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HOCHSCHILD HOMOLOGY IN A BRAIDED TENSOR CATEGORY

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ABSTRACT. An r-algebra is an algebra A over k equipped with a Yang-Baxter operator $R: A \otimes A \to A \otimes A$ such that $R(1 \otimes a) = a \otimes 1$, $R(a \otimes 1) = 1 \otimes a$, and the quasitriangularity conditions $R(m \otimes I) = (I \otimes m)(R \otimes I)(I \otimes R)$ and $R(I \otimes m) = (m \otimes I)(I \otimes R)(R \otimes I)$ hold, where $m: A \otimes A \to A$ is the multiplication map and $I: A \to A$ is the identity. R-algebras arise naturally as algebra objects in a braided tensor category of k-modules (e.g., the category of representations of a quantum group). If $m = mR^2$, then A is both a left and right module over the braided tensor product $A^e = A \hat{\otimes} A^{\mathrm{op}}$, where A^{op} is simply A equipped with the "opposite" multiplication map $m^{\mathrm{op}} = mR$. Moreover, there is an explicit chain complex computing the braided Hochschild homology $H^R(A) = \mathrm{Tor}^{A^e}(A, A)$. When $m = mR$ and $R^2 = \mathrm{id}_{A \otimes A}$, this chain complex admits a generalized shuffle product, and there is a homomorphism from the r-commutative differential forms $\Omega_R(A)$ to $H^R(A)$.

1. INTRODUCTION

It has been known for some time that many constructions in commutative algebra can be profitably extended to "supercommutative" algebras, that is, $\mathbf{Z}_2$-graded algebras satisfying $ab = (-1)^{\deg a \deg b} ba$. To do so, one follows the simple rule that the twist map

$$a \otimes b \mapsto b \otimes a$$

should be replaced everywhere by the graded twist map

$$a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a.$$

This rule appears frequently in physics, where even and odd variables are used to describe bosonic and fermionic degrees of freedom, respectively.

More recently, developments in mathematical physics have led to a further generalization of commutativity, in which the role of the symmetric group is taken instead by the braid group. The twist map is thus replaced by an arbitrary Yang-Baxter operator

$$a \otimes b \mapsto R(a \otimes b)$$

satisfying certain compatibility relations with multiplication. More precisely, suppose that A is a unital algebra over a commutative ring k, and let $m: A \otimes A \to A$ denote the multiplication map and $I: A \to A$ the identity map. Then

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A is an *r-algebra* if there is an *r-structure* $R\colon A \otimes A \to A \otimes A$, an invertible linear map satisfying: (1) the *Yang-Baxter equation*:

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),$$

(2) the *quasitriangularity conditions*:

$$R(m \otimes I) = (I \otimes m)(R \otimes I)(I \otimes R), \quad R(I \otimes m) = (m \otimes I)(I \otimes R)(R \otimes I),$$

and (3) the following conditions for all $a \in A$:

$$R(1 \otimes a) = a \otimes 1, \quad R(a \otimes 1) = 1 \otimes a.$$

We say that A is "r-commutative" if in addition $m = mR$. For example, every $\mathbf{Z}_2$-graded algebra is an r-algebra with

$$R(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a,$$

and then r-commutativity reduces to supercommutativity. Many other interesting noncommutative algebras are also r-commutative. These include quantum groups [6, 7], quantum matrix algebras [7], quantum vector spaces [20, 22, 23, 31, 33], noncommutative tori [4, 26], the Weyl and Clifford algebras, and certain universal enveloping algebras. R-commutative algebras also arise quite generally as semiclassical limits in the quantization of Poisson algebras.

As noted by Manin [20, 21], r-algebras may be defined simply as algebra objects in braided tensor categories of k-modules. Roughly speaking, a braided tensor category is a category with tensor products such that for any objects U, V there is a "braiding"

$$R_{U,V}\colon U \otimes V \to V \otimes U,$$

an isomorphism satisfying axioms generalizing the properties of the twist map. The concept of a braided tensor category has its roots in Mac Lane's work [18] on the category-theoretic foundations of associativity and commutativity, and a certain class, the Tannakian categories, have long found wide application in number theory and other subjects [5, 27]. More recently, braided tensor categories have been seen to arise naturally in a variety of closely related situations. First, given a k-module V and an isomorphism $R\colon V \otimes V \to V \otimes V$ satisfying the Yang-Baxter equation, one may canonically construct a braided tensor category of k-modules containing V for which $R_{V,V} = R$ [17]. Second, the category of representations of a quantum group over k is a braided tensor category of k-modules, with the universal R-matrix serving to define the braiding [19, 30]. Third, certain braided tensor categories give rise to invariants of knots and three-manifolds [10, 25]. Fourth, quantum field theories give rise to braided tensor categories of vector spaces over $\mathbf{C}$ [9]. These field theories, in turn, are closely related to statistical mechanics models [32].

Manin has proposed an approach to noncommutative geometry in which one generalizes standard constructions of commutative algebra to algebras in a braided tensor category. The author has taken this up [1], studying an analog of differential forms that is applicable to any "strong" r-commutative algebra, that is, one with $R^2 = \mathrm{id}_{A \otimes A}$. Certain nonstrong cases are also understood [2]. This approach to noncommutative geometry may at first seem orthogonal to that taken by Connes [4], Tsygan [29], Loday-Quillen [24] and others, in which one works with cyclic Hochschild chains to define cyclic homology as an analog

of de Rham cohomology for noncommutative algebras. Here, however, we begin a unification of these approaches by defining a generalization of Hochshild homology for algebras in a braided tensor category. This extends the previous generalization of Hochschild homology to supercommutative algebras.

The general strategy is to systematically replace the twist map wherever it appears by the appropriate braiding. We begin by associating to any r-algebra A an "opposite" r-algebra A^{op} with multiplication map $m^{\mathrm{op}} = mR$. The algebra A then becomes a left module over $A^e = A \widehat{\otimes} A^{\mathrm{op}}$, where $\widehat{\otimes}$ denotes the "braided tensor product." We construct a flat resolution of A as an A^e-module when A is flat over k. This enables us to compute $\mathrm{Tor}^{A^e}(E, A)$ for a right A^e-module E by means of an explicit chain complex. The algebra A itself is a right A^e-module when A is "weakly r-commutative," that is, $m = mR^2$. This allows the construction of the "braided Hochschild homology" $H^R(A) = \mathrm{Tor}^{A^e}(A, A)$. In particular, when A is strong and r-commutative, there is a product on $H^R(A)$ induced by a generalized shuffle product on the corresponding chain complex. Recall that in the commutative case there is a homomorphism from the differential forms over A to the Hochschild homology of A, which is an isomorphism when A is smooth [11]. Here we construct a homomorphism from the r-commutative generalization of differential forms to $H^R(A)$.

The author is indebted to Ezra Getzler for raising the idea of generalizing Hochschild cohomology to algebras in braided tensor categories, and for pointing out the difficulties. The author would also like to thank Minhyong Kim for many helpful conversations.

2. Algebras in braided tensor categories

In earlier work on r-commutative algebra [1, 2], we worked directly from the definition given above, avoiding the introduction of category theory. In homological considerations, however, a small investment in the language of tensor categories is amply repaid.

A *monoidal category* is a category $\mathscr{C}$ equipped with a bifunctor $\otimes: \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, a functorial isomorphism

$$A_{U,V,W}: U \otimes (V \otimes W) \to (U \otimes V) \otimes W$$

called the *associativity constraint*, and an object 1 equipped with functorial isomorphisms

$$l_V: 1 \otimes V \to V, \quad r_V: V \otimes 1 \to V,$$

satisfying:

(1) the *pentagon axiom*, namely that all diagrams

$$\begin{tikzcd}
U \otimes (V \otimes (W \otimes X)) \arrow[r, "A"] \arrow[d, "\mathrm{id} \otimes A"'] & (U \otimes V) \otimes (W \otimes X) \arrow[r, "A"] & (U \otimes (V \otimes W)) \otimes X \\
U \otimes ((V \otimes W) \otimes X) \arrow[rr, "A"] & & (U \otimes (V \otimes W)) \otimes X \arrow[u, "A \otimes \mathrm{id}"']
\end{tikzcd}$$

commute, and

(2) The *triangle axiom*, namely that all diagrams

$$\begin{tikzcd} (U \otimes 1) \otimes V \arrow[rr, "A"] \arrow[dr, "r \otimes \mathrm{id}"'] & & U \otimes (1 \otimes V) \arrow[dl, "\mathrm{id} \otimes l"] \\ & U \otimes V & \end{tikzcd}$$

commute.

