $(N(\theta_{1} \otimes I_{r}))/N_{m,\alpha, \gamma}^{0} \simeq N(\theta_{1})/C_{1}$ since $(N(\theta): C_{G}(R)R)_{r} = 1$. So $(N(\theta_{1}): C_{1})_{r} = 1$ and the block of $C_{1}$ containing $\theta_{1}$ has defect group $R_{1}$. By (3H) $G_{1} = G_{\Gamma}, R_{1} = R_{\Gamma}$, $\theta_{1} = \theta_{\Gamma}$, and $(R, \theta)$ has type $\Gamma$.

Following the remark above we can get a corollary.

(3K). Let $V$ be a symplectic or even dimensional orthogonal space, $G = I(V), G_{0} = I_{0}(V), G_{0} = I_{0}(V), b$ be blocks of $G$ with defect $D$ and $D'$ respectively such that $[V, D] = V = [V, D']$. Let $b$ and $b'$ be root blocks of $B$ and $B'$ respectively, $b^{G_{0}} \subseteq \mathbb{S}_{\epsilon}(G_{0}, (s)), b^{G_{0}} \subseteq \mathbb{S}_{\epsilon}(G_{0}, (s'))$, where $s$ and $s'$ are semisimple $r'$-elements of $G_{0}$. Then $B = B'$ if and only if $s$ and $s'$ are conjugate in $I(V^*)$, where $V^*$ is the underlying space of $G_{0}$.

Proof. Since $D$ is radical in $G$, a primary element of $D$ exists and then $G$ has an $r$-subgroup of form $R_{m, 0, 0}$ for some $m \geq 1$. By [12], $[1.12]$ and $(5.2)$, $V$ has dimension $2em$ and type $\epsilon^{m}$ if $V$ is orthogonal.

Suppose $s$ and $s'$ are conjugate in $I(V^*)$, so that $s^*$ and $s'^*$ are conjugate in $G$ by definition. By (3F) $D$ and $D'$ are conjugate with Sylow $r$-subgroups of $C_{G}(s^*)$ and $C_{G}(s'^*)$, respectively, so that they are conjugate in $G$. We may suppose $D = D'$.
By (2D) \( V \) and \( D \) have a corresponding decomposition,

\[
V = V_1 \perp V_2 \perp \cdots \perp V_t, \quad D = D_1 \times D_2 \times \cdots \times D_t,
\]

where \( D_i \) is a basic subgroup of \( I(V_i) \). Let \( \theta \) and \( \theta' \) be a corresponding decomposition,

\[
V = V_1 \perp V_2 \perp \cdots \perp V_t, \quad D = D_1 \times D_2 \times \cdots \times D_t,
\]

where \( D_i \) is a basic subgroup of \( I(V_i) \). Let \( \theta \) and \( \theta' \) be a corresponding decomposition,

\[
V = V_1 \perp V_2 \perp \cdots \perp V_t, \quad D = D_1 \times D_2 \times \cdots \times D_t,
\]

where \( D_i \) is a basic subgroup of \( I(V_i) \). Let \( \theta \) and \( \theta' \) be a corresponding decomposition,
or 2. If \((N(\theta) : N_0(\theta)) = 2\), then \(\theta^x = \theta\) for some \(x \in G\) of determinant \(-1\). So \((bG_0)^x = bG_0\) and thus \((bG_0)^g = bG_0\) for all \(g \in G\). This is impossible. Thus \(N(\theta) = N_0(\theta)\) and then \(m_{\chi \pm 1}(s) = 0\) by [12, (7B) and (7C)]. It follows that \(C_{G_0}(s) = C_{G_0}^*(s)\), so there exists \(x \in I(V^*)\) of determinant \(-1\) such that \(s^x\) and \(s\) are not conjugate in \(G_0^*\). Let \(D_x\) be a Sylow \(r\)-subgroup of \(C_{G_0}(s^x)\), and \(y^* \in Z(D_x)\) primary. Thus \(D_x\) and \(D\) are conjugate in \(G\), and \(s^x \in C_G(y^*) \approx GL(m, eq^e)\) for some \(m \geq 1\). Let \(y\) be a primary element of a Sylow \(r\)-subgroup of \(Z(C_G(y^*))\). Then \(C_{G_0}^*(y) = C_{G_0}^*(y)^*\) and \((y)\) is conjugate in \(I(V^*)\) with the subgroup generated by a dual of \(y^*\), so \(y^k\) is a dual of \(y^*\) for some integer \(k \geq 1\) and \(\langle y \rangle = \langle y^k \rangle\) by \(|y^k| = ra\). By the remark of \((3E)\) we may suppose \(s^x\) lies in the \(r\)-section containing \(y^k\) and \(s^x \in C_G(y^*)^*\). There exists a block \(b_x\) of \(C_{G_0}(y^*)\) labeled by \((s^x, -)\), so that \((\langle y^* \rangle, b_x)\) is a Brauer pair of \(G_0\) labeled by \((\langle y^* \rangle, s^x, -)\) and \(b_{G_0}^{(y^*)} \subseteq \mathcal{S}(G_0, (s^x))\) by \((3C)\). Since \(s\) and \(s^x\) are conjugate in \(G_0^*\), it follows that \(bG = B = bG_0\) by the first half of the proof and so \(B\) covers \(b_{G_0}^{(y^*)}\) since \(s\) and \(s^x\) are not conjugate in \(G_0^*\). This completes the proof.

4. Weights for classical groups

In this section we count the number of \(B\)-weights for a block \(B\) of finite classical groups. Given \(\Gamma \in \mathcal{F}'\) and integer \(d \geq 0\), let \(V_{r,d}\) be a unitary space of dimension \(r^d e_r d_r\) over \(F_q^d\), or a symplectic or orthogonal space over \(F_q\) given by \((3.12)\). Denote \(G = G_0 = U(V_{r,d})\) in the case \(V_{r,d}\) is unitary, and \(G = I(V_{r,d})\), \(G_0 = I_0(V_{r,d})\) in the remaining cases. Let \(0 < y < d\), and \(c = (c_1, c_2, \ldots, c_l)\) a sequence of nonnegative integers such that \(d - y = c_1 + c_2 + \cdots + c_l\). In addition, let

\[
R = R_{m, c, r, y} = I_{c_1} \cdot \cdots \cdot I_{c_l},
\]

be a basic subgroup of \(G\), \(C = C_G(R)\), and \(N = N_G(R)\). Then \(C = C_{r} \otimes I_{y} \otimes I_{e}\), where \(I_{c_i}\) and \(I_{e}\) are identity matrices of orders \(r^y\) and \(r^{c_1 + c_2 + \cdots + c_l}\) respectively. Define \(\theta\) on \(C\) by \(\theta(c \otimes I_{y} \otimes I_{e}) = \theta(c)\) for \(c \in C_{r}\). Then \(\theta\) is an irreducible character of \(C\) and \((R, \theta)\) is of type \(\Gamma\). Regard \(\theta\) as a character of \(CR\) trivial on \(R\). Then the block \(b\) of \(CR\) containing \(\theta\) has defect group \(R\) and the Brauer pair \((R, b)\) of \(G_0\) has label \((R, x_{r, -})\), where \(b\) is regarded as a block of \(C\), and \(x_{r} = r^d e_r\) in the case \(G\) is unitary and \(x_{r}\) is given by \((3.14)\) in the remaining cases. Let \(V_{m, r, \Gamma, y}\) be the underlying space of \(R_{m, r, \Gamma, y}\), \(G_{m, r, \Gamma, y} = U(V_{m, r, \Gamma, y})\) in the case \(V_{m, r, \Gamma, y}\) is unitary, or \(I(V_{m, r, \Gamma, y})\) in the remaining case. If \(\theta_{r} \otimes I_{\Gamma}\) is the character of \(G_{m, r, \Gamma, y}(R_{m, r, \Gamma, y}) = C_{r} \otimes I_{\Gamma}\) defined by \((\theta_{r} \otimes I_{\Gamma})(c \otimes I_{\Gamma}) = \theta_{r}(c)\) for \(c \in C_{r}\) and \(N(\theta_{r} \otimes I_{\Gamma})\) is its stabilizer in \(G_{m, r, \Gamma, y}(R_{m, r, \Gamma, y})\), then by \((2.2)\) or \((2.5)\)

\[
N(\theta) = \frac{(N(\theta_{r} \otimes I_{\Gamma})/R_{m, r, \Gamma, y}) \otimes N_{S_{(\psi)}(A_{c})}}{N(\theta_{r} \otimes I_{\Gamma})/R_{m, r, \Gamma, y} \times GL(c_1, r) \times \cdots \times GL(c_l, r)}.
\]

Thus the characters \(\psi\) in \(\text{Irr}^0(N(\theta), \theta)\) are parametrized by \((l + 1)\)-tuples \((\psi_0, \psi_1, \ldots, \psi_l)\), where \(\psi_0 \in \text{Irr}^0(N(\theta_{r} \otimes I_{\Gamma})), \theta_{r} \otimes I_{\Gamma})\) and \(\psi_i\) is an irreducible character of \(GL(c_i, r)\) of defect 0 for \(i \geq 1\). Necessarily, \(\psi_i\) are one of the \(r - 1\) Steinberg characters of \(GL(c_i, r)\) for \(i \geq 1\). By \((3A)\) or \((3I)\) there are
\( \beta_{r\Gamma r} \) such characters \( \psi_0 \), so that there are \( \beta_{r\Gamma r}(r-1)^l \) such characters \( \psi \), where \( \beta_{r\Gamma} = 1 \) or 2 according as \( \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2 \) or \( \Gamma \in \mathcal{F}_0 \). Thus there are \( \beta_{r\Gamma r}(r-1)^l \) \( b^G \)-weights of the form \((R, I(\psi))\).

(4A). Let \( V = V_{r, d} \), \( B \) a block of \( G \) with defect group \( D \) and root block \( b \) such that \([V, D] = V\) and \( b^G \subseteq \mathcal{F}_r(G_0, (x_\Gamma))\). Then there are exactly \( \beta_{r\Gamma r}d^d \) \( B \)-weights \((R, \varphi)\), where \( R \) runs over the basic subgroups of \( G \) with degree \( \beta_{r\Gamma r}d^d \).

