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A BANACH SPACE NOT CONTAINING c_0, l_1 OR A REFLEXIVE SUBSPACE

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ABSTRACT. An infinite-dimensional Banach space is constructed which does not contain c_0, l_1 or an infinite-dimensional reflexive subspace. In fact, it does not even contain l_1 or an infinite-dimensional subspace with a separable dual.

An old result of James [2] asserts that a Banach space with an unconditional basis is either reflexive or has a subspace isomorphic to one of c_0 or l_1. This suggests a natural problem, which has been considered by several authors: does every Banach space contain c_0, l_1 or a reflexive subspace? James's result yields a positive answer for any space containing an unconditional basic sequence, so the problem was thrown into sharper focus by the recent construction [1] of a space without one. In this paper we adapt the construction of [1]. We shall draw attention to the differences and similarities later. We also show that our space has no subspace with a separable dual. Since a theorem of Johnson and Rosenthal [5] states that a subspace of a separable dual space either has a reflexive subspace or a nonseparable dual, this is only a slightly stronger result. However, our proof is direct.

This second statement should be compared with results of James [4] and Lindenstrauss and Stegall [6]. They independently constructed separable spaces not containing l_1 but with nonseparable duals, answering in the negative a question of Banach. The space in this paper can therefore be regarded as a hereditary version of those spaces. (This is true, to some extent, not just of the result, but also of the construction.)

The paper is self-contained, but will be easier to read by those familiar with the techniques of [1] and indeed of [8], Schlumprecht's construction of an arbitrarily distortable space, which lies at the heart of the construction here as well as that of [1]. The main difficulty of this result is the proof of Lemma 4 below. This proof is postponed until after the lemma is used to prove the main result. Thus the reader who wishes to understand the main ideas of the construction without wading through pages of technical argument can simply stop reading when the proof of Lemma 4 starts.

We shall begin by giving the definition of our norm, which is fairly complicated.

First, let $f\colon \mathbf{R} \to \mathbf{R}$ be the function $x \mapsto \sqrt{\log_2(x+1)}$, and note that f has the following properties:

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(i) $f(1) = 1$ and $f(x) < x$ for every $x > 1$;
(ii) f is strictly increasing and tends to infinity;
(iii) $\lim_{x\to\infty} x^{-q} f(x) = 0$ for every $q > 0$;
(iv) the function $xf(x)^{-2}$ is concave and nondecreasing;
(v) $f(xy) \le f(x)f(y)$ for every $x, y \ge 1$.

Let c_{00} be the vector space of sequences of real numbers all but finitely many of which are zero, and let the standard basis of c_{00} be written $\mathbf{e}_1, \mathbf{e}_2, \dots$. An *interval* $E \subset \mathbf{N}$ is a subset of the form $\{a, a+1, a+2, \dots, b\}$ for some $a \le b$, $a, b \in \mathbf{N}$. Given an interval E, let the letter E also stand for the projection from c_{00} to c_{00} defined by $\sum_{i=1}^{\infty} a_i \mathbf{e}_i \mapsto \sum_{i\in E} a_i \mathbf{e}_i$. Given two intervals $E, F \subset \mathbf{N}$, write $E < F$ if $\max E < \min F$. The *support* of a vector $x = \sum_{i=1}^{\infty} x_i \mathbf{e}_i \in c_{00}$ is $\operatorname{supp}(x) = \{i : x_i \ne 0\}$. This is always a finite set. We define the *range* of a vector, written $\operatorname{ran}(x)$, to be the smallest interval containing $\operatorname{supp}(x)$. Given $x, y \in c_{00}$, we shall write $x < y$ for the statement $\operatorname{ran}(x) < \operatorname{ran}(y)$. If $x_1 < \dots < x_N$, then we say that $x_1, \dots, x_N$ are *successive*.

Let $\mathbf{Q}$ be the subset of c_{00} consisting of sequences of rationals in the interval $[-1, 1]$. Let $J \subset \mathbf{N}$ be a set such that, if $m < n$ and $m, n \in J$, then $\log\log\log\log\log n \ge 1000m$. Let us also suppose that $f(j) \ge 10^{103}$ for every $j \in J$. Let σ be an injection from the set of finite sequences of successive elements of $\mathbf{Q}$ to J.

Let $X = (c_{00}, \|\cdot\|)$ be any normed space such that the standard basis is bimonotone. For every $m \in \mathbf{N}$, define $A_m^*(X)$ to be the set of linear functionals on X of the form $f(m)^{-1}(x_1^* + \dots + x_m^*)$, where $x_1^*, \dots, x_m^*$ are successive members of c_{00} and $\|x_i^*\| \le 1$ for each i. A *special sequence* of functionals on X is defined to be a sequence of the form $z_1^*, \dots, z_M^*$, where $M \in \mathbf{N}$, the z_i^* are successive, $z_1^* \in A_m^* \cap \mathbf{Q}$ for some $m \in J$ and, for $2 \le i \le M$, we have $z_i^* \in A^*_{\sigma(z_1^*, \dots, z_{i-1}^*)} \cap \mathbf{Q}$. A *special functional* on X is defined to be a functional of the form $E(z_1^* + \dots + z_M^*)$ such that $z_1^*, \dots, z_M^*$ is a special sequence. To any special sequence $z_1^*, \dots, z_M^*$ we can associate a sequence of integers $n_1, \dots, n_M \in J$ such that $z_1^* \in A_{n_1}^*$ and $n_i = \sigma(z_1^*, \dots, z_{i-1}^*)$ for $2 \le i \le M$. The first number n_1 is not necessarily uniquely determined, but $n_2, \dots, n_M$ certainly are. Given a special functional $z^* = E(z_1^* + \dots + z_M^*)$, we say that $Z \subset J$ is an *associated set* for z^* if we can pick such a sequence $n_1, \dots, n_M$ associated to the sequence $z_1^*, \dots, z_M^*$ and Z consists of those n_i for which $E \cap \operatorname{ran}(z_i^*) \ne \varnothing$. A collection of special functionals $w_1^*, \dots, w_N^*$ is called *disjoint* if we can choose for them disjoint associated sets $Z_1, \dots, Z_N$.

We are now ready to define our norm. We shall define it as the limit of a sequence of norms on c_{00}. First, let X_0 be defined by $\|x\|_{X_0} = \|x\|_\infty$. For $n \ge 1$, define X_n by

$$\|x\|_{X_n} = \|x\|_{X_{n-1}} \vee \sup\left\{ f(N)^{-1} \sum_{i=1}^{N} \|E_i x\|_{X_{n-1}} : N \ge 2,\ E_1 < \dots < E_N \right\}$$
$$\vee \sup\left(\sum_{i=1}^{M} |x_i^*(x)|^2 \right)^{1/2}$$

where the second supremum ranges over all sequences $x_1^*, \dots, x_M^*$ of disjoint special functionals on X_{n-1}.