A monoidal category $\mathscr{C}$ is said to be *braided* if for all objects U, V there exist functorial isomorphisms

$$R_{U,V} \colon U \otimes V \to V \otimes U,$$

satisfying the *hexagon axioms*, namely that all diagrams

$$\begin{tikzcd} (X \otimes Y) \otimes Z \arrow[r, "R"] \arrow[d, "A^{-1}"'] & Z \otimes (X \otimes Y) \\ X \otimes (Y \otimes Z) \arrow[d, "\mathrm{id} \otimes R"'] & (Z \otimes X) \otimes Y \arrow[u, "A^{-1}"'] \\ X \otimes (Z \otimes Y) \arrow[r, "A"] & (X \otimes Z) \otimes Y \arrow[u, "R \otimes \mathrm{id}"'] \end{tikzcd}$$

and

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "R"] \arrow[d, "A"'] & (Y \otimes Z) \otimes X \\ (X \otimes Y) \otimes Z \arrow[d, "R \otimes \mathrm{id}"'] & Y \otimes (Z \otimes X) \arrow[u, "A"'] \\ (Y \otimes X) \otimes Z \arrow[r, "A^{-1}"] & Y \otimes (X \otimes Z) \arrow[u, "\mathrm{id} \otimes R"'] \end{tikzcd}$$

commute. If $\mathscr{C}$ is a braided monoidal category such that $R_{V,U} \circ R_{U,V} = \mathrm{id}_{U \otimes V}$ for all objects U, V, we say $\mathscr{C}$ is a *symmetric* monoidal category. In a symmetric monoidal category, either of the two hexagon axioms implies the other.

Given a commutative ring k, we define a *tensor category of k-modules* to be a monoidal category $\mathscr{C}$ equipped with a faithful functor F to the category of k-modules such that: (1) $F(U \otimes V) = F(U) \otimes F(V)$, (2) $F(A_{U,V,W})$ is the natural isomorphism of k-modules,

$$F(U) \otimes (F(V) \otimes F(W)) \xrightarrow{\sim} (F(U) \otimes F(V)) \otimes F(W),$$

(3) $F(1) = k$, (4) $F(l_V)$ and $F(r_V)$ are the natural isomorphisms of k-modules,

$$k \otimes F(V) \xrightarrow{\sim} F(V), \quad F(V) \otimes k \xrightarrow{\sim} F(V).$$

A *braided* (resp. *symmetric*) *tensor category* of k-modules is a braided (resp. symmetric) monoidal category equipped with a faithful functor F to the category of k-modules such that properties (1)–(5) above hold. The point is that

while the associativity constraint A in $\mathscr{C}$ is mapped to the usual associativity constraint for k-modules, this need not hold of the braiding R. It is also important that while distinct objects in $\mathscr{C}$ may correspond to the same k-module, distinct morphisms in $\mathscr{C}$ correspond to distinct k-module morphisms.

Henceforth we shall work in a fixed braided tensor category $\mathscr{V}$ of k-modules, where k is some commutative ring. Because the associativity constraint in $\mathscr{V}$ maps to the standard one in the category of k-modules, and the pentagon axiom allows us to reparenthesize tensor products in $\mathscr{V}$ in a coherent manner, we simply omit parentheses around tensor products in $\mathscr{V}$, and omit all mention of the associativity constraint. We also identify objects and morphisms in $\mathscr{V}$ with their images under F when this will cause no confusion.

One can easily prove generalized forms of the Yang-Baxter equation and the quasitriangularity conditions in a braided tensor category. The quasitriangularity conditions state that, for any objects U, V, W, X in $\mathscr{V}$ and morphism $f\colon U \otimes V \to X$, the following diagrams commute:

$$\begin{tikzcd}
U \otimes V \otimes W \arrow[r, "f\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes R"'] & X \otimes W \arrow[dd, "R"] \\
U \otimes W \otimes V \arrow[d, "R\otimes\mathrm{id}"'] & \\
W \otimes U \otimes V \arrow[r, "\mathrm{id}\otimes f"] & W \otimes X \arrow[d, equal] \\
W \otimes U \otimes V \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "R\otimes\mathrm{id}"'] & W \otimes X \arrow[dd, "R"] \\
U \otimes W \otimes V \arrow[d, "\mathrm{id}\otimes R"'] & \\
U \otimes V \otimes W \arrow[r, "f\otimes\mathrm{id}"] & X \otimes W
\end{tikzcd}$$

These conditions follow from hexagon axiom together with the functoriality of the braiding. There is also a dual version for morphisms $f\colon X \to U \otimes V$, which appears in the definition of a quasitriangular Hopf algebra, but we will not need this here. The Yang-Baxter equation follows from the first quasitriangularity condition upon taking $X = V \otimes U$, $f = R_{U,V}$. It states that, for any objects U, V, W in $\mathscr{V}$, the following diagram commutes:

$$\begin{tikzcd}
 & U \otimes V \otimes W \arrow[dl, "R\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes R"] & \\
V \otimes U \otimes W \arrow[d, "\mathrm{id}\otimes R"'] & & U \otimes W \otimes V \arrow[d, "R\otimes\mathrm{id}"] \\
V \otimes W \otimes U \arrow[dr, "R\otimes\mathrm{id}"'] & & W \otimes U \otimes V \arrow[dl, "\mathrm{id}\otimes R"] \\
 & W \otimes V \otimes U &
\end{tikzcd}$$

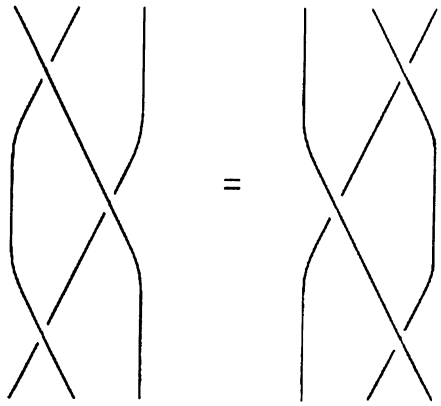

FIGURE 1. The Yang-Baxter equation

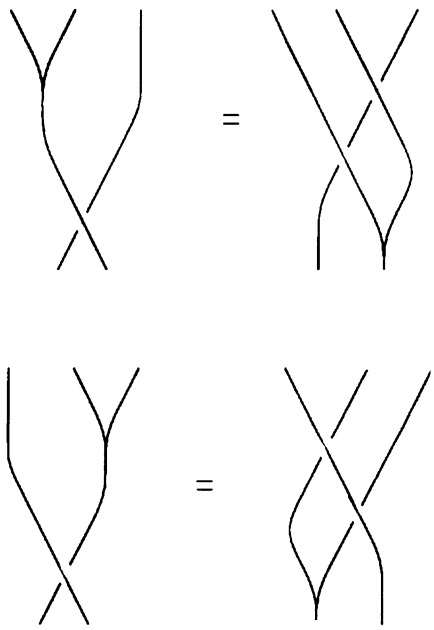

FIGURE 2. The quasitriangularity conditions

Using quasitriangularity for the morphism $l\colon k \otimes k \to k$ and property (4) of tensor categories of k-modules, it follows that

$$R_{k,V}(x \otimes v) = v \otimes x$$

and

$$R_{V,k}(v \otimes x) = x \otimes v$$

for all $x \in k$ and $v \in V$. From these it also follows that

$$l_V R_{V,k} = r_V, \quad r_V R_{k,V} = l_V.$$

It proves very convenient to compute in a braided tensor category using pictures. Calculations of this sort have long been used in knot theory, especially by Kauffman [15, 16]. Their category-theoretic foundations have recently been explored by Joyal and Street [12, 13]. If we denote the braiding

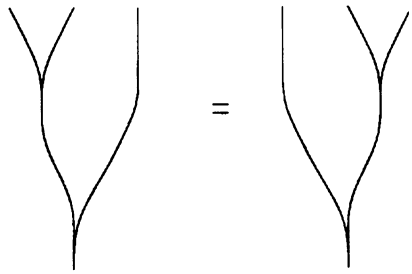

FIGURE 3. Associativity

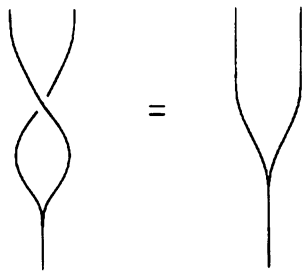

FIGURE 4. R-commutativity

$R_{U,V}: U \otimes V \to V \otimes U$ by the crossing of two strands and denote identity morphisms by a vertical line, then the Yang-Baxter equation may be diagrammed as in Figure 1. If in addition we denote the morphism $f: U \otimes V \to W$ by the joining of two strands, the quasitriangularity conditions may be diagrammed as in Figure 2.

