

## THE HEXAGONAL PACKING LEMMA AND THE RODIN SULLIVAN CONJECTURE

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**ABSTRACT.** The Hexagonal Packing Lemma of Rodin and Sullivan [6] states that  $s_n \rightarrow 0$  as  $n \rightarrow \infty$ . Rodin and Sullivan conjectured that  $s_n = O(1/n)$ . This has been proved by Z-Xu He [2]. Earlier, the present author proved the conjecture under some additional restrictions [1].

In the following we are able to remove these restrictions, and thus give an alternative proof of the RS conjecture. The proof is based on our previous article [1]. It is completely different from the proof of He, and it is mainly based on discrete potential theory, as developed by Rodin for the hexagonal case [4].

### 1. INTRODUCTION

The Hexagonal Packing Lemma plays an important role in the proof of Rodin and Sullivan of Thurston's conjecture that his scheme converges to the Riemann mapping.

It was suggested by Rodin and Sullivan [4, 5, 6] that  $s_n = O(1/n)$ . This property is important for investigation of the behavior of the derivatives. More specifically,  $s_n = O(1/n)$  implies the convergence of the "circle function"  $r_\varepsilon$  to the modulus of the derivative of the Riemann mapping function [5, 6].

It is easy to show that one cannot get anything better than  $s_n = O(1/n)$  (cf. [2]).

In what follows, we prove the RS conjecture. The proof is based on our previous paper [1]. The main additional tool in the present work is an assertion of a "mean convergence" which is, in a way, a form of discrete  $L^2$  convergence. This "mean convergence" allows us to remove the restriction in [1] and thus get the full result by an induction process.

We use freely the standard notations and definitions introduced by the researchers working in the area (cf. [4, 5, 6]). In particular,  $HCP_N$  denotes the first  $N$  generations of the regular hexagonal circle packing, with all radii equal to 1.  $HCP'_N$  denotes a circle packing that is combinatorially isomorphic to  $HCP_N$ .

In addition  $HL(h)$  is the hexagonal lattice of mesh  $h$ .  $HL(h, N)$  denotes the subset of  $HL(h)$  consisting of lattice points of generations  $\leq N$  (see [4]).

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Also in the following we shall use the notation  $\overline{s_N}$  instead of  $s_N$  in order to emphasize the fact that the additional restriction (2.1) is added.

2. PRELIMINARY RESULTS

We first state our main result in [1] in a slightly different form, which is more suitable for our purposes.

**Theorem 2.1** [1]. *Let  $HCP'_N$  be a circle packing combinatorially isomorphic to  $HCP_N$ . Then there exist two numerical constants  $\rho > 0$ ,  $A > 0$  with the following property: If*

$$(2.1) \quad \exp(-3\rho) \leq r \leq \exp(3\rho)$$

for each of the radii of the circles in  $HCP'_N$ , then for any natural  $N$

$$(2.2) \quad \overline{s_N} < A/N.$$

Also:

$$(2.3) \quad \rho \leq \min(\log 2, A).$$

It is worthwhile to note that the above theorem is, in fact, a statement on a certain class of functions. More specifically, if both  $u$  and  $1/u$  are subharmonic discrete functions on  $HL(1, N)$  and if, in addition, the “ $\rho$  condition”

$$(2.4) \quad \exp(-3\rho) \leq u \leq \exp(3\rho)$$

is satisfied, then (2.2) holds. In this more general setting, the above theorem is sharp, in the sense that the “ $\rho$  condition” cannot be removed.

Indeed, consider the subharmonic function

$$(2.5) \quad u = u(x, y) = \exp(3\rho x/N)$$

defined on  $HL(1, N)$ . It is an easy matter to show that both  $u$  and  $1/u$  are subharmonic discrete functions. If  $x$  is an integer and  $-N \leq x \leq N$ ,  $u$  satisfies the “ $\rho$  condition” (2.4) on  $HL(1, N)$ . Hence, it follows by Theorem 2.1 that  $u(1)/u(0) = \exp(3\rho/N) = O(1/N)$ . On the other hand, it is clear that the “ $\rho$  condition” cannot be omitted since, if instead of (2.5), one considers the function  $u(x, y) = \exp(Mx/N)$ , then  $u$  and  $1/u$  are still subharmonic, but  $u(1)/u(0)$  depends on  $M$ , and thus (2.2) cannot be satisfied for an arbitrary large  $M$ .

It is therefore clear that in order to prove the RS conjecture without the restriction of the “ $\rho$  condition”, one needs additional information arising from the particular geometry of the present situation, in addition to the property of discrete subharmonicity of the “circle function”  $r$  and its reciprocal  $1/r$ .

In the sequel we consider, for some positive integer  $l$ , the six configurations  $HCP_N$ ,  $HCP_{2N}$ ,  $HCP_{lN}$ ,  $HCP'_N$ ,  $HCP'_{2N}$ ,  $HCP'_{lN}$ .

