INFINITE FAMILIES OF ISOMORPHIC NONCONJUGATE FINITELY GENERATED SUBGROUPS

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Abstract. Let \( \langle \ , \rangle : L \times L \to \mathbb{Z} \) be a nondegenerate symmetric bilinear form on a finitely generated free abelian group \( L \) which splits as an orthogonal direct sum \( (L, \langle \ , \rangle) \cong (L_1, \langle \ , \rangle) \perp (L_2, \langle \ , \rangle) \perp (L_3, \langle \ , \rangle) \) in which \( (L_1, \langle \ , \rangle) \) has signature \((2, 1)\), \( (L_2, \langle \ , \rangle) \) has signature \((n, 1)\) with \( n \geq 2 \), and \( (L_3, \langle \ , \rangle) \) is either zero or indefinite with \( \text{rk}_{\mathbb{Z}}(L_3) \geq 3 \). We show that the integral automorphism group \( \text{Aut}_{\mathbb{Z}}(L, \langle \ , \rangle) \) contains an infinite family of mutually isomorphic finitely generated subgroups \( (\Gamma_\sigma)_{\sigma \in \Sigma} \), no two of which are conjugate. In the simplest case, when \( L_3 = 0 \), the groups \( \Gamma_\sigma \) are all normal subdirect products in a product of free groups or surface groups. The result can be seen as a failure of the rigidity property for subgroups of infinite covolume within the corresponding Lie group \( \text{Aut}_{\mathbb{Z}}(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle \ , \rangle \otimes 1) \).

0. Introduction

The following question arose from the joint work of Ebeling and Okonek on diffeomorphisms of algebraic surfaces.

Question. Let \( \langle \ , \rangle : L \times L \to \mathbb{Z} \) be a nondegenerate symmetric bilinear form on a finitely generated free abelian group \( L \). When, if ever, does there exist an infinite family of isomorphic finitely generated subgroups \( (\Gamma_\sigma)_{\sigma \in \Sigma} \) of \( \text{Aut}_{\mathbb{Z}}(L, \langle \ , \rangle) \) such that for \( \sigma \neq \tau \), \( \Gamma_\sigma \) is not conjugate to \( \Gamma_\tau \) in \( \text{Aut}_{\mathbb{Z}}(L, \langle \ , \rangle) \)?

In this paper, we establish the existence of such infinite families \( (\Gamma_\sigma)_{\sigma \in \Sigma} \) of nonconjugate isomorphic finitely generated subgroups when \( (L, \langle \ , \rangle) \) splits as an orthogonal direct sum

\[
(L, \langle \ , \rangle) \cong (L_1, \langle \ , \rangle) \perp (L_2, \langle \ , \rangle) \perp (L_3, \langle \ , \rangle)
\]

in which \( (L_1, \langle \ , \rangle) \) has signature \((2, 1)\), \( (L_2, \langle \ , \rangle) \) has signature \((n, 1)\) with \( n \geq 2 \), and \( (L_3, \langle \ , \rangle) \) is either zero or indefinite with \( \text{rk}_{\mathbb{Z}}(L_3) \geq 3 \). The parameter set \( \Sigma \) may be thought of as an infinite subset of \( \text{Aut}_{\mathbb{Z}}(L, \langle \ , \rangle) \).

The construction of the groups \( (\Gamma_\sigma)_{\sigma \in \Sigma} \) uses a variation on the methods of our earlier paper [3]; in addition, the main theorem of [3] is needed to show finite generation. In §1, we recall some basic facts about orthogonal groups and integral quadratic forms. The necessary results from [3] are reviewed in §§2–3, and the families \( (\Gamma_\sigma)_{\sigma \in \Sigma} \) are constructed in §4 (Theorems 4.4 and 4.5).
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1. INTEGRAL QUADRATIC FORMS AND THEIR ARITHMETIC SUBGROUPS

Let \((L, \langle , \rangle): L \times L \to \mathbb{Z}\) be a nondegenerate symmetric integral bilinear form on a free abelian group \(L\) of finite rank \(m\), say. \((L, \langle , \rangle)\) is said to be isotropic (over \(\mathbb{Z}\)) when there exists a nonzero element \(x \in L\) such that \(\langle x, x \rangle = 0\); otherwise \((L, \langle , \rangle)\) is said to be anisotropic. Put \(\Gamma = \text{Aut}_\mathbb{Z}(L, \langle , \rangle)\). The associated real form \(\langle , \rangle: L \otimes \mathbb{R} \times L \otimes \mathbb{R} \to \mathbb{R}\) is diagonalisable as

\[
\sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i,
\]

assigning to \((L, \langle , \rangle)\) the signature \((p, q)\) where \(p + q = m\); \(\Gamma\) imbeds as a discrete subgroup of finite covolume in the group \(\text{Aut}_\mathbb{R}(L \otimes \mathbb{R}, \langle , \rangle) \cong O(p, q)\), and acts properly discontinuously as a group of isometries of the symmetric space of \(O(p, q)\). Moreover, \(\Gamma\) is cocompact precisely when \((L, \langle , \rangle)\) is anisotropic. (When \((L, \langle , \rangle)\) is indefinite, a classical theorem of Meyer [5] asserts that for \((L, \langle , \rangle)\) to be anisotropic it is necessary that \(m \leq 4\).)

