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EIGENVALUES AND EIGENSPACES FOR THE TWISTED DIRAC OPERATOR OVER $SU(N, 1)$ AND $Spin(2N, 1)$

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ABSTRACT. Let X be a symmetric space of noncompact type whose isometry group is either $SU(n, 1)$ or $Spin(2n, 1)$. Then the Dirac operator $\mathbf{D}$ is defined on L^2-sections of certain homogeneous vector bundles over X. Using representation theory we obtain explicitly the eigenvalues of $\mathbf{D}$ and describe the eigenspaces in terms of the discrete series.

1. INTRODUCTION

Let G be a connected real reductive Lie group. From now on we fix a maximal compact subgroup K of G. Let $g_0 = k_0 \oplus p_0$ be the Cartan decomposition of the Lie algebra of G, with k_0 the Lie algebra of K, and let h_0 be a Cartan subalgebra of k_0. We denote by g, k, p, h the complexifications of g_0, k_0, p_0, h_0, and let $\Phi(h, g)$ be the root system of (g, h). Let Φ_k and Φ_n be the compact and noncompact rootspaces of $\Phi(h, g)$ respectively; fix $\Phi^+ = \Phi_k^+ \cup \Phi_n^+$, a positive root system; and denote by ρ one-half of the sum of the positive roots of $\Phi(h, g)$.

Let (τ, V) be a representation of K. We denote

$$C^\infty(G/K, V) = \{ f : G \to V, \quad C^\infty \mid f(gk) = \tau(k)^{-1} f(g) \quad \forall k \in K\},$$
$$L^2(G/K, V) = \{ f : G \to V \mid f(gk) = \tau(k)^{-1} f(g) \quad \forall k \in K, \ \|f\|_2^2 < \infty\}$$

where $\| \ \|_2$ is the L^2-norm with respect to a fixed Haar measure. Both spaces are representations of G under the left regular action.

Let V_σ be an irreducible representation of K with maximal weight σ relative to Φ_k^+. The Dirac operator defines a map

$$\mathbf{D}: L^2(G/K, V_\sigma \otimes S) \quad \to \quad L^2(G/K, V_\sigma \otimes S)$$

as in (3.1). $\mathbf{D}$ is an elliptic essential selfadjoint G-invariant operator.

In this paper the eigenvalues of the Dirac operator are explicitly obtained for $G = SU(n, 1)$ and $Spin(2n, 1)$, and with σ far from the walls of the Weyl chambers. In additions, the respective eigenspaces are expressed as a finite

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sum of discrete series using the Harish-Chandra parametrization of the discrete series. To obtain this we derive specific results for these groups which say when a discrete series occurs in $L^2(G/K, V_\sigma \otimes S)$; furthermore, its multiplicity is a power of two. For the case of $G = Sp(2, \mathbb{R})$, we give examples of discrete series which occur in $L^2(G/K, V_\sigma \otimes S)$ with multiplicity different from a power of two. In general, we show that each discrete series occurring in an eigenspace for a nonzero eigenvalue has even multiplicity. For the kernel the multiplicity is one.

2. Notation

In this section we fix notation and give some known results.

2.1. Let G be a connected real reductive Lie group and, from now on, let K denote a fixed maximal compact subgroup of G. Assume that the rank of G is equal to the rank of K. Let $g_0 = k_0 \oplus p_0$ be the Cartan decomposition of the Lie algebra of G, with k_0 the Lie algebra of K; and let h_0 be a Cartan subalgebra of k_0. Because of the rank condition h_0 is also a Cartan subalgebra of G. The complexification of any Lie algebra is denoted without the subscript. So if $\Phi(h, g)$ is the root system of g (resp. h) and $\Phi(h, k)$ that of k (resp. h), then $\Phi(h, k) \subset \Phi(h, g)$. $\Phi(h, k) = \Phi_k$ is called the set of compact roots of $\Phi(h, g)$. The complement of Φ_k is called the set of noncompact roots and is denoted by Φ_n. Let Φ_k^+ be a fixed positive root system of Φ_k. One can choose a subset Φ_n^+ of Φ_n such that $\Phi^+ = \Phi_k^+ \cup \Phi_n^+$ is a positive root system of $\Phi(h, g)$. The choice of Φ_n^+ is not unique: there are exactly $|W_G|/|W_K|$ choices, where W_G is the Weyl group of g and W_K is that of k. When necessary, we will say explicitly which choice will be taken.