**Proof.** (1) Suppose \( R = R_{r, \alpha, \gamma, c} \) is a basic subgroup of \( G \), \((R, \varphi)\) is a \( B \)-weight, and \( \varphi \) covers the irreducible character \( \theta \) of \( G(G(R))R \). Then the block \( b \) of \( G(G(R))R \) containing \( \theta \) has a defect group \( R \) and \( b^G = B \). By (3B) or (3J)(a) \((R, \theta)\) has type \( \Delta \) for some \( \Delta \in \mathcal{F}' \) and \((R, b)\) has label \((R, x_\Delta, -)\), where \( b \) is regarded as a block of \( G(G(R)) \). If \( V \) is unitary, then \( \Delta = \Gamma \) by [7, (3.2)]. Suppose \( V \) is a symplectic or orthogonal space. Let \((D', b')\) be a maximal pair containing \((R, b)\), so that \( b'^G = B \). As a block of \( G(G(D')D') \), \( b' \) is also a root block of \( B \) and \( b'^G = b^G \subseteq \mathcal{F}_r(G_0, (x_\Delta)) \) by (3C). Since \( D' \) is a defect group of \( B \), \( D' \) and \( D \) are conjugate in \( G \) and so \([V, D'] = V \). By (3K) \( x_\Delta \) and \( x_\Gamma \) are conjugate in \( I(V^*) \), where \( V^* \) is the underlying space of \( G^*_F \). Thus \( \Delta = \Gamma \) and \( m = m_r, \alpha = \alpha_r, \gamma + c_1 + c_2 + \cdots + c_1 = d \).

The number of different sequences \( c = (c_1, c_2, \ldots, c_i) \) such that
\[
d'(R_{r, \alpha, \gamma, c}) = \beta_{r\Gamma r}d^d \quad \text{and} \quad l(R_{r, \alpha, \gamma, c}) = l
\]
is \((d-\gamma-1)_{l-1}\). Here \( 1 \leq l \leq d - \gamma \) when \( d - \gamma \geq 1 \); \( l = 0 \) when \( d = \gamma \), and \((\gamma)\) is interpreted as \(1\). There are \( \beta_{r\Gamma r}(r-1)^l \) characters \( \varphi \) associated with \( R_{r, \alpha, \gamma, c} \), so that there are
\[
\beta_{r\Gamma r} \sum_{\gamma=0}^d \sum_{l=0}^{d-\gamma} \binom{d - \gamma - 1}{l - 1} (r-1)^l = \beta_{r\Gamma r}d^d,
\]
characters associated with \( R_{r, \alpha, \gamma, c} \)’s.

(2) Suppose \( V \) is a symplectic or orthogonal space. By (3J)(b) the pair \((R, \theta)\) of type \( \Gamma \) is determined uniquely up to conjugacy in \( N_G(R) \), so that there are \( \beta_{r\Gamma r}d^d \) \( B \)-weights \((R, \varphi)\). Suppose \( V \) is a unitary space and \((R, b')\) is another Brauer pair of \( G \) such that \( b'^G = B \), and \( \theta' \) is the canonical character of \( b' \), where \( R = R_{r, \alpha, \gamma, c} \). Then \((R, \theta')\) has type \( \Gamma \), \( C = C_{G(R)} = C_{\Gamma} \otimes I_\gamma \otimes I_c \), and \( \theta' \) has the form \( \theta_{\Gamma} \otimes I_\gamma \otimes I_c \), where \( \theta_{\Gamma} \) is an irreducible character of \( C_{\Gamma} \). If \( b_{\Gamma} \) is the block of \( C_{\Gamma} \) containing \( \theta_{\Gamma} \), then \( b_{\Gamma}^{Gr} = B_{\Gamma} \) and both \( B_{\Gamma} \) and \( b_{\Gamma} \) have a defect group \( R_{\Gamma} \). By definition \( b_{\Gamma}^{Gr} = B_{\Gamma} \) and \( b_{\Gamma} \) has a defect group \( R_{\Gamma} \). Thus \( b_{\Gamma}^{w} = b_{\Gamma} \) for some \( w \in N_{\Gamma} \) by Brauer First Main Theorem. A similar proof to that of (3J)(b) shows that \( \theta' = \theta_n \) for some \( n \in N_G(R) \). Thus (4A) follows in this case.

**Remark.** In the notation of (4A), suppose \( V \) is orthogonal, \( G = I(V) \), and \( G_0 = I_0(V) \). If \((R, \theta)\) has type \( \Gamma \), then \( |N(\theta) : N_0(\theta)| = \beta_{r\Gamma} \) and for each \( \psi \in \text{Irr}^0(N(\theta), \theta) \), the restriction \( \psi|_{N_0(\theta)} \) of \( \psi \) to \( N_0(\theta) \) is irreducible, where \( N_0(\theta) = N(\theta) \cap G_0 \). Indeed in the notation above \( \psi = (\psi_0, \psi_1, \ldots, \psi_i) \) as a character of \( N(\theta)/R \), where \( \psi_0 \in \text{Irr}^0(N(\theta_\Gamma \otimes I_\gamma), \theta_\Gamma \otimes I_\gamma) \), and \( \psi_i \) is an irreducible character of \( \text{GL}(c_i, r) \) of defect 0 for \( i \geq 1 \). Let \( N_0(\theta_\Gamma \otimes I_\gamma) \) be the subgroup of \( N(\theta_\Gamma \otimes I_\gamma) \) of determinant 1. Then \( |N(\theta_\Gamma \otimes I_\gamma) : N_0(\theta_\Gamma \otimes I_\gamma)| = \beta_{r\Gamma} \)
and the restriction of $\psi_0$ to $N_0(\theta_{\Gamma} \otimes I_{\Gamma})$ is irreducible by the remark of (31). Thus by (4.1) $|N(\theta): N_0(\theta)| = \beta_{\Gamma}$. Now the restriction of $\psi$ to

$$H = (N_0(\theta_{\Gamma} \otimes I_{\Gamma})/R_{m_{\Gamma}}) \times \text{GL}(c_1, r) \times \text{GL}(c_2, r) \times \cdots \times \text{GL}(c_t, r)$$

is irreducible. Since $N_0(\theta)/R \geq H$, $\psi|_{N_0(\theta)/R}$ is irreducible, and so $\psi|_{N_0(\theta)}$ is irreducible.

Given $\Gamma \in \mathcal{F}'$ and integer $w_{\Gamma} \geq 1$, let $G = U(V)$ or $I(V)$ and $G_0 = G$ or $I_0(V)$, where in the former case $V$ is a unitary space of dimension $w_{\Gamma} r_{\Gamma} d_{\Gamma}$ over $\mathbb{F}_q$, in the latter case $V$ is a symplectic or orthogonal space over $\mathbb{F}_d$ such that $\dim V = w_{\Gamma} r_{\Gamma} d_{\Gamma}$ and if $V$ is orthogonal, then $\eta(V) = e^{w_{\Gamma} r_{\Gamma} d_{\Gamma}}$ or $e_{w_{\Gamma} r_{\Gamma} d_{\Gamma}}$ according as $\Gamma \in \mathcal{F}_0$ or $\Gamma \in \mathcal{F} \cup \mathcal{F}_2$. Thus if $V$ is unitary, then $s = w_{\Gamma} r_{\Gamma} \Gamma$ is a semisimple element of $G$ and $C_G(s) \cong \text{GL}(w_{\Gamma} r_{\Gamma} d_{\Gamma}, q)$, so that $G$ has a block $B$ labeled by $(s, -)$ and a defect group $D$ of $B$ acts fixed-point freely on $V$ since we may suppose $D$ is a Sylow $r$-subgroup of $C_G(s)$. In the remaining case, a semisimple element $s^*$ in $G_0$ exists such that $m_{\Gamma}(s^*) = w_{\Gamma} r_{\Gamma} \tau_{\Gamma}$ and $\eta_{\Gamma}(s^*) = \eta(V)$, so that a primary element of a Sylow $r$-subgroup of $C_G(s^*)$ exists and by the remark of (3E) a dual $s$ of $s^*$ exists in $G_0$ which is uniquely determined in $I(V^*)$ up to conjugacy, where $V^*$ is the underlying space of $G_0$. Moreover, by (3K) $s$ uniquely determines a block $B$ of $G$ which covers a block in $\mathcal{F}(G_0, (s))$ and whose defect group acts fixed-point freely on $V$.

For each $\Gamma \in \mathcal{F}'$ and integer $d \geq 0$, let $\mathcal{E}_{\Gamma, d} = \{\varphi_{\Gamma, d, i, j} : 1 \leq i \leq \beta_{w_{\Gamma} r_{\Gamma}} d_{\Gamma}, 1 \leq j \leq r_{\Gamma}^d\}$ be the set of characters associated with basic subgroups of $G = U(V_{\Gamma, d})$ or $I(V_{\Gamma, d})$ in (4A).

(4B). With the preceding notation, let $B$ be a block of $G$ with defect group $D$ and root block $b$ such that $[V, D] = V$ and $b^{G_0} \subseteq \mathcal{E}(G_0, (s))$. Then the number of $B$-weights is the number $f_{\Gamma}$ of assignments

$$\prod_{d \geq 0} \mathcal{E}_{\Gamma, d} \rightarrow \{ \text{r-cores} \}, \quad \varphi_{\Gamma, d, i, j} \mapsto \kappa_{\Gamma, d, i, j},$$

such that

$$\sum_{d \geq 0} \beta_{w_{\Gamma} r_{\Gamma}} d_{\Gamma} \sum_{i=1}^{r_{\Gamma}^d} \sum_{j=1}^{r_{\Gamma}^d} |\kappa_{\Gamma, d, i, j}| = w_{\Gamma}.$$