When the signature of \((L, \langle , \rangle)\) is \((2, 1)\), the corresponding symmetric space is the upper half-plane, so that \(\Gamma\) is a Fuchsian group. When \((L, \langle , \rangle)\) is isotropic, \(\Gamma\) contains a nonabelian free subgroup of finite index. When \((L, \langle , \rangle)\) is anisotropic, \(\Gamma\) contains, as a subgroup of finite index, a surface group \(\Sigma^+_g\); that is, the fundamental group of an orientable surface of genus \(g \geq 2\), having a presentation of the form

\[
\Sigma_g^+ = \left\langle X_1, \ldots, X_g, Y_1, \ldots, Y_g : \prod_{r=1}^{g} X_r Y_r X_r^{-1} Y_r^{-1} \right\rangle.
\]

We summarise these observations.

**Proposition 1.1.** Let \(\Gamma\) be the automorphism group of a nondegenerate integral quadratic form of signature \((2, 1)\); then \(\Gamma\) is finitely generated, and

(i) \(\Gamma\) contains a surface subgroup of finite index when \((L, \langle , \rangle)\) is anisotropic;

(ii) \(\Gamma\) contains a nonabelian free subgroup of finite index when \((L, \langle , \rangle)\) is isotropic.

Let \(G\) be a linear algebraic group defined and semisimple over \(\mathbb{Q}\); we may take \(G\) to be imbedded \(G_\mathbb{Q} \subset G\mathcal{L}_n(\mathbb{Q})\). By an arithmetic subgroup of \(G\), we mean a subgroup \(\Gamma\) of \(G_\mathbb{R}\) which is commensurable with \(G_\mathbb{Z} = G_\mathbb{Q} \cap G\mathcal{L}_n(\mathbb{Z})\). This does not depend on the particular imbedding \(G_\mathbb{Q} \subset G\mathcal{L}_n(\mathbb{Q})\) chosen. Moreover, for such a subgroup \(\Gamma\), \(G_\mathbb{R}/\Gamma\) has finite invariant volume. Let \(\overline{\Delta} \subset G_\mathbb{C}\) denote the Zariski closure of a subgroup \(\Delta \subset G_\mathbb{R}\).

We begin by observing the following, where \([\Gamma, \Gamma]\) denotes the commutator subgroup of \(\Gamma\).
Proposition 1.2. Let $G$ be a linear algebraic group defined and semisimple over $\mathbb{Q}$, with the property that $G_{i, \mathbb{R}}$ is noncompact for each $\mathbb{Q}$-simple factor $G_i$. If $\Gamma$ is an arithmetic subgroup of $G$ then $[\Gamma, \Gamma] = G_c$.

Proof. We first consider the case where $G$ is $\mathbb{Q}$-simple. By Borel’s Density Theorem in the form of [1], $\Gamma = G_c$, and since $G_c$ is nonabelian, $\Gamma$ is also nonabelian; hence $[\Gamma, \Gamma]$ is nontrivial. $\Gamma$ normalises $[\Gamma, \Gamma]$, so that $\Gamma$ normalises $[\Gamma, \Gamma]$. However, since $\Gamma = G_c$, $[\Gamma, \Gamma]$ is a normal complex algebraic subgroup of $G_c$. Moreover, since $[\Gamma, \Gamma]$ is the Zariski closure of a subset $[\Gamma, \Gamma]$ of $G_{\mathbb{Q}}$, then by Weil’s Rationality Criterion [8], $[\Gamma, \Gamma]$ is defined over $\mathbb{Q}$. The assertion that $[\Gamma, \Gamma] = G_c$ now follows from the fact that $G$ is $\mathbb{Q}$-simple and $[\Gamma, \Gamma]$ is nontrivial.

In general, $G$ is isogenous with the product of its $\mathbb{Q}$-simple factors $G_1 \times \cdots \times G_n$, so that $\Gamma$ contains, with finite index, a subgroup of the form $\Gamma_1 \times \cdots \times \Gamma_n$, where $\Gamma_i$ is an arithmetic subgroup of $G_i$. Hence $[\Gamma_1, \Gamma_1] \times \cdots \times [\Gamma_n, \Gamma_n]$ is contained in $[\Gamma, \Gamma]$, and the result follows easily from the special case already considered. \(\square\)

For any field $k$, let $O(n, k)$ denote the group of automorphisms of the standard symmetric bilinear form

$$\langle , \rangle : k^n \times k^n \rightarrow k; \quad \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i,$$

and let $\mathcal{O}(n, k)$ denote the Lie algebra of $O(n, k)$,

$$\mathcal{O}(n, k) = \{ A \in M_n(k) : A^T + A = 0 \}.$$

The obvious isomorphism $k^{n_1} \oplus \cdots \oplus k^{n_f} \cong k^{n_1 + \cdots + n_f}$ induces injections

$$\mathcal{O}(n_1, k) \oplus \cdots \oplus \mathcal{O}(n_f, k) \subset \mathcal{O}(n_1 + \cdots + n_f, k),$$

and

$$O(n_1, k) \times \cdots \times O(n_f, k) \subset O(n_1 + \cdots + n_f, k).$$

Proposition 1.3. $\mathcal{O}(n_1, \mathbb{C}) \oplus \cdots \oplus \mathcal{O}(n_f, \mathbb{C})$ is a self-normalising Lie subalgebra of $\mathcal{O}(n_1 + \cdots + n_f, \mathbb{C})$ provided that each $n_i \geq 2$.