We now introduce some additional notation:  $HCP_{N,q}$  will denote the regular hexagonal configuration, but with radius  $q$  and not 1.  $f_{lN}$  denotes the  $k$ -qc ( $k$  quasi conformal) mapping from the carrier [5, 6] of  $HCP_{lN, N-1}$  onto the carrier of  $HCP'_{lN}$ .

Later on we will say occassionally that  $f_{lN}$  maps  $HCP_{lN, N-1}$  onto  $HCP'_{lN}$ , in those places where there is no danger of ambiguity. We recall a few important facts [5, 6] about the method of constructing functions similar to the function  $f_{lN}$ . The center of each circle is mapped onto the center of the image circle by

the isomorphic correspondence. The map  $f_{IN}$  is then extended by the barycentric coordinates. Using the Ring Lemma [6] one knows that there is a uniform  $k$  which is independent of  $N$  and  $l$ , s.t. the constructed map is  $k$ -qc. We also note that  $f_{IN}(0) = 0$  is not a restriction and this will be assumed throughout.

Later we consider the function  $g_{IN} = f_{IN}/f_{IN}(1)$  with the normalization

$$(2.6) \quad g_{IN}(0) = 0, \quad g_{IN}(1) = 1.$$

Also, the function  $g_{IN}$  is  $k$ -qc, with the same  $k$  as  $f_{IN}$ .

As  $f_{IN}$  maps  $HCP_{IN,N-1}$  onto  $HCP'_{IN}$ , the function  $g_{IN}$  maps  $HCP_{IN,N-1}$  onto  $HCP^*_{IN}$  which denotes the configuration  $HCP'_{IN}$  "divided" by  $f_{IN}(1)$ . Of course, each of the radii of circles in  $HCP'_{IN}$  is divided by  $f_{IN}(1)$ , to get the corresponding value of the radius of the image circle in  $HCP^*_{IN}$ .

The above normalization (2.6) is needed for constructing a normal family of  $k$ -qc mappings. In any case, there is a simple invariance property which is of importance: since  $g_{IN} = f_{IN}/f_{IN}(1)$ , and the relation is only an expansion (or contraction) by  $f_{IN}(1)^{-1}$ , the ratio between two radii is not changed. Later on we will use this invariance property, while dealing with statements about ratio of two radii.

We now recall a known result that will be needed to establish the "normal family" property of the class  $\{g_{IN}\}$ . For the convenience of the reader we bring the proof of this result.

**Lemma 2.1.** *Let  $G$  be a family of  $k$ -qc mappings in a plane domain  $D$ . Let  $a, b$  be two distinct points in  $D$ , s.t.  $f(a) = a, f(b) = b$ , for each  $f \in G$ . Also assume that for each  $f \in G$ , a certain fixed value (say  $\infty$ ) is omitted. Then  $G$  is a normal family.*

*Proof.* Consider the class  $G$  restricted to  $D \setminus \{a\}$ .  $G$  is a family of  $k$ -qc mappings omitting the two values  $\{a, \infty\}$  in  $D \setminus \{a\}$ . Hence,  $G$  is a normal family in  $D \setminus \{a\}$  [3, p. 73]. Thus, from a given sequence  $\{f_n\}$  in  $G$ , we can choose a subsequence  $\{f_{n_p}\}$  s.t.  $f_{n_p}$  converges uniformly on each compact subset of  $D \setminus \{a\}$ . Now, consider the sequence  $\{f_{n_p}\}$  restricted to  $D \setminus \{b\}$ . This set of functions omit the two values  $\{b, \infty\}$ . Thus, again, we can find a subsequence, say  $\{f^s_{n_p}\} \subset \{f_{n_p}\}$  that converges uniformly on each compact subdomain of  $D \setminus \{b\}$ . Hence  $\{f^s_{n_p}\}$ , which is a subsequence of  $\{f_{n_p}\}$  by construction, converges uniformly on each compact subdomain of  $D$ , and the proof is complete.

Note that the limit function is  $k$ -qc mapping with the same  $k$ , and it cannot be a constant on  $D$ , since the family has two different fixed points.

### 3. THE "MEAN CONVERGENCE" PROPERTY

**Lemma 3.1.** *Let  $\{g_{IN}\}$  be defined as above with the normalization (2.6).  $g_{IN}$  maps  $HCP_{IN,N-1}$  on  $HCP^*_{IN}$ . Then, given  $\varepsilon > 0$  and a compact domain  $D \subset \mathbb{C}$ , there exist  $N_0 = N(\varepsilon, D), l_0 = l(\varepsilon, D)$  s.t.*

$$(3.1) \quad |g_{IN}(z) - z| < \varepsilon, \quad \text{if } z \in D, N \geq N_0, l \geq l_0.$$

Also,  $N_0, l_0$  do not depend on the particular sequence  $\{g_{IN}\}$ .