Now we claim that $\|x\|_{X_n} \le \|x\|_1$ for every n. This is certainly true when

$n = 0$. If it is true for $n = k$ then $\|x^*\|_{X_k^*} \geq \|x^*\|_\infty$ for every $x^* \in c_{00}$. It follows that $\|x^*\|_\infty \leq f(m)^{-1}$ for every $x^* \in A_m^*(X_k)$. Given a sequence $x_1^*, \dots, x_M^*$ of disjoint special functionals on X_k, we can find disjoint associated sets $Z_1, \dots, Z_M \subset J$. We know that $\|x_i^*\|_\infty \leq f(\min Z_i)^{-1}$, so

$$\left(\sum_{i=1}^M |x_i^*(x)|^2\right)^{1/2} \leq \sum_{i=1}^M |x_i^*(x)| \leq \|x\|_1 \sum_{i=1}^M \|x_i^*\|_\infty$$
$$\leq \|x\|_1 \sum_{j\in J} f(j)^{-1} \leq \|x\|_1 .$$

It follows easily that $\|\cdot\|_{X_{k+1}}$ is also dominated by $\|\cdot\|_1$.

It is also clear that if X and Y are two normed spaces on c_{00} such that $\|x\|_X \leq \|x\|_Y$ for every $x \in c_{00}$, then every sequence of disjoint special functionals on X is also such a sequence on Y. This implies that $\|\cdot\|_{X_0}, \|\cdot\|_{X_1}, \|\cdot\|_{X_2}, \dots$ is an increasing sequence of norms. Since they are bounded above by $\|\cdot\|_1$, they tend to a limit, giving a space $X = (c_{00}, \|\cdot\|)$. Strictly speaking, we will be interested in the completion of this space, but it is more convenient for the time being to consider the incomplete space X.

It is easy to check that every $x \in X$ satisfies the equation

$$\|x\| = \|x\|_\infty \vee \sup\left\{ f(N)^{-1} \sum_{i=1}^N \|E_i x\| : N \geq 2,\ E_1 < \dots < E_N \right\}$$
$$\vee \sup \left(\sum_{i=1}^M |x_i^*(x)|^2\right)^{1/2}$$

where the second supremum is over all sequences $x_1^*, \dots, x_M^*$ of disjoint special functionals on X. Note in particular that the standard basis of X is bimonotone.

We shall now state and prove some lemmas about X. The first three are very similar to lemmas proved by Schlumprecht and slightly adapted in [1]. First, we say that $x \in X$ is an *l_{1+}^n-average with constant C* if $\|x\| = 1$ and $x = \sum_{i=1}^n x_i$ for some sequence of successive nonzero vectors $x_1, \dots, x_n$ such that $\|x_i\| \leq Cn^{-1}$ for every i. An *l_{1+}^n-vector* is simply a nonzero multiple of an l_{1+}^n-average. That is, it is a vector x that can be written as $\sum_{i=1}^n x_i$ for some sequence of successive nonzero vectors $x_1, \dots, x_n$ such that $\|x_i\| \leq Cn^{-1}\|x\|$ for each i.

Lemma 1. *For every $n \in \mathbf{N}$ and $C > 1$ there exists N such that, for any sequence $x_1, \dots, x_N$ of successive nonzero vectors in X, the subspace generated by $x_1, \dots, x_N$ contains an l_{1+}^n-average with constant C.*

Proof. Suppose the result is false. Without loss of generality the x_i all have norm one. Let k be an integer such that $k \log C > \log f(n^k)$ (such an integer exists because of property (iii) of the function f), let $N = n^k$ and let $x = \sum_{i=1}^N x_i$. For every $0 \leq i \leq k$ and every $1 \leq j \leq n^{k-i}$, let $x(i, j) = \sum_{t=(j-1)n^i+1}^{jn^i} x_t$. Thus $x(0, j) = x_j$, $x(k, 1) = x$ and, for $1 \leq i \leq k$, each $x(i, j)$ is a sum of n successive $x(i-1, j)$'s. By our assumption, no $x(i, j)$ is an l_{1+}^n-vector with constant C. It follows easily by induction that

$\|x(i,j)\| \le C^{-i}n^i$, and, in particular, that $\|x\| \le C^{-k}n^k = C^{-k}N$. However, it also follows easily from the definition of the norm on X that $\|x\| \ge Nf(N)^{-1}$. This is a contradiction, by choice of k. □

Note that we have proved the slightly stronger result that if the x_i have norm one then there is an interval $E \subset \{1, 2, \dots, N\}$ such that $\sum_{i\in E} x_i$ is an l_{1+}^n-vector. The technique used to prove this lemma is essentially due to R. C. James [3].

Lemma 2. *Let $M, N \in \mathbf{N}$ and $C \ge 1$, let x be an l_{1+}^N-vector with constant C, and let $E_1 < \cdots < E_M$ be a sequence of intervals. Then*

$$\sum_{j=1}^{M} \|E_j x\| \le C(1 + 2M/N)\|x\|.$$

Proof. For convenience, let us normalize so that $\|x\| = N$ and $x = \sum_{i=1}^N x_i$, where $x_1 < \cdots < x_N$ and $\|x_i\| \le C$ for each i. Given j, let A_j be the set of i such that $\operatorname{supp}(x_i) \subset E_j$, and let B_j be the set of i such that $E_j(x_i) \ne 0$. By the triangle inequality and the fact that the basis is bimonotone,

$$\|E_j x\| \le \left\| \sum_{i\in B_j} x_i \right\| \le C(|A_j| + 2).$$

Since $\sum_{j=1}^M |A_j| \le N$, we get

$$\sum_{j=1}^{M} \|E_j x\| \le C(N + 2M)$$

which gives the result, because of our normalization. □

The next definition is extremely important, as it was in [1]. We shall say that a sequence $x_1 < \cdots < x_N$ is a *rapidly increasing sequence of l_{1+}-averages*, or R.I.S., *of length N with constant $1+\epsilon$* if, for each k, x_k is an $l_{1+}^{n_k}$-average with constant $1+\epsilon$, $n_1 \ge 2^{2^N}$ and, for $k = 2, 3, \dots, N$, we have $\epsilon\sqrt{f(n_k)} \ge |\operatorname{ran}(x_1 + \cdots + x_{k-1})|$.