Proof. It clearly suffices to show that $\mathcal{O}(m, \mathbb{C}) \oplus \mathcal{O}(n, \mathbb{C})$ is a self-normalizing Lie subalgebra of $\mathcal{O}(m + n, \mathbb{C})$ provided that $m, n \geq 2$; the general case follows easily by induction. Thus suppose that $\alpha \in M_{m+n}(\mathbb{C})$ has the property

$$[\alpha, \xi] \in \mathcal{O}(m, \mathbb{C}) \oplus \mathcal{O}(n, \mathbb{C})$$

for all $\xi \in \mathcal{O}(m, \mathbb{C}) \oplus \mathcal{O}(n, \mathbb{C})$.

We may write $\alpha, \xi$ in block form:

$$\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \xi = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

where $X \in \mathcal{O}(m, \mathbb{C})$ and $Y \in \mathcal{O}(n, \mathbb{C})$ so that

$$[\alpha, \xi] = \begin{bmatrix} [A, X] & BY - XB \\ CX - YC & [D, Y] \end{bmatrix}.$$

The condition that $[\alpha, \xi] \in \mathcal{O}(m, \mathbb{C}) \oplus \mathcal{O}(n, \mathbb{C})$ implies that $BY - XB = 0$ and $CX - YC = 0$. However, if $BY - XB = 0$ for all $X \in \mathcal{O}(m, \mathbb{C})$ and all $Y \in \mathcal{O}(n, \mathbb{C})$, then we may take $Y$ to be the zero matrix and, since $m \geq 2$,
For any group $G$ and subgroup $H$, we denote by $N_G(H)$ the normaliser of $H$ in $G$. When $G$ is an algebraic group and $H$ is an algebraic subgroup, $N_G(H)$ is also an algebraic subgroup of $G$. In particular, the normaliser $N(n_1, \ldots, n_f)$ of $O(n_1, \mathbb{C}) \times \cdots \times O(n_f, \mathbb{C})$ in $O(n_1 + \cdots + n_f, \mathbb{C})$, is an algebraic subgroup of $O(n_1 + \cdots + n_f, \mathbb{C})$. It follows that $N(n_1, \ldots, n_f)$ is a complex Lie group; moreover, when each $n_i \geq 2$, it follows from Proposition 1.3 that $N(n_1, \ldots, n_f)$ has the same identity component as $O(n_1, \mathbb{C}) \times \cdots \times O(n_f, \mathbb{C})$. Since any linear algebraic group over $\mathbb{C}$ has only finitely many connected components [2, p. 86]), we see that

**Corollary 1.4.** Let $N(n_1, \ldots, n_f)$ be the normaliser of $O(n_1, \mathbb{C}) \times \cdots \times O(n_f, \mathbb{C})$ in $O(n_1 + \cdots + n_f, \mathbb{C})$; then $O(n_1, \mathbb{C}) \times \cdots \times O(n_f, \mathbb{C})$ has finite index in $N(n_1, \ldots, n_f)$ provided that each $n_i \geq 2$.

**Proposition 1.5.** Let $L$ be a finitely generated free abelian group, and let $(\langle , \rangle)$: $L \times L \to \mathbb{Z}$ be a nondegenerate symmetric integral bilinear form which splits as a direct sum

$$
(L, \langle , \rangle) \cong (L_1, \langle , \rangle) \perp (L_2, \langle , \rangle) \perp \cdots \perp (L_f, \langle , \rangle)
$$

where $f \geq 2$, and each $\text{rk}_\mathbb{Z}(L_i) \geq 2$. Let $G$ (resp. $G_i$) be the linear algebraic group whose group of $k$-rational points is $\text{Aut}_k(L \otimes k, \langle , \rangle)$, (resp. $\text{Aut}_k(L_i \otimes k, \langle , \rangle)$), and let

$$
H = G_1 \times \cdots \times G_f \subset G;
$$

then $N_G(H) \cap \text{Aut}_\mathbb{Z}(L, \langle , \rangle)$ contains $\text{Aut}_\mathbb{Z}(L_1, \langle , \rangle) \times \cdots \times \text{Aut}_\mathbb{Z}(L_f, \langle , \rangle)$ as a subgroup of finite index.

**Proof.** Put $\lambda_i = \text{rk}_\mathbb{Z}(L_i)$, and $\Lambda = \sum \lambda_i$. $H$ and $N_G(H)$ are both linear algebraic subgroups of $G$, defined over $\mathbb{Q}$, and the groups of real points, $H_\mathbb{R}$ and $(N_G(H))_\mathbb{R}$ respectively, are Lie groups possessing only finitely many connected components. Observe that $G_\mathbb{C}$ (respectively $G_i, \mathbb{C}$) is isomorphic to $O(\Lambda, \mathbb{C})$ (respectively $O(\lambda_i, \mathbb{C})$), so that, by Corollary 1.4, $H_\mathbb{C}$ is a subgroup of finite index in $(N_G(H))_\mathbb{C}$. Thus the identity components of the corresponding real groups are equal; that is, $H_{\mathbb{R}, 0} = (N_G(H))_{\mathbb{R}, 0}$. The conclusion follows since $\text{Aut}_\mathbb{Z}(L_1, \langle , \rangle) \times \cdots \times \text{Aut}_\mathbb{Z}(L_f, \langle , \rangle)$ and $N_G(H) \cap \text{Aut}_\mathbb{Z}(L, \langle , \rangle)$ are both arithmetic in $N_G(H)$, and $N_G(H) \cap \text{Aut}_\mathbb{Z}(L, \langle , \rangle)$ contains $\text{Aut}_\mathbb{Z}(L_1, \langle , \rangle) \times \cdots \times \text{Aut}_\mathbb{Z}(L_f, \langle , \rangle)$. $\square$

