THE STRUCTURE OF THE SET OF SINGULAR POINTS OF A CODIMENSION 1 DIFFERENTIAL SYSTEM ON A 5-MANIFOLD

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Abstract. Generic modules $V$ of vector fields tangent to a 5-dimensional smooth manifold $M$, generated locally by four not necessarily linearly independent fields $X_1, X_2, X_3, X_4$, are considered. Denoting by $\omega$ the 1-form $X_4 \wedge X_3 \wedge X_2 \wedge X_1 \wedge \Omega$ conjugated to $V$ ($\Omega$ is a fixed local volume form on $M$), the loci of singular behavior of $V$: $M_{\text{deg}}(V) = \{p \in M | \omega(p) = 0\}$ and $M_{\text{sing}}(V) = \{p \in M | \omega \wedge (d\omega)(p) = 0\}$ are handled. The local classification of this pair of sets is carried out (outside a curve and a discrete set in $M_{\text{deg}}(V)$) up to a smooth diffeomorphism. In the most complicated case, around points of a codimension 3 submanifold of $M$, $M_{\text{sing}}(V)$ turns out to be diffeomorphic to the Cartesian product of $\mathbb{R}^2$ and the Whitney’s umbrella in $\mathbb{R}^3$.

1

We are going to consider generic differential systems of codimension 1 in the tangent bundle over a $C^\infty$ manifold $M$ of dimension 5. Unfortunately, this notion in different papers is used in many senses. We mean by it the module generated locally, over the ring of smooth functions on $M$, by four vector fields (the fields may happen to be linearly dependent at some points).

The equivalent framework for this investigation is that of singularities of $k$-tuples of vector fields on $\mathbb{R}^n$ set in $[JP]$ (here $k = 4$, $n = 5$). The paper can be considered as a continuation of similar research in dimension 3 (included primarily in $[JP]$) and in dimension 4 $[MR, M1-M3]$. The study will be local, so that we shall often use the language of germs of sets, functions, vector fields, etc.

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We shall assume once and for all that all the considered objects are $C^\infty$ smooth, and this will not be additionally inserted in the statements. A point $p \in M$ is of interest to us when the germ at $p$ of the considered system $V$ is not equivalent to the Darboux model

\[
\text{span} \left( \frac{\partial}{\partial y} - u \frac{\partial}{\partial x}, \frac{\partial}{\partial z} - v \frac{\partial}{\partial x}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right),
\]

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or, equivalently, to the Pfaffian equation $dx + u dy + v dz = 0$ (throughout this paper we use $x, y, z, u, v$ rather than $x_1, x_2, y_1, y_2, z$).

In other words, if (all this locally) $\Omega$ is a volume form on $M$ and $X_1, X_2, X_3, X_4$ generate $V$, then one can define a 1-form conjugated to $V$, $\omega(\cdot) := \Omega(X_1, X_2, X_3, X_4, \cdot)$, which vanishes at every point $p$ where $\dim V(p) \leq 3$. Such $\omega$ is not defined invariantly, but the Pfaffian equation it represents already is. So we are interested in either (a) $\omega(p) = 0$ or (b) $\omega(p) \neq 0$ and $\omega \wedge (d\omega)^2|_p = 0$. The latter means that the class of the Pfaffian equation $\omega = 0$ at $p$ is not 0 (regarding this notion, see [F, Ma]).

Throughout this paper the union of the geometric loci of (a) and (b) is denoted by $M_{\text{sing}}(V)$, and the locus of (a) alone by $M_{\text{deg}}(V)$.

Note. We consider only typical degenerations, i.e., those that are unavoidable under arbitrary small perturbations of $V$. Therefore we assume that $\dim V(p) = 3$ for every $p \in M_{\text{deg}}(V)$ (the falling of $\dim V(p)$ by 2 is already a codimension 6 feature and is not typical by Transversality theorem, see [AGV, Ma]).

The normal forms of the smooth classification of germs of generic $V$'s were found at generic points of $M_{\text{sing}}(V) \setminus M_{\text{deg}}(V)$ (i.e., in codimension 1) by Martinet [Ma], and recently at points of certain 2-dimensional surface included in it (where, typically, the next degeneration having codimension 3 materializes) by Zhitomirskii [Z1, Z2]. All these normal forms are simple, i.e., moduleless; such normal forms are called local models.