Lemma 3. *Let $0 < \epsilon < 1/2$, let $x_1, \dots, x_N$ be a R.I.S. of length N with constant $1+\epsilon$, and let $x = \sum_{i=1}^N x_i$. Let $M \ge f^{-1}(6N/\epsilon)$, and let $E_1 < \cdots < E_M$. Then $f(M)^{-1}\sum_{j=1}^M \|E_j x\| \le 1 + 2\epsilon$.*

Proof. For each i let n_i be maximal such that x_i is an $l_{1+}^{n_i}$-average with constant $1+\epsilon$. We obtain three estimates for $\sum_{j=1}^M \|E_j x_i\|$.

First, it follows directly from the definition of the norm that $\sum_{j=1}^M \|E_j x_i\| \le f(M)$. Second, we know that it is at most $f(|\operatorname{supp}(x_i)|)$ and by Lemma 2 it is at most $(1+\epsilon)(1 + 2M/n_i)$.

Let t be maximal such that $n_t \le M$. Then, if $i < t$, we have $n_{i+1} \le M$. Hence, by the definition of a R.I.S., we have

$$\sum_{i=1}^{t-1} f(|\operatorname{supp}(x_i)|) \le \epsilon\sqrt{f(M)}$$

and hence

$$\sum_{i=1}^{t-1}\sum_{j=1}^{M}\|E_j x_i\| \le \epsilon\sqrt{f(M)}.$$

Using this and the other two inequalities, we find that

$$\begin{aligned}\sum_{j=1}^{M}\sum_{i=1}^{N}\|E_j x_i\| &\le \epsilon\sqrt{f(M)} + f(M) + (N-t)(1+\epsilon)(1+2M/n_{t+1})\\ &\le f(M) + 3N(1+\epsilon).\end{aligned}$$

Since $f(M) \ge 6N/\epsilon$, the result follows. □

We shall now state the main lemma of this paper and deduce from it that X does not contain c_0, l_1 or a reflexive subspace. We will then prove the main lemma.

Lemma 4. *Let $M \in J$, and let $x_1, \dots, x_M$ be a R.I.S. of length M with constant $3/2$. Then there exists a choice of signs $\epsilon_1, \dots, \epsilon_M$ such that $\|\sum_{i=1}^{M}\epsilon_i x_i\| < 100Mf(M)^{-1}$.*

Once we have proved this lemma, the proof of our main theorem is fairly straightforward. First, if the completion of X contains a subspace that is c_0, l_1 or reflexive, then it must contain such a subspace generated by a block basis. Let us call it Y and the block basis $y_1, y_2, \dots$. Since $\|\sum_{i=1}^{N} y_i\| \ge Nf(N)^{-1}$ we know that Y cannot be c_0. Let $x_1, \dots, x_M$ be a R.I.S. in Y with constant $3/2$ and length $M \in J$. Then by Lemma 4 there is a sequence of signs $\epsilon_1, \dots, \epsilon_M$ such that $\|\sum_{i=1}^{M}\epsilon_i x_i\| < 100Mf(M)^{-1}$. This shows that $y_1, y_2, \dots$ is not $(f(M)/100)$-equivalent to the unit vector basis of l_1. Since M can be arbitrarily large, Y is not l_1.

To show that Y is not reflexive is slightly more complicated, but still easy. Given an integer $M \in J$, define an M-pair as follows. Let $u_1, \dots, u_M$ be a R.I.S. of length M and constant $3/2$. By changing signs if necessary, let the norm of $\sum_{i=1}^{M} u_i$ be at most $100Mf(M)^{-1}$. For each i let u_i^* be a support functional for u_i such that $\operatorname{ran}(u_i^*) \subset \operatorname{ran}(u_i)$. Now let v be the vector $\sum_{i=1}^{M} u_i/\|\sum_{i=1}^{M} u_i\|$ and let v^* be the functional $f(M)^{-1}(u_1^* + \cdots + u_M^*)$. Then $v^*(v) = Mf(M)^{-1}/\|\sum_{i=1}^{M} u_i\| > 1/100$ and $v^* \in A_M$. After a perturbation, we can also get that $v^* \in \mathbf{Q}$, while keeping v^* in A_M^* and keeping the estimate for $v^*(v)$. Such a pair (v, v^*) is what we mean by an M-pair.

We can clearly choose an M-pair (v, v^*) for any integer M in such a way that $v \in Y$, and we can also make $\min\operatorname{supp}(v)$ as large as we like. It follows that we can find a sequence of pairs (v_i, v_i^*) such that $v_1 < v_2 < \cdots$ are successive elements of Y and, for each $i > 1$, (v_i, v_i^*) is a $\sigma(v_1^*, \dots, v_{i-1}^*)$-pair. This ensures that $v_1^*, v_2^*, \dots$ is a special sequence.

We now claim that the linear functional w^* defined by

$$x \mapsto \lim_{N\to\infty}\sum_{n=1}^{N} v_n^*(x)$$

is continuous on the completion of X. This is true because the limit certainly exists for any $x \in X$ and is bounded on $B(X)$, since the functionals used in

the limit are all special functionals and hence have norm at most 1. Thus the functional can be extended to the completion. However, $w^*(v_n) \geq 1/100$ for every n, so the basic sequence $v_1, v_2, \ldots$ is not shrinking. (For basic facts about bases and reflexivity see [2] or [7].) This shows that Y has a nonreflexive subspace and hence is not reflexive.

The same argument shows that Y does not have a separable dual. Indeed, suppose that $z_1^*, z_2^*, \ldots$ were a dense subset of Y^*. Then we could pick a block basis $x_1, x_2, \ldots$ of $y_1, y_2, \ldots$ such that $z_i^*(x_j) = 0$ for every $j \geq i$ and hence such that every sequence of successive vectors generated by $x_1, x_2, \ldots$ tended weakly to zero. However, as above, we can find a block basis $v_1, v_2, \ldots$ of $x_1, x_2, \ldots$ and a functional w^* such that $w^*(v_n) \geq 1/100$ for every $n \in \mathbf{N}$. This argument is standard and forms part of the proof of a more general result of Johnson and Rosenthal [5] stated in the introduction.