Now let us turn to the study of algebras in a braided tensor category. We will only develop the ingredients necessary to set up Hochschild homology. To begin with, we say that A is an *algebra in* $\mathscr{V}$ if A is an object in $\mathscr{V}$ equipped with a *multiplication map* $m \in \mathrm{Hom}_{\mathscr{V}}(V \otimes V, V)$ satisfying associativity:

$$m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m)$$

and a *unit map* $\iota \in \mathrm{Hom}_{\mathscr{V}}(k, V)$ satisfying

$$m(\iota \otimes \mathrm{id}_A) = l_A, \quad m(\mathrm{id}_A \otimes \iota) = r_A.$$

It is easy to check that an algebra in $\mathscr{V}$ is indeed an r-algebra as defined in the introduction, taking $R = R_{A,A}$. Conversely, given an r-algebra over a commutative ring k, we may regard it as an algebra in some braided tensor category of k-modules, using a construction of Lyubashenko [17].

We say that an algebra A in $\mathscr{V}$ is *r-commutative* if $m = mR_{A,A}$. Using the joining of two strands to denote the multiplication map, associativity corresponds to Figure 3, while r-commutativity corresponds to Figure 4. Even when A is not r-commutative, $mR_{A,A}$ is an associative product on A. In fact:

Lemma 1. *Let A be an algebra in V. Let A^{op} denote A with the multiplication map $m^{\mathrm{op}} = mR_{A,A}$ and the same unit map ι. Then A^{op} is an algebra in $\mathscr{V}$.*

Proof. First we check associativity, by a four-step calculation shown in Figure 5. Writing $R = R_{A,A}$ and $\mathrm{id}_A = I$, by definition we have $m^{\mathrm{op}}(m^{\mathrm{op}} \otimes I) = mR(m \otimes I)(R \otimes I)$.

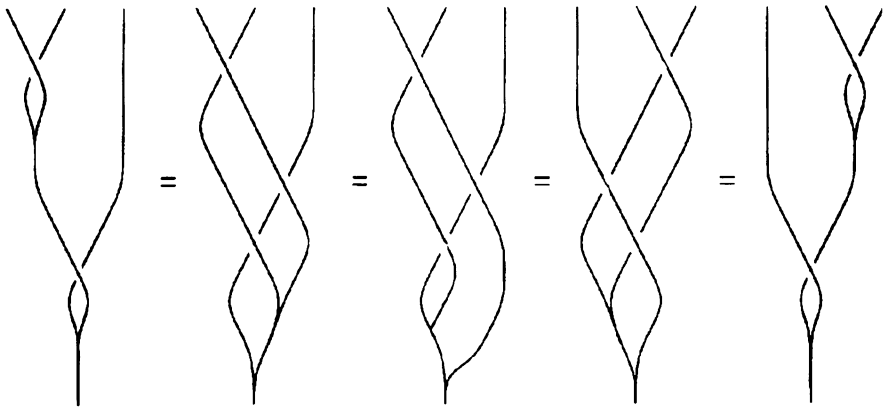

FIGURE 5. Associativity of the opposite algebra

Step 1: By quasitriangularity the latter equals

$$m(I \otimes m)(R \otimes I)(I \otimes R)(R \otimes I).$$

Step 2: By associativity this equals

$$m(m \otimes I)(R \otimes I)(I \otimes R)(R \otimes I).$$

Step 3: By the Yang-Baxter equation this equals

$$m(m \otimes I)(I \otimes R)(R \otimes I)(I \otimes R).$$

Step 4: By quasitriangularity this equals

$$mR(I \otimes m)(I \otimes R),$$

which equals $m^{\mathrm{op}}(I \otimes m^{\mathrm{op}})$, as desired. Checking that ι is a unit map for m^{op} is straightforward:

$$m^{\mathrm{op}}(\iota \otimes I) = m(I \otimes \iota)R_{k,A} = r_A R_{k,A} = l_A,$$

and

$$m^{\mathrm{op}}(I \otimes \iota) = m(\iota \otimes I)R_{A,k} = l_A R_{A,k} = r_A. \quad \square$$

The r-algebra A^{op} is called the *opposite algebra* of A. If the braiding $R_{A,A}$ is the twist map this reduces to the opposite algebra as usually defined. Another basic construction is the "braided tensor product" of algebras.

Lemma 2. *Let A and B be algebras in $\mathscr{V}$ with products m_A, m_B and unit maps ι_A, ι_B, respectively. Endowed with the multiplication map*

$$M = (m_A \otimes m_B)(\mathrm{id}_A \otimes R_{B,A} \otimes \mathrm{id}_B)$$

and the unit map $\iota = \iota_A \otimes \iota_B$, $A \otimes B$ is an algebra in $\mathscr{V}$. (Here we are identifying $\iota_A \otimes \iota_B \colon k \otimes k \to A \otimes B$ with a map from k to $A \otimes B$.)

Proof. First we check associativity, as shown in Figure 6.

Step 1: Starting with

$$M(\mathrm{id}_{A\otimes B} \otimes M) = (m_A \otimes m_B)(\mathrm{id}_A \otimes R_{B,A} \otimes \mathrm{id}_B)$$
$$(\mathrm{id}_{A\otimes B} \otimes m_A \otimes m_B)(\mathrm{id}_{A\otimes B\otimes A} \otimes R_{B,A} \otimes \mathrm{id}_B),$$

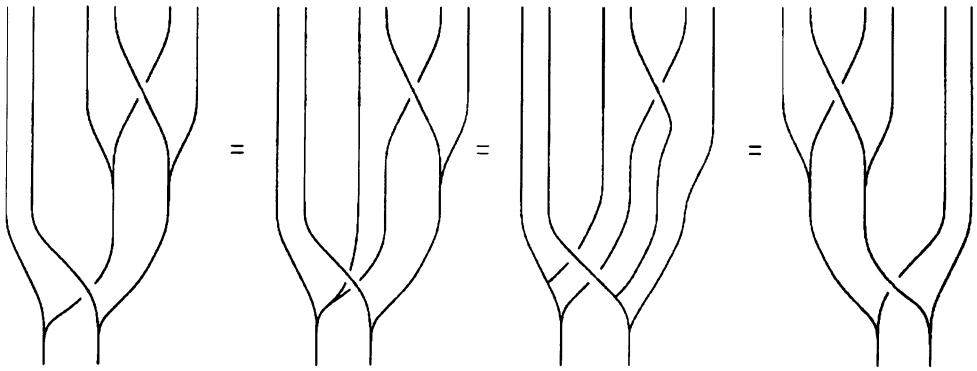

FIGURE 6. Associativity of the braided tensor product

and using quasitriangularity, we obtain

$$(m_A \otimes m_B)(\mathrm{id}_A \otimes m_A \otimes \mathrm{id}_B \otimes m_B)(\mathrm{id}_{A^{\otimes 2}} \otimes R_{B,A} \otimes \mathrm{id}_{B^{\otimes 2}})(\mathrm{id}_A \otimes R_{B,A} \otimes R_{B,A} \otimes \mathrm{id}_B).$$

Step 2: Using associativity twice, we obtain

$$(m_A \otimes m_B)(m_A \otimes \mathrm{id}_A \otimes m_B \otimes \mathrm{id}_B)(\mathrm{id}_{A^{\otimes 2}} \otimes R_{B,A} \otimes \mathrm{id}_{B^{\otimes 2}})(\mathrm{id}_A \otimes R_{B,A} \otimes R_{B,A} \otimes \mathrm{id}_B).$$

Step 3: Using quasitriangularity again, this equals

$$(m_A \otimes m_B)(\mathrm{id}_A \otimes R_{B,A} \otimes \mathrm{id}_B)(m_A \otimes m_B \otimes \mathrm{id}_{A\otimes B})(\mathrm{id}_A \otimes R_{B,A} \otimes \mathrm{id}_{B\otimes A\otimes B})$$

or $M(M \otimes \mathrm{id}_{A\otimes B})$.

To show that ι is a unit map for M we note that

$$M(\iota \otimes \mathrm{id}_{A\otimes B}) = (m_A \otimes m_B)(\iota_A \otimes (\mathrm{id}_A \otimes \iota_B)R_{k,A} \otimes \mathrm{id}_B) = L_{A\otimes B},$$

and similarly $M(\mathrm{id}_{A\otimes B} \otimes \iota) = r_{A\otimes B}$. □

Given two algebras $A, B \in \mathscr{V}$, we call the algebra $A \otimes B$ constructed in Lemma 2 the *braided tensor product* of A and B. We write this algebra as $A \widehat{\otimes} B$ to emphasize that it need not be isomorphic to the usual tensor product of the algebras A and B. The braided tensor product generalizes the well-known $\mathbf{Z}_2$-graded tensor product of $\mathbf{Z}_2$-graded algebras. Braided tensor products are already visible in a number of other contexts as well. For example, noncommutative tori are braided tensor products of algebras of Laurent polynomials. (Geometrically speaking, one could say that a noncommutative torus is a braided product of circles.) Also, the algebra of differential forms on a quantum vector space with Hecke-type relations is the braided tensor product of an "r-symmetric" and an "r-exterior" algebra [2].