Proof. Let $(R, \varphi)$ be a $B$-weight of $G$, $C = C_G(R)$, and $N = N_G(R)$. Then there exists a block $b$ of $CR$ with defect group $R$ such that $b^{G_0} = B$ and $\varphi \in b^N$. We may suppose $Z(D) \leq Z(R) \leq R \leq D$. Let $z$ be a primary element of $D$ defined by the remark of (2D). Then $z \in Z(D)$ and $[V, z] = V$, so that $C_V(R) = 0$. Thus in the decomposition (2B) or (2D) of $R$, we may suppose

$$R = R_1^{d_1} \times R_2^{d_2} \times \cdots \times R_u^{d_u},$$

where $R_i$'s are distinct nontrivial basic subgroups and $R_i$ appears $d_i$ times as a component of $R$. Let $V_i$ be the underlying space of $R_i$, $G_i = U(V_i)$ or $I(V_i)$ according as $V_i$ is or is not a unitary space, $C_i = C_G(R_i)$, and $N_i = N_G(R_i)$. Then $C = C_1^{d_1} \times C_2^{d_2} \times \cdots \times C_u^{d_u}$. Let $\theta$ be the canonical character of $b$, so that we may suppose $\theta = \prod_{i=1}^u \theta_i^{d_i}$, where $\theta_i$ is an irreducible character of $C_iR_i$ trivial on $R_i$. Let $z_i$ be the restriction of $z$ on $V_i$ and $K_i = C_{G_i}(z_i)$ for all $i$. Then $K_j$ and $\prod_{i=1}^u K_i^{d_i}$ are a regular subgroup of $I_0(V_j)$ and $G_0$,
so that $\prod_{i=1}^{u}(K_i^{*})^{d_i}$ is embedded as a regular subgroup of $G_0^*$. If $(R_i, s_i, \tau_i)$ is a label of the Brauer pair $(R_i, \theta_i)$, then $s_i \in K_i^{*}$, $\prod_{i=1}^{u} s_i^{d_i} \in \prod_{i=1}^{u}(K_i^{*})^{d_i}$, and so $(R, \prod_{i=1}^{u} s_i^{d_i}, \tau_i)$ is a label of the Brauer pair $(R, b^\alpha)$. Thus $s_i$ and $\chi_i$ are conjugate in $I_0(V_i)^*$, $(R_i, \theta_i)$ has type $\Gamma$, and $R_i = R_{\alpha \Gamma_, \alpha_i, \gamma_i, \tau_i}$ for some $\gamma_i$ and $\tau_i$. It is clear that

$$N(\theta) = \prod_{i=1}^{u}N(\theta_i) \cdot S(d_i),$$

where $N(\theta_i)$ is the stabilizer of $\theta_i$ in $N_i$. In particular, if $\psi \in \text{Irr}^0(N(\theta), \theta)$, then $\psi = \prod_{i=1}^{u} \psi_i$, where $\psi_i$ is an irreducible character of $N(\theta_i)S(d_i)$ covering $\theta_i^{d_i}$. Moreover, $\psi_i$ has defect 0 as a character of

$$N(\theta_i) \cdot S(d_i)/R_i^{d_i} \cong (N(\theta_i)/R_i) \cdot S(d_i).$$

Let $\text{Irr}^0(N(\theta_i), \theta_i) = \{\phi_{i,j}: 1 \leq j \leq \beta_{\Gamma}r_{\Gamma}(r-1)^{(R_i)}\}$. As shown in the proof of [3, (2C)], the irreducible characters of defect 0 of $(N(\theta_i)/R_i) \cdot S(d_i)$ covering $\theta_i^{d_i}$ are in bijection with assignments $\phi_{i,j} \mapsto \kappa_{i,j}$ of characters to $r$-cores such that $\sum_{j \geq 1} |\kappa_{i,j}| = d_i$. Thus the irreducible characters of $\text{Irr}^0(N(\theta), \theta)$ are in bijection with assignments $\phi_{i,j} \mapsto \kappa_{i,j}$ of characters to $r$-cores such that

$$\sum_{i=1}^{u} (\deg R_i) \sum_{j \geq 1} |\kappa_{i,j}| = \beta_{\Gamma}r_{\Gamma}d_{\Gamma}w_{\Gamma}.$$  

For fixed $d \geq 0$, the number of irreducible characters associated with basic groups of degree $\beta_{\Gamma}r_{\Gamma}d_{\Gamma}r^d$ is $\beta_{\Gamma}r_{\Gamma}r^d$. Let $\mathcal{G}_{\Gamma,d} = \{\phi_{\Gamma,d,i,j}: 1 \leq i \leq \beta_{\Gamma}r_{\Gamma}, 1 \leq j \leq r^d\}$ be the set of these characters. Then the number of $B$-weights is the number of assignments

$$\bigcup_{d \geq 0} \mathcal{G}_{\Gamma,d} \rightarrow \{r\text{-cores}\}, \quad \phi_{\Gamma,d,i,j} \mapsto \kappa_{\Gamma,d,i,j},$$

such that

$$\sum_{d \geq 0} \beta_{\Gamma}r_{\Gamma}d_{\Gamma}r^d \sum_{i=1}^{\beta_{\Gamma}r_{\Gamma}} r^d \sum_{j=1}^{r_{\Gamma}} |\kappa_{\Gamma,d,i,j}| = \beta_{\Gamma}r_{\Gamma}d_{\Gamma}w_{\Gamma}.$$  

This induces the required condition of (4B).

(4C). With the preceding notation, let $G = O(V)$ be an orthogonal group, $G_0 = SO(V)$, and $R$ a radical subgroup of $G$ such that $[V, R] = V$. Let $(R, b)$ a Brauer pair of $G_0$ labeled by $(R, s, \tau)$ and $\theta$ the canonical character of $b$. Then $|N(\theta): N_0(\theta)| = \beta_{\Gamma}$ and the restriction $\psi|_{N_0(\theta)}$ of each $\psi \in \text{Irr}^0(N(\theta), \theta)$ to $N_0(\theta)$ is irreducible, where $N_0(\theta) = N(\theta) \cap G_0$.

Proof. In the notation above $R = R_{d_1} \times R_{d_2} \times \cdots \times R_{d_u}$, $V_i$ is the underlying space of $R_i$, $C = C_G(R) = \prod_{i=1}^{u} C_i^{d_i}$, and $\theta = \prod_{i=1}^{u} \theta_i^{d_i}$, where $\theta_i$ is an irreducible character of $C_i = C_{O(V_i)}(R_i)$ for $i \geq 1$. Each $(R_i, \theta_i)$ has type $\Gamma$. Let $N(\theta_i)$ and $N_0(\theta_i)$ be the stabilizers of $\theta_i$ in $N_{O(V_i)}(R_i)$ and $N_{SO(V_i)}(R_i)$ respectively. By the remark of (4A), $|N(\theta_i): N_0(\theta_i)| = \beta_{\Gamma}$ and so $|N(\theta): N_0(\theta)| = \beta_{\Gamma}$ since $N(\theta) = \prod_{i=1}^{u} N(\theta_i) \cdot S(d_i)$. If $\psi \in \text{Irr}^0(N(\theta), \theta)$, then $\psi = \prod_{i=1}^{u} \psi_i$, where $\psi_i$ is an irreducible character of $N(\theta_i) \cdot S(d_i)$ covering $\theta_i^{d_i}$. Moreover,
\( \psi_i \) has defect 0 as a character of \( N(\theta_i) \triangleleft S(d_i)/R_i^d \simeq (N(\theta_i)/R_i) \triangleleft S(d_i) \). Let \( N_0(\theta_i^d) \) be the subgroup of \( N(\theta_i) \triangleleft S(d_i) \) of determinant 1. It then suffices to show that the restriction of \( \psi_i \) to \( N_0(\theta_i^d) \) is irreducible. Thus we may suppose \( u = 1 \) and \( d = d_1 \), so that \( \theta = \theta_i^d \) and \( N(\theta) = N(\theta_i) \triangleleft S(d) \). Since \( |N(\theta) : N_0(\theta)| \leq 2 \), \( \psi_i|_{N_0(\theta)} \) is irreducible if and only if \( N(\theta) \) stabilizes an irreducible constituent of \( \psi_i|_{N_0(\theta)} \).

Let \( T = N(\theta_1), H = N(\theta) = T \triangleleft S(d) \), \( X = T^d \) the base subgroup of \( H \), \( H_0 = N_0(\theta) \), and \( X_0 \) the subgroup of \( X \) of determinant 1. Then \( H = X \rtimes S(d) \) and \( H_0 = X_0 \rtimes S(d) \). We may suppose \(|H : H_0| = 2 \) and hence \(|T : T_0| = 2 \), where \( T_0 = T \cap I_0(V_1) \). Moreover, \((R_1, \theta_1)\) has type \( \Gamma \) and the restriction of each character in \( \text{Irr}^0(T, \theta_1) \) to \( T_0 \) is irreducible by the remark of \((4A)\). As shown in the proof of \([3, (2B)]\) (cf. also \([17, 5.20]\)), the irreducible characters of \( H \) can be obtained as follows: Let \( \text{Irr} \ T = \{\xi^1, \xi^2, \ldots, \xi^l\} \) be the complete set of irreducible characters of \( T \), and \( \xi \) an irreducible character of \( X \). Then \( m = (m_1, m_2, \ldots, m_l) \) is called the type of \( \xi \) if \( m_i \) is the multiplicity of \( \xi^i \) as a factor of \( \xi \). The stabilizer of \( \xi \) in \( H \) is \( XS_m \), and \( \xi \) can be extended to an irreducible character \( \tilde{\xi} \) of \( XS_m \) (see \([17, 5.13]\)), where \( S_m \) is the Young subgroup of \( S(d) \) of type \( m \). By Clifford theory, all irreducible characters of \( XS_m \) covering \( \xi \) have form \( \tilde{\xi} \zeta \) and \( \text{Ind}_{X}^{H}(\tilde{\xi}\zeta) \) is irreducible, where \( \zeta \) is an irreducible character of \( XS_m \) trivial on \( X \). Moreover, these characters \( \{\text{Ind}_{X}^{H}(\tilde{\xi}\zeta)\} \) consist of a complete set of irreducible characters of \( H \) as \( \xi \) runs over the representatives of conjugacy \( H \)-classes of \( \text{Irr} X \), and, while \( \xi \) is fixed, \( \xi \) runs over irreducible characters of \( S_m \), where \( m \) is the type of \( \xi \) (see \([17, 5.20]\)). In particular, \( \text{Ind}_{X}^{H}(\tilde{\xi}\zeta) \) has defect 0 as a character of \( H/R \) if and only if \( \xi \) has defect 0, and \( \xi \) has defect 0 as a character of \( X/R \). If \( \text{Ind}_{X}^{H}(\tilde{\xi}\zeta) \in \text{Irr}^0(H, \theta) \), then we may suppose \( \xi \) covers \( \theta \).