*Proof.* By Lemma 2.1 applied for the family  $\{g_{IN}\}$ , it is clear that  $\{g_{IN}\}$  form a normal family for each disc of radius  $R > 0$  provided  $l$  is large enough.

This follows at once from the fact that the domain of definition of  $g_{lN}$  is  $HCP_{lN, N-1}$ , and it contains a disc of radius  $l$ , as can be easily seen.

If Lemma 3.1 is not valid for some  $D$ , there exist an  $\varepsilon > 0$  and two sequences  $\{l_j\}_{j=1}^\infty, \{N_j\}_{j=1}^\infty$  s.t.  $l_j \rightarrow \infty, N_j \rightarrow \infty$  as  $j \rightarrow \infty$  and a sequence  $\{g_{l_j N_j}\}$  for which  $\max_{z \in D} |g_{l_j N_j}(z) - z| \geq \varepsilon$ . By taking subsequences and using the standard diagonal process, we may assume that  $g_{l_j N_j}(z)$  converges uniformly in any compact subdomain of  $\mathbb{C}$ . Also, since  $s_n \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that  $g_{l_j N_j}$  tends to a univalent analytic mapping. (We omit the details of the proof of this fact. Instead of the reader is referred to [6] where an almost identical situation occurs, and the limiting process is used to show the convergence of the "circle functions"  $r_\varepsilon(z)$  to the Riemann map of a given plane domain onto the unit disc). Continuing with our argument, we can conclude now that the limit function must be a univalent analytic map on  $\mathbb{C}$ . Such a map is necessarily linear, and must be of the form  $\alpha z + \beta$ , for some constants  $\alpha, \beta$ . But  $\{0, 1\}$  are two fixed points for each  $g_{l_j N_j}$ . Hence, the same is true for the limit function. We thus end up with the conclusion that the limiting function must be the identity map. Thus,  $g_{l_j N_j}$  converges uniformly in any compact subdomain of  $\mathbb{C}$  to the identity function  $I(z) \equiv z$ . This contradicts the condition  $\max |g_{l_j N_j}(z) - z| \geq \varepsilon$  and the proof is complete.

We now state another variant of Lemma 3.1, in a form that will be more convenient for applications. For this aim we take a particular  $D$  in Lemma 3.1, namely, the disc  $\{z, |z| \leq 5\}$ . Also, consider the function  $\varphi_{lN}(z) = N g_{lN}(z/N)$  which maps  $HCP_{lN}$  on the configuration  $HCP_{lN}^*$  "multiplied" by  $N$ . We denote this "expanded" domain by  $HCP_{lN}^{**}$ . It is also clear that

$$(3.2) \quad \varphi_{lN}(0) = 0, \quad \varphi_{lN}(N) = N.$$

We now have

**Lemma 3.2.** *Let  $\{\varphi_{lN}\}$  be defined as above with the normalization (3.2).  $\varphi_{lN}$  maps  $HCP_{lN}$  on  $HCP_{lN}^{**}$ . Then, given  $\varepsilon > 0$  there exist  $N_0 = N(\varepsilon), l_0 = l(\varepsilon)$ , s.t. for  $N \geq N_0, l_0 \geq l$ ,*

$$(3.3) \quad |\varphi_{lN}(z) - z| < \varepsilon N, \quad \text{if } |z| \leq 5N.$$

Also,  $N_0, l_0$  do not depend on the particular choice of  $\{\varphi_{lN}\}$ .

The proof follows at once from Lemma 3.1, by taking  $D$  as the disc  $|z| \leq 5$  and using  $\varphi_{lN} = N g_{lN}(z/N)$ .

We shall need also the following elementary trigonometric assertion.

**Lemma 3.3** (see Figure 1). *Let  $\{K_j\}_1^3$  be three mutually tangent discs with disjoint interior. Denote, further, the radii of these discs by  $\{r_j\}_1^3$ , and their centers by  $\{a_j\}_1^3$ , respectively. Let the triangle with the three vertices  $\{a_j\}_1^3$  be denoted by  $T$ , and its area by  $S(T)$ . Denote by  $S$  the area of those parts of the discs covered by  $T$ . Then, given any  $\varepsilon > 0$ , we may find  $\delta = \delta(\varepsilon)$ , s.t. if  $1 - \delta < r_j/r_k < 1 + \delta, \{j, k\} \subset \{1, 2, 3\}$  then*

$$(3.4) \quad \frac{\pi\sqrt{3}}{6}(1 - \varepsilon) < S/S(T) < \frac{\pi\sqrt{3}}{6}(1 + \varepsilon).$$