One surprise is that the braided tensor product of r-commutative algebras in $\mathscr{V}$ need not be r-commutative, although this is always true when $\mathscr{V}$ is symmetric. More precisely:

Lemma 3. *Let A and B be r-commutative algebras in $\mathscr{V}$ and suppose $R_{B,A}R_{A,B} = \mathrm{id}_{A\otimes B}$. Then $A \widehat{\otimes} B$ is r-commutative.*

Proof. The proof is shown in Figure 7. Let M denote the multiplication map for $A \widehat{\otimes} B$. Then we need to show that $M = MR_{A\otimes B, A\otimes B}$. By repeated use of the hexagon axioms we have

$$R_{A\otimes B, A\otimes B} = (\mathrm{id}_A \otimes R_{A,B} \otimes \mathrm{id}_B)(R_{A,A} \otimes R_{B,B})(\mathrm{id}_A \otimes R_{B,A} \otimes \mathrm{id}_B).$$

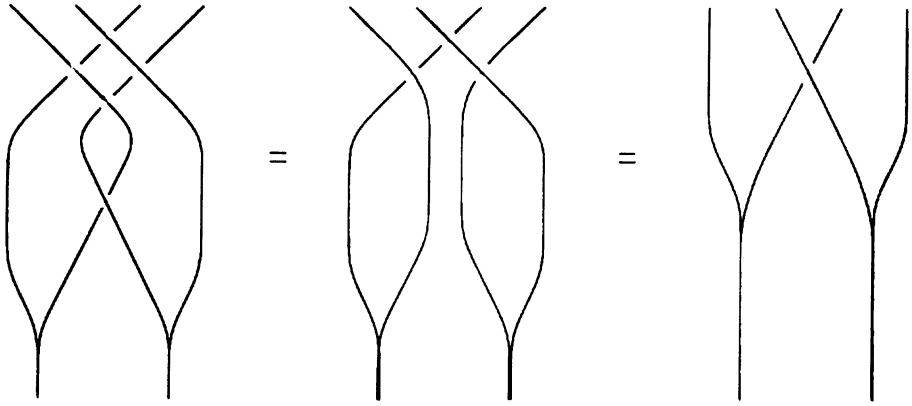

FIGURE 7. R-commutativity of the braided tensor product of r-commutative algebras

Using the fact that $R_{B,A}R_{A,B} = \mathrm{id}_{A\otimes B}$, it follows that

$$MR_{A\otimes B,A\otimes B} = (m_A \otimes m_B)(R_{A,A} \otimes R_{B,B})(\mathrm{id}_A \otimes R_{B,A} \otimes \mathrm{id}_B).$$

Using the r-commutativity of A and B, it follows that

$$MR_{A\otimes B,A\otimes B} = (m_A \otimes m_B)(\mathrm{id}_A \otimes R_{B,A} \otimes \mathrm{id}_B) = M. \quad \square$$

Given an algebra A in $\mathscr{V}$, we define a *left A-module in $\mathscr{V}$* to be an object E of $\mathscr{V}$ together with a *left action*, a morphism $\alpha \in \mathrm{Hom}_{\mathscr{V}}(A \otimes E, E)$ satisfying

$$\alpha(\mathrm{id}_A \otimes \alpha) = \alpha(m \otimes \mathrm{id}_E) \quad \text{and} \quad \alpha(\iota \otimes \mathrm{id}_E) = l_E.$$

Given two left A-modules (E, α) and (F, β) in $\mathscr{V}$, we define a *left A-module morphism in $\mathscr{V}$* to be a morphism $f \in \mathrm{Hom}_{\mathscr{V}}(E, F)$ such that $f\alpha = \beta(\mathrm{id}_A \otimes f)$. Right A-modules and right A-module morphisms in $\mathscr{V}$ are defined analogously.

Let A^e denote the braided tensor product $A \hat{\otimes} A^{\mathrm{op}}$.

Lemma 4. *Let A be an algebra in $\mathscr{V}$. Then A is a left A^e-module in $\mathscr{V}$ with the left action $\lambda \in \mathrm{Hom}_{\mathscr{V}}(A^e \otimes A, A)$ given by $\lambda = m(\mathrm{id}_A \otimes mR_{A,A})$.*

Proof. Let R denote $R_{A,A}$ and let M denote the multiplication map of A^e:

$$M = (m \otimes mR)(\mathrm{id}_A \otimes R \otimes \mathrm{id}_A).$$

We need to show that $\lambda(\mathrm{id}_{A^e} \otimes \lambda) = \lambda(M \otimes \mathrm{id}_A)$. The proof of this is shown in Figure 8. We begin by considering

$$\begin{aligned}\lambda(\mathrm{id}_{A^e} \otimes \lambda) &= m(\mathrm{id}_A \otimes mR)(\mathrm{id}_{A^e} \otimes m(\mathrm{id}_A \otimes mR)) \\ &= m(\mathrm{id}_A \otimes m)(\mathrm{id}_A \otimes R(\mathrm{id}_A \otimes m))(\mathrm{id}_{A^{\otimes 3}} \otimes mR).\end{aligned}$$

Step 1: By quasitriangularity this equals

$$\begin{aligned}&m(\mathrm{id}_A \otimes m)(\mathrm{id}_A \otimes (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes R)(R \otimes \mathrm{id}_A))(\mathrm{id}_{A^{\otimes 3}} \otimes mR) \\ &\quad = m(\mathrm{id}_A \otimes m)(\mathrm{id}_A \otimes m \otimes \mathrm{id}_A)(\mathrm{id}_{A^{\otimes 2}} \otimes R(\mathrm{id}_A \otimes m))(\mathrm{id}_A \otimes R \otimes R).\end{aligned}$$

Step 2: By quasitriangularity this equals

$$m(\mathrm{id}_A \otimes m)(\mathrm{id}_A \otimes m \otimes \mathrm{id}_A)(\mathrm{id}_{A^{\otimes 2}} \otimes (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes R)(R \otimes \mathrm{id}_A))(\mathrm{id}_A \otimes R \otimes R).$$

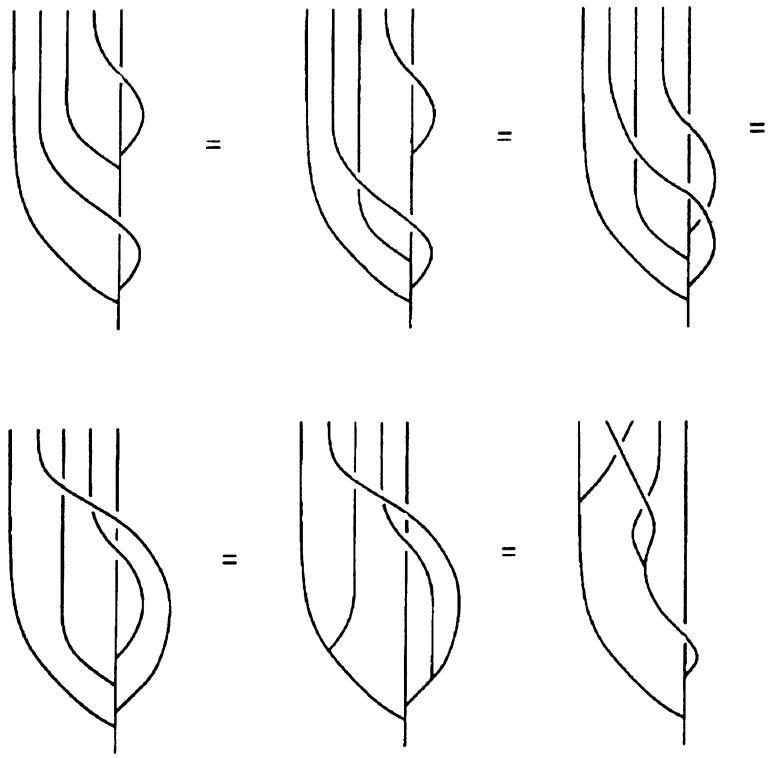

FIGURE 8. A is a left A^e-module

Step 3: By the Yang-Baxter equation this equals

$$m(\mathrm{id}_A \otimes m)(\mathrm{id}_A \otimes m \otimes \mathrm{id}_A)(\mathrm{id}_{A^{\otimes 2}} \otimes mR \otimes \mathrm{id}_A)$$
$$(\mathrm{id}_{A^{\otimes 3}} \otimes R)(\mathrm{id}_{A^{\otimes 2}} \otimes R \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes R \otimes \mathrm{id}_{A^{\otimes 2}}).$$

Step 4: Using associativity three times, this equals

$$m(\mathrm{id}_A \otimes m)(m \otimes \mathrm{id}_A \otimes m)(\mathrm{id}_{A^{\otimes 2}} \otimes R \otimes \mathrm{id}_A)(\mathrm{id}_{A^{\otimes 3}} \otimes R)$$
$$(\mathrm{id}_{A^{\otimes 2}} \otimes R \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes R \otimes \mathrm{id}_{A^{\otimes 2}}).$$

Step 5: By quasitriangularity this equals

$$m(\mathrm{id}_A \otimes m)(m \otimes R)(\mathrm{id}_{A^{\otimes 2}} \otimes m \otimes \mathrm{id}_A)(\mathrm{id}_{A^{\otimes 2}} \otimes R \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes R \otimes \mathrm{id}_{A^{\otimes 2}})$$
$$= m(\mathrm{id}_A \otimes mR)(m \otimes mR \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes R \otimes \mathrm{id}_{A^{\otimes 2}}).$$

The latter expression equals $\lambda(M \otimes \mathrm{id}_A)$, as was to be shown.