As regards $M_{\text{deg}}(V)$, which typically has codimension 2 (see Proposition in §7), even at its generic points normal forms are unknown. This is in contradistinction to dimensions 3 and 4, where local models at such points of the respective loci of $\dim V$ falling by 1 were found by Jakubczyk and Przytycki [JP] in the case of $\dim M = 3$, and by Mormul and Roussarie [MR] for $\dim M = 4$.

We suspect that in dimension 5 at points of $M_{\text{deg}}(V)$ there are no models, and moreover normal forms contain functional parameters. Yet already the problem of classification of the pair of germs of sets $M_{\text{sing}}(V), M_{\text{deg}}(V)$ turns out to be interesting. The paper is devoted to this problem.

**Main Theorem.** For a generic differential system $V$ on $M$ there exist subsets of $M_{\text{deg}}(V)$: a curve $M_1$ and a set of isolated points $M_2$ such that the germ of $(M_{\text{sing}}(V), M_{\text{deg}}(V))$ at any point of $M_{\text{sing}}(V) \setminus (M_1 \cup M_2)$ is equivalent to one of the following germs:

(A) germ of 4-manifold, $\varnothing$;
(B) germ of 3-manifold, $M_{\text{deg}}(V) = M_{\text{sing}}(V)$;
(C) germ of stratified manifold with strata of dimensions 4, 4, 3 (the last in the intersection of closures of the first and second), $M_{\text{deg}}(V)$ = the 3-dimensional stratum (see Figure 1);
(D) the germ of the Whitney's umbrella $\times \mathbb{R}^2$, (its handle) $\times \mathbb{R}^2$ (see Figure 2).

In other words, in suitable coordinates $x, y, z, u, v$, the pair of sets $M_{\text{sing}}(V), M_{\text{deg}}(V)$ is locally given by the equations
(A) \[ M_{\text{sing}} = \{ x = 0 \}, \quad M_{\text{deg}} = \emptyset; \]
(B) \[ M_{\text{sing}} = M_{\text{deg}} = \{ x = y = 0 \}; \]
(C) \[ M_{\text{sing}} = \{ xy = 0 \}, \quad M_{\text{deg}} = \{ x = y = 0 \}; \]
(D) \[ M_{\text{sing}} = \{ x^2 = y^2 \}, \quad M_{\text{deg}} = \{ x = y = 0 \}. \]
In cases (B)-(D), in the mentioned coordinates,

\[ V(0) = \text{span}(\partial/\partial x, \partial/\partial y, \partial/\partial z). \]

As we said in §3, in cases (B)-(D) local models probably do not exist. Nevertheless, we are able then to simplify (locally) \( V \) significantly. This, among other things, will come in handy in proving the Main Theorem in §11.

Let \( m_{x,y} \) stand for the ideal of germs at \( 0 \in \mathbb{R}^5 \) of functions vanishing on \( \{x = y = 0\} \); \( m_{x,y}^k \) is its \( k \)th power. For brevity we write the same symbol for the set of germs at \( 0 \) of 1-forms with coefficients in \( m_{x,y}^k \). In the sequel \( j_{x,y}^k \) will denote the natural projection \( \mathcal{F}_0^5 \to \mathcal{F}_0^5/m_{x,y}^{k+1} \) (\( \mathcal{F}_0^5 \) = the ring of germs at \( 0 \) of smooth functions on \( \mathbb{R}^5 \)). We shall apply \( j_{x,y}^k \) to germs of 1-forms in the natural sense, too.

The mentioned simplified description of \( V \) (a normal form) is given in terms of the conjugated form \( \omega \) (cf. §2).