Suppose \( \xi \in \text{Irr}^0(X, \theta) \). Then the restriction \( \xi|_{X_0} = \xi|_{X_0} \) is irreducible since \( \xi|_{X_0} \) is irreducible by the remark of \((4A)\). Let \( K \) be the stabilizer of \( \xi \) in \( H_0 \). Then \( X_0S_m \leq K \), where \( m \) is the type of \( \xi \). We claim \( X_0S_m = K \). Indeed if there exists \( x \in K \setminus X_0S_m \), then we may suppose \( x \in S(d) \setminus S_m \), \( \xi^x \neq \xi \), and \( \xi^x|_{X_0} = \xi_0 \), since \( H_0 = X_0S(d) \) and the stabilizer \( \xi \) is \( XS_m \). In particular, \( d > 1 \). Thus \( \xi_i \neq \xi_i^x \) for some \( i \), \( j \) components \( \xi_i \) and \( \xi_i^x \) of \( \xi \) and \( \xi^x \) respectively and so \( \xi_i(h) \neq \xi_i^x(h) \) for some \( h \in T \). Since \( \xi|_{X_0} = \xi^x|_{X_0}, h \) has determinant \(-1 \). Let \( w = \text{diag}(w_1, w_2, \ldots, w_d) \in X \) such that \( w_i = h = w_j \), for some \( j = i \), and \( w_k = 1 \) for \( k \neq i, j \). Then \( w \in X_0 \) and \( \zeta(w) = \xi(w) = \xi^x(w) \). But the \( i \)th components of \( \xi(w) \) and \( \xi^x(w) \) are \( \xi_i(h) \) and \( \xi_i^x(h) \) respectively. This is impossible and the claim holds.

Since \( \tilde{\xi} \) is an extension of \( \xi \) to \( XS_m \), it follows \( \tilde{\xi}|_{X_0} = \xi_0 \) and hence \( \tilde{\xi}|_{X_0S_m} = \xi_0 \) is an extension of \( \xi_0 \) to \( X_0S_m \). By Clifford theory again, each irreducible character of \( X_0S_m \) covering \( \xi_0 \) has the form \( \tilde{\xi}_0\chi \), where \( \chi \) is an irreducible character of \( X_0S_m \) trivial on \( X_0 \), and each irreducible character of \( H_0 \) covering \( \xi_0 \) has the form \( \text{Ind}_{X_0S_m}(\tilde{\xi}_0\chi) \). Now for \( \psi \in \text{Irr}^0(H, \theta), \psi = \text{Ind}_{X}^{H}(\tilde{\xi}\zeta) \) for some irreducible character \( \xi \) of \( X \) with defect 0 as a character of \( X/R \), and \( \psi|_{X_0} = \xi_0 \) is irreducible. Thus there is an irreducible constituent \( \psi_0 \) of \( \psi|_{H_0} \) covering \( \xi_0 \) and so \( \psi_0 = \text{Ind}_{X_0S_m}(\tilde{\xi}_0\chi) \). We claim that \( \psi_0 = \psi_0 \) for any \( \tau \in X \). Indeed this is true for \( \tau \in X_0 \) and we may suppose \( \tau \) has
determinant $-1$. Since $|X_{Sm}| : X_{0Sm}| \leq 2$, $\tau$ normalizes $X_{0Sm}$ and for $x$, $h \in H_0$, we have $h^{-1}x \in X_{0Sm}$ if and only if $h^x \in X_{0Sm}$ since $h^{-1}x = h^{x(x^{-1}r^{-1}x)}$ and $x^{-1}r^{-1}x \in X$. If $h^{-1}x \in X_{0Sm}$, then $(\hat{\xi}_0 \chi)(h^{-1}x) = (\hat{\xi}_0 \chi)^\tau(h^x)$, where $\tau = x^{-1}rxx \in X$. Since $\hat{\xi}_0 |X_{0Sm}| = \hat{\xi}_0$ is irreducible and $\chi$ is trivial on $X_0$, $\hat{\xi}_0^g = \hat{\xi}_0$ and $\chi^g = \chi$ for any $g \in X$. Therefore $(\hat{\xi}_0 \chi)^\tau(h^{x}) = (\hat{\xi}_0 \chi)(h^x)$ and so $(\hat{\xi}_0 \chi)(h^{-1}x) = (\hat{\xi}_0 \chi)(h^x)$, for any $h$, $x \in H_0$. Thus $\psi_0 = \psi_0$ and so $\psi|_{H_0} = \psi_0$ is irreducible. This proves (4C).

We now prove the main theorem of unitary groups.

(4D). Let $V$ be a unitary space over $\mathbb{F}_q$, $G = U(V)$, $B$ be a block of $G$ with label $(s, \kappa)$, $\prod_\Gamma s(\Gamma)$ the primary decomposition of $s$, $\sum_\Gamma V(\Gamma)$ the corresponding orthogonal decomposition of $V$, and $w_\Gamma$ the integer such that $\dim V(\Gamma) = d_\Gamma |\kappa_\Gamma| = d_\Gamma \kappa r w_\Gamma$. Then the following hold:

1. The number of $B$-weights of $G$ is $\prod_\Gamma f_\Gamma$, where $f_\Gamma$ is given by (4B). In particular, $f_\Gamma$ is the number of $e_\Gamma$-tuples $(\kappa_1, \kappa_2, \ldots, \kappa_{e_\Gamma})$ of partitions $\kappa_i$ such that $\sum_{i=1}^{e_\Gamma} |\kappa_i| = \omega_\Gamma$.

2. The number of $B$-weights of $G$ is the number $l(B)$ of irreducible modular characters in $B$.

Proof. Let $R$ be a radical subgroup of $G$ and $V = V_0 \perp V_+$, where $V_0 = C_R(R)$ and $V_+ = [V, R]$. Then $R = R_0 \times R_+$, where $R_0 = \langle 1_{V_0} \rangle$ and $R_+ \leq U(V_+)$. Let $C = C_{G}(R)$, $N = N_{G}(R)$, so that $C = C_0 \times C_+$, $N = N_0 \times N_+$, where $C_0 = N_0 = U(V_0)$, $C_+ = C_{U(V_+)}(R_+)$ and $N_+ = N_{U(V_+)}(R_+)$. Suppose $b$ is a block of $CR$ with defect group $R$ and $b^G = B$. Then $b = b_0 \times b_+$, where $b_0$ is a block of $C_0R_0 = U(V_0)$ of defect 0, and $b_+$ is a block of $C_+R_+$ with defect group $R_+$. The canonical character $\theta$ of $b$ decomposes as $\theta_0 \times \theta_+$, where $\theta_0$ and $\theta_+$ are the canonical characters of $b_0$ and $b_+$ respectively. Thus $N(\theta) = N_0 \times N_+(\theta_+)$, where $N_+(\theta_+)$ is the stabilizer of $\theta_+$ in $N_+$.

Suppose $(R, I(\psi))$ is a $B$-weight of $G$, for some $\psi \in \text{Irr}^0(N(\theta), \theta)$. Clearly $\psi = \psi_0 \times \psi_+$ for character $\psi_0$ of $N_0$ and $\psi_+ \in \text{Irr}^0(N(\theta_+), \theta_+)$. Since $\psi_0$ is a character of $N_0 = C_0$ covering $\theta_0$, it follows that $\psi_0 = \theta_0$. The correspondence $(R, I(\psi)) \mapsto (R_+, I_+(\psi_+))$, where $\psi = \theta_0 \times \psi_+$ and $R_+ = U(V_+)$ is a bijection from $\{(R, I(\psi)) : \psi \in \text{Irr}^0(N(\theta), \theta)\}$ to $\{(R_+, I_+(\psi_+)) : \psi_+ \in \text{Irr}^0(N(\theta_+), \theta_+)\}$.

By a theorem of Broué-Puig, [7, 3.2], we may suppose $s = s_0 \times s_+$ such that $s_0 \in C_0$, $s_+ \in C_+$, $(s_0, \kappa)$ is the label of $b_0$, and $(s_+, -)$ is the label of $b^U(V_+)$. In the correspondence above, $(R_+, I_+(\psi_+))$ is a $b^U(V_+)$-weight. So the number of $B$-weights in $G$ is the number of $b^U(V_+)$-weights in $U(V_+)$. Thus we may suppose $V = V_+$.

Let $R = \prod_{i=1}^t R_i$ and $V = \bigoplus_{i=1}^t V_i$ be the decompositions of (2B), and let $C = \prod_{i=1}^t C_i$ and $\theta = \prod_{i=1}^t \theta_i$, where $C_i = C_{U(V_i)}(R_i)$ and $\theta_i$ is a character of $C_i$. Since the block $b_i$ of $C_iR_i$ containing $\theta_i$ has a defect group $R_i$, $(R_i, \theta_i)$ has type $\Gamma$ for a unique $\Gamma \in \mathcal{F}$ by (3B). Moreover, if $(R_i, t_i, -)$ is the label of $(R_i, b_i)$, then $(R, \prod_{i} t_i, -)$ is the label of Brauer pair $(R, b)$ of $G$, where $b_i$ and $b$ are regarded as blocks of $C_i$ and $C$ respectively. By [7, (3.2)] $(R, s, -)$ is also a label of $(R, b)$, so that $s$ and $\prod_{i} t_i$ are conjugate in $G$. Let $R(\Gamma) = \prod_{i} R_i$, $C(\Gamma) = \prod_{i} C_i$, $\theta(\Gamma) = \prod_{i} \theta_i$, and $t(\Gamma) = \prod_{i} t_i$, where $i$ runs over all $1 \leq i \leq t$ such that $(R_i, \theta_i)$ is of type $\Gamma$. Then $R = \prod_{\Gamma} R(\Gamma)$,
\( \theta = \prod_{\Gamma} \theta(\Gamma), \quad C = \prod_{\Gamma} C(\Gamma), \) and \( \prod_{\Gamma} t(\Gamma) \) is a primary decomposition of \( s \) in \( G \). We may suppose \( s(\Gamma) = t(\Gamma) = t, \) so that \( N(\theta) = \prod_{\Gamma} N(\theta(\Gamma)) \), where \( N(\theta(\Gamma)) \) is the stabilizer of \( \theta(\Gamma) \) in \( U_{U(\Gamma)}(R(\Gamma)) \).