This concludes the less technical part of the paper. The rest of the paper is devoted to proving Lemma 4. We will need some preliminary lemmas before we begin the proof in earnest. The first is similar to, but more complicated than, Lemma 3. We need a few more definitions. A *special combination* is a functional of the form $\sum_{i=1}^N a_i x_i^*$ where $\sum_{i=1}^N |a_i|^2 = 1$ and $x_1^*, \ldots, x_N^*$ is a sequence of disjoint special functionals. A particularly simple sort of special combination can be defined as follows. Pick distinct integers $j_1, \ldots, j_N \in J$, pick $x_i^* \in A_{j_i}^*$ (recall that the sets A_m^* were defined at the same time as special sequences, etc.), let $E_1, \ldots, E_N$ be any sequence of intervals, let $\sum_{i=1}^N |a_i|^2 = 1$, and let $x^* = \sum_{i=1}^N a_i E_i(x_i^*)$. We shall call these *basic* special combinations. Thus, a basic special combination is one where the special sequences used to build it have length at most 1.

For the proof of the next lemma and of the main lemma later it will be convenient to make one other definition. Let $x_1 < \cdots < x_M$ be a R.I.S. with constant $1+\epsilon$, for some $\epsilon > 0$. For each i, let n_i be maximal such that x_i is an $l_{1+}^{n_i}$-average with constant $1+\epsilon$ and let us write it out as $x_i = x_{i1} + \cdots + x_{in_i}$, where $\|x_{ij}\| \leq (1+\epsilon)n_i^{-1}$ for each j. Given an interval $E \subset \mathbf{N}$, let $i = i_E$ and $j = j_E$ be respectively minimal and maximal such that Ex_i and Ex_j are nonzero, and let $r = r_E$ and $s = s_E$ be respectively minimal and maximal such that Ex_{ir} and Ex_{js} are nonzero. Define the *length* $\lambda(E)$ of the interval E to be $j_E - i_E + (s_E/n_{j_E}) - (r_E/n_{i_E})$. Thus the length of E is the number of x_i's contained in E, allowing for fractional parts. Obviously this definition depends on the R.I.S. in question, but it will always be clear from the context which one is being considered.

It is easy to check that if $E_1 < \cdots < E_M$ and $E = \bigcup E_i$ then $\sum \lambda(E_i) \leq \lambda(E)$. It follows from the triangle inequality and the lower bound for n_1 in the definition of a R.I.S. that if $x = x_1 + \cdots + x_M$ with $x_1, \ldots, x_M$ a R.I.S. with constant $1+\epsilon$, then $\|Ex\| \leq (1+\epsilon)(\lambda(E) + 2^{2^{-M}})$.

Lemma 5. *Let $l \in J$, and let $x_1, \ldots, x_M$ be a R.I.S. of length M with constant $3/2$ such that $\exp\sqrt{\log l} \leq M \leq l$. Suppose also that $x = x_1 + \cdots + x_M$ is an $l_{1+}^{M'}$-vector with constant 2 for some $M' \geq \log M$. Let $x^* = \sum_{i=1}^N a_i x_i^*$ be a basic special combination. Then $|x^*(x)| \leq 10^{-100}\|x\| + 2Mf(M)^{-1}$.*

Proof. There exist distinct integers $k_1, \ldots, k_N \in J$ such that, for each i, x_i is an interval projection of some functional in $A_{k_i}^*$. Suppose that there is some

i such that $k_i = l$. Then $x_i^* = f(l)^{-1}E(y_1^* + \cdots + y_l^*)$ for some sequence $y_1^*, \ldots, y_l^*$ of successive norm-one functionals and some interval E. Let $E_r = \operatorname{ran}(y_r^*)$ for each r. Then certainly

$$\begin{aligned}|x_i^*(x)| &\le f(l)^{-1}\sum_{r=1}^{l}\|E_r x\| \le f(l)^{-1}\left(2^{2^{-M}}l + (3/2)\sum_{r=1}^{l}\lambda(E_r)\right)\\ &\le f(l)^{-1}(1+3M/2) \le 2Mf(M)^{-1}.\end{aligned}$$

Let us now suppose that $\sum_{i=1}^N a_i x_i^*$ is a basic combination and no k_i is equal to l. It follows that, for each i, either $k_i \le \log\log\log l$ or $k_i \ge \exp\exp\exp l$.

For each $j = 1, 2, \ldots, M$, let n_j be the greatest integer such that x_j is an $l_{1+}^{n_j}$-average with constant $3/2$, and, for each $i = 1, 2, \ldots, N$, let t_i be the greatest value of j such that $k_i \ge n_j$ (or zero if there is no such j). We shall now examine the effect of the functional x_i^* on x.

We have

$$x_i^*\left(\sum_{j=1}^{M} x_j\right) = x_i^*\left(\sum_{j=1}^{t_i-1} x_j\right) + x_i^*(x_{t_i}) + x_i^*\left(\sum_{j=t_i+1}^{M} x_j\right).$$

When $j < t_i$, we know that $n_{j+1} \le k_i$ which implies, by the definition of a R.I.S., that $|\operatorname{supp}(x_1 + \cdots + x_{t_i-1})| \le \sqrt{f(k_i)}$. The first part of the right-hand side is therefore at most $\sqrt{f(k_i)}\|x_i^*\|_\infty \le f(k_i)^{-1/2}$ in modulus. As for the third part, if we temporarily set $y_i = \sum_{j=t_i+1}^M x_j$, we find that it is at most

$$\sup\left\{f(k_i)^{-1}\sum_{r=1}^{k_i}\|E_r y_i\| : E_1 < \cdots < E_{k_i}\right\}$$

in magnitude. Given an optimal sequence $E_1, \ldots, E_{k_i}$, at most M of the intervals contains part of more than one x_j. We now consider two cases.

If $k_i \ge M$ then $k_i \ge \exp\exp\exp M$, so the best splitting is no greater than the greatest possible value of $f(k_i)^{-1}\sum_{j=t_i+1}^M\sum_{r=1}^{s_j}\|E_{jr}x_j\|$ such that $\sum_{j=t_i+1}^M s_j \le k_i + M$. However, when $j \ge t_i + 1$ in this case, we certainly know that $n_j \ge k_i + M$, so the greatest possible value is, by Lemma 2, at most

$$\sum_{j=t_i+1}^{M} f(k_i)^{-1}(1 + 2n_j^{-1}(k_i + M)) \le 3(M - t_i)f(k_i)^{-1} \le f(k_i)^{-1/2}.$$

On the other hand, if $k_i \le M$, then $k_i \le \log\log\log M$. Using Lemma 2 again and the fact that x is an $l_{1+}^{M'}$-average, we have

$$\sum_{r=1}^{N_i}\|E_r y_i\| \le \sum_{r=1}^{N_i}\|E_r x\| \le 2(1 + 2k_i/M')\|x\|,$$

giving $|x^*(y_i)| \le 6f(k_i)^{-1}\|x\|$.