**Theorem on normal form.** Let \( p \in M_{\text{deg}}(V) \setminus M_1 \). There exist coordinates \( x, y, z, u, v \) (vanishing at \( p \)) s.t. the germ of \( \omega \) at \( p \) has the form

\[
(1) \quad x \, du + y \, dv + f \, dz, \quad f \in m_{x,y}^2.
\]

Observe that in these coordinates

\[
(2) \quad M_{\text{deg}}(V) = \{x = y = 0\}.
\]

**Corollary 1.** At any \( p \in M_{\text{deg}}(V) \), \( j_p^1(\omega \wedge (d\omega)^2) = 0 \).

Indeed, in the normal form coordinates

\[
(3) \quad \omega \wedge (d\omega)^2 = 2(x f_x + y f_y - f) \, dx \wedge dy \wedge dz \wedge du \wedge dv,
\]

and \( f, x f_x, y f_y \in m_{x,y}^2 \). (Here and in the sequel the symbol of a function followed by a lowercase letter subscript denotes the respective function's partial derivative.) \( \square \)

**Remark 1.** In the normal form coordinates Corollary 1 can be written compactly as \( j_{x,y}^1(\omega \wedge (d\omega)^2) = 0 \).

**Corollary 2.** Let \( p \in M_{\text{deg}}(V) \setminus M_1 \). There exist coordinates \( x, y, z, u, v \) such that the germ of \( V \) at \( p \) is generated by vector fields \( \partial/\partial x, \partial/\partial y, \partial/\partial z + f_1 \partial/\partial u + f_2 \partial/\partial v, y \partial/\partial u - x \partial/\partial v \), where \( f_1, f_2 \in m_{x,y} \).

**Proof.** Let \( \omega \) be the 1-form conjugated to \( V \) and \( x, y, z, u, v \) be the coordinates of n.f. (1). We can write \( f = -xf_1 - yf_2 \) with \( f_1, f_2 \in m_{x,y} \). Any v.f. \( \xi \) from \( V \),

\[
\xi = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + A_4 \frac{\partial}{\partial u} + A_5 \frac{\partial}{\partial v},
\]

satisfies \( \omega(\xi) \equiv 0 \) or, equivalently, \( x A_4 + y A_5 - (x f_1 + y f_2) A_3 \equiv 0 \). The latter relation implies \( A_4 - f_1 A_3 = y g \) and \( A_5 - f_2 A_3 = -x g \) for some function \( g \).

Thus

\[
\xi = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \left( \frac{\partial}{\partial z} + f_1 \frac{\partial}{\partial u} + f_2 \frac{\partial}{\partial v} \right) + g \left( y \frac{\partial}{\partial u} - x \frac{\partial}{\partial v} \right). \quad \square
\]

**Remark 2.** It follows from Corollary 2 that the generators of \( V \) and their first order Lie brackets span at \( 0 \) the full 5-dimensional tangent space. (This prop-
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Property of $V$ is weaker than the transversality assumed in §8 and occurs also without that transversality.)

We postpone the proof of the theorem on normal form till §12; the final part of it occupies §13.

6

In proving the Main Theorem we shall represent $V$ by a triple $(\omega_1, \omega_2, Y)$, where $\omega_1$, $\omega_2$ are 1-forms, and $Y$ is a vector field.

Observation 1. For $p \in M_{\deg}(V)$ there exist germs at $p$ of a vector field $Y$ and of independent Pfaffian forms $\omega_1$ and $\omega_2$ such that

\[(4) \quad \omega := Y \wedge (\omega_1 \wedge \omega_2)\]

is conjugated to $V$ and $M_{\deg}(V) = \{\omega_1(Y) = \omega_2(Y) = 0\}$.

Passing to the $(\omega_1, \omega_2, Y)$’s is purposeful, since any $\omega$ conjugated to $V$ is highly nontypical among all differential 1-forms (it vanishes on a “big” set $M_{\deg}(V)$), while the objects in the triple $(\omega_1, \omega_2, Y)$ do not vanish at any point.

Proof. Let $V$ be generated by four vector fields $X_1$, $X_2$, $X_3$, $Y$, three of which are independent (for instance, $X_1$, $X_2$, $X_3$). The distribution spanned by them can be described by a Pfaffian system $\omega_1 = \omega_2 = 0$, and then $V = \text{span}(Y, \ker \omega_1 \cap \ker \omega_2)$. Now the statements of the observation are easy to verify. □

It seems to us that the above representation of $\omega$ is of its own interest and can be applied in many situations.

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Our first step in the proof of the Main Theorem consists in ensuring that

Proposition. For $V$ generic $M_{\deg}(V)$ is, if not empty, a smooth codimension 2 submanifold of $M$.