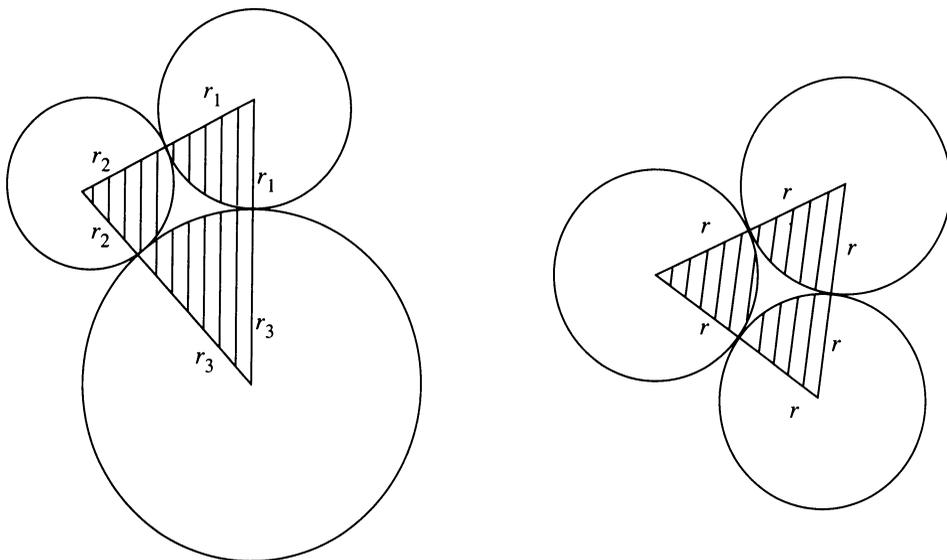


FIGURE 1

*Proof.* For the regular case we have  $S/S(T) = \pi\sqrt{3}/6$ , as follows easily by an elementary argument. To get Lemma 3.3, we can either make a simple calculation, or use an obvious continuity argument.

Our aim now is to present the main result in this section. In what follows, consider the configurations  $HCP_N$ ,  $HCP_{2N}$ ,  $HCP_{IN}$ . Also,  $f_{IN}$ ,  $g_{IN}$ ,  $\varphi_{IN}$  are the functions constructed above. Hence  $\varphi_{IN}$  satisfies (3.2) and maps  $HCP_{IN}$  onto  $HCP_{IN}^{**}$ . We have

**Theorem 3.1** (Discrete “mean convergence”). *Given  $\varepsilon > 0$  and any sequence  $\{\varphi_{IN}\}$  defined as above, there exist  $N_1 = N_1(\varepsilon)$ ,  $l_1 = l_1(\varepsilon)$ , s.t.*

$$(3.5) \quad \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} (1 - r_{jk})^2 < \varepsilon^2 N^2$$

for  $N \geq N_1$ ,  $l \geq l_1$ , where the  $\{r_{jk}\}$  are the radii of the circles in the configuration  $HCP_{IN}^{**}$ . Also,  $N_1$ ,  $l_1$  do not depend on the particular sequence  $\{\varphi_{IN}\}$ .

*Proof.* We first emphasize the fact that  $N_1$ ,  $l_1$  depend only on  $\varepsilon$ , and not on the particular sequence  $\{\varphi_{IN}\}$ . Also, we point out that the disc  $\{z, |z| \leq 5N\}$ , appearing in (3.3), contains the configuration  $HCP_{2N}$ .

Our aim now is to compare the map  $\varphi_{IN}$  and the identity map  $I(z) \equiv z$ . For  $I(z)$  the left-hand side of (3.5) is obviously zero since all the radii are equal to one. For the map  $\varphi_{IN}$  we rewrite the left-hand side of (3.5):

$$(3.6) \quad \sum_{N+1}^{2N} \sum_1^{6j} (1 - r_{jk})^2 = \left( \sum_{N+1}^{2N} \sum_1^{6j} r_{jk}^2 - \sum_{N+1}^{2N} \sum_1^{6j} 1 \right) + 2 \left( \sum_{N+1}^{2N} \sum_1^{6j} 1 - \sum_{N+1}^{2N} \sum_1^{6j} r_{jk} \right).$$

Next, we plan to show that each of the two expressions on the right-hand side of (3.6) is  $o(N^2)$ . This will be done with the aid of Lemmas 3.2 and 3.3.