Since $k_1, \ldots, k_N$ are distinct, we get

$$\sum_{i=1}^{N}\left|\sum_{j\ne t_i} x_i^*(x_j)\right| \le 2\sum_{s\in J} f(s)^{-1/2} + 6\sum_{s\in J} f(s)^{-1}\|x\| \le 10^{-101}\|x\|.$$

On the other hand, $\sum_{i=1}^N a_i x_i^*(x_{t_i})$ is at most $(\sum_{i=1}^N |x_i^*(x_{t_i})|^2)^{1/2}$ which equals $(\sum_{j=1}^M \sum_{t_i=j} |x_i^*(x_j)|^2)^{1/2}$. But $\sum_{t_i=j} |x_i^*(x_j)|^2 \le 1$ for every j, so this is at most $\sqrt{M}$ which is less than $10^{-101} M f(M)^{-1} \le 10^{-101}\|x\|$.

It follows in this case that $|x^*(x)| \le 10^{-100}\|x\|$. It follows from our two calculations that in general $|x^*(x)| \le 10^{-100}\|x\| + 2Mf(M)^{-1}$, as stated. □

Some later arguments will make use of the following easy Chebyshev-type lemma.

Lemma 6. *Let $\epsilon > 0$ and $\delta = \sqrt{2\epsilon}$. Let $\sum_{i=1}^N a_i^2 \le 1$, let $\sum_{i=1}^N b_i^2 \le 1$, and let $\sum_{i=1}^N a_i b_i \ge 1 - \epsilon$. Then there exists a subset $A \subset \{1, 2, \dots, N\}$ such that $\sum_{i\in A} a_i^2 \ge 1 - \delta$ and, for every $i \in A$, we have $1 - \sqrt{\delta} \le b_i/a_i \le 1 + \sqrt{\delta}$.*

Proof. First, we have $\sum_{i=1}^N (a_i - b_i)^2 \le 2\epsilon$. Suppose that there exists $A \subset \{1, 2, \dots, N\}$ such that $\sum_{i\in A} a_i^2 \ge \delta$ and $(a_i - b_i)^2 > \delta a_i^2$ for every $i \in A$. Then

$$\sum_{i=1}^N (a_i - b_i)^2 \ge \sum_{i\in A} (a_i - b_i)^2 > \delta \sum_{i\in A} a_i^2 \ge \delta^2 = 2\epsilon$$

contradicting the first estimate. It follows that there exists a subset $A \subset \{1, 2, \dots, N\}$ such that $\sum_{i\in A} a_i^2 \ge 1 - \delta$ and $(a_i - b_i)^2 \le \delta a_i^2$ for every $i \in A$. This implies that $|a_i - b_i| \le \sqrt{\delta}|a_i|$ for each $i \in A$ which implies the lemma. □

The next lemma is very well known and has been used extensively in the local theory of Banach spaces.

Lemma 7. *Let $f\colon \{-1, 1\}^n \to \mathbf{R}$ be a function that is 1-Lipschitz with respect to the Hamming distance on $\{-1, 1\}^n$, let $\mathbf{P}$ be the uniform distribution on $\{-1, 1\}^n$, and let $\mathbf{M}$ be the median of f. Then*

$$\mathbf{P}[|f(\epsilon) - \mathbf{M}| \ge \delta n] \le 2\exp(-\delta^2 n/2).$$

We are now ready to prove the main lemma.

Proof of Lemma 4. We shall prove the following stronger statement. There is a choice of signs $\epsilon_1, \dots, \epsilon_M$ such that, for every interval E,

$$\left\| E\left(\sum_{i=1}^M \epsilon_i x_i\right)\right\| < 100\lambda(E) f(M) f(\lambda(E))^{-2}.$$

Indeed, suppose this statement is false. We shall derive a contradiction in several stages. First, define a seminorm $|||\cdot|||$ (actually it is easy to see that it is a norm) on X by

$$|||x||| = \sup\{|x^*(x)| : x^* \text{ is a special combination}\}.$$

Also, for the rest of the paper, let $\epsilon = 10^{-50}$.

Step 1. There exists an interval $A \subset \{1, 2, \dots, M\}$ of cardinality

$$N \ge 20 \exp\sqrt{\log M}$$

such that, with probability at least M^{-2} (over $\{-1, 1\}^A$) the following statements are true:

(i) $|||\sum_{i\in E}\epsilon_i x_i||| \geq (1-\epsilon)\|\sum_{i\in A}\epsilon_i x_i\| \vee (1-\epsilon)100Nf(M)f(N)^{-2}$;
(ii) for every subinterval $B \subset A$ we have

$$\left\|\sum_{i\in B}\epsilon_i x_i\right\| < 100|B|f(M)f(|B|)^{-2}.$$

Proof. For every $\varepsilon \in \{-1, 1\}^M$ there must be a minimal interval E such that $\|Ex(\varepsilon)\| \geq 100\lambda(E)f(M)f(\lambda(E))^{-2}$, where $x(\varepsilon)$ stands for the vector $\sum_{i=1}^M \epsilon_i x_i$. (Recall also that $\lambda(E)$ is the length of the interval E defined before Lemma 5.) Now $\|Ex(\varepsilon)\| \leq (3/2)\lambda(E)+2$, so this tells us that $(3/2)\lambda(E)+2 > 100\lambda(E)f(M)f(\lambda(E))^{-2}$ which implies that $\lambda(E) \geq 20\exp\sqrt{\log M}$.

First, we shall show that, for such an E, we have $\|Ex(\varepsilon)\| = |||Ex(\varepsilon)|||$, i.e., $x(\varepsilon)$ is normed by a special combination. Indeed, if this is *not* the case, then we can find a sequence of intervals $F_1 < \cdots < F_k$ with $\bigcup_{i=1}^k F_i = E$ such that, writing $x = x(\varepsilon)$,

$$\|Ex\| = f(k)^{-1}\sum_{i=1}^k \|F_i x\|.$$

Setting $\lambda_i = \lambda(F_i)$ and $\lambda = \lambda(E)$, we know that $\sum_{i=1}^k \lambda_i \leq \lambda$. By Lemma 3, we also know that $k \leq f^{-1}(6M/\epsilon)$. We also know that, for each i, $\|F_i x\| \leq (3/2)\lambda + 2^{2^{-M}}$. It follows that $\sum_{i=1}^k \|F_i x\| \leq (3/2)\lambda_i + 2^{2^{-M/2}}$. Since $\|Ex\| > 100\lambda f(M)f(\lambda)^{-2}$ this tells us that

$$((3/2)\lambda + 2^{2^{-M/2}})f(k)^{-1} > 100\lambda f(M)f(\lambda)^{-2} > 100\lambda f(\lambda)^{-1}.$$

If $k \geq \lambda^{1/100}$ then this is a contradiction.