Proof. We use description (4) and consider the set $Q_1$ of 1-jets at $0 \in \mathbb{R}^3$ of 3-tuples $(\omega_1, \omega_2, Y)$. Let its subset $\tilde{Q}_1$ be given by equations

\[\omega_1(Y)(0) = \omega_2(Y)(0) = 0, \quad d(\omega_1(Y)) \wedge d(\omega_2(Y))|_0 = 0.\]

Clearly $\tilde{Q}_1$ has codimension 6 in $Q_1$. The standard use of Transversality theorem gives that for generic $V$ its 1-jet is nowhere included in $\tilde{Q}_1$. This implies the conclusion of the Proposition. □

Remark 3. The theory mentioned in §1, developed in [JP], yields, among many other things, that the locus of $\dim V = k - 1$ is smooth for a generic codimension $n - k$ differential system in $\mathbb{R}^n$.

8

Here is the definition (in invariant terms) of the sets $M_1, M_2 \subset M_{\deg}(V)$ occurring in the Main Theorem.

A point $p$ is included in $M_1$ iff $V(p)$ is not transversal to $M_{\deg}(V)$; a point $p$ is included in $M_2$ iff $j^2_p(\omega \wedge (d\omega)^2) = 0$.
Now we are going to prove

(i) $M_1$ is a smooth curve.

The proof is based on the following

**Lemma 1.** For a generic system $V$, at any $p \in M_{\text{deg}}(V)$, $V(p)$ is transversal to $M_{\text{deg}}(V) \iff \text{rank}(d\omega)^2(p) = 4$.

*Proof.* We take the $\omega$ from Observation 1. Computing at a point of $M_{\text{deg}}(V)$ and using Observation 1, we get $d\omega = d(\omega_1(Y)) \wedge \omega_2 - d(\omega_2(Y)) \wedge \omega_1$. Thus $(d\omega)^2 = -2\omega_1 \wedge \omega_2 \wedge d(\omega_1(Y)) \wedge d(\omega_2(Y))$. The condition rank$(d\omega)^2 = 4$ means that the kernels of the 1-forms entering the above formula intersect one another as sparingly as possible.

Because for $p \in M_{\text{deg}}(V)$, $V(p) = \ker \omega_1(p) \cap \ker \omega_2(p)$, and since, in view of $d(\omega_1(Y)) \wedge d(\omega_2(Y)) \neq 0$, $M_{\text{deg}}(V)$ is a smooth manifold and $T_p M_{\text{deg}}(V) = \ker d(\omega_1(Y)) \cap \ker d(\omega_2(Y))$ (cf. Proposition in §7 and Observation 1), we conclude that so intersect each other $V(p)$ and $T_p M_{\text{deg}}(V)$, which means transversality.

The implication $\Rightarrow$ uses the same arguments and the condition of smoothness of $M_{\text{deg}}(V)$ valid for a generic system $V$. □

**Note.** The statement on the right-hand side of Lemma 1 is here equivalent to $(d\omega)^2(p) \neq 0$.

*Proof of (i).* Let $p \in M_1$. By Lemma 1 rank$(d\omega)^2(p) < 4$, which means that the 1-jet at $p$ of $(\omega_1, \omega_2, Y)$ satisfies the conditions

$$\omega_1(Y)(p) = 0, \quad \omega_2(Y)(p) = 0, \quad \omega_1 \wedge \omega_2 \wedge d(\omega_1(Y)) \wedge d(\omega_2(Y))|_p = 0.$$

It is clear that these conditions distinguish a (stratified) codimension 4 manifold in the space $Q_1$ of 1-jets. One can show that its singular points form a set of codimension 6 in $Q_1$. Therefore (i) follows from Transversality theorem. □

In turn, we can show that

(ii) $M_2$ consists of isolated points.

*Proof.* $M_2$ is defined invariantly (see §8), and we prefer to work in the coordinates giving (1). Let

$$f = A(z, u, v)x^2 + B(z, u, v)xy + C(z, u, v)y^2 + \hat{f}, \quad \hat{f} \in m^3_{x, y}.$$

Then in view of (2) and (3) $M_2$ is given locally by the equations

$$x = y = A = B = C = 0.$$

The application of Transversality theorem completes the proof. □

11. Proof of the Main Theorem

(A) This is the well-known case: for $V$ generic in the sense of being transversal (after natural identifications) to the stratification $C$ of 1-jets of the Pfaffian equations on $M$, constructed in [Ma, p. 136], $M_{\text{sing}}(V) \setminus M_{\text{deg}}(V)$ is a smooth codimension 1 submanifold of $M$. 