### 2. Normal subdirect products

By a product structure on a group $G$ we mean a finite sequence $\mathcal{F} = (G_r)_{1 \leq r \leq n}$ of (nontrivial) normal subgroups of $G$ such that $G$ is the internal direct product $G = G_1 \circ \cdots \circ G_n$; that is, each $g \in G$ can be expressed uniquely as a product $g = g_1 \cdots g_n$ with $g_i \in G_i$. For a group $G$ having a product structure $\mathcal{F} = (G_r)_{1 \leq r \leq n}$, we identify $G$ with the external direct product $\prod_{j=1}^n G_j$. Let $\pi_i: \prod_{j=1}^n G_j \to G_i$ be the projection onto the $i$th factor; a subgroup $H$
of \( \prod_{i=1}^{n} G_i \) is a subdirect product of \( G \) (or more accurately, of \( \mathcal{F} \)) when \( \pi_i(H) = G_i \) for each \( i \). Let \( S(G_1, \ldots, G_n) \) the set of normal subdirect products of \( G_1 \circ \cdots \circ G_n \); that is, subdirect products which are also normal subgroups.

For any group \( H \), let \( \nu: H \to H^{ab} \) denote the canonical map onto the abelianisation \( H^{ab} = H/[H, H] \). To any product structure \( \mathcal{F} = (G_r)_{1 \leq r \leq n} \), we may associate its abelianisation \( \mathcal{F}^{ab} = (G_r^{ab})_{1 \leq r \leq n} \). Moreover, the abelianisation map \( \nu: G_1 \circ \cdots \circ G_n \to G_1^{ab} \circ \cdots \circ G_n^{ab} \) induces a mapping

\[
\nu^{-1}: S(G_1^{ab}, \ldots, G_k^{ab}) \to S(G_1, \ldots, G_k)
\]

by means of \( H \mapsto \nu^{-1}(H) \). We have shown elsewhere [3, Proposition 1.2] that

**Proposition 2.1.** For any product structure \( \mathcal{F} = (G_r)_{1 \leq r \leq n} \)

\[
\nu^{-1}: S(G_1^{ab}, \ldots, G_k^{ab}) \to S(G_1, \ldots, G_n)
\]

is bijective.

The following result of [3, Corollary 3.6] is important in the sequel.

**Theorem 2.2.** Let \( H \) be a normal subdirect product of \( G_1 \circ \cdots \circ G_n \). Then \( H \) is finitely generated (as a group, not merely as a normal subgroup) if and only if each \( G_i \) is finitely generated.

The conclusion of Theorem 2.2 is false if the assumption of normality on \( H \) is dropped.

### 3. A CONSTRUCTION FOR ABELIAN SUBDIRECT PRODUCTS

Let \( B \) be an infinite finitely generated abelian group. By an oriented splitting for \( B \), we shall mean a triple \( X \) of the form \( X = (M_X, N_X, e_X) \), where \( B/\text{Tor}(B) = M_X \oplus N_X \) in which \( N_X \) is free of rank \( 1 \), and \( e_X \in N_X \) is a generator. We denote by \( \mathcal{S}(B) \) the set of oriented splittings of \( B \). Clearly the group \( \text{Aut}(B/\text{Tor}(B)) \) acts transitively on \( \mathcal{S}(B) \). Since \( \text{Tor}(B) \) is a characteristic subgroup of \( B \), there is a natural epimorphism \( \text{Aut}(B) \to \text{Aut}(B/\text{Tor}(B)) \), from which we see that \( \text{Aut}(B) \) also acts transitively on \( \mathcal{S}(B) \).

We now consider subdirect products of abelian groups; it is more convenient to write our groups additively, and to confuse direct products with direct sums. Thus suppose that \( A = A_1 \oplus A_2 \) where \( A_1 \) is a free abelian group of rank \( r_1 \geq 2 \), and \( A_2 \) is a finitely generated abelian group such that \( A_2/\text{Tor}(A_2) \) has rank \( r_2 \geq 1 \).

Let \( X = (M_X, N_X, e_X) \) be an oriented splitting for \( A_1 \), and \( Y = (M_Y, N_Y, e_Y) \) an oriented splitting for \( A_2/\text{Tor}(A_2) \). Let \( \delta(X, Y) \) denote the subgroup of \( A_1 \oplus A_2/\text{Tor}(A_2) \) defined by

\[
\delta(X, Y) = M_X \oplus (e_X + e_Y) \oplus M_Y,
\]

and let \( \Delta(X, Y) \) denote the preimage of \( \delta(X, Y) \) in \( A = A_1 \oplus A_2 \), under the natural mapping

\[
\psi: A_1 \oplus A_2 \to A_1 \oplus A_2/\text{Tor}(A_2).
\]

It is easy to see that \( \Delta(X, Y) \) is a (necessarily normal) subdirect product of \( A_1 \oplus A_2 \). The group \( \text{Aut}(A_1, A_2) \) of product preserving automorphisms of \( A_1 \oplus A_2 \) acts naturally on \( S(A_1, A_2) \). Since \( \text{Aut}(A_1) \) imbeds naturally in \( \text{Aut}(A_1, A_2) \), by extending its natural action on \( A_1 \) via the identity on \( A_2 \), we see that
$\text{Aut}(A_1)$ also acts naturally on $S(A_1, A_2)$. On taking $\Delta = \Delta(X, Y)$ for some suitable oriented splittings $X = (M_X, N_X, \varepsilon_X)$ and $Y = (M_Y, N_Y, \varepsilon_Y)$ for $A_1$ and $A_2/\text{Tor}(A_2)$ respectively, we obtain

**Theorem 3.1.** Let $A_1, A_2$ be finitely generated abelian groups such that $A_1$ is free abelian of rank $r_1 \geq 2$, and $A_2/\text{Tor}(A_2)$ has rank $r_2 \geq 1$. Then there is a subdirect product $\Delta \subset A_1 \oplus A_2$, and an infinite subset $\Theta \subset \text{Aut}(A_1)$ such that $\theta(\Delta) \neq \sigma(\Delta)$ for $\theta, \sigma \in \Theta$ such that $\theta \neq \sigma$.