On the other hand, by minimality of E, we also know that

$$\|F_i x\| \leq 100\lambda_i f(M)f(\lambda_i)^{-2}.$$

This tells us that

$$f(k)^{-1}\sum_{i=1}^k 100\lambda_i f(M)f(\lambda_i)^{-2} > 100\lambda f(M)f(\lambda)^{-2}.$$

Since $x/f(x)^2$ is concave and $\lambda_1 + \cdots + \lambda_k = \lambda$, Jensen's inequality gives us that

$$f(k)^{-1}k.(\lambda/k).f(M)f(\lambda/k)^{-2} > \lambda f(M)f(\lambda)^{-2}$$

which implies that $f(k)f(\lambda/k)^2 < f(\lambda)^2$. But if $2 \leq k < \lambda^{1/100}$ this gives us that $f(2)f(\lambda^{99/100})^2 < f(\lambda)^2$ which is clearly false, by the definition of the function f.

This establishes that $\|Ex\| = |||Ex|||$. Now let $A = \{i\colon \operatorname{ran}(x_i) \subset E\}$. We shall abuse notation in the following way. When it is understood that an interval A is a subset of $\{1, 2, \ldots, M\}$, we shall use the letter A to refer to the projection $\sum_{i=1}^m \epsilon_i x_i \mapsto \sum_{i\in A}\epsilon_i x_i$. Then, setting $N = |A|$, we have

$$|||Ax||| \geq |||Ex||| - 2 \geq (1-\epsilon).100f(M)f(N)^{-2}.$$

Since $\|Ex\| = |||Ex|||$ and the basis of X is bimonotone, it is also clear that $|||Ax||| \geq \|Ax\| - 2 \geq (1-\epsilon)\|Ax\|$. If $B \subset A$ is any subinterval, then, by the minimality of E, we have $\|Bx\| \leq 100|B|f(M)f(|B|)^{-2}$.

Now, there are at most M^2 such intervals $A \subset \{1, 2, \dots, M\}$ and there is one for each collection of signs $(\epsilon_i)_{i=1}^M$. Hence, some interval A is used at least $M^{-2}2^M$ times. If $|A| = N$, then for at least $M^{-2}2^N$ choices of $\varepsilon \in \{-1, 1\}^A$ parts (i) and (ii) of the claim hold.

Before stating the next step, we shall introduce some notation. Let $K = \lceil \log N \rceil$ and let $B_1 < \cdots < B_{5K}$ be subintervals of A, each of cardinality between $(1-\epsilon)N/5K$ and $(1+\epsilon)N/5K$. Let $v_i = \sum_{j \in B_i} \epsilon_j x_j$ and for $r = 1, 2, 3, 4, 5$ let $u_r = \sum_{i=(r-1)K+1}^{rK} v_i$. Thus the u_r and the v_i are variables depending on the $(\epsilon_j : j \in A)$.

Step 2. There exists a choice of signs $(\epsilon_i : i \in A)$ such that

$$\left|\!\left|\!\left| \sum_{j=1}^{5} \eta_j u_j \right|\!\right|\!\right| > (1-\epsilon).100Nf(M)f(N)^{-2}$$

for every choice of signs $\eta_1, \dots, \eta_5$, and also such that for each i

$$\|v_i\| \le (1+3\epsilon).20Nf(M)/Kf(N)^2.$$

Proof. For a fixed choice of $\eta_1, \dots, \eta_5$ we know that $|\!|\!| \sum_{j=1}^5 \eta_j u_j |\!|\!|$ is a 2-Lipschitz function on $\{-1, 1\}^A$. Hence, by Lemma 7,

$$\mathbf{P}\left[\left| \left|\!\left|\!\left| \sum_{j=1}^{5} \eta_j u_j \right|\!\right|\!\right| - \mathbf{M} \left|\!\left|\!\left| \sum_{j=1}^{5} \eta_j u_j \right|\!\right|\!\right| \right| > \frac{\epsilon}{20}.100Nf(M)f(N)^{-2}\right] \le \exp\left(-\frac{1}{2}\left(\frac{5\epsilon}{2f(N)}\right)^2 N\right).$$

Since M^{-2} is greater than this, Step 1 implies that

$$\mathbf{M} \left|\!\left|\!\left| \sum_{j=1}^{5} \eta_j u_j \right|\!\right|\!\right| \ge \left(1 - \epsilon - \frac{\epsilon}{20}\right).100Nf(M)f(N)^{-2},$$

and hence, by Lemma 7 again,

$$\mathbf{P}\left[\left|\!\left|\!\left| \sum_{j=1}^{5} \eta_j u_j \right|\!\right|\!\right| < (1-2\epsilon).100Nf(M)f(N)^{-2}\right] \le \exp\left(-\left(\frac{\epsilon}{40f(N)}\right)^2 N\right).$$

Similarly,

$$\mathbf{P}[\|v_i\| > (1+\epsilon).100(1+\epsilon)(N/5K)f(M)f(N/5K)^{-2}] \le \exp\left(-\left(\frac{\epsilon}{40f(N)}\right)^2 \frac{N}{5K}\right).$$

But $(1+\epsilon)^2 f(N/5K)^{-2} \le (1+3\epsilon)f(N)^{-2}$, so

$$\mathbf{P}[\|v_i\| > (1+3\epsilon).100(N/5K)f(M)f(N)^{-2}] \le \exp\left(-\left(\frac{\epsilon}{40f(N)}\right)^2 \frac{N}{5K}\right).$$

We have $5K+32$ events that we want to occur simultaneously, and the probability of each individual event failing is at most $\exp(-(\epsilon/40f(N))^2(N/5K))$ which is less than $(5K+32)^{-1}$. This proves the second step.