Now consider the case $p \in M_{\mathrm{deg}}(V) \setminus (M_1 \cup M_2)$. We are using the normal form \((1)\) coordinates. Putting $g := xf_x + yf_y - f$ and using \((5)\), one has
\[
g = A(z, u, v)x^2 + B(z, u, v)xy + C(z, u, v)y^2 + \hat{f}, \quad \hat{f} \in \mathbb{R}^3.
\]
By \((3)\), $(M_{\mathrm{sing}}(V), M_{\mathrm{deg}}(V)) = \{(g = 0), \{x = y = 0\}\}$. Now to prove the Main Theorem it suffices to prove that there exists a local diffeomorphism $\Phi$ preserving the manifold $\{x = y = 0\}$ and such that $g \circ \Phi = \mp(x^2 + y^2)$, or $g \circ \Phi = xz$, or $g \circ \Phi = x^2 - y^2z$. As we are outside $M_2$, $j^2 g \neq 0$. We can assume $p = 0 \in \mathbb{R}^3$. Suppose at first that
\[
(6) \quad \begin{vmatrix}
2A & B \\
B & 2C
\end{vmatrix} (0) \neq 0.
\]
Two subcases are possible:

(B) the Hessian is positively or negatively defined, and

(C) \((6)\) holds and the Hessian is neither positively nor negatively defined.

By the Morse lemma with parameters [AGV] we can claim the existence of a coordinate change
\[
(7) \quad x \to x + \phi(x, y, z), \quad y \to y + \psi(x, y, z), \quad \phi, \psi \in \mathbb{R}^3
\]
simplifying the function $g$ to $x^2 + y^2$ or $-x^2 - y^2$ in case (B) and to $xy$ in (C). Such transformation \((7)\) preserves \((2)\), and one obtains normal forms (B) and (C) in §4.

Now suppose that condition \((6)\) is violated. Since $j^2 g \neq 0$, we can assume that $A(0) \neq 0$, and further that $A(z, u, v) = 1$. Using again the Morse lemma with parameters (the parameters are now $y, z, u, v$) we can claim the existence of a coordinate change of form \((7)\) bringing the function $g$ to the form
\[
(8) \quad g = x^2 + y^2\tau(z, u, v) + y^3\nu(y, z, u, v), \quad \tau(0) = 0.
\]
For a generic differential system $d\tau(0) \neq 0$ (the condition $d\tau(0) = 0$ together with violating \((6)\) and the inclusion of the source point $0$ in $M_{\mathrm{deg}}(V)$ give the degeneration of codimension 6, nontypical by Transversality theorem). This condition is a bit stronger than the following one: the set of points $p \in M_{\mathrm{deg}}(V)$ such that the germ at $p$ of a pair $(M_{\mathrm{sing}}(V), M_{\mathrm{deg}}(V))$ is not equivalent either to normal form (B) or (C) in §4 is a smooth 2-dimensional submanifold. Therefore there exists a transformation of the form
\[
(9) \quad z \to \eta_1(z, u, v), \quad u \to \eta_2(z, u, v), \quad v \to \eta_3(z, u, v)
\]
simplifying expression \((8)\) of $g$ to
\[
(10) \quad g = x^2 - y^2z + y^3\tilde{\nu}(y, z, u, v).
\]
A transformation of the form
\[
z \to z + y\alpha(y, z, u, v)
\]
reduces $\tilde{\nu}$ in \((10)\) to
\[
\tilde{\nu} = \tilde{\nu}(y, z + y\alpha, u, v) - \alpha.
\]
By the implicit function theorem we can ensure $\tilde{\nu} = 0$ by choosing a suitable function $\alpha$. Thus we arrive at normal form (D) in §4.
Finally, it is clear that normalizing the equation for $M_{\text{sing}}(V)$ above we preserve the description of $V(0) = \text{span}(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ valid for system $V$ in normal form (cf. Corollary 2). The proof of the Main Theorem is complete. \(\square\)

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The proof of the theorem on normal form will be split into several assertions in this section and the next.