To complete the proof that λ is a left action it suffices to note that $\iota \otimes \iota \colon k \to A \otimes A$ is the unit map for A^e, and

$$\begin{aligned}\lambda(\iota \otimes \iota \otimes \mathrm{id}_A) &= m(\mathrm{id}_A \otimes mR)(\iota \otimes \iota \otimes \mathrm{id}_A)\\ &= m(\iota \otimes m(\mathrm{id}_A \otimes \iota)R_{k,A})\\ &= m(\iota \otimes r_A R_{k,A}) = m(\iota \otimes l_A) = l_A. \quad \square\end{aligned}$$

Unfortunately, there is no right action of A^e on A analogous to the left action unless we introduce a further condition. Let us say that an r-algebra A in $\mathscr{V}$ is *weakly r-commutative* if the multiplication map $m \colon A \otimes A \to A$ satisfies $m = mR^2_{A,A}$. Note that this is the case if A is either r-commutative or *strong*, i.e., $R^2_{A,A} = \mathrm{id}_{A \otimes A}$.

Lemma 5. *Let A be a weakly r-commutative algebra in $\mathscr{V}$. Then A is a right A^e-module in $\mathscr{V}$ with the right action $\rho \in \mathrm{Hom}_{\mathscr{V}}(A \otimes A^e, A)$ given by*

$$\rho = mR_{A,A}(m \otimes \mathrm{id}_A).$$

Proof. Writing R for $R_{A,A}$ and M for the multiplication map of A^e, we need to show that $\rho(\rho \otimes \mathrm{id}_{A^e}) = \rho(\mathrm{id}_A \otimes m)$. We begin by using weak r-commutativity to write

$$\rho(\rho \otimes \mathrm{id}_{A^e}) = mR^{-1}(m \otimes \mathrm{id}_A)(mR^{-1}(m \otimes \mathrm{id}_A) \otimes \mathrm{id}_{A^{\otimes 2}}).$$

By a sequence of six steps shown in Figure 9, this equals

$$mR^{-1}(\mathrm{id}_A \otimes m \otimes mR^{-1})(\mathrm{id}_{A^{\otimes 2}} \otimes R \otimes \mathrm{id}_A).$$

These steps use: (1) associativity, (2) quasitriangularity, (3) quasitriangularity, (4) associativity (twice), (5) the Yang-Baxter equation, and (6) quasitriangularity. By weak r-commutativity the result is equal to

$$mR(\mathrm{id}_A \otimes m \otimes mR)(\mathrm{id}_{A^{\otimes 2}} \otimes R \otimes \mathrm{id}_A),$$

or $m(\mathrm{id}_A \otimes m)$.

The proof that the unit map $\iota \otimes \iota \colon k \to A^e$ satisfies $\rho(\mathrm{id}_A \otimes \iota \otimes \iota) = r_A$ is similar to the calculation at the end of the proof of Lemma 4. □

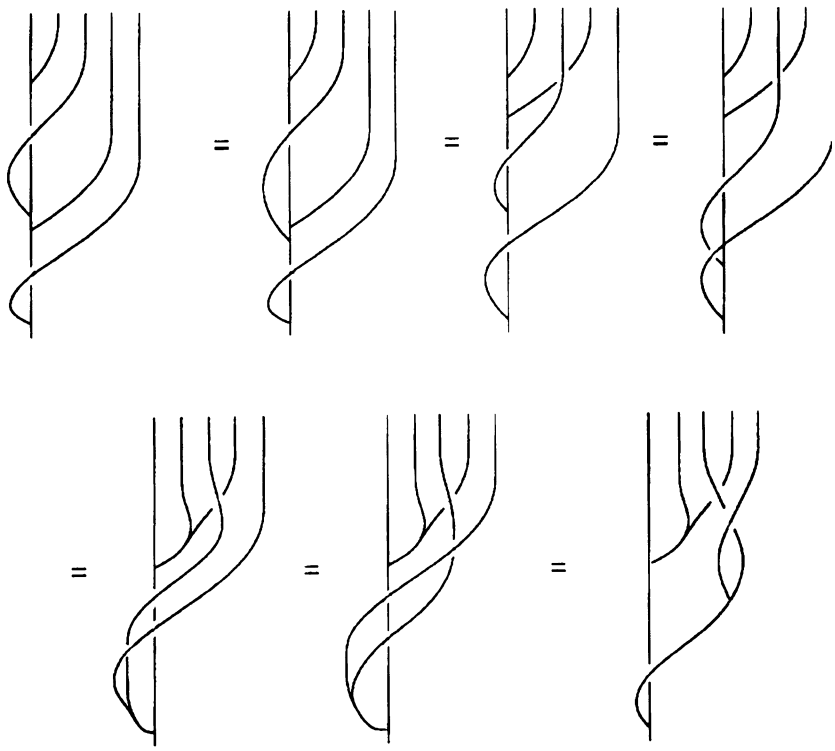

FIGURE 9. A is a right A^e-module if A is weakly r-commutative

3. Braided Hochschild homology

In this section we generalize the usual construction of Hochschild homology to an arbitrary braided tensor category. Let k be a commutative ring, let $\mathscr{V}$ be a fixed braided tensor category of k-modules, and let A be an algebra in $\mathscr{V}$. We begin by constructing a flat resolution of A as a left A^e-module, under the assumption that A is flat as a k-module.

For $n \geq 1$, let $\alpha \in \mathrm{Hom}_{\mathscr{V}}(A^e \otimes A^{\otimes n}, A^{\otimes n})$ be given by

$$(a \otimes b) \otimes (a_1 \otimes \cdots \otimes a_n) \mapsto (m \otimes \mathrm{id}_{A^{\otimes(n-2)}} \otimes m)(\mathrm{id}_A \otimes R_{A, A^{\otimes n}})(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n).$$

Theorem 1. *Suppose $\mathscr{V}$ a braided tensor category of k-modules, and A an algebra in A. The map α is a left action of A^e on $A^{\otimes n}$. If A is flat over k, $A^{\otimes n}$ is a flat left A^e-module for $n \geq 2$. Defining $d_n \in \mathrm{Hom}_{\mathscr{V}}(A^{\otimes n}, A^{\otimes(n-1)})$ for $n \geq 2$ by*

$$d_n(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n,$$

then

$$\cdots \to A^{\otimes 3} \xrightarrow{d_3} A^{\otimes 2} \xrightarrow{d_2} A \to 0$$

is a long exact sequence of left A^e-module morphisms in $\mathscr{V}$.

Proof. First, define K_n for $n \geq -1$ by

$$K_n = \begin{cases} A^e \otimes A^{\otimes n}, & n \geq 1, \\ A^e, & n = 0, \\ A, & n = -1. \end{cases}$$

We give K_n the structure of a left A^e-module in $\mathscr{V}$ as follows. Define $\beta \in \mathrm{Hom}_{\mathscr{V}}(A^e \otimes K_n, K_n)$ by

$$\beta = \begin{cases} M \otimes \mathrm{id}_{A^{\otimes n}}, & n \geq 1, \\ M, & n = 0, \\ \lambda, & n = -1, \end{cases}$$

where M is a multiplication map for A^e, and λ is the left action of A^e on A. Then β is a left action of A^e on K_n by Lemmas 1, 2, and 4. Note that if A is a flat k-module, K_n is a flat A^e-module for $n \geq 0$. Next, define $T \in \mathrm{Hom}_{\mathscr{V}}(K_n, A^{\otimes(n+2)})$ for $n \geq -1$ by

$$T = \begin{cases} \mathrm{id}_A \otimes R_{A, A^{\otimes n}}, & n \geq 1, \\ \mathrm{id}_{A^e}, & n = 0, \\ \mathrm{id}_A, & n = -1. \end{cases} \tag{1}$$

A calculation shows that $\alpha = T\beta(\mathrm{id}_{A^e} \otimes T^{-1})$. It follows that α is a left action of A^e on $A^{\otimes n}$, and that if A is flat then $A^{\otimes n}$ is a flat left A^e-module for $n \geq 2$.