To estimate the expression  $2(\sum \sum 1 - \sum \sum r_{jk})$  we use Lemma 3.2, taking  $\varepsilon^2/24$  instead of  $\varepsilon$ . Hence, from (3.3), and a simple geometric consideration

$$(3.7) \quad \sum_{k=1}^{6j} r_{jk} > 6j - \frac{1}{4}\varepsilon^2 N, \quad N+1 \leq j \leq 2N, \quad \forall N \geq N_0 \left( \frac{\varepsilon^2}{24} \right), \quad \forall l \geq l_0 \left( \frac{\varepsilon^2}{24} \right),$$

which implies

$$\sum_{j=N+1}^{2N} \sum_{k=1}^{6j} r_{jk} > \sum_{j=N+1}^{2N} \left( \sum_{k=1}^{6j} 1 - \frac{\varepsilon^2}{4} N \right)$$

and

$$(3.8) \quad 2 \left( \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} 1 - \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} r_{jk} \right) < \frac{\varepsilon^2}{2} N^2, \\ \forall N \geq N_0 \left( \frac{\varepsilon^2}{24} \right), \quad \forall l \geq l_0 \left( \frac{\varepsilon^2}{24} \right).$$

For the estimation of the other expression appearing on the right-hand side of (3.6), first observe that  $\pi \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} r_{jk}^2$  is the area of the set of discs belonging to  $\text{HCP}_{2N}^{**} \setminus \text{HCP}_N^{**}$ . Similarly,  $\pi \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} 1^2$  is the area of the set of discs belonging to  $\text{HCP}_{2N} \setminus \text{HCP}_N$ . Next, compare the area of the two carriers of  $\text{HCP}_{2N} \setminus \text{HCP}_N$  and  $\text{HCP}_{2N}^{**} \setminus \text{HCP}_N^{**}$ . We now prefer to proceed in a less formal way and leave aside part of the somewhat long (but elementary) computational details.

Using, once more, Lemma 3.2, we have for  $N, l \rightarrow \infty$

$$(3.9) \quad |\text{area of carrier } (\text{HCP}_{2N}^{**} \setminus \text{HCP}_N^{**}) \\ - \text{area of carrier } (\text{HCP}_{2N} \setminus \text{HCP}_N)| = o(N^2).$$

Also note that each one of the circles in  $\text{HCP}_{2N}^{**}$  is surrounded by a subpacking of  $\text{HCP}_N^{**}$  that contains at least  $(l-2)N$  generations. By taking  $(l-2)N$  large enough, we can make  $s_{(l-2)N}$  as small as we please. (Indeed, this follows from the RS theorem, i.e.,  $s_n \rightarrow 0$  for  $n \rightarrow \infty$ .) Our aim is now to combine this fact with Lemma 3.3. For this, take  $K_{j_1}, K_{j_2}, K_{j_3}$  (in place of the  $K_j$ ) that are mutually tangent and belong to the configuration  $\text{HCP}_{2N}^{**} \setminus \text{HCP}_N^{**}$ . Using what has been said about the convergence of  $s_{(l-2)N}$  to zero, it is clear from (3.4) that

$$(3.10) \quad S_{j_1 j_2 j_3} = S(T_{j_1 j_2 j_3}) \cdot \frac{\pi\sqrt{3}}{6} (1 + o(1))$$

with an obvious meaning of the notations.

Note that  $o(1) = O(s_{(l-2)N})$  is uniform for all triplets  $\{K_{j_1}, K_{j_2}, K_{j_3}\}$  belonging to  $\text{HCP}_{2N}^{**} \setminus \text{HCP}_N^{**}$ . Also note that the number of triangles in the carrier of  $\text{HCP}_{2N}^{**} \setminus \text{HCP}_N^{**}$  is  $O(N^2)$ . On summation we have from (3.10):

$$(3.11) \quad \pi \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} r_{jk}^2 = \text{area of carrier } (\text{HCP}_{2N}^{**} \setminus \text{HCP}_N^{**}) \cdot \frac{\pi\sqrt{3}}{6} (1 + o(1)).$$

$$(3.12) \quad \pi \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} 1^2 = \text{area of carrier } (\text{HCP}_{2N} \setminus \text{HCP}_N) \cdot \frac{\pi\sqrt{3}}{6} (1 + o(1)).$$

From (3.9), (3.11), (3.12),

$$\pi \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} r_{jk}^2 - \pi \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} 1^2 = o(N^2).$$

Writing this “quantitatively” we get

For any  $\varepsilon > 0$  there exist  $\bar{N}_0 = \bar{N}_0(\varepsilon)$ ,  $\bar{l}_0 = \bar{l}_0(\varepsilon)$  s.t. if  $N \geq \bar{N}_0$ ,  $l \geq \bar{l}_0$

$$(3.13) \quad 2 \left( \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} r_{jk}^2 - \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} 1^2 \right) < \frac{\varepsilon^2}{2} N^2.$$

From (3.8) and (3.13) the proof of Theorem 3.1 follows by taking  $N_1 = N_1(\varepsilon) = \max(N_0, \bar{N}_0)$ ,  $l_1 = l_1(\varepsilon) = \max(l_0, \bar{l}_0)$ .