Each \( \psi = \prod_{\Gamma} \psi(\Gamma) \), for \( \psi \in \text{Irr}^0(N(\theta), \theta) \) and \( \psi(\Gamma) \in \text{Irr}^0(N(\theta(\Gamma)), \theta(\Gamma)) \). Let \( B(\Gamma) \) be a block of \( C(\Gamma) \) containing \( \theta(\Gamma) \), and \( B(\Gamma) = b(\Gamma)U(\Gamma) \). Then \( B(\Gamma) \) is labeled by \( (s(\Gamma), -) \) and \( (R(\Gamma), I(\psi(\Gamma))) \) is a \( B(\Gamma) \)-weight. Conversely, if \( B(\Gamma) \) is a block of \( U(\Gamma) \) with label \( (s(\Gamma), -) \) and \( (R(\Gamma), \varphi(\Gamma)) \) is a \( B(\Gamma) \)-weight, then there exists a block \( b(\Gamma) \) of \( C(\Gamma) \) with defect group \( R(\Gamma) \) and the canonical character \( \theta(\Gamma) \) such that \( b(\Gamma)U(\Gamma) = B(\Gamma) \) and \( \varphi(\Gamma) = I(\psi(\Gamma)) \) for some \( \psi(\Gamma) \in \text{Irr}^0(N(\theta(\Gamma)), \theta(\Gamma)) \). Let \( R = \prod_{\Gamma} R(\Gamma) \), \( \theta = \prod_{\Gamma} \theta(\Gamma), \) \( b = \prod_{\Gamma} b(\Gamma), \) and \( \psi = \prod_{\Gamma} \psi(\Gamma) \). Then \( \psi \in \text{Irr}^0(N(\theta), \theta), \) \( b^G = B, \) and \( (R, I(\psi)) \) is a \( B(\Gamma) \)-weight. By (4B) the number of \( B(\Gamma) \)-weights of \( U(\Gamma) \) is \( f^G \) and so the number of \( B \)-weights of \( G \) is \( \prod_{\Gamma} f^G \). By [3, (1A)] \( f^G \) is also the number of \( \eta \)-tuples \((\kappa_1, \kappa_2, \ldots, \kappa_\eta) \) of partitions \( \kappa_i \) such that \( \sum_{i=1}^\eta |\kappa_i| = \eta \). This last number is also the number of partitions with \( \eta \)-core \( \kappa \) and \( \eta \)-weight \( \eta \). So \( \prod_{\Gamma} f^G \) is the number \( l(B) \) of irreducible modular characters in \( B \) by [11, (8A)]. This completes the proof.

(4E). Let \( q \) be a power of an odd prime, \( V \) be a symplectic or even dimensional orthogonal space over \( \mathbb{F}_q \), \( G = I(V), \) \( G_0 = I_0(V), \) \( B \) is a block of \( G \) with defect group \( D \) and root block \( b \) such that \( [V, D] = V \) and \( b^G_0 \subseteq G_0 \) for some \( s \in G_0^\ast \). Let \( s^* \) be a dual of \( s \) in \( G_0 \) and \( m_f(s^*) = \eta \beta \epsilon \eta \), where \( \epsilon \) is an integer and \( \beta = 1 \) or 2 according as \( \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2 \) or \( \Gamma \in \mathcal{F}_0 \). Then the number of \( B \)-weights is \( \prod_{\Gamma} f^{G_0}_\Gamma \), where \( f^{G_0}_\Gamma \) is given by (4B). In particular, the number \( f^{G_0}_\Gamma \) is the number of \( \beta \epsilon \eta \)-tuples \((\kappa_1, \kappa_2, \kappa_\beta \epsilon \eta) \) of partitions \( \kappa_i \) such that \( \sum_{i=1}^{\beta \epsilon \eta} |\kappa_i| = \eta \).

Proof. Let \( (R, \varphi) \) be a \( B(\Gamma) \)-weight of \( G \), \( C = C_G(R), \) and \( N = N_G(R) \). Then there is a block \( b \) of \( CR \) with defect group \( R \) and the canonical character \( \theta \) such that \( b^G = B \) and \( \varphi = I(\psi) \) for some \( \psi \in \text{Irr}^0(N(\theta), \theta) \). We may suppose \( Z(D) \leq Z(R) \leq R \leq D \), so that \( [V, R] = V \).

Let \( R = \prod_{i=1}^u R_i \) and \( V = \sum_{i=1}^u V_i \) be the decompositions of \( (2D) \), and let \( C = \prod_{i=1}^u C_i \), and \( \theta = \prod_{i=1}^u \theta_i \), where \( C_i = C_i(V_i)(R_i) \) and \( \theta_i \) is a character of \( C_iR_i \) for all \( i \). The block \( b_i \) of \( C_iR_i \) containing \( \theta_i \) has defect group \( R_i \). We claim that there is a weight \((R_i, \chi_i)\) of \( I(V_i) \) such that \( \chi_i \) covers \( \theta_i \), namely there is an irreducible character \( \chi_i \) of \( N_i/R_i \) which covers \( \theta_i \) and whose defect is 0, where \( N_i = N_{I(V_i)}(R_i) \). Thus by (3J)(a) \((R_i, \theta_i)\) has type \( \Gamma \) for some \( \Gamma \in \mathcal{F}_i \). To prove the claim we rewrite the decomposition of \( R \) as \( \prod_{j=1}^u R_j^{d_j} \), where \( R_j \)'s are distinct basic subgroups and \( R_j \) appears \( d_j \)-times as a component of \( R \). Then

\[
N = \prod_{j=1}^u N_j \cdot S(d_j).
\]

Thus \( \varphi = \prod_{j=1}^u \varphi_j \) and \((R_j^{d_j}, \varphi_j)\) is a weight of \( I(U_j) \), where \( U_j \) is the underlying space of \( R_j^{d_j} \). So we may suppose \( u = 1 \) and \( d = d_1 \). Thus \( R = R_1^{d_1} \), \( N = N_1 \cdot S(d) \), and \( \varphi \) is a character of defect 0 of \( N/R \simeq (N_1/R_1)^d \cdot S(d) \). As shown in the proof of (4C), the restriction of \( \varphi \) to the base group \((N_1/R_1)^d \) of
$N/R$ has a constituent $(\xi_1, \xi_2, \ldots, \xi_d)$ covering $\theta$ and each $\xi_i$ has defect 0 as character of $N_i/R_i$. Thus $\xi_i$ covers $\theta_i$ and the claim holds.

Let $(R_i, t_i, -)$ be the label of Brauer pair $(R_i, b_i)$. As shown in the proof of (4B), $(R, \prod_{i=1}^t t_i, -)$ is a label of $(R, b)$ and $b_0 \subseteq \mathcal{E}(G_0, (\prod_{i=1}^t t_i))$. If $V^*$ is the underlying space of $G_0^*$, then $s$ and $\prod_{i=1}^t t_i$ are conjugate in $I(V^*)$ by (3K).

Let $R(\Gamma) = \prod_i R_i$, $V(\Gamma) = \sum_i V_i$, $C(\Gamma) = \prod_i C_i$, $\theta(\Gamma) = \prod_i \theta_i$, and $t(\Gamma) = \prod_i t_i$, where $i$ runs over $1 \leq i \leq t$ such that $(R_i, t_i)$ is of type $\Gamma$. Then $R = \prod R(\Gamma)$, $V = \sum_i V(\Gamma)$, $C = \prod R C(\Gamma)$, $\theta = \prod \theta(\Gamma)$, and $\prod_i t(\Gamma)$ is conjugate with $s$ in $I(V^*)$. It is clear that $N(\theta) = \prod R N(\theta(\Gamma))$, where $N(\theta(\Gamma))$ is the stabilizer of $\theta(\Gamma)$ in $N_i(V(\Gamma))(R(\Gamma))$. A similar proof to the last paragraph of (4D) shows that the number of $B$-weights is $\prod f^*_r$ and by [3, (1A)] $f_\Gamma$ is the number of $\beta_r$-tuple of partitions $\kappa_i$ such that $\sum_o |\kappa_i| = w^\Gamma$. This completes the proof.

Remark. With the assumption of (4E), let $G = O(V)$, $G_0 = SO(V)$, $R$, $\varphi$ a $B$-weight of $G$, and $\theta$ an irreducible character of $C = C_G(R)$ covered by $\varphi$. Then $|N(\theta) : N_0(\theta)| = 1$ as 2 according as $m_{x_{\pm 1}}(s) = 0$ or $m_{x_{\pm 1}}(s) \neq 0$. Moreover, for each $\psi \in \text{Irr}^0(N(\theta), \theta)$, the restriction $\psi|_{N_0(\theta)}$ is irreducible, where $N_0(\theta) = N(\theta) \cap G_0$. Indeed in the notation above $R = \prod R(\Gamma)$, $V = \sum_i V(\Gamma)$, $\theta = \prod \theta(\Gamma)$, $N(\theta) = \prod N(\theta(\Gamma))$, and $s = \prod t(\Gamma)$. Thus $\psi = \prod \psi(\Gamma)$ for some $\psi(\Gamma) \in \text{Irr}^0(N(\theta(\Gamma)), \theta(\Gamma))$. Since $[V, R] = V$, it follows that $[V(\Gamma), R(\Gamma)] = V(\Gamma)$. If $b(\Gamma)$ is the block of $C_{O(V(\Gamma))}(R(\Gamma))$ containing $\theta(\Gamma)$, then the Brauer pair $(R(\Gamma), \theta(\Gamma))$ has label $(R(\Gamma), t(\Gamma), -)$. Thus $\psi(\Gamma)$ is irreducible, where $N_0(\theta(\Gamma)) = (N(\theta(\Gamma)) \cap SO(V(\Gamma))$. So $|N(\theta) : N_0(\theta)| = 1$ or 2 according as $m_{x_{\pm 1}}(s) = 0$ or $m_{x_{\pm 1}}(s) \neq 0$, and $\psi|_{N_0(\theta)}$ is irreducible.

(4F). Let $q$ be a power of an odd prime, $G = \text{Sp}(2n, q) = \text{Sp}(V)$, $B$ a block of $G$ contained in $\mathcal{E}(G(s))$ for some semisimple $r$'-element $s$ of $G^* = SO(2n + 1, q)$. Let $D$ be a defect group of $B$, $V_0 = C_Y(D)$, $V_+ = [V, D]$, so that $V = V_0 \perp V_+$, and let $s = s_0 \times s_+$ be the corresponding decomposition in $G^*$. Then $m_r(s) - m_r(s_0) = w^\Gamma f^*_r$ for some $w^\Gamma f^*_r \geq 0$, where $\beta_r = 1$ or 2 according as $\Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2$ or $\Gamma \in \mathcal{F}_0$. The number of $B$-weights is $\prod f^*_r$, where $f^*_r$ is given by (4B). In particular, $f^*_r$ is the number of $\beta_r$-tuples $(\kappa_1, \kappa_2, \ldots, \kappa_{\beta_r+1})$ of partitions $\kappa_i$ such that $\sum_{i=1}^{\beta_r+1} |\kappa_i| = w^\Gamma$.