Let us now fix a choice of signs ε satisfying the conditions of the previous step, so that $u_1, \dots, u_5$ and $v_1, \dots, v_{5K}$ are now fixed vectors. Note that u_j is the sum of a R.I.S. of length approximately $N/5$ (certainly at least $\exp\sqrt{\log M}$) and constant $3/2$. Also, since

$$\|u_j\| \geq |||u_j||| \geq (1-\epsilon).20Nf(M)f(N)^{-2}$$

and $u_j = \sum_{i=(j-1)K+1}^{jK} v_i$ with $\|v_i\| \leq (1+3\epsilon).(20N/K)f(M)f(N)^{-2}$, we have that u_j is an l_{1+}^K-average with constant $(1+5\epsilon)$. This remark will be useful later when we shall apply Lemma 5 to the vector u_4.

Before moving on to the next step, we shall need some more notation. First, we know that there must exist special combinations $x^* = \sum_{i=1}^n a_i x_i^*$ and $y^* = \sum_{j=1}^m b_j y_j^*$ such that

$$5\alpha \leq |x^*(u_1+u_2+u_3+u_4+u_5)| \leq 5\alpha(1+3\epsilon)(1-\epsilon)^{-1}$$

and

$$5\alpha \leq |y^*(u_1+u_2-u_3+u_4+u_5)| \leq 5\alpha(1+3\epsilon)(1-\epsilon)^{-1}$$

while

$$\|u_j\| \leq \alpha(1+3\epsilon)(1-\epsilon)^{-1} \quad \text{for each } j.$$

Here α stands for $(1-\epsilon).20Nf(M)f(N)^{-2}$.

There are four cases for the signs of $x^*(u_1+u_2+u_3+u_4+u_5)$ and $y^*(u_1+u_2-u_3+u_4+u_5)$. We shall only look at the case when both are positive. The other cases are similar. Let us define probability measures μ and ν on $\{1, 2, \dots, n\}$ and $\{1, 2, \dots, m\}$ by $\mu(A) = \sum_{i\in A}|a_i|^2$ and $\nu(B) = \sum_{j\in B}|b_j|^2$. Let us also define a sequence of signs $\eta_1, \dots, \eta_5$ by $\eta_3 = -1$ and otherwise $\eta_r = 1$.

Step 3. Let $\delta = \sqrt{100\epsilon}$. Then there exist C and D with $\mu(C) \geq 1-5\delta$ and $\nu(D) \geq 1-5\delta$ such that, for every $i \in C$, $j \in D$ and $1 \leq r \leq 5$, we have

$$(1-\sqrt{\delta})(1+5\epsilon)a_i\alpha \leq x_i^*(u_r) \leq (1+\sqrt{\delta})(1+5\epsilon)a_i\alpha$$

and

$$(1-\sqrt{\delta})(1+5\epsilon)b_j\alpha \leq \eta_r y_j^*(u_r) \leq (1+\sqrt{\delta})(1+5\epsilon)b_j\alpha.$$

Proof. We know that $x^*(u_r) \leq \alpha(1+3\epsilon)(1-\epsilon)^{-1} \leq \alpha(1+5\epsilon)$ for each r, so $x^*(u_r) \geq \alpha(1-20\epsilon)$ for each r (since we know $x^*(u_r) \leq \alpha(1+5\epsilon)$ and $|x^*(u_1+\cdots+u_5)| \geq 5\alpha$. In other words, $\sum_{i=1}^n a_i x_i^*(u_r) \geq \alpha(1-20\epsilon)$ while $\sum_{i=1}^n |x_i^*(u_r)|^2 \leq \alpha^2(1+5\epsilon)^2$. The existence of C now follows from Lemma 6 applied once for each u_r. Similarly, we get the set D.

Let us take a closer look at the x_i^* and y_j^*. Each x_i^* is of the form $E_i(x_{i1}^* + \cdots + x_{ip_i}^*)$ for some special sequence $x_{i1}^*, \dots, x_{ip_i}^*$. Let k_i be minimal such that $\operatorname{ran}(x_{ik_i}^*) \cap \operatorname{ran}(u_5) \neq \varnothing$. Then it may be the case that $\operatorname{ran}(x_{ik_i}^*) \cap \operatorname{ran}(u_3) = \varnothing$ or it may not. Let us define C_1 to be $\{i : \operatorname{ran}(x_{ik_i}^*) \cap \operatorname{ran}(u_3) \neq \varnothing\}$. Similarly, let $D_1 = \{j : \operatorname{ran}(y_{jl_j}^*) \cap \operatorname{ran}(u_3) \neq \varnothing\}$, where l_j is minimal such that $\operatorname{ran}(y_{jl_j}^*) \cap \operatorname{ran}(u_5) \neq \varnothing$.

Step 4. $\max\{\mu(C_1), \mu(D_1)\} \leq 1/50$.

Proof. Let $C_2 = \{1, 2, \dots, n\}\backslash C_1$ and $D_2 = \{1, 2, \dots, n\}\backslash D_1$. Then

$$\sum_{i=1}^{n} a_i x_i^*(u_4) = \sum_{i\in C_1} a_i x_i^*(u_4) + \sum_{i\in C_2} a_i x_i^*(u_4).$$

Writing U_r for $\operatorname{ran}(u_r)$, we have that $U_4(\sum_{i\in C_1} a_i x_i^*)$ is $\mu(C_1)^{1/2}$ multiplied by a basic special combination. We also remarked earlier that u_4 satisfied the conditions required in Lemma 5 (with $N/5$ replacing M and M replacing l). Hence, we may apply that lemma and deduce that

$$\left|\sum_{i\in C_1} a_i x_i^*(u_4)\right| \le \mu(C_1)^{1/2}(10^{-100}\|u_4\| + (2N/5).f(N/5)^{-1}).$$

Now

$$\|u_4\| \ge |||u_4||| \ge \alpha(1-20\epsilon) \ge (1-25\epsilon).20Nf(M)f(N)^{-2},$$

so this is at most $\mu(C_1)^{1/2}(10^{-100} + 1/45)\|u_4\|$.

Meanwhile, $|\sum_{i\in C_2} a_i x_i^*(u_4)| \le \mu(C_2)^{1/2}\|u_4\|$. Since $\sum_{i=1}^{n} a_i x_i^*(u_4) \ge \alpha(1-20\epsilon) \ge (1-25\epsilon)\|u_4\|$, we find that $(1/20)\mu(C_1)^{1/2} + \mu(C_2)^{1/2} \ge 1 - 25\epsilon$. We also know that $\mu(C_1) + \mu(C_2) = 1$. If $\mu(C_1) \ge 1/50$ then $\mu(C_2) \le 49/50$ which implies that $\mu(C_2)^{1/2} \le 99/100$. This tells us that $(99/100) + (1/20\sqrt{50}) \ge 1 - 25\epsilon$, which is false. The argument for D_1 is similar.