4. CONNECTION BETWEEN THE “MEAN CONVERGENCE” AND THE “ $\rho$  CONDITION”

In the following, whenever  $\rho$  and  $A$  are mentioned, they are the specific numerical constants appearing in §2. We further denote

$$(4.1) \quad \varepsilon_1 = \varepsilon(\rho, A) = \rho^2/4A.$$

Hereafter,  $\varepsilon_1$  will mean this specific numerical constant.

From (3.5) we have

$$(4.2) \quad \sum_{j=N+1}^{2N} \sum_{k=1}^{6j} (1 - r_{jk})^2 < \varepsilon_1^2 N^2$$

provided  $N \geq N_1(\varepsilon_1)$ ,  $l \geq l_1(\varepsilon_1)$ .

Now let  $N_1$  be any specific natural  $N$  satisfying

$$(4.3) \quad N_1 > \max(N_1(\varepsilon_1), 2A/\rho).$$

Next, we take  $l \geq l_1(\varepsilon_1)$  sufficiently large s.t. in addition we also have  $s_{(l-2)N_1} < A/N_1$ . It will be convenient to add the trivial restriction  $l \geq 4$ . Now, putting all these conditions together, take a specific  $l$  with these restrictions and denote it by  $l_1$ . Hence,

$$(4.4) \quad s_{(l-2)N_1} < A/N_1, \quad \forall l \geq l_1 \geq \max(l_1(\varepsilon_1), 4).$$

Again, from now on whenever  $N_1$ ,  $l_1$  are mentioned, they will mean these specific natural numbers. After making all these preliminaries, we are able to present our key result in this paper.

**Theorem 4.1.** *Let  $\rho$ ,  $A$ ,  $\varepsilon_1$ ,  $N_1$ ,  $L_1$  be the numbers defined above. In addition let  $N_k = 2N_{k-1} = 2^{k-1}N_1$  for any natural  $k$ . Also, consider the functions  $\{f_{l_1 N_k}\}$ ,  $\{g_{l_1 N_k}\}$ ,  $\{\varphi_{l_1 N_k}\}$  as we have defined them.  $\varphi_{l_1 N_k}$  maps  $\text{HCP}_{l_1 N_k}$  onto  $\text{HCP}_{l_1 N_k}^{**}$  and satisfies (3.2) for  $N = N_k$ ,  $l = l_1$ . Then the radii of the circles in  $\text{HCP}_{2N_k}^{**}$  must satisfy the “ $\rho$  condition” (2.1).*

*Proof.* Before turning to the proof we point out two facts. First, it is enough to consider the “ $\rho$  condition” only for the border circles of a given configuration. Indeed, it is quite obvious that the maximum principle holds for a discrete subharmonic function. Hence, if we apply this to the functions  $r$ ,  $1/r$  [1] we

get that the “circle function”  $r$  attains both its max and min value on the set of border circles. Thus, if (2.1) is satisfied for the  $m$ th generation of a configuration  $HCP'_m$ , it is satisfied for all generations of lower order, i.e., for all radii of circles appearing in  $HCP'_m$ .

The other fact is a simple (but useful) observation concerning a certain invariance property mentioned already in §2.

If we consider a certain configuration, say,  $HCP'_m$ , for some natural  $m$ , and “expand” it by a constant  $\lambda > 0$ , then, for the new configuration  $\lambda \times HCP'_m$ , the ratio  $r_j/r_k$  is transformed to  $\lambda r_j/\lambda r_k$  which is the same. Hence, any statement about  $s_m$ , which is verified for  $HCP'_m$ , is also verified for  $\lambda \times HCP'_m$ . Thus, given arbitrary sequence  $\{f_{lN}\}$  constructed as above, we may consider instead the sequence  $\{\varphi_{lN}\}$  w.l.o.g.

We now turn to the proof, which will be by induction. First, consider the case  $k = 1$ . With  $N = N_1$ , (4.2) takes the form

$$(4.5) \quad \sum_{j=N_1+1}^{2N_1} \sum_{l=1}^{6j} (1 - r_{jl})^2 < \varepsilon_1^2 N_1^2.$$

Consider, now, the configuration  $HCP_{l_1 N_1}^{**}$  and its subconfiguration  $HCP_{2N_1}^{**}$ . By our choice of  $l_1$  we have the validity of (4.4). Our aim is to show that the “ $\rho$  condition” (2.1) is satisfied for the radii of the border circles of  $HCP_{2N_1}^{**}$ . Assume the contrary, namely, that for some circle  $K_{p, 2N_1}$ , which is a border circle of  $HCP_{2N_1}^{**}$ , at least one of the following holds: either

$$(4.6) \quad r_{p, 2N_1} \leq e^{-3\rho}$$

or

$$(4.6)' \quad r_{p, 2N_1} \geq e^{3\rho}.$$

We now show that each of the two assumptions (4.6), (4.6)' leads to a contradiction. For this purpose, first observe that

$$(4.7) \quad r_j/r'_j < 1 + A/N_1$$

for each two neighboring circles of  $HCP_{2N_1}^{**}$ . Indeed, since  $HCP_{2N_1}^{**}$  is a subconfiguration of  $HCP_{l_1 N_1}^{**}$ , each circle of  $HCP_{2N_1}^{**}$  is surrounded by at least  $(l_1 - 2)N_1$  generations, which establishes (4.4), and thus  $r_j/r'_j < 1 + s_{(l_1-2)N_1} < 1 + A/N_1$  which is (4.7).