Exactness of the sequence

$$\cdots \to A^{\otimes 3} \xrightarrow{d_3} A^{\otimes 2} \xrightarrow{d_2} A \to 0$$

is well-known [3], so the only thing left to show is that the maps d_n are morphisms of A^e-modules. It suffices to show that each map

$$a_1 \otimes \cdots \otimes a_n \mapsto a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

for $i = 1, \ldots, n-1$, is a morphism of A^e-modules in $\mathscr{V}$. For $1 < i < n-1$, this follows directly from quasitriangularity, while for $i = 1$ and $i = n-1$ one must also use associativity. □

Given an algebra A in $\mathscr{V}$ we define the *acyclic braided Hochschild complex* $K(A)$ to be the chain complex of left A^e-modules $(K_n(A), b')$ $(n \geq 0)$, where

$$K_n(A) = A^e \otimes A^{\otimes n},$$

taking $A^{\otimes 0} = k$, and

$$b'_n = T d_{n+2} T^{-1},$$

with $T\colon A^{\otimes(n+2)} \to K_n$ given by equation (1). Note that this complex really depends only on the structure of A as an r-algebra, not on the braided tensor category in which A is an algebra.

Corollary 1. *The chain complex of left A^e-modules $(A^{\otimes(n+2)}, d_{n+2})$ (where $n \geq 0$) is isomorphic to $K(A)$ by $T\colon A^{\otimes(n+2)} \to K_n(A)$.*

Proof. This follows from the proof of Theorem 1. □

Corollary 2. *Let A be an algebra in $\mathscr{V}$, and let E be a right A^e-module in $\mathscr{V}$. Then there is a natural isomorphism between $\mathrm{Tor}_n^{A^e}(E, A)$ and $H_n(E \otimes_{A^e} K(A))$.*

Proof. This follows from Theorem 1 using homological algebra [3]. □

Now suppose that A is weakly r-commutative. Then A has the structure of a right A^e-module in $\mathscr{V}$ by Lemma 5. Note that in this case the chain complex of Corollary 1 computing $\mathrm{Tor}_n^{A^e}(A, A)$ depends only on the structure of A as an r-algebra. We call this complex the *braided Hochschild complex* of A, $C(A) = (C_n(A), b_n)$, where

$$C_n(A) = A \otimes_{A^e} K_n(A)$$

and $b_n = \mathrm{id}_A \otimes b'_n$. We call the homology of this complex the *braided Hochschild homology* of A, which we denote as $H^R(A)$.

Next we define a shuffle product on the braided Hochschild complex when A is a strong r-commutative algebra. Together with the differential of $C(A)$, this will make $C(A)$ into a differential graded algebra (with differential of degree -1). Let s_i, $1 \leq i < n$, denote the standard generators of the braid group B_n, which satisfy the relations

$$s_i s_j = s_j s_i, \quad |i-j| \geq 2, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

Let $\pi\colon B_n \to S_n$ denote the canonical surjection onto the symmetric group, which maps s_n to the elementary transposition σ_n. Given $\beta \in B_n$, we write simply $\mathrm{sign}(\beta)$ for $\mathrm{sign}(\pi(\beta))$. The notion of "shuffles" generalizes from the symmetric group to the braid group as follows. Suppose that $p + q = n$. Beginning with p red strands to the left of q blue strands, a braid in which at every crossing a red strand crosses over and to the right of a blue strand is called a (p, q)-*shuffle*. More algebraically, we define $\beta \in B_n$ to be a (p, q)-shuffle if it is of the form

$$(s_{1+i_1} \cdots s_2 s_1)(s_{2+i_2} \cdots s_3 s_2) \cdots (s_{p+i_p} \cdots s_{p+1} s_p)$$

where $-1 \leq i_j \leq n-j-1$, $i_j \leq i_{j+1}$, and for $i_j = -1$ we interpret the (empty) product $s_{j+i_j} \cdots s_{j+1} s_j$ as $1 \in B_n$. We denote the set of (p, q)-shuffles

by $sh(p, q)$. The canonical map from B_n to the symmetric group S_n defines a one-to-one correspondence between $sh(p, q)$ and the elements of S_n that are normally referred to as (p, q)-shuffles.

Now let V be an object in $\mathscr{V}$. There is a unique representation of B_n on $V^{\otimes n}$ such that

$$s_i(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes R(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n.$$

Let $TV = \bigoplus_{n \geq 0} V^{\otimes n}$. We define the *shuffle product* $[\cdot\,;\cdot]$ on TV as follows:

Lemma 6. *If V is an object in $\mathscr{V}$, there is a graded associative product on TV given by*

$$[v_1 \otimes \cdots \otimes v_p\,; w_1 \otimes \cdots \otimes w_q] = \sum_{\beta \in sh(p,q)} \operatorname{sign}(\beta)\beta(v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q).$$

Proof. First note that given p, q, r with $p + q + r = n$, there are inclusions of B_p, B_q, and B_r in B_n, given by $s_i \mapsto s_i$, $s_i \mapsto s_{i+p}$, and $s_i \mapsto s_{i+p+q}$, respectively. Similarly, there are inclusions of B_{p+q} and B_{q+r} in B_n given by $s_i \mapsto s_i$ and $s_i \mapsto s_{i+p}$. Identifying by these inclusions, one can show using the braid group relations that

$$sh(p+q, r)sh(p, q) = sh(p, q+r)sh(q, r).$$

Associativity of product given above then follows as for the standard shuffle product.

Lemma 7. *Suppose that V is an object in $\mathscr{V}$ with $R^2_{V,V} = \mathrm{id}_{V \otimes V}$. Let $m\colon TV \otimes TV \to TV$ be given by $m(x \otimes y) = [x\,; y]$. Then if $x \in V^{\otimes p}$ and $y \in V^{\otimes q}$,*

$$m(x \otimes y) = (-1)^{pq} m R_{V^{\otimes p}, V^{\otimes q}}(x \otimes y).$$

Proof. Since $R^2_{V,V}$ is the identity, the representation of B_n on $V^{\otimes n}$ factors through a representation of S_n and the proof follows the usual proof of graded commutativity of the shuffle product [28]. □

Identifying $K(A)$ with $A^e \widehat{\otimes} TA$, where TA is given the shuffle product, $K(A)$ inherits the structure of an algebra in $\mathscr{V}$. Note that $K(A)$ is a graded algebra.

Theorem 2. *Let A be a strong r-commutative algebra over a commutative ring k, let $K = K(A)$ be the acyclic braided Hochschild complex of A, and let m denote the multiplication map for K. Then K is a differential graded algebra with differential b' of degree -1, and for all $x \in K_p$ and $y \in K_q$,*

$$m(x \otimes y) = (-1)^{pq} m R_{K_p, K_q}(x \otimes y). \tag{2}$$

Proof. The algebra A^e is r-commutative by Lemma 3. Equation (2) then follows from Lemma 7 and the method of proof of Lemma 3.

We will show that for all $x \in K_p$ and $y \in K_q$,

$$b'(xy) = b'(x)y + (-1)^p x b'(y). \tag{3}$$

Note that if this is known for given values of p, q, then

$$b' m R_{K_q, K_p}(x \otimes y) = m(b' \otimes \mathrm{id} + (-1)^p \,\mathrm{id} \otimes b') R_{K_q, K_p}(x \otimes y)$$

for all $x \in K_q$ and $y \in K_p$. By equation (2) and quasitriangularity, it follows that

$$b'(xy) = b'(x)y + (-1)^q x b'(y).$$

Thus in what follows we assume $p \leq q$.

By Theorem 1 and Corollary 1, b' is A^e-linear. Using this together with quasitriangularity, it follows that equation (3) for given x, y implies

$$b'((ex)(fy)) = b'(ex)(fy) + (-1)^p (ex) b'(fy)$$

for all $e, f \in A^e$. Thus it suffices to prove equation (3) for $x, y \in K$ of the form

$$x = (1 \otimes 1) \otimes a_1 \otimes \cdots \otimes a_p,$$
$$y = (1 \otimes 1) \otimes a_{p+1} \otimes \cdots \otimes a_{p+q}.$$

We prove this by induction on $n = p + q$, following Seibt's argument in the standard case [28].