### 4. Infinite Families of Nonconjugate Isomorphic Imbeddings

Let $\Lambda_1$ be a nonabelian free group of finite rank $m \geq 2$, and let $\Lambda_2$ be a finitely generated group such that $\Lambda_2^{\text{ab}}$ is infinite. Put $A_i = \Lambda_i^{\text{ab}}$ for $i = 1, 2$.

Since $A_1 \cong \mathbb{Z}^m$ and $A_2$ maps epimorphically onto $\mathbb{Z}$, we may apply Theorem 3.1 to obtain the existence of a faithfully indexed family $(\Theta(\Lambda))_{\theta \in \Theta}$ of normal subdirect products of $A_1 \oplus A_2$, where $\theta$ ranges over some infinite subset $\Theta$ of $\text{Aut}(A_1) \cong \text{GL}_m(\mathbb{Z})$. As we have seen, $\nu^{-1}: S(A_1^{\text{ab}}, A_2^{\text{ab}}) \rightarrow S(\Lambda_1, \Lambda_2)$ is bijective. Put $\Gamma = \nu^{-1}(\Delta)$; then $\Gamma$ is a normal subdirect product of $\Lambda_1 \times \Lambda_2$, and so is finitely generated by Theorem 2.2. Furthermore, the group $\text{Aut}(\Lambda_1) \times \text{Aut}(\Lambda_2)$ acts naturally on subgroups of $\Lambda_1 \times \Lambda_2$, and the orbit of $\Gamma$ under this action consists entirely of normal subdirect products of $\Lambda_1 \times \Lambda_2$. In fact, we need only consider the subgroup $\text{Aut}(\Lambda_1) \cong \text{Aut}(\Lambda_1) \times \{1\}$ of $\text{Aut}(\Lambda_1) \times \text{Aut}(\Lambda_2)$. Since $\Lambda_1$ is free, by a theorem of Nielsen [7], for each automorphism $\theta$ of $A_1 = \Lambda_1^{\text{ab}}$ we may choose a lifting of $\theta$ to an automorphism $\hat{\theta}$ of $\Lambda_1 \cong \Lambda_1 \times \{1\}$. Put $\Gamma_{\theta} = \hat{\theta}(\Gamma)$. It is clear that $\Gamma_{\theta}$ is isomorphic to $\Gamma$. We may summarise our progress so far thus:

**Theorem 4.1.** Let $\Lambda_1$ be a nonabelian free group of finite rank $m \geq 2$, and let $\Lambda_2$ be a finitely generated group which maps epimorphically onto $\mathbb{Z}$; then there is a subset $\Theta \subset \text{Aut}(\Lambda_1)$ which parametrises an infinite family $(\Gamma_{\theta})_{\theta \in \Theta}$ of mutually isomorphic finitely generated normal subdirect products of $\Lambda_1 \times \Lambda_2$ with the property that $\Gamma_{\theta} \neq \Gamma_{\sigma}$ for $\theta \neq \sigma$.

The analogue of Theorem 4.1 in which $\Lambda_1$ is replaced by the fundamental group of a closed orientable surface is also true; we proceed to outline the necessary variations.

Let $\Sigma_g^+$ denote the closed orientable surface of genus $g \geq 2$, and let $\Sigma_g^+$ denote its fundamental group;

$$\Sigma_g^+ = \left\langle X_1, \ldots, X_g, Y_1, \ldots, Y_g : \prod_{r=1}^{g} X_rY_rX_r^{-1}Y_r^{-1} \right\rangle.$$

We may identify the abelianisation $H_1(\Sigma_g^+; \mathbb{Z})$ of $\Sigma_g^+$ with $\mathbb{Z}^{2g}$, and the intersection form on $\Sigma_g^+$ gives rise to a nondegenerate skew-symmetric bilinear form $\langle \ , \, \rangle : \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \rightarrow \mathbb{Z}$. With this identification, symplectic automorphisms of $\mathbb{Z}^{2g}$ lift back to automorphisms of $\Sigma_g^+ = \pi_1(\Sigma_g^+)$, with transvections lifting back to Dehn twists.

Let $\{e_1, \ldots, e_g, \phi_1, \ldots, \phi_g\}$ be the standard symplectic basis for $\langle \ , \, \rangle$; that is,

$$\langle e_i, e_j \rangle = \langle \phi_i, \phi_j \rangle = 0; \quad \langle e_i, \phi_j \rangle = \delta_{ij}.$$
In constructing subdirect products in $A_1 \oplus A_2$, as in §3, where now $A_1 = H_1(\mathbb{Z}^g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$, we take our “basepoint splitting” $X$ of $A_1$ to be of the form $A_1 = M_X \oplus N_X$, where $\text{Span}_\mathbb{Z}\{e_1, \ldots, e_g\} \subset M_X$ and $N_X \subset \{\phi_1, \ldots, \phi_g\}$. There is an infinite set of such splittings which we parametrise by suitable elements of the group $\text{Sp}_{2g}(\mathbb{Z})$. With these modifications, we obtain the following analogue of Theorem 4.1.