**Lemma 2.** If $(d\omega)^2(0) \neq 0$ then $\omega$ is reducible to $x\,du + y\,dv + \tau$, where $\tau \in m_{x,y}^2$.

**Proof.** Let us choose such coordinates that (2) holds. Then obviously $j_{x,y}^0 \omega = 0$ and one has an expansion

$$j_{x,y}^1 \omega = (x A_{11} + y A_{12}) dx + (x A_{21} + y A_{22}) dy + (x A_{31} + y A_{32}) dz + (x A_{41} + y A_{42}) du + (x A_{51} + y A_{52}) dv,$$

where $A_{ij} = A_{ij}(z, u, v)$. Now we consider two 1-forms on $\mathbb{R}^3(z, u, v)$:

$$\mu_i := A_{3i}(z, u, v) dz + A_{4i}(z, u, v) du + A_{5i}(z, u, v) dv, \quad i = 1, 2.$$

The transformation

$$x \to \alpha_{11}(z, u, v) x + \alpha_{12}(z, u, v) y,$$
$$y \to \alpha_{21}(z, u, v) x + \alpha_{22}(z, u, v) y$$

brings $j_{x,y}^1 \omega$ to the form

$$(\tilde{A}_{11} x + \tilde{A}_{12} y) dx + (\tilde{A}_{21} x + \tilde{A}_{22} y) dy + x \tilde{\mu}_1 + y \tilde{\mu}_2,$$

where

$$\tilde{\mu}_1 = \alpha_{11} \mu_1 + \alpha_{12} \mu_2, \quad \tilde{\mu}_2 = \alpha_{21} \mu_1 + \alpha_{22} \mu_2.$$

Direct calculation shows that the assumption $(d\omega)^2(0) \neq 0$ means exactly $\mu_1 \wedge \mu_2 \neq 0$ (the reason we have introduced $\mu_1, \mu_2$). Apply any transformation (9) that straightens in $\mathbb{R}^3$ the line field $\ker \mu_1 \wedge \mu_2$ to $\text{span}(\partial/\partial z)$. Understanding it as the coordinate change in $\mathbb{R}^5$, one has then $\text{span}(\mu_1, \mu_2) = \text{span}(du, dv)$ in a neighbourhood of $0 \in \mathbb{R}^5$. This can be improved, using (12), to $\tilde{\mu}_1 = du$, $\tilde{\mu}_2 = dv$, yielding

$$j_{x,y}^1 \omega = (x \tilde{A}_{11} + y \tilde{A}_{12}) dx + (x \tilde{A}_{21} + y \tilde{A}_{22}) dy + x du + y dv.$$

The change of coordinates

$$u \to u - x \tilde{A}_{11} - y \tilde{A}_{21}, \quad v \to v - x \tilde{A}_{12} - y \tilde{A}_{22}$$

eventually annihilates $j_{x,y}^1(\omega - x\,du - y\,dv)$. \(\square\)

Lemma 2 will serve as the premise for $k = 1$ in the inductive argument justifying Corollary 3 (see below). The following will constitute the induction step.
Lemma 3. For any \( k \geq 1 \) and \( \omega \) satisfying \((d\omega)^2(0) \neq 0\) the jet \( j^k_{x,y,\omega} \) is reducible to form (1).

Proof. Suppose that for certain \( k \geq 1 \) \( j^k_{x,y,\omega} \) is already reduced to form (1). We shall show that \( j^{k+1}_{x,y,\omega} \) can be so reduced, too. Let \( m^{(k)}_{x,y} \) be the set of all function germs of the form \( \sum c_{ij}(z,u,v)x^iy^j \), where \( c_{ij} \) are germs at 0 \( \in \mathbb{R}^3 \) of smooth functions. Write

\[
j^{k+1}_{x,y,\omega} = j^k_{x,y,\omega} + A_1 dx + A_2 dy + A_3 dz + A_4 du + A_5 dv,
\]

where \( A_i \in m^{(k+1)}_{x,y} \). We are going to take new coordinates

\[
x + \varphi, \quad y + \psi, \quad z, \quad u + \gamma, \quad v + \mu, \quad \text{where} \quad \varphi, \psi, \gamma, \mu \in m^{(k+1)}_{x,y}.
\]