Proof. Let $(D, b)$ be a maximal Brauer pair of $G$ containing $(1, B)$, and $\theta$ be the canonical character of $b$. Then $D = D_0 \times D_+$, $b = b_0 \times b_+$, and $\theta = \theta_0 \times \theta_+$, where $D_0 = \langle 1 \rangle_{V_0} \leq \text{Sp}(V_0)$, $D_+ \leq \text{Sp}(V_+)$, $b_0, b_+$ are blocks of $\text{Sp}(V_0)$ and $C_{\text{Sp}(V_0)}(D_+)$ respectively, and $\theta_0 \in b_0$, $\theta_+ \in b_+$. Let $(R, \varphi)$ be a $B$-weight of $G$, $C = C_G(R)$, and $N = N_G(R)$. Then there is a block $b$ of $CR$ with defect group $R$ and canonical character $\theta$ such that $b^G = B$ and $\varphi = I(\psi)$ for some $\psi \in \text{Irr}^0(N(\theta), \theta)$. We may suppose $Z(D) \leq Z(R) \leq R \leq D$. Thus $C_Y(R) = V_0$, $[V, R] = V_+$, so that $R = R_0 \times R_+$, $C = C_0 \times C_+$, $N = N_0 \times N_+$, where $R_0 = D_0$, $R_+ \leq \text{Sp}(V_+)$, $C_0 = N_0 = \text{Sp}(V_0)$, $C_+ = C_{\text{Sp}(V_0)}(R_+)$, and $N_+ = N_{\text{Sp}(V_0)}(R_+)$. Let $b = b_0 \times b_+$ and $\theta = \theta_0 \times \theta_+$ be the corresponding decompositions. Then $b_0$ is a block of $C_0R_0 = \text{Sp}(V_0)$ of defect 0, $b_+$ is a block of $C_+R_+$ with defect group $R_+$, and $\theta_0 \in b_0$, $\theta_+ \in b_+$. We claim $\theta_0 = \theta_0$. Indeed let $(D', b'_+)$ be a
maximal Brauer pair of Sp(V+0) containing (R+, b+), b' = b0 x b'+, and D' = D0 x D'+. Then (D', b') is a maximal Brauer pair of Sp(V0) x Sp(V+) containing (R, b). If N(D', b') is the stabilizer of (D', b') in the normalizer N_G(D') of D', then (D', b') is maximal in G if and only if (D', b') is maximal in N(D', b'). Since N_G(D') ≤ Sp(V0) x Sp(V+), (D', b') is maximal in N(D', b') and then maximal in G containing (1, B). By the Brauer First Main Theorem, (D, b)^g = (D', b') for some g ∈ G, so that (θ0 x θ+)^g = θ0 x θ'+, where θ'+ is the canonical character of b'. Since D = D0 x D+ and D' = D0 x D'+, it follows g ∈ Sp(V0) x Sp(V+), and so g = g0 x g+, for g0 ∈ Sp(V0) and g+ ∈ Sp(V+). Thus θ0 = θ0 and θ'+ = θ'+. Moreover, b^Sp(V+) = b^Sp(V+) = b^Sp(V+).

It is clear that N(θ0) = N0 x N(θ+), where N0 = Sp(V0) and N(θ+) is the stabilizer of θ+ in N+. If ψ ∈ Irr^0(N(θ), θ), then ψ = ψ0 x ψ+, where ψ0 is an irreducible character of N0 = C0 covering θ0, and ψ+ ∈ Irr^0(N(θ+), θ+), so that ψ0 = θ0 = θ0. The correspondence (R, I(ψ)) ↔ (R+, I+(ψ+)), where ψ = ψ0 x ψ+ and I+(ψ+) = Ind_N_0(N(θ+), (θ+)), is clearly a bijection form {(R, I(ψ)) : ψ ∈ Irr^0(N(θ), θ)} to {(R+, I+(ψ+)) : ψ+ ∈ Irr^0(N(θ+), θ+)}. Since (R+, I+(ψ+)) is a b^Sp(V+)-weight, the number of B-weights in G is the number of b^Sp(V+)-weights in Sp(V+). Thus (4E) implies (4F).

In the following, we consider special orthogonal groups. If G = SO(2n + 1, q), then by Fong and Srinivasan, [12, (10B)], a block B of G is labeled by a pair (s, κ), where s is a semisimple r'-element in a dual group G* of G, κ = \prod \kappa_T is a product of symbols or partitions κ_T according as \Gamma ∈ \mathcal{F}_0 or \Gamma ∈ \mathcal{F}_1 ∪ \mathcal{F}_2 such that each κ_T is the r_T-core of either a symbol with rank \lceil \frac{1}{2} m_T(s) \rceil and odd defect, or a partition of m_T(s) according as \Gamma ∈ \mathcal{F}_0 or \Gamma ∈ \mathcal{F}_1 ∪ \mathcal{F}_2. Moreover, by [12, (12A)], B ⊆ Irr(G, (s)).

(4G). Let q be a power of an odd prime, G = SO(V) = SO(2n + 1, q), B a block of G with label (s, κ), \prod_{i=1}^r f_i a primary decomposition of s in G* = Sp(2n, q), and let \nu_T an integer such that m_T(s) = |\kappa_T| + e_T \nu_T if \Gamma ∈ \mathcal{F}_1 ∪ \mathcal{F}_2, and m_T(s) = 2 rank κ_T + 2e_T \nu_T if \Gamma ∈ \mathcal{F}_0. Then the following hold:

(1) The number of B-weights of G is \prod_{f_i} f_i, where \nu_T is the number of \beta f_T-tuples (κ_1, κ_2, ..., \kappa_{r_f_T}) of partitions κ_i such that \sum_{i=1}^{r_f_T} |\kappa_i| = \nu_T, and \beta = 1 or 2 according as \Gamma ∈ \mathcal{F}_1 ∪ \mathcal{F}_2 or \Gamma ∈ \mathcal{F}_0.

(2) The number of B-weights of G is |B ∩ Irr(G, (s))|.

Proof. Let \widetilde{G} = O(V), so that \widetilde{G} = (−1_V) × G, and let \widetilde{B} = 1 × B be a block of \widetilde{G}, where 1 is the principal block of (−1_V). Let (R, ϕ) be a B-weight of G, N = N_G(R), and \widetilde{N} = N_{\widetilde{G}}(R), so that \widetilde{N} = (−1_V) × N. There exists a block b of N such that ϕ ∈ b and b^G = B. Let \tilde{b} = 1 × b and \tilde{ϕ} = 1_{(−1_V)} × ϕ, where 1_{(−1_V)} is the principal character of (−1_V). Thus \tilde{ϕ} ∈ \tilde{b}, \tilde{b}^G = \tilde{B}, and (R, \tilde{ϕ}) is a \tilde{B}-weight of \widetilde{G}. The correspondence (R, ϕ) ↔ (R, \tilde{ϕ}) is clearly a bijection from B-weights to \tilde{B}-weights. Thus the number of B-weights in G is the number of \tilde{B}-weights in \widetilde{G}.

Let (D, b) be a maximal Brauer pair of \widetilde{G} containing (1, \tilde{B}), θ the canonical character of \tilde{b}, V_0 = C_\Gamma(D), V_+ = [V, D]. Then V = V_0 ⊥ V_+ and V_+ is an even dimensional orthogonal space since D is radical. In addition, let \widetilde{G}_0 = O(V_0), \widetilde{G}_0 = SO(V_0), \widetilde{G}_+ = O(V_+), and G_+ = SO(V_+). Then
$D = D_0 \times D_+ = \tilde{b} = b_0 \times \tilde{b}_+, \ \tilde{\vartheta} = \tilde{\vartheta}_0 \times \tilde{\vartheta}_+, \ \text{where} \ D_0 = \langle 1_{V_0} \rangle \leq \tilde{G}_0, \ D_+ \leq \tilde{G}_+, \ b_0, \ \tilde{b}_+ \text{ are blocks of } \tilde{G}_0, \ C^{-+}(D_+) \text{ respectively, and } \tilde{\vartheta}_0 \in \tilde{b}_0, \ \tilde{\vartheta}_+ \in \tilde{b}_+.$

Now the proof of (4F) can be applied here with $G$ replaced by $\tilde{G}$, $B$ by $\tilde{B}$, $\vartheta$ by $\tilde{\vartheta}$, $b$ by $\tilde{b}$, and some obvious modifications. Thus the number of $\tilde{B}$-weights in $\tilde{G}$ is the number of $\tilde{b}_+^{-+}$-weights in $\tilde{G}_+$. Moreover, $\tilde{b}_+$ is a root block of $\tilde{b}_+^{-+}$ and $\tilde{b}_+^{-+} \subseteq \mathcal{E}_\Gamma(G_+, (s_+))$. Since $C^{-+}(D) = (\langle 1_{V_0} \rangle) \times C_G(D)$ and $\tilde{\vartheta}_0 = 1 \times \tilde{\vartheta}_0$ for $\tilde{\vartheta}_0 \in \tilde{b}_0$. Since $C^{-+}(D_+) = C_G(D_+)$, $\tilde{b}_+$ is a block of $C_G(D_+)$ and then $b_0 \times \tilde{b}_+$ is a root block of $B$. Here $b_0 \times \tilde{b}_+$ is regarded as a block of $C_G(D)$. As shown in the proof of [12, (12A)], $(s_0, \kappa)$ is the label of $\vartheta_0$, so that $m_{\Gamma}(s) = |\kappa_1| + m_{\Gamma}(s_+)$ if $\Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2$, and $m_{\Gamma}(s) = 2$ rank $\kappa + m_{\Gamma}(s_+)$ if $\Gamma \in \mathcal{F}_0$. Thus $m_{\Gamma}(s_+) = m_{\Gamma}(s_+^*) = w_{\Gamma}^{-} \beta_\Gamma e_{\Gamma}$, where $s_+^*$ is a dual of $s_+$ in $G_+$. So (4G)(1) follows from (4E).