Next, choose a natural  $m$  satisfying

$$(4.8) \quad \rho/2A \leq m/N_1 \leq \rho/A \leq 1.$$

Such an  $m$  actually exists since by (4.3)  $\rho/A - \rho/2A = \rho/2A > 1/N_1$  (also,  $\rho \leq A$  (2.3)). So now, make the assumption (4.6) for some circle  $K_{p, 2N_1}$  which is a border circle of  $HCP_{2N_1}^{**}$ . Using (4.7) successively, we now show that for all circles that are not “too far” from  $K_{p, 2N_1}$  the weaker condition  $r_j \leq e^{-\rho}$  follows. Indeed, take different “walks” starting at the “base”  $K_{p, 2N_1}$ . Each “walk” is of at most  $2m$  “steps”. For each circle of radius  $r_j$  appearing in such a “walk” we have

$$r_j \leq \left(1 + \frac{A}{N_1}\right)^{2m} r_{p, 2N_1} \leq \left(1 + \frac{A}{N_1}\right)^{2m} e^{-3\rho} < e^{2mA/N_1 - 3\rho} \leq e^{2\rho - 3\rho} = e^{-\rho}$$

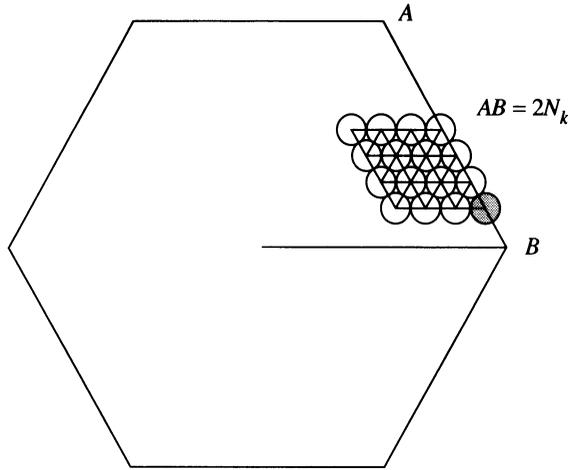


FIGURE 2

where we have used (4.6), (4.7) and (4.8). Since  $m \leq N_1$  by (4.8), it follows that among such circles there is a rhombus containing  $m^2$  circles arising from different “walks” starting in the “base”  $K_{\rho, 2N_1}$  and of at most  $2m$  “steps” (see Figure 2, where the case  $m = 4$  is illustrated. Note also that  $2m \leq 2N_1 \leq 2N_k$ ).

On the other hand, we get from (4.5) by summation on the  $m^2$  circles:

$$\varepsilon_1 N_1^2 > \sum_{N_1+1}^{2N_1} \sum_1^{6j} (1 - r_{jk})^2 > m^2(1 - e^{-\rho})^2 = m^2(e^\rho - 1)^2 e^{-2\rho} > m^2 \rho^2 / 4.$$

The last inequality follows from (2.3). Using this, combined with (4.8) we get  $\varepsilon_1^2 > (m/N_1)^2 \rho^2 / 4 \geq \rho^4 / 16A^2$ . But  $\varepsilon_1 = \rho^2 / 4A$  by (4.1) and this leads to a contradiction to the assumption (4.6). Assuming now (4.6)' instead of (4.6), and choosing  $m$  by a similar procedure, we have

$$\begin{aligned} r_j &> r_{\rho, 2N_1} (1 + A/N_1)^{-2m} \geq e^{3\rho} (1 + A/N_1)^{-2m} \\ &> e^{3\rho} e^{-2mA/N_1} \geq e^{3\rho - 2\rho} = e^\rho. \end{aligned}$$

Hence,  $\varepsilon_1^2 N_1^2 > m^2 (e^\rho - 1)^2 > \rho^2 m^2$ . It follows that

$$\rho^4 / 16A^2 = \varepsilon_1^2 > \rho^2 m^2 / N_1^2 \geq \rho^4 / 4A^2,$$

which is, again, a contradiction. Hence, the treatment of the case  $k = 1$  is complete.