Denoting by \( T \) the right-hand side of (13), \( j^{k+1}_{x,y,\omega} \) assumes in these coordinates the form

\[
T + (xyx + ypx) dx + (xyy + ypy) dy + \varphi du + \psi dv.
\]

So it suffices to solve the system

\[
\begin{align*}
\varphi + A_3 &= \psi + A_4 = 0, \\
xyx + yux + A_1 &= 0, \\
xyy + yuy + A_2 &= 0.
\end{align*}
\]

Obviously \( \varphi = -A_3, \psi = -A_4 \). By putting \( R := xy + yu \) we reduce (15) to the system of three equations for \( R, \gamma, \mu : \)

\[
\begin{align*}
Rx - \gamma + A_1 &= 0, \\
Ry - \mu + A_2 &= 0, \\
R &= xy + yu,
\end{align*}
\]

\( \gamma, \mu \in m^{(k+1)}_{x,y} \). This can be written briefly as

\[
R = x(Rx + A_1) + y(Ry + A_2), \quad R \in m^{(k+2)}_{x,y}
\]

(having such \( R \), we take \( \gamma := Rx + A_1, \mu := Ry + A_2 \). As \( xa_1 + ya_2 = \sum \sum b_{ij}(z,u,v)x^iy^j \), we can give an explicit solution to (16):

\[
R = \sum (1 - i - j)^{-1}b_{ij}(z,u,v)x^iy^j. \quad \square
\]

We denote by \( m_{x,y} \) the ideal of germs of functions vanishing on \( \{x = y = 0\} \) together with all their partial derivatives, and also the set of 1-form germs having such coefficients, and the set of respective vector field germs, too. Consequently, \( j^\infty_{x,y} \) is defined analogously to \( j^k_{x,y} \) (see §5).

Corollary 3. \( \omega \) as in Lemma 3 is reducible to \( x du + y dv + f dz + \tau \), where \( \tau \) is a 1-form, \( \tau \in m_{x,y} \), and \( f \in m^2_{x,y} \).

Proof. Using Lemma 2 as the departure point \( (k = 1) \) we reduce inductively consecutive jets \( j^k_{x,y,\omega} \), applying Lemma 3 at each step. Taking into account the character of normalizing transformations (14) and the fact that, after passing to a fixed representative of \( \omega \), (16) is solvable in an independent of \( k \) neighbourhood of \( 0 \in \mathbb{R}^3(z,u,v) \), there exists a formal in \( x \), \( y \) transformation of \( \mathbb{R}^5 \), having as coefficients (of its series in \( x \), \( y \)) smooth functions of \( z, u, v \) defined in a common neighbourhood of \( 0 \), which reduces \( j^\infty_{x,y,\omega} \) to form (1). Now it suffices to apply the Whitney extension theorem (see [W]). \( \square \)
To prove the theorem on normal form we must still prove

**Lemma 4.** Let \( f \in \mathbb{M}^2_{x,y} \), \( \tau \) be a 1-form, \( \omega \in \mathbb{M}^\infty_{x,y} \). Then the 1-form \( \omega = x\,du + y\,dv + f\,dz + \tau \) is reducible to the form \( x\,du + y\,dv + \hat{\tau}\,dz \), \( \hat{\tau} \in \mathbb{M}^2_{x,y} \).

**Proof.** We use some modifications of the homotopy method [Z2, Chapter 1, §3]. Let \( \omega := x\,du + y\,dv \). Introduce also the truncation operator \( P \) sending every 1-form \( \kappa_1\,dx + \kappa_2\,dy + \kappa_3\,dz + \kappa_4\,du + \kappa_5\,dv \) (\( \kappa_i \) are functions of \( x, y, z, u, v \)) to \( \kappa_1\,dx + \kappa_2\,dy + \kappa_4\,du + \kappa_5\,dv \), and the family of forms \( \omega_t := \omega + t(f\,dz + \tau) \), \( t \in [0, 1] \). Consider the equation

\[
(17) \quad P(X_t \, \omega_t + d(X_t \, \omega_t) + f\,dz + \tau) = 0
\]

for an unknown family of vector fields \( X_t \). (We shall require additionally that \( X_t \, dz = 0 \).) Equation (17) can be equivalently written as

\[
(18) \quad P(X_t \, \omega_t + d(X_t \, \omega_t) + \tau) = 0.
\]