Finally, there exists a bijection between $\mathcal{E}(G, (s))$ and $\mathcal{E}(C_{G^*}(s)^{\star}, (1))$. By [12, (12A)] and [19, Proposition 14] the number given by (1) is the number of the characters of $\mathcal{E}(G, (s)) \cap B$.

**Remark.** (1) Suppose $G = SO(2n + 1, q)$ and $r$ is a good prime. Then by [13, 5.1] $l(B) = |B \cap \mathcal{E}(G, (s))|$, so that $l(B)$ is the number of $B$-weights.

(2) By a result of Fong and Olsson (unpublished), if $G = SO(2n + 1, q)$ and $r$ is odd, then $l(B) = |B \cap \mathcal{E}(G, (s))|$ and this is the number of $B$-weights.

(4H). Let $q$ be a power of an odd prime, $G = SO^\pm(2n, q) = SO(V)$, $B$ is a block of $G$ with defect group $D$ and root block $b$ such that $B \subseteq \mathcal{E}(G, (s))$ for some semisimple $r'$-element $s$ of $G^* = SO^\pm(2n, q)$, and let $V_0 = C_{V_0}(D)$, $V_+ = [V, D]$, so that $V = V_0 \perp V_+$. Let $s = s_0 \times s_+$, $\vartheta = \vartheta_0 \times \vartheta_+$ be the corresponding decompositions, where $\vartheta$ is the canonical character of $b$. If $m_{\Gamma}(s_+) = w_{\Gamma}^{-} \beta_\Gamma e_{\Gamma}$ for some $w_{\Gamma} \geq 0$, then denote $f_{\Gamma}$ the number of $\beta_\Gamma e_{\Gamma}$-tuples $(\kappa_1, \kappa_2, \ldots, \kappa_{\beta_\Gamma e_{\Gamma}})$ of partitions $\kappa_i$ such that $\sum_{i=1}^{\beta_\Gamma e_{\Gamma}} |\kappa_i| = w_{\Gamma}$, where $\beta_{\Gamma} = 1$ or 2 according as $\Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2$ or $\Gamma \in \mathcal{F}_0$. Then the following hold:

(1) If either $m_{\Gamma}(s_+) = 0$ or $\vartheta_0^0 \neq \vartheta_0$ for some $\vartheta_0 \in O(V_0)$ of determinant $-1$, then the number of $B$-weights is $\prod_{\Gamma} f_{\Gamma}$.

(2) Suppose $m_{\Gamma}(s_+) \neq 0$. If either $V_0 = 0$ or $\vartheta_0^0 = \vartheta_0$ for any $\vartheta_0 \in O(V_0)$ of determinant $-1$, then the number of $B$-weights is $\frac{1}{2} \prod_{\Gamma} f_{\Gamma}$.

**Proof.** Let $\tilde{G} = O(V), \ \tilde{G}_0 = O(V_0), \ G_0 = SO(V_0), \ G_+ = O(V_+), \ G_+ = SO(V_+)$, and $D = D_0 \times D_+$, where $D_0 = \langle 1_{V_0} \rangle$ and $D_+ \leq G_+$. In addition, let $b_+$ be a block of $C_{G_+}(D_+)D_+$ containing $\vartheta_+$, and $b_+^{G_+} \subseteq \mathcal{E}_\Gamma(G_+, (s_+'))$ for some semisimple $r'$-element $s_+'$ of $G_+^*$. Then $(D_+, s_+', -)$ is a label of Brauer pair $(D_+, b_+)$. But $(D_+, s_+', -)$ is also a label of $(D_+, b_+)$, and so $s_+', s_+'$ are conjugate in $G_+^*$. Let $(R, \varphi)$ be a $B$-weight, $C = C_G(R), \ \tilde{C} = C_{\tilde{G}}(R), \ N = N_G(R), \ \tilde{N} = N_{\tilde{G}}(R)$. Then there exists a block $b$ of $CR$ with defect group $R$ and canonical character $\vartheta$ such that $b^G = B$ and $\varphi = I(\psi)$ for some $\psi \in Irr^0(N(\theta), \theta)$. We may suppose $Z(D) \leq Z(R) \leq R \leq D$, so that $R = R_0 \times R_+$, $C = G_0 \times C_+, \ \tilde{C} = \tilde{G}_0 \times C_+, \ N = \langle \tau, G_0 \times N_+ \rangle$, and $\tilde{N} = \tilde{G}_0 \times \tilde{N}_+$, where $R_0 = D_0$, $R_+ = D_+$.
WEIGHTS FOR CLASSICAL GROUPS

\[ R_+ \leq G_+, \quad C_+ = C_{G_+}(R_+), \quad N_+ = N_{G_+}(R_+), \quad \tilde{N}_+ = N_{\tilde{G}_+}(R_+), \quad \text{and} \quad \tau = \tau_0 \times \tau_+ \]

with \( \tau_0 \in \tilde{G}_0, \quad \tau_+ \in \tilde{G}_+ \) of determinants \(-1\). Thus \( \tilde{N} = (\tau_0, N), \quad \theta = \theta_0 \times \theta_+ \),
and \( b = b_0 \times b_+ \), where \( b_0 \) is a block of \( G_0 \) of defect 0, \( b_+ \) is a block of \( C_+ R_+ \) with defect group \( R_+, \quad \theta_0 \in \theta_0, \quad \text{and} \quad \theta_+ \in \theta_+ \).

Let \((D'_+, b'_+)\) be a maximal Brauer pair of \( \tilde{G}_+ \) containing \((R_+, b_+)\), where \( b_+ \) is regarded as a block of \( C_+ \). Let \( D' = D_0 \times D'_+, \quad b' = b_0 \times b'_+ \). A similar proof to that of \((4F)\) shows that \((D', b')\) is a maximal Brauer pair of \( G \) containing \((R, b)\), where \( b \) is regarded as a block of \( C \). So \((D, b)_{\theta} = (D', b')\) for some \( \theta \in G \) by the Brauer First Main Theorem. Thus \( g = g_0 \times g_+ \) for \( g_0 \in \tilde{G}_0 \) and \( g_+ \in \tilde{G}_+ \). If \( \det(g_0) = -1 \), then we replace \( b \) by \( b'^* \) and \( \theta_0 \) by \( \theta'^*_0 \). We may suppose \( g_0 \in \tilde{G}_0 \) and \( g_+ \in \tilde{G}_+ \). Since \((\theta_0 \times \theta_+)^* = \theta_0 \times \theta_+^* \),
it follows that \( \theta_0 = \theta_0^* \) and \( \theta_+^* = \theta_+^* \), where \( \theta_+^* \) is the canonical character of \( b_+^* \). It follows that \( b_+^{G_+} = b_+^G \), so that \( b_+^{\tilde{G}_+} = b_+^{\tilde{G}} \) and we may suppose \((R_+, s_+, -)\) is a label of \((R_+, b_+)\). Replacing \( R \) by \( R_0 \times R_+^{-1} \) and \( b \) by \( b_0 \times b_+^{\sigma_0} \), we may suppose \((R, b) \leq (D, b)\).

(1) Suppose \( m_{X \pm 1}(s_+) = 0 \). Set \( \tilde{B}_+ = b_+^{\tilde{G}_+} \), so that \( b_+ \) is a root block of \( \tilde{B}_+ \) and \( D_+ \) is a defect group of \( \tilde{B}_+ \). We shall show that the number of \( B \)-weights in \( G \) is the number of \( B_+ \)-weights in \( \tilde{G}_+ \).

Let \( N(\theta_+) \) and \( \tilde{N}(\theta_+) \) be the stabilizers of \( \theta_+ \) in \( N_+ \) and \( \tilde{N}_+ \) respectively.
By the remark of \((4E)\) \( N(\theta_+) = \tilde{N}(\theta_+) \).
Since \( N(\theta) = G_0 \times N(\theta_+) \), it follows that \( \psi = \theta_0 \times \psi_+ \) for some \( \psi_+ \in \text{Irr}^0(\tilde{N}(\theta_+), \theta_+) \).
Then \((R_+, I_+^0(\psi_+))\)
is a \( \tilde{B}_+ \)-weight of \( \tilde{G}_+ \), where \( I_+^0(\psi_+) = \text{Ind}_{\tilde{N}(\theta_+)}^{\tilde{N}(\theta_+)}(\psi_+) \).
Conversely, suppose \((R_+, \varphi_+)\) is a \( \tilde{B}_+ \)-weight, where \( R_+ \) is a radical subgroup of \( \tilde{G}_+ \).
Then \([V_+, R_+] = V_+ \) and there exists a block of \( C_+ R_+ \) with defect group \( R_+ \) and canonical character \( \theta_+ \) such that \( \varphi_+ = I_+^0(\psi_+) \) for some \( \psi_+ \in \text{Irr}^0(\tilde{N}(\theta_+), \theta_+) \) and \( b_+^{\tilde{G}_+} = \tilde{B}_+ \), where \( C_+, \tilde{N}_+ \) are given before, \( \tilde{N}(\theta_+) \) is the stabilizer of \( \theta_+ \) in \( \tilde{N}_+ \), and \( I_+ \) is defined as before. By the remark of \((4E)\) \( \tilde{N}(\theta_+) \leq G_+ \).
Let \( \theta = \theta_0 \times \theta_+ \), \( R = D_0 \times R_+, \quad \psi = \theta_0 \times \psi_+ \), \( b \) a block of \( C_{G}(R) \) containing \( \theta \),
and \( \varphi \in \text{Irr}^0(N(\theta), \theta) \). We may suppose \((R_+, b_+) \leq (D_+, b_+) \), so that \((R, b) \leq (D, b) \).
Thus \( b^G = B \) and \((R, I(\psi))\) is a \( B \)-weight.
The correspondence \((R, I(\psi)) \mapsto (R_+, I_+^0(\psi_+))\), where \( R = D_0 \times R_+ \) and \( \psi = \varphi_0 \times \psi_+ \) is clearly a bijection from \( \{(R, I(\psi)) : \psi \in \text{Irr}^0(N(\theta), \theta)\} \) to \( \{(R_+, I_+^0(\psi_+)) : \psi_+ \in \text{Irr}^0(\tilde{N}(\theta_+), \theta_+)\} \).
So the number of \( B \)-weights is the number of \( B_+ \)-weights, and it is \( \prod \pi_1 \gamma_1 \) by \((4E)\).