**Theorem 4.2.** Let $\Lambda_1$ be a surface group of genus $g \geq 2$, and let $\Lambda_2$ be a finitely generated group which maps epimorphically onto $\mathbb{Z}$; then there is a subset $\Theta \subset \text{Sp}_{2g}(\mathbb{Z})$ which parametrises an infinite family $(\Gamma_\theta)_{\theta \in \Theta}$ of mutually isomorphic finitely generated normal subdirect products of $\Lambda_1 \times \Lambda_2$ with the property that $\Gamma_\theta \neq \Gamma_{\alpha}$ for $\theta \neq \alpha$.

**Theorem 4.3.** Let $(\langle, \rangle): L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group $L$, such that $(L, \langle, \rangle)$ splits as an orthogonal direct sum

$$(L, \langle, \rangle) \cong (L_1, \langle, \rangle) \perp (L_2, \langle, \rangle),$$

where $(L_1, \langle, \rangle)$ has signature $(2, 1)$, and $\text{Aut}_\mathbb{Z}(L_2, \langle, \rangle)$ has a subgroup of finite index which maps epimorphically onto $\mathbb{Z}$. Then there exists an infinite family $(\Gamma_\sigma)_{\sigma \in \Sigma}$ of mutually isomorphic finitely generated nonconjugate subgroups of $\text{Aut}_\mathbb{Z}(L_1, \langle, \rangle) \times \text{Aut}_\mathbb{Z}(L_2, \langle, \rangle)$.

**Proof.** $\text{Aut}_\mathbb{Z}(L_1, \langle, \rangle)$ is a finitely generated linear group, and so, by Selberg’s Theorem, has a torsion free subgroup, $\Lambda_i$ say, of finite index. Suppose that $(L_1, \langle, \rangle)$ has signature $(2, 1)$; if $(L_1, \langle, \rangle)$ is isotropic, then $\Lambda_1$ is free, whilst if $(L_1, \langle, \rangle)$ is anisotropic, then $\Lambda_1$ is a surface group. Either way, if $\Lambda_2$ maps epimorphically onto $\mathbb{Z}$, we may apply the results of Theorems 4.1 and 4.2 to conclude that there is an infinite family, $(\Gamma_\sigma)_{\theta \in \Theta}$, of mutually isomorphic finitely generated normal subdirect products of $\Lambda_1 \times \Lambda_2$. Moreover, since the family $(\Gamma_\sigma)_{\theta \in \Theta}$ consists of normal subgroups of $\Lambda_1 \times \Lambda_2$, we see that no $\Gamma_\sigma$ is conjugate in $\Lambda_1 \times \Lambda_2$ to any $\Gamma_{\alpha}$ for $\theta \neq \alpha$.

Since $\Lambda_1 \times \Lambda_2$ has finite index in $\text{Aut}_\mathbb{Z}(L_1, \langle, \rangle) \times \text{Aut}_\mathbb{Z}(L_2, \langle, \rangle)$, each $\Gamma_{\sigma}$ is conjugate in $\text{Aut}_\mathbb{Z}(L_1, \langle, \rangle) \times \text{Aut}_\mathbb{Z}(L_2, \langle, \rangle)$ to at most finitely many $\Gamma_{\alpha}$. In particular, we may choose an infinite subfamily $(\Gamma_{\sigma})_{\sigma \in \Sigma}$ so that no two distinct elements are conjugate in $\text{Aut}_\mathbb{Z}(L_1, \langle, \rangle) \times \text{Aut}_\mathbb{Z}(L_2, \langle, \rangle)$. $\square$

Although not conjugate in $\text{Aut}_\mathbb{Z}(L_1, \langle, \rangle) \times \text{Aut}_\mathbb{Z}(L_2, \langle, \rangle)$, subgroups in the family $(\Gamma_{\sigma})_{\sigma \in \Sigma}$ just constructed may become conjugate in $\text{Aut}_\mathbb{Z}(L, \langle, \rangle)$. We show, however, that for each $\tau \in \Sigma$, the set $\{\sigma \in \Sigma: \Gamma_{\sigma} \text{ is conjugate to } \Gamma_{\tau} \text{ in } \text{Aut}_\mathbb{Z}(L, \langle, \rangle)\}$ is finite.

**Theorem 4.4.** Let $(\langle, \rangle): L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group $L$, such that $(L, \langle, \rangle)$ splits as an orthogonal direct sum

$$(L, \langle, \rangle) \cong (L_1, \langle, \rangle) \perp (L_2, \langle, \rangle)$$

where $(L_1, \langle, \rangle)$ has signature $(2, 1)$, and $\text{Aut}_\mathbb{Z}(L_2, \langle, \rangle)$ has a subgroup of finite index which maps epimorphically onto $\mathbb{Z}$. Then there exists an infinite family $(\Gamma_\omega)_{\omega \in \Omega}$ of mutually isomorphic finitely generated subgroups of $\text{Aut}_\mathbb{Z}(L, \langle, \rangle)$ such that $\Gamma_\omega$ is not conjugate, in $\text{Aut}_\mathbb{Z}(L, \langle, \rangle)$, to $\Gamma_\mu$ for $\omega \neq \mu$. 
Proof. Let $G$ (resp. $G_i$) be the linear algebraic group whose group of $k$-rational points is $\text{Aut}_k(L \otimes k, \langle , \rangle)$ (resp. $\text{Aut}_k(L_i \otimes k, \langle , \rangle)$), and let $\mathbb{H} = G_1 \times G_2 \subset G$. Let $\Gamma_\sigma$, $\Gamma_i$ be subgroups from the family constructed in Theorem 4.3, and suppose that for some $g \in \text{Aut}_Z(L, \langle , \rangle)$, $g\Gamma_\sigma g^{-1} = \Gamma_i$. Since $\Gamma_\sigma$, $\Gamma_i$ are normal subdirect products of $\Lambda_1 \times \Lambda_2$, then by [3],

$$[\Lambda_1, \Lambda_1] \times [\Lambda_2, \Lambda_2] \subset \Gamma_\sigma \cap \Gamma_i.$$ 