We proceed, now, with the induction proof. So, assume that our statement has been established for  $N_k$ . If  $\varphi_{l_1 N_k}$  maps  $\text{HCP}_{l_1 N_k}$  on  $\text{HCP}_{l_1 N_k}^{**}$ , the induction assumption is that the “ $\rho$  condition” is fulfilled for  $\text{HCP}_{2N_k}^{**}$ .

Using Theorem 2.1, we have from (2.2) for  $N = 2N_k$

$$(4.9) \quad S_{l_1 N_k} \leq \bar{S}_{2N_k} < A / 2N_k = A / N_{k+1}.$$

Consider now  $\varphi_{l_1 N_{k+1}}$  that maps  $\text{HCP}_{l_1 N_{k+1}}$  onto  $\text{HCP}_{l_1 N_{k+1}}^{**}$ . For the subconfiguration  $\text{HCP}_{2N_{k+1}}$  each border circle may be considered as the zero generation for a packing with  $l_1 N_{k+1} - 2N_{k+1} = (l_1 - 2)N_{k+1}$  generations. Using  $l_1 \geq 4$  (from (4.4)) and  $N_{k+1} = 2N_k$ , it follows that  $l_1 N_{k+1} - 2N_{k+1} \geq l_1 N_k$ . Hence,

at least  $l_1 N_k$  generations surround each of the circles that belong to  $\text{HCP}_{2N_{k+1}}$ . Thus, from (4.9),

$$(4.10) \quad r_j/r'_j < 1 + A/N_{k+1}$$

for each two neighboring circles of  $\text{HCP}_{2N_{k+1}}^{**}$ . Also, by substituting  $N = N_{k+1} \geq N_1$  in (4.2)

$$(4.11) \quad \sum_{j=N_{k+1}}^{2N_{k+1}} \sum_{l=1}^{6j} (1 - r_{jl})^2 < \varepsilon_1^2 N_{k+1}^2.$$

The analysis now is identical with the previous treatment for  $N_1$ . Indeed, (4.11) replaces (4.5) and (4.10) replaces (4.7), and thus, the “ $\rho$  condition” is established for  $N_{k+1}$  which completes the induction process and the proof of the theorem.

### 5. PROOF OF THE RODIN SULLIVAN CONJECTURE

The RS conjecture [6] asserts that  $s_n = O(1/n)$  for  $n \rightarrow \infty$ . As proved by Rodin in [5] (see also [6]) this result implies the uniform convergence of the “circle function” ( $r_\varepsilon$ , in Rodin’s notation) to the modulus of the derivative of the Riemann map.

It is obviously enough to prove that  $s_n < B/n$  for  $n \geq n_0$  where  $B$  is an absolute constant, and  $n_0$  is some fixed natural number. Hence, it will be enough to prove

**Theorem 5.1.** *Let  $A, l_1, N_1$ , be the numerical constants defined above. Then*

$$(5.1) \quad s_n < l_1 A/n, \quad \forall n \geq l_1 N_1.$$

*Proof.* The proof is an easy consequence of Theorem 4.1 combined with Theorem 2.1.

We first consider the case  $n = l_1 N_k$  and then treat the general case  $l_1 N_k \leq n \leq l_1 N_{k+1}$ . So, let  $n = l_1 N_k$ ,  $N_k = 2^{k-1} N_1$  where  $k$  is an arbitrary natural number.

Also, consider any hexagonal packing  $\text{HCP}'_{l_1 N_k}$  which is combinatorially equivalent to  $\text{HCP}_{l_1 N_k}$ . Using the invariance property discussed previously, we may replace w.l.o.g.  $\text{HCP}'_{l_1 N_k}$  by the corresponding  $\text{HCP}_{l_1 N_k}^{**}$ . From Theorem 4.1 we conclude that the radii of the circles in the subconfiguration  $\text{HCP}_{2N_k}^{**}$  must satisfy the “ $\rho$  condition” (2.1). Now, using Theorem 2.1 applied for  $\text{HCP}_{2N_k}^{**} \subset \text{HCP}_{l_1 N_k}^{**}$  we have from (2.2)

$$(5.2) \quad S_{l_1 N_k} \leq \bar{S}_{2N_k} < A/2N_k = (l_1 A/2)/l_1 N_k,$$

which implies (5.1) for  $n = l_1 N_k$ , even with a better constant. Next, take any  $n$  s.t.  $n \geq l_1 N_1$ . Then, for some natural  $k \geq 1$ ,

$$(5.3) \quad l_1 N_k \leq n \leq l_1 N_{k+1} = 2l_1 N_k.$$

Hence, from (5.2), (5.3)

$$s_n \leq S_{l_1 N_k} < (l_1 A/2)/l_1 N_k = l_1 A/2l_1 N_k \leq l_1 A/n,$$

which is the statement of Theorem 5.1.

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