Suppose \( \varphi_0^0 = \tilde{\varphi}_0 \) for some \( \sigma_0 \in \tilde{G}_0 \) of determinant \(-1\). Then there are two irreducible characters \( \varphi_0^0 \) and \( \varphi_0^0' \) of \( G_0 \) covering \( \varphi_0 \). Let \( \varphi' = \varphi_0^0 \times \varphi_0^0', \quad \varphi'' = \varphi_0^0 \times \varphi_0^0', \quad \text{and} \quad b', \ b'' \) the blocks of \( C_{\tilde{G}}(D) \) containing \( \varphi', \varphi'' \) respectively.
Then \( \varphi', \varphi'' \) are not conjugate in \( N_{\tilde{G}}(D) = \tilde{G}_0 \times N_{\tilde{G}_+}(D_+, b_+) \), so \( b'\tilde{G} \) and \( b''\tilde{G} \) are two blocks of \( \tilde{G} \). We shall show that the number of \( b'\tilde{G} \)-weights is the number of \( B \)-weights.

Suppose \((R, \varphi)\) is a \( B \)-weight. In the notation above, \( N = (\tau, G_0 \times N_+)\)
and \( \widetilde{N} = \langle \tau_0, G_0 \times \widetilde{N}_+ \rangle \), where \( \tau = \tau_0 \times \tau_+ \) with \( \tau_0 \in \widetilde{G}_0, \tau_+ \in \widetilde{G}_+ \) of determinants \(-1\). Moreover, we may suppose \((R, b) \leq (D, b)\) and \(\theta_0 = \theta_0\). Let \(\widehat{b} = \theta_0' \times \theta_+\) and \(\widehat{b}\) the block of \(\tilde{C}\) containing \(\widehat{\theta}\). Then \((R, \widehat{b}) \leq (D, b')\) and \(\widehat{b} \mathcal{C} = b' \mathcal{C}\). Conversely, if \((R, \phi)\) is a weight of \(b' \mathcal{C}\), then there exists a block \(b\) of \(\mathcal{C}R\) with defect group \(R\) and canonical character \(\widehat{\theta}\) such that \(\widehat{b} \mathcal{C} = b' \mathcal{C}\) and \(\phi \in \text{Irr}(\widetilde{N}, \widehat{\theta})\), where \(\tilde{C}\) is defined before. Then \(b = b_0 \times b_+\) and \(\widehat{\theta} = \theta_0' \times \theta_+\), where \(b_0\) and \(b_+\) are blocks of \(\widetilde{G}_0\) and \(\widetilde{G}_+\) respectively and \(\theta_0 \in b_0\) and \(\theta_+ \in b_+\). As shown in the proof of \((4F)\), we may suppose \(\theta_0 = \theta_0'\) and \((R, \widehat{b}) \leq (D, b')\). Let \(\theta = \theta_0 \times \theta_+\) and \(b\) the block of \(\mathcal{C}\) containing \(\theta\). Then \((R, b) \leq (D, b)\). In addition, each character \(\phi \in \text{Irr}^0(N, \theta)\) or \(\phi \in \text{Irr}^0(\widetilde{N}, \widehat{\theta})\) covers a character of \(\text{Irr}^0(G_0 \times N_+, \theta)\) and each character of \(\text{Irr}^0(\widetilde{N}_+, \theta')\) decomposes as \(\theta_0 \times \phi_+\) for some \(\phi_+ \in \text{Irr}(\mathcal{C}+\theta)\). So it suffices to show that the number of \(b' \mathcal{C}\)-weights of the form \((R, \phi)\) with \(\phi\) covering \(\theta_0 \times \phi_+\) is the number of \(B\)-weights of the form \((R, \phi)\) with \(\phi\) covering \(\theta_0 \times \phi_+\). It is equivalent to show that the number of irreducible characters in \(b' \mathcal{C}\) covering \(\theta_0 \times \phi_+\) is the number of irreducible characters in \(b' \mathcal{C}\) covering \(\theta_0 \times \phi_+\) since \((\mathcal{N} : N) = (\mathcal{N} : G_0 \times N_+) = 2\).

If \(\tau_+\) stabilizes \(\phi_+\), then there are two irreducible characters \(\phi_+\) and \(\phi_+\) of \(\widetilde{N}_+\) covering \(\phi_+\), so that there are four irreducible characters \(\theta_0' \times \phi_+, \theta_0' \times \phi_+\), \(\theta_0'' \times \phi_+\), and \(\theta_0'' \times \phi_+\) of \(\widetilde{N} = \widetilde{G}_0 \times \widetilde{N}_+\) covering \(\theta_0 \times \phi_+\). Moreover, exactly two of them \(\theta_0' \times \phi_+\), and \(\theta_0' \times \phi_+\) cover \(\theta_0 \times \phi_+\) and both lie in \(b' \mathcal{C}\) by \([10, V 3.10]\). Since \(\tau = \tau_0 \times \tau_+\) stabilizes \(\phi_+\), there are two irreducible characters of \(\mathcal{N}\) covering \(\theta_0 \times \phi_+\) and lying in \(b' \mathcal{C}\). It follows that both \(\widehat{b} \mathcal{N}\) and \(\widehat{b} \mathcal{N}\) have two irreducible characters covering \(\theta_0 \times \phi_+\), so that the number of \(b' \mathcal{C}\)-weights is the number of \(B\)-weights.

If \(\tau_+\) does not stabilize \(\phi_+\), then there are two irreducible characters \(\theta_0' \times (\phi_+ + \phi_+\) and \(\theta_0' \times (\phi_+ + \phi_+\) of \(\widetilde{N}\) covering \(\theta_0 \times \phi_+\) and only the first lies in \(\widehat{b} \mathcal{N}\). Since \((\theta_0 \times \phi_+) \tau \neq \theta_0 \times \phi_+\), \(\mathcal{N}\) has only one irreducible character covering \(\theta_0 \times \phi_+\) and lying in \(b' \mathcal{C}\). So both \(\widehat{b} \mathcal{N}\) and \(\widehat{b} \mathcal{N}\) has one irreducible character covering \(\theta_0 \times \phi_+\). Thus the number of \(b' \mathcal{C}\)-weights is the number of \(B\)-weights.

A similar proof to that of \((4F)\) can be applied here with \(G\) replaced by \(\widetilde{G}\), \(B\) by \(\mathcal{C}\), \(b\) by \(\mathcal{C}\), \(\theta\) by \(\phi\), and some obvious modifications, so that the number of \(b' \mathcal{C}\)-weights is the number of \(B\)-weights. By \((4E)\) the number of \(b' \mathcal{C}\)-weights is \(\Pi_{\tau \in G} f_{\tau}\) and this is the number of \(B\)-weights. This completes the proof of \((1)\).

(2) Suppose \(m_{x^\pm_1(s_+)} \neq 0\) and \((R, \phi)\) is a \(B\)-weight. In the notation above, suppose \(\widetilde{N}(\theta)\) and \(\mathcal{N}(\theta)\) are the stabilizers of \(\theta\) in \(\widetilde{N}\) and \(\mathcal{N}\) respectively.

If \(V_0 = 0\), then \((\widetilde{N}(\theta) : \mathcal{N}(\theta)) = 2\) and \(|\text{Irr}^0(\widetilde{N}(\theta), \tau)| = 2|\text{Irr}^0(\mathcal{N}(\theta), \tau)|\) by the remark of \((4E)\). So the number of \(B\)-weights is \(\frac{1}{2} \Pi_{\tau \in G} f_{\tau}\) by \((4E)\).

Suppose \(V_0 \neq 0\) and \(\theta_0^{-1} \neq \theta_0\) for some \(\tau_0 \in \widetilde{G}_0\) of determinant \(-1\). By the proof above, we may suppose \(\theta = \theta_0 \times \theta_+\) for some character \(\phi_+\) of \(\mathcal{C}_+\) and \((R, b) \leq (D, b)\). Let \(\mathcal{N}(\theta_+)\) and \(\mathcal{N}(\theta_+)\) be the stabilizers of \(\theta_+\) in \(\widetilde{N}_+\) and \(\mathcal{N}_+\) respectively. Then \(\widetilde{N}(\theta) = G_0 \times \mathcal{N}(\theta_+)\) and \(\mathcal{N}(\theta) = G_0 \times \mathcal{N}(\theta_+)\),
so that by the remark of (4E), \(|\text{Irr}^0(\widetilde{N}(\theta_+), \theta_+)| = 2|\text{Irr}^0(N(\theta_+), \theta_+)|\). Thus 
\(|\text{Irr}^0(\widetilde{N}(\theta), \theta)| = 2|\text{Irr}^0(N(\theta), \theta)|\) since each character \(\psi\) of \(\text{Irr}^0(\widetilde{N}(\theta), \theta)\) and each \(\psi\) of \(\text{Irr}^0(N(\theta), \theta)\) decomposes as \(\psi = \theta_0 \times \psi_+\) and \(\psi = \theta_0 \times \psi_+\) for some \(\psi_+ \in \text{Irr}^0(\widetilde{N}(\theta_+), \theta_+)\) and \(\psi_+ \in \text{Irr}^0(N(\theta_+), \theta_+)\). Let \(b'\) be the block of \(C^{\sim}(D)D\) containing \(\vartheta' = (\theta_0 + \theta_0^\circ) \times \theta_+\) and \(b\) the block of \(C\) containing \(\vartheta = (\theta_0 + \theta_0^\circ) \times \theta_+\). Since \((R, b) \leq (D, b)\) in \(G\), it follows that 
\((R, b') \leq (D, b')\) in \(\widetilde{G}\), so that \(b^{\sim}G = b^G\). Thus the number of \(B\)-weights is half of the number of \(b^{\sim}G\)-weights. A similar proof to that of (4F) can be applied here with \(G\) replaced by \(\widetilde{G}\), \(B\) by \(b^\sim G\), \(b\) by \(b'\), \(\vartheta\) by \(\vartheta'\), and some obvious modifications, so that the number of \(b^\sim G^+\)-weights is the number of \(b^G\)-weights. By (4E) the number of \(b^\sim G^+\)-weights is \(\prod_{\Gamma} f_{\Gamma}\) and so the number of \(B\)-weights is \(\frac{1}{2} \prod_{\Gamma} f_{\Gamma}\). This completes the proof.

**References**


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