Step 5. There exist $C_3 \subset C \cap C_2$ and $D_3 \subset D \cap D_2$ such that $\mu(C_3)$ and $\nu(D_3)$ both exceed $19/20$ and a bijection $\phi\colon C_3 \to D_3$ such that, for every $i \in C_3$, the special functionals $U_5 x_i^*$ and $U_5 y_{\phi(i)}^*$ are not disjoint.

Proof. Without loss of generality all a_i and b_j are nonzero. If $i \in C$ this implies that $x_i^*(u_r) \ne 0$ for every r, and in particular that $\operatorname{ran}(x_i^*) \cap \operatorname{ran}(u_1) \ne \varnothing$. This remark will be useful later. Similarly, if $j \in B$, then $\operatorname{ran}(y_j^*) \cap \operatorname{ran}(u_1) \ne \varnothing$.

If $i \in C \cap C_2$, then $\operatorname{ran}(x_{i,k_i}^*) \cap \operatorname{ran}(u_3) = \varnothing$, but $\operatorname{ran}(x_i^*) \cap \operatorname{ran}(u_3) \ne \varnothing$. It follows that $E_i(x_{i,k_i-1}^*) \ne 0$, so the associated set of $U_5 x_i^*$ is uniquely determined. Let

$$C_4 = \{i \in C \cap C_2\colon U_5 x_i^* \text{ and } U_5 y_j^* \text{ are disjoint for every } j \in D \cap D_2\}.$$

By the definition of C_4, we know that

$$\sum_{i\in C_4} |x_i^*(u_5)|^2 + \sum_{j\in D\cap D_2} |y_j^*(u_5)|^2 \le \|u_5\|^2.$$

From the properties of C and D we can deduce that

$$(1-\sqrt{\delta})^2(1+5\epsilon)^2\alpha^2\mu(C_4) + (1-\sqrt{\delta})^2(1+5\epsilon)^2\alpha^2\nu(D\cap D_2) \le (1+5\epsilon)^2\alpha^2.$$

Hence,

$$(1-\sqrt{\delta})^2\mu(C_4) + (1-5\delta-(1/50))(1-\sqrt{\delta})^2 \le 1,$$

which implies that $\mu(C_4) \le 1/40$. Let $C_3 = (C \cap C_2)\backslash C_4$. Then $\mu(C_3) \ge (39/40) - (1/50) - 5\delta \ge 19/20$ and, for every $i \in C_3$, there exists some $j \in D\cap D_2$ (which must be unique) such that the associated sets of $U_5 x_i^*$ and $U_5 y_j^*$ are not disjoint. By the same argument for the y_j^* we can define a set D_3. It is easy to see that C_3 and D_3 have the properties claimed above.

Step 6. Let $V = \operatorname{ran}(u_2 + u_3)$. If $i \in C_3$, then $Vx_i^* = Vy_{\phi(i)}^*$.

Proof. Setting $j = \phi(i)$, we know that $E_i x_{i,k_i-1}^* \neq 0$ and similarly $F_j y_{j,l_j-1}^* \neq 0$ (where $y_j^* = F_j(y_{j1}^* + \cdots + y_{jq_j}^*)$). We also know that $\sigma(x_{i1}^*, \ldots, x_{i,k_i-1}^*) = \sigma(y_{j1}^*, \ldots, y_{j,l_j-1}^*)$. It follows from the definition of a special sequence and the fact that σ is an injection that $k_i = l_j$ and $x_{it}^* = y_{jt}^*$ for every $t < k_i$. Finally, we know that $U_1 x_i^*$ and $U_1 y_j^*$ are both nonzero and that $U_3 \cap \operatorname{ran}(x_{i,k_i}^*) = U_3 \cap \operatorname{ran}(y_{j,l_j}^*) = \varnothing$. Putting all these facts together tells us that $Vx_i^* = Vy_j^*$ as claimed.

The contradiction. The set C_3 is certainly not empty. Let $i \in C_3$. Then since $C_3 \subset C$ it follows from Step 3 that $\alpha/2 \leq x_i^*(u_2)/a_i \leq 2\alpha$. Since $x_i^*(u_2) = y_{\phi(i)}^*(u_2)$ and $D_3 \subset D$, we also get $\alpha/2 \leq x_i^*(u_2)/b_{\phi(i)} \leq 2\alpha$. This implies that $b_{\phi(i)}/a_i \geq 1/4$. By the same argument we have $\alpha/2 \leq x_i^*(u_3)/a_i \leq 2\alpha$ and $\alpha/2 \leq -x_i^*(u_3)/b_{\phi(i)} \leq 2\alpha$. This implies that $b_{\phi(i)}/a_i \leq -1/4$. (Recall that we restricted our attention to nonzero a_i and b_j.) We have arrived at the contradiction we promised. □

Remarks. 1. It is not hard to see that the space constructed in this paper has a predual, and that this predual does not contain c_0 or a boundedly complete basic sequence. Since the predual certainly does not contain l_1, it also gives a space not containing c_0, l_1 or a reflexive subspace.

2. An overview of the proof of Lemma 4 might be helpful. The aim of the first two steps is to obtain an almost isometric copy of l_1^n for a suitable n, in which every vector is normed by a special combination. The remaining steps are designed to show that this cannot happen. There is a technical problem about basic special combinations, which is dealt with in Step 4. The main point of Steps 3 to 6 is to show that the special combinations norming $u_1 + \cdots + u_5$ and $u_1 + u_2 - u_3 + u_4 + u_5$ are roughly equal on u_2, u_3 and u_4, which clearly cannot happen. Of key importance is that any term in a special sequence determines the whole of the sequence up to that point, but this fact is harder to apply than it was in [1], where one did not deal with combinations of special sequences.

3. The most important difference between the problem solved in this paper and that solved in [1] is that reflexivity is an infinite-dimensional phenomenon, whereas the property of not containing an unconditional basic sequence is equivalent to saying that every infinite-dimensional subspace contains arbitrarily large finite-dimensional (block) subspaces of a certain kind. The resulting need to consider infinite special sequences is a serious difficulty, dealt with by the l_2-sum in the definition of the norm and the nonexplicitness of Lemma 4.

4. There are possible extensions of the main result. For example, James constructed a nonreflexive space with nontrivial type and cotype. It seems likely that the construction of this paper could be adapted to give a space with nontrivial type and cotype and no reflexive subspace (answering a question of Casazza). Another hereditary version of a James space would be the following—a uniformly nonoctahedral space with no reflexive subspace. Again, such a space probably exists, but a proof is unlikely to be pleasant.

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