The cases $n = 0, 1$ are trivial. For $n = 2$, the only nontrivial case occurs for $p = q = 1$, where

$$xy = 1 \otimes 1 \otimes (a_1 \otimes a_2 - R(a_1 \otimes a_2)),$$

hence by r-commutativity

$$b'(xy) = a_1 \otimes 1 \otimes a_2 + 1 \otimes R^{-1}_{A,A}(a_1 \otimes a_2) - R_{A^{\otimes 2},A}(1 \otimes a_1 \otimes a_2) - 1 \otimes a_1 \otimes a_2.$$

On the other hand,

$$b'(x)y = a_1 \otimes 1 \otimes a_2 - 1 \otimes a_1 \otimes a_2$$

and

$$-xb'(y) = -R_{A^{\otimes 2},A}(1 \otimes a_1 \otimes a_2) + 1 \otimes R_{A,A}(a_1 \otimes a_2),$$

so using the fact that A is strong, equation (3) holds in this case.

Now assuming the inductive hypothesis for $n - 1 \geq 2$, we will prove it for n. First, though, let $s\colon K_k \to K_{k+1}$ be given by

$$s(a_{-1} \otimes a_0 \otimes \cdots \otimes a_k) = (1 \otimes R^{-1}_{A,A}(a_{-1} \otimes a_0) \otimes a_1 \otimes \cdots \otimes a_k).$$

An easy calculation (not using the fact that A is strong or r-commutative) shows that s is a contracting homotopy for K, i.e.,

$$sb' + b's = \mathrm{id}_K. \tag{4}$$

Next, note that the left A^e-module structure of K and the natural homomorphism $A \to A^e$ make K into a left A-module, and that the product in K satisfies $a(uv) = (au)v$ for all $a \in A$ and $u, v \in K$.

Now let

$$\overline{x} = (1 \otimes 1) \otimes a_2 \otimes \cdots \otimes a_p \in K_{p-1},$$
$$\overline{y} = (1 \otimes 1) \otimes a_{p+2} \otimes \cdots \otimes a_n \in K_{q-1}.$$

We prove the induction hypothesis, equation (3), using the following identities, which we verify below:

$$xy = s(a_1 \overline{x} y + (-1)^p x (a_{p+1} \overline{y})), \tag{5}$$
$$b'(x)y = a_1 \overline{x} y - s(a_1 b'(\overline{x}) y + (-1)^p b'(x) a_{p+1} \overline{y}), \tag{6}$$
$$xb'(y) = x a_{p+1} \overline{y} + s(a_1 \overline{x} b'(y) - (-1)^p x a_{p+1} b'(\overline{y})). \tag{7}$$

Using equations (4) and (5),

$$b'(xy) = a_1\overline{x}y + (-1)^p a_{p+1}x\overline{y} - sb'(a_1\overline{x}y + (-1)^p x a_{p+1}\overline{y}).$$

Using the fact that b' is a left A-module morphism, together with the induction hypothesis, we have

$$\begin{aligned} b'(xy) = a_1\overline{x}y + (-1)^p x a_{p+1}\overline{y} - s(a_1 b'(\overline{x})y) + (-1)^p s(a_1\overline{x}b'(y)) \\ - (-1)^p s(b'(x)a_{p+1}\overline{y}) - s(x a_{p+1} b'(\overline{y})). \end{aligned}$$

Equation (3) now follows from equations (6) and (7).

To prove equation (5), note that

$$s(a_1\overline{x}y) = (1 \otimes 1) \otimes \sum_{\beta \in sh(p-1,\, q)} \operatorname{sign}(\beta) a_1 \otimes \beta(a_2 \otimes \cdots \otimes a_n)$$

and

$$s(x a_{p+1}\overline{y}) = (1 \otimes 1) \otimes \sum_{\beta \in sh(p,\, q-1)} \operatorname{sign}(\beta)(\mathrm{id}_A \otimes \beta)(R_{A^{\otimes p},\, A} \otimes \mathrm{id}_{A^{\otimes(q-1)}})(a_1 \otimes \cdots \otimes a_n).$$

Let $i\colon B_{n-1} \to B_n$ denote the inclusion given by

$$i(s_k) = s_{k+1}.$$

Define the one-to-one map $j\colon B_{n-1} \to B_n$ (not a homomorphism) by

$$j(\beta) = \beta s_1 \cdots s_p.$$

We have

$$s(a_1\overline{x}y) = (1 \otimes 1) \otimes \sum_{\beta \in sh(p-1,\, q)} \operatorname{sign}(\beta) i(\beta)(a_1 \otimes \cdots \otimes a_n),$$

$$s(x a_{p+1}\overline{y}) = (1 \otimes 1) \otimes \sum_{\beta \in sh(p,\, q-1)} \operatorname{sign}(\beta) j(\beta)(a_1 \otimes \cdots \otimes a_n).$$

Note that $i(sh(p-1, q))$ and $j(sh(p, q-1))$ are contained in $sh(p, q)$. Moreover, $i(sh(p-1, q))$ and $j(sh(p, q-1))$ are disjoint, since if $s = \pi(\beta) \in S_n$ is the permutation corresponding to $\beta \in i(sh(p-1, q))$ then $s(1) = 1$, while if $s = \pi(\beta)$ for $\beta \in j(sh(p, q-1))$ then $s(p+1) = 1$. Since $sh(p, q)$ has $(p+q)!/p!q!$ elements, a counting argument shows that $sh(p, q)$ is the disjoint union of $i(sh(p-1, q))$ and $j(sh(p, q-1)$. Using the fact that

$$\operatorname{sign}(i(\beta)) = \operatorname{sign}(\beta), \quad \operatorname{sign}(j(\beta)) = (-1)^p \operatorname{sign}(\beta),$$

it follows that

$$s(a_1\overline{x}y + (-1)^p x a_{p+1}\overline{y}) = (1 \otimes 1) \otimes \sum_{\beta \in sh(p,\, q)} \operatorname{sign}(\beta)\beta(a_1 \otimes \cdots \otimes a_n) = xy$$

as desired.

To conclude, we prove (6), omitting the proof of (7) as it is analogous. Note first that

$$\begin{aligned} b'(x) - a_1\overline{x} = \sum_{i=1}^{p-1} (-1)^i \otimes 1 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \\ + (-1)^p \otimes R_{A^{\otimes(p-1)},\, A}(a_1 \otimes \cdots \otimes a_p). \end{aligned}$$

(Here and below powers of -1 are taken as elements of A.) Thus

(8)

$$b'(x)y - a_1\overline{x}y$$
$$= \sum_{\beta\in sh(p-1,q)} \operatorname{sign}(\beta)\left\{\sum_{i=1}^{p-1}(-1)^i \otimes 1 \otimes \beta(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^p \right.$$
$$\left. \otimes (\mathrm{id}_A \otimes \beta)(R_{A^{\otimes(p-1)},A} \otimes \mathrm{id}_{A^{\otimes q}})(a_1 \otimes \cdots \otimes a_n)\right\}.$$

Define $i\colon B_{n-2} \to B_{n-1}$ to be the homomorphism given by $i(s_k) = s_{k+1}$, and define $j\colon B_{n-2} \to B_{n-1}$ by $j(\beta) = i(\beta)s_1 \cdots s_{p-1}$. Since $sh(p-1, q)$ is the disjoint union of $i(sh(p-2, q))$ and $j(sh(p-1, q-1))$ we may write the right side of (8) as the sum of two terms,

$$X = \sum_{\beta\in sh(p-2,q)} \operatorname{sign}(\beta)\left\{-1 \otimes 1 \otimes a_1 a_2 \otimes \beta(a_3 \otimes \cdots \otimes a_n)\right.$$
$$+ \sum_{i=2}^{p-1}(-1)^i \otimes 1 \otimes a_1 \otimes \beta(a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$
$$\left. + (-1)^p \otimes (\mathrm{id}_{A^{\otimes 2}} \otimes \beta)(R_{A^{\otimes(p-1)},A} \otimes \mathrm{id}_{A^{\otimes q}})(a_1 \otimes \cdots \otimes a_n)\right\}$$

and

$$Y = \sum_{\beta\in sh(p-1,q-1)} \operatorname{sign}(\beta)\left\{1 \otimes (\mathrm{id}_{A^{\otimes 2}} \otimes \beta)(R_{A^{\otimes p},A} \otimes \mathrm{id}_{A^{\otimes(q-1)}})\right.$$
$$(R_{A^{\otimes(p-1)},A} \otimes \mathrm{id}_{A^{\otimes q}})(a_1 \otimes \cdots \otimes a_n) + \sum_{i=1}^{p-1}(-1)^{i+p} \otimes 1$$
$$\left. \otimes (\mathrm{id}_A \otimes \beta)(R_{A^{\otimes p},A} \otimes \mathrm{id}_{A^{\otimes(q-1)}})(a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)\right\}.$$

It now suffices to check that the first term equals $s(a_1 b'(\overline{x})y)$, while the second equals $s(b'(x)a_{p+1}\overline{y})$. □

Noting that the braided Hochschild complex $C(A) = A \otimes_{A^e} K(A)$ is the quotient of $A \otimes K(A)$ by an ideal, when the latter is regarded as the braided tensor product $A\widehat{\otimes}K(A)$, we obtain a product on $C(A)$.