**Claim.** If there exists a smooth family \( X_t \in \mathbb{M}^\infty_{x,y} \) depending on \( t \) and satisfying (18) and such that \( X_t \, dz = 0 \), then Lemma 4 holds. (Compare the classical variant of the homotopy method [AGV].) In order to substantiate the Claim, consider the family of diffeomorphisms \( \phi_t \) defined by

\[
\frac{d\phi_t}{dt} = X_t(\phi_t), \quad \phi_0 = \text{id}.
\]

Then

\[
\frac{d}{dt}(P(\phi_t^* \omega_t)) = P\left( \frac{d}{dt}(\phi_t^* \omega_t) \right) = P(\phi_t^*(L_{X_t} \omega_t + \frac{d\omega_t}{dt})
= P(\phi_t^*(X_t \, dz + d(X_t \, \omega_t) + f\,dz + \tau),
\]

where \( L_{X_t} \omega_t \) is the Lie derivative of \( \omega_t \) along the field \( X_t \). Equation (17) implies that \( L_{X_t} \omega_t + d\omega_t/dt \in \ker P \) for all \( t \). By virtue of \( X_t \, dz = 0 \), also \( \phi_t^*(L_{X_t} \omega_t + d\omega_t/dt) \) is always included in \( \ker P \); hence \( d(P(\phi_t^* \omega_t))/dt \equiv 0 \).

This infers \( P(\phi_t^* \omega_t = P(\phi_0^* \omega_0) \), or \( \phi_t^* \omega - \omega = \hat{\tau}\,dz \). Noticing that \( \phi_t = \text{id} + \psi_t \), \( \psi_t \in \mathbb{M}^\infty_{x,y} \), the assumption \( f \in \mathbb{M}^2_{x,y} \) obviously implies \( \hat{\tau} \in \mathbb{M}^2_{x,y} \), proving the Claim.

Now we are going to show that the premise in the Claim holds. Seeking \( X_t \) in the form \( f_1, i\,\partial/\partial x + f_2, i\,\partial/\partial y + f_3, i\,\partial/\partial u + f_4, i\,\partial/\partial v \), \( f_i, t \in \mathbb{M}^\infty_{x,y} \), (18) boils down to a system of five equations for the \( f_i, t \)'s and \( R_t := X_t \, \omega_t \), the last defining identity being itself the fifth equation which, on writing \( \tau = \tau_1\,dx + \tau_2\,dy + \tau_3\,du + \tau_4\,dv \) and computing \( X_t \, \omega_t \) explicitly, assumes the form

\[
R_t = t\tau_1 f_1, t + t\tau_2 f_2, t + (x + t\tau_3) f_3, t + (y + t\tau_4) f_4, t.
\]

The unknowns \( f_i, t \) can be eliminated from the first four equations (they can be expressed via the first order partial derivatives of \( R_t \)), after which we arrive at one equation for \( R_t \):

\[
(19) \quad R_t - x(R_t)x - y(R_t)y + \Theta_t(R_t) = a_t,
\]

where \( \Theta_t \) is a family of vector fields, \( a_t \) is a family of function germs, \( \Theta_t \in \mathbb{M}^\infty_{x,y} \), \( a_t \in \mathbb{M}^\infty_{x,y} \). (Observe that (19) is, to some extent, similar to (16), but the occurring flat function and vector field make the great difference.)
The fields $-x \partial / \partial x - y \partial / \partial y + \Theta_t$, are hyperbolic on the manifold $\{x = y = 0\}$ (this manifold is attracting for the respective dynamical system). Thanks to that, by virtue of the Belitskii results (see [Bl, B2] and references in the latter\(^1\)), (19) has a smooth family of solutions $R_t \in m_{x,y}^\infty$ depending on $t$. The proof of Lemma 4 is finished. \(\square\)

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\(^1\) Various results by Belitskii on the solvability of singular partial differential equations can be found in [Z2, Chapter 1, §6], where they are collected in a compact form.