Since $(L_1, \langle , \rangle)$ has signature $(2, 1)$, it follows that $G_1$ is Q-simple, and $G_{1,R}$ is noncompact. The condition that $\text{Aut}_Z(L_2, \langle , \rangle)$ has a subgroup of finite index which maps epimorphically onto $\mathbb{Z}$ implies that $(L_2, \langle , \rangle)$ is indefinite, and that $\text{rk}_Z(L_2) \geq 3$. If $\text{rk}_Z(L_2) = 4$ then $G_2$ is Q-simple, and $G_{2,R}$ is noncompact. If $\text{rk}_Z(L_2) = 4$ then either $G_2$ is Q-simple, and $G_{2,R}$ is noncompact, or $G_2$ is a product $L_1 \times L_2$ where $L_1$ and $L_2$ are both Q-simple, and $L_{1,R}, L_{2,R}$ are both noncompact. Either way, if $L$ is a Q-simple factor of $G_1 \times G_2$, then $L_R$ is noncompact; applying (1.2) we conclude that $[\Lambda_1, \Lambda_1] = G_i$. Thus $[\Lambda_1, \Lambda_1] \times [\Lambda_2, \Lambda_2] = \mathbb{H}$. It now follows that $g \in N_\mathbb{H}(\mathbb{H}) \cap \text{Aut}_Z(L, \langle , \rangle).$

Denote the index of $\text{Aut}_Z(L_1, \langle , \rangle) \times \text{Aut}_Z(L_2, \langle , \rangle)$ in $N_\mathbb{H}(\mathbb{H}) \cap \text{Aut}_Z(L, \langle , \rangle)$ by $\alpha$. For each $\tau \in \Sigma$, the set $C_\tau = \{ \sigma \in \Sigma : \Gamma_\sigma \text{ is conjugate to } \Gamma_\tau \text{ in } \text{Aut}_Z(L, \langle , \rangle) \}$ has cardinality bounded by $\alpha$. By (1.5), $\alpha$ is finite, so that each $C_\tau$ is finite. Let $\Omega$ be a subset of $\Sigma$ obtained by choosing exactly one element from each $C_\tau$; then $\Omega$ is infinite, and the family $(\Gamma_\omega)_{\omega \in \Omega}$ consists of isomorphic finitely generated subgroups of $\text{Aut}_Z(L, \langle , \rangle)$, and has the desired property that $\Gamma_\omega$ is not conjugate, in $\text{Aut}_Z(L, \langle , \rangle)$, to $\Gamma_\mu$ for $\omega \neq \mu$. □

Analogously, we show

**Theorem 4.5.** Let $(\langle , \rangle) : L \times L \rightarrow \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group $L$, such that $(L, \langle , \rangle)$ splits as an orthogonal direct sum

$$(L, \langle , \rangle) \cong (L_1, \langle , \rangle) \perp (L_2, \langle , \rangle) \perp (L_3, \langle , \rangle)$$

where $(L_1, \langle , \rangle)$ has signature $(2, 1)$, $\text{Aut}_Z(L_2, \langle , \rangle)$ has a subgroup of finite index which maps epimorphically onto $\mathbb{Z}$, and where $(L_3, \langle , \rangle)$ is indefinite with $\text{rk}_Z(L_3) \geq 3$. Then there exists an infinite family $(\Delta_\omega)_{\omega \in \Omega}$ of mutually isomorphic finitely generated nonconjugate subgroups of $\text{Aut}_Z(L, \langle , \rangle)$.

**Proof.** Let $(\Gamma_\sigma)_{\sigma \in \Sigma}$ be the family of mutually isomorphic finitely generated nonconjugate subgroups of $\text{Aut}_Z(L_1, \langle , \rangle) \times \text{Aut}_Z(L_2, \langle , \rangle)$ constructed in Theorem 4.3, and for each $\sigma \in \Sigma$, put

$$\Delta_\sigma = \Gamma_\sigma \times \text{Aut}_Z(L_3, \langle , \rangle) \subset \text{Aut}_Z(L_1, \langle , \rangle) \times \text{Aut}_Z(L_2, \langle , \rangle) \times \text{Aut}_Z(L_3, \langle , \rangle).$$

Let $G$ (resp. $G_i$) be the linear algebraic group whose group of $k$-rational points is $\text{Aut}_k(L \otimes k, \langle , \rangle)$ (resp. $\text{Aut}_k(L_i \otimes k, \langle , \rangle)$), and let $\mathbb{H} = G_1 \times G_2 \times G_3 \subset G$. As in the proof of Theorem 4.4, we obtain $[\Lambda_1, \Lambda_1] \times [\Lambda_2, \Lambda_2] = G_1 \times G_2$.