Corollary 3. *Let A be a strong r-commutative algebra over a commutative ring k, let $C = C(A)$ be the braided Hochschild complex of A, and let m denote the multiplication map for C. Then C is a differential graded algebra with differential b of degree -1, and for all $x \in C_p$ and $y \in C_q$,*

$$m(x \otimes y) = (-1)^{pq} m R_{C_p,C_q}(x \otimes y).$$

Proof. This follows from Theorem 2. □

4. Hochschild homology and differential forms

Let A be a strong r-commutative r-algebra with r-structure R. By the results of the previous section the braided Hochschild homology $H^R(A)$ is a graded algebra with product induced by the shuffle product on the chain complex $C(A)$. In this section we recall the definition of the r-commutative differential forms $\Omega_R(A)$ and construct a natural homomorphism from them to $H^R(A)$.

First, let $\Omega_u(A)$ be the *universal differential calculus* over A [4, 14]. This is a differential graded algebra with $\Omega_u^0(A) = A$ such that for any differential graded algebra Ω and any homomorphism $f\colon A \to \Omega^0$, there exists a unique differential graded algebra morphism $\tilde{f}$ for which the following diagram commutes:

$$\begin{array}{ccc} \Omega_u(A) & \xrightarrow{\tilde{f}} & \Omega \\ \uparrow & & \uparrow \\ A & \xrightarrow{f} & \Omega^0 \end{array}$$

Let m denote the multiplication map for $\Omega_u(A)$. We define the differential graded algebra of *r-commutative differential forms* over A, $\Omega_R(A)$, to be the quotient of $\Omega_u(A)$ by the differential ideal generated by dl together with the range of the map

$$m(\mathrm{id}_A \otimes d - (d \otimes \mathrm{id}_A)R)\colon A \otimes A \to \Omega_u^1(A).$$

These have been treated before when k is a field [1], and most of their properties carry over to the present situation. In particular, if m now denotes the multiplication map for $\Omega_R(A)$, we have

$$adb = m(d \otimes \mathrm{id}_A)R(a \otimes b),$$

hence

$$dadb = -m(d \otimes d)R(a \otimes b)$$

for all $a, b \in A$, generalizing the usual identities which hold for differential forms:

$$adb = (db)a, \quad dadb = -dbda.$$

Theorem 3. *Let A be a strong r-commutative r-algebra. Then there is a unique graded algebra hommomorphism $f\colon \Omega_R(A) \to H^R(A)$,*

$$f(a) = [a], \qquad f(da) = [1 \otimes a].$$

Here $[a]$ denotes the class in $H_0^R(A)$ of a regarded as an element of $C_0(A) \cong A$, and $[1 \otimes a]$ denotes the class in $H_1^R(A)$ of a regarded as an element of $C_1(A) \cong A^{\otimes 2}$.

Proof. First, note that $C_n(A) = A \otimes_{A^e} A^e \otimes A^{\otimes n}$ is isomorphic to $A^{\otimes(n+1)}$ under the map

$$a \otimes (b \otimes c) \otimes x \mapsto \rho(a \otimes (b \otimes c)) \otimes x,$$

where $a, b, c \in A$ and $x \in A^{\otimes n}$. (An inverse is given by the map $a \otimes x \mapsto a \otimes (1 \otimes 1) \otimes x$.) In what follows we will identify $C_n(A)$ with $A^{\otimes(n+1)}$ under this isomorphism. Note in particular that the boundary of $a \otimes b \otimes c \in C_2(A)$ is

$$ab \otimes c - a \otimes bc + ac^i \otimes b_i,$$

where for any $b, c \in A$ we write $R(b \otimes c) = \sum_i c^i \otimes b_i$, and suppress explicit mention of the summation by the Einstein summation convention.

Next, note that $\Omega_R(A)$ may be defined as the graded algebra over $A = \Omega_R^0(A)$ generated by elements da for $a \in A$ subject to the relations:

$$d(\alpha a + b) = \alpha\, da + db, \quad d(ab) = adb + (db)a, \quad d1 = 0,$$
$$a\, db = m(d \otimes \mathrm{id}_A)R(a \otimes b), \quad da\, db = -m(d \otimes d)R(a \otimes b),$$

for $a, b \in A$, $\alpha \in k$. (The first two relations serve to define $\Omega_u(A)$, and the last three define $\Omega_R(A)$ as a quotient of the universal differential calculus.) Thus $f\colon \Omega_R(A) \to H^R(A)$ is clearly unique if it exists, and existence reduces to showing that

$$[1 \otimes (\alpha a + b)] = \alpha[1 \otimes a] + [1 \otimes b], \tag{9}$$
$$[1 \otimes ab] = [a][1 \otimes b] + [1 \otimes a][b], \tag{10}$$
$$[1 \otimes 1] = 0, \tag{11}$$
$$[a][1 \otimes b] = [1 \otimes b^i][a_i], \tag{12}$$
$$[1 \otimes a][1 \otimes b] = -[1 \otimes b^i][1 \otimes a_i]. \tag{13}$$

Equation (9) is trivial. For equation (10), note that

$$a(1 \otimes b) + (1 \otimes a)b - 1 \otimes ab = a \otimes b + b^i \otimes a_i - 1 \otimes ab$$

in $C_1(A)$ is the boundary of $1 \otimes a \otimes b \in C_2(A)$. Similarly, the boundary of $1 \otimes 1 \otimes 1 \in C_2(A)$ is $1 \otimes 1$, proving equation (11). Equation (12) holds because $(1 \otimes b^i)a_i = a \otimes b = a(1 \otimes b)$ in $C_1(A)$. Equation (13) also holds at the level of chains:

$$(1 \otimes a)(1 \otimes b) = 1 \otimes a \otimes b - 1 \otimes b^i \otimes a_i = -(1 \otimes b^i)(1 \otimes a_i). \quad \square$$

In fact, the homomorphism $f\colon \Omega_R(A) \to H^R(A)$ is an isomorphism in degrees 0 and 1. Given an r-algebra let $[A, A]_R$ denote the *r-commutator*, the span in A of elements of the form $m(\mathrm{id}_{A\otimes A} - R)(a \otimes b)$.

Corollary 4. *Let A be an r-algebra. Then $H_0^R(A) \cong A/[A, A]_R$. If A is strong and r-commutative then f is an isomorphism from $\Omega_R^0(A) = A$ to $H_0^R(A)$, and from $\Omega_R^1(A)$ to $H_1^R(A)$.*

Proof. Identifying $C_n(A)$ with $A^{\otimes(n+1)}$ as in the proof of Theorem 3, the cycles in $C_0(A)$ are identified with A, while the boundaries are identified with $[A, A]_R$. Thus when A is strong and r-commutative, $f\colon \Omega_R^0(A) \to H_0^R(A)$ is an isomorphism.

Also, when A is strong and r-commutative, we may define $g\colon C_1(A) \to \Omega_R^1(A)$ by $g(a \otimes b) = adb$. Since boundaries in $C_1(A)$ are spanned by elements of the form $ab \otimes c - a \otimes bc + ac^i \otimes b_i$, which are annihilated by g, g serves to define an inverse to f. $\square$

It would be interesting to find smoothness conditions on A that would imply that f is an isomorphism in all degrees, thus generalizing the work of Hochschild, Kostant and Rosenberg [11].

5. Conclusions

One could proceed to develop the theory of braided Hochschild and (in the strong case) cyclic homology for r-algebras, but clearly it is more urgent to

calculate it in certain cases and develop an understanding of its significance. The lack of "stability" under deformations of ordinary Hochschild homology is well-known. For noncommutative tori this was observed by Connes [4] (and see also [26]), while for quantum groups this was shown by Feng and Tsygan [8]. In both these cases the algebras in question are r-commutative, so it would be interesting to see if working with braided Hochschild homology would increase stability. We know $H_i^R(A)$ for $i = 0, 1$ for a variety of algebras including noncommutative tori and the Weyl algebra, by Corollary 4 and the results of [1], and in these cases stability is indeed enhanced.

Some of our constructions could be placed more firmly in a categorical framework by assuming that the category $\mathscr{V}$ admits direct sums, kernels and cokernels, and so on. Various extensions of the notion of a braided category appear in the literature, such as "rigid quasitensor categories" and "braided compact closed categories", so adopting such an approach would largely be a matter of choosing among the available definitions.

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