If $\text{rk}_Z(L_3) \neq 4$ then $G_3$ is Q-simple, and $G_{3,R}$ is noncompact. If $\text{rk}_Z(L_3) = 4$, then, since $(L_3, \langle , \rangle)$ is indefinite, either the identity component of $G_{3,R}$ is isomorphic to $\text{SO}(3, 1)$ and $G_3$ is Q-simple, or $G_{3,R}$ is locally isomorphic to a product $\text{SO}(2, 1) \times \text{SO}(2, 1)$; either way, if $L$ is a Q-simple factor of $G$,
then $L_{\mathbb{R}}$ is noncompact, so that we may apply Proposition 1.2 to conclude that $[\Lambda_3, \Lambda_3] = G_3$, and

$$[\Lambda_1, \Lambda_1] \times [\Lambda_2, \Lambda_2] \times [\Lambda_3, \Lambda_3] = G_1 \times G_2 \times G_3.$$  

As in the proof of Theorem 4.4, for each $\tau \in \Sigma$, the cardinality of the set

$$C_\tau = \{ \sigma \in \Sigma : \Delta_\sigma \text{ is conjugate to } \Delta_\tau \text{ in } \text{Aut}_\mathbb{Z}(L, \langle , \rangle) \}$$

is bounded by the index, $\alpha$, of $\text{Aut}_\mathbb{Z}(L_1, \langle , \rangle) \times \text{Aut}_\mathbb{Z}(L_2, \langle , \rangle) \times \text{Aut}_\mathbb{Z}(L_3, \langle , \rangle)$ in $N_G(H) \cap \text{Aut}_\mathbb{Z}(L, \langle , \rangle)$. By Proposition 1.5, $\alpha$ is finite, so that each $C_\tau$ is finite. Let $\Omega$ be a subset of $\Sigma$ obtained by choosing exactly one element from each $C_\tau$; then $\Omega$ is infinite, and the family $(\Delta_\omega)_{\omega \in \Omega}$ consists of isomorphic finitely generated subgroups of $\text{Aut}_\mathbb{Z}(L, \langle , \rangle)$, and has the desired property that $\Delta_\omega$ is not conjugate, in $\text{Aut}_\mathbb{Z}(L, \langle , \rangle)$, to $\Delta_\mu$ for $\omega \neq \mu$.  

The referee points out that the condition “$\text{Aut}_\mathbb{Z}(L_2, \langle , \rangle)$ has a subgroup of finite index which maps epimorphically onto $\mathbb{Z}$”, is precisely the same as requiring that $(L_2, \langle , \rangle)$ have signature $(n, 1)$ for some $n \geq 2$. Indeed, if $\text{Aut}_\mathbb{Z}(L_2, \langle , \rangle)$ has a subgroup $\Gamma$ of finite index which maps epimorphically onto $\mathbb{Z}$, then $H_1(\Gamma, \mathbb{Z})$ is infinite, so that, by Kazhdan's Theorem [4], $(L_2, \langle , \rangle)$ has signature $(n, 1)$ for some $n \geq 2$. Conversely, Millson [6, §4] has shown that for any nondegenerate integral quadratic form $(L, \langle , \rangle)$ of signature $(n, 1)$, with $n \geq 2$, there exists a subgroup $\Gamma$ of finite index in $\text{Aut}_\mathbb{Z}(L, \langle , \rangle)$ for which $H_1(\Gamma, \mathbb{Z})$ is infinite; in particular, $\Gamma$ maps epimorphically onto $\mathbb{Z}$. Combining this observation with Theorems 4.4 and 4.5, we obtain

**Corollary 4.6.** Let $(\langle, \rangle) : L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group $L$, such that $(L, \langle, \rangle)$ splits as an orthogonal direct sum

$$(L, \langle, \rangle) \cong (L_1, \langle, \rangle) \perp (L_2, \langle, \rangle) \perp (L_3, \langle, \rangle)$$

where $(L_1, \langle, \rangle)$ has signature $(2, 1)$, $(L_2, \langle, \rangle)$ has signature $(n, 1)$ for some $n \geq 2$, and either $L_3 = 0$ or $(L_3, \langle, \rangle)$ is indefinite with $\text{rk}_\mathbb{Z}(L_3) \geq 3$. Then there exists an infinite family $(\Delta_\omega)_{\omega \in \Omega}$ of mutually isomorphic finitely generated nonconjugate subgroups of $\text{Aut}_\mathbb{Z}(L, \langle, \rangle)$.

Our concern in this paper has been with conjugacy of subgroups within the discrete group $\text{Aut}_\mathbb{Z}(L, \langle, \rangle)$. From a different viewpoint, our results can be seen as a failure of the rigidity property for subgroups of infinite covolume within the corresponding Lie group $\text{Aut}_\mathbb{R}(L \otimes_\mathbb{Z} \mathbb{R}, \langle, \rangle \otimes 1)$; recall that the groups $\Gamma_\tau$ we construct all have infinite index in $\text{Aut}_\mathbb{Z}(L, \langle, \rangle)$. If $G$ is a noncompact linear almost simple Lie group with $\text{rank}_\mathbb{R} \geq 2$, then in consequence of the super-rigidity theorem of Margulis, when $\Delta$ is a discrete subgroup of finite covolume in $G$ there are only finitely many $G$-conjugacy classes of discrete finitely generated subgroups isomorphic to $\Delta$. The arguments of the present paper can be extended to show that under the hypothesis “$\Delta$ is finitely generated, discrete, of infinite covolume in $G$”, the number of $G$-conjugacy classes of discrete finitely generated subgroups isomorphic to $\Delta$ becomes infinite in general. We will pursue this idea more fully elsewhere.
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