Denote by

$$\rho_k = \frac{1}{2} \sum_{\alpha \in \Phi_k^+} \alpha, \qquad \rho_n = \frac{1}{2} \sum_{\alpha \in \Phi_n^+} \alpha$$

and by $\rho = \rho_k + \rho_n$. When ρ is not analytically integral in G, fix a twofold cover of G, which will be also denoted by G without causing confusion, and call K the inverse image of K.

2.2. The Killing form is defined at g_0 by

$$B(X, Y) = \operatorname{Trace}(\operatorname{ad} X \operatorname{ad} Y).$$

Its restriction to h is nondegenerate and negative definite, so $-B(\ ,\ )$ is an inner product on h_0 which gives one on ih_0. Let $(ih_0)'$ be the real dual of ih_0 and denote by $(\ ,\ )$ the inner product at $(ih_0)'$ which comes from the Killing form. Also, B is positive definite in p_0 and the K-representation on p_0 is orthogonal.

Because of the last condition of (2.1), the representation

$$K \to SO(p_0) \simeq SO(\dim p_0)$$

given by the adjoint representation lifts to the universal cover $Spin(p_0)$ of $SO(p_0)$; that is, the usual spin representation S of $Spin(p_0)$ gives rise to a K-module. Let (s, S) denote this K-module.

2.3. Let (π, H) be a representation of G on the Hilbert space H. Without lost of generality we can suppose that $\pi(K)$ acts by unitary operators. Hence H is an orthogonal sum of irreducible representations of K as a K-module

$$H = \bigoplus_{\tau \in \hat{K}} m(\tau) V_\tau$$

where $\hat{K}$ is the set of equivalence classes of irreducible representations of K; the multiplicity $m(\tau)$ is a nonnegative integer or $+\infty$. The subspace $m_\tau V_\tau$ is the isotypic K-submodule of type τ of (π, H). It is usually denoted by $H[\tau]$.

We say that (π, H) is an admissible representation if $\pi(K)$ acts by unitary operators and m_τ is finite for all $\tau \in \hat{K}$.

An admissible representation (π, H) is a discrete series if it is irreducible and all its matrix coefficients $g \to \langle \pi(g)u, v\rangle$ (with $u, v \in V_K$) are square integrable.

All discrete series can be parametrized by weights $\lambda \in (ih_0)'$, the dual of ih_0, such that λ is nonsingular (i.e., $(\lambda, \alpha) \neq 0 \quad \forall \alpha \in \Phi(h, g)$), and $\lambda + \rho$ is integral (i.e., $\lambda(H) \in 2\pi i \mathbb{Z}$, $\forall H \in ih_0$ such that $\exp H = 1$). The discrete series H_λ of parameter (or Harish-Chandra parameter) λ has infinitesimal character χ_λ, and two discrete series are equivalent if and only if their parameters are conjugate by an element of the Weyl group of K.

2.4. Let $f \in C^\infty(G/K, V)$ or $f \in L^2(G/K, V)$ and consider the action of G given by

$$\pi(g)f(x) = f(g^{-1}x).$$

We also require the action of the elements of g_0 as left-invariant differential operators, that is, if $X \in g_0$

$$X f(x) = \left.\frac{d}{dt}\right|_{t=0} f(x \exp tX).$$

Now if $Z = X + iY \in g$, we define $Zf = Xf + iYf$. Then each $D \in (\mathscr{U}(g) \otimes \mathrm{End}(V))^K$ defines a left-invariant differential operator on $C^\infty(G/K, V)$ [Wa, Chapter 5]. G acts on $(\mathscr{U}(g) \otimes \mathrm{End}(V))^K$ by $\mathrm{Ad} \otimes$ (repres. of K on $\mathrm{End}(V)$)

2.5. If $\{X_i\}$ is an orthonormal base of g (with respect to the Killing form), the Casimir element Ω is defined by

$$\Omega = \sum X_i \bar{X}_i.$$

It is known that Ω belongs to the center of $\mathscr{U}(g)$. The Casimir operator acts on a discrete series H_λ by the constant $\|\lambda\|^2 - \|\rho\|^2$. An explicit expression for the Casimir can be computed as follows. Let $\{H_i\}$ be an orthonormal basis of ih_0, and for each $\alpha \in \Phi(h, g)$, let

$$g_\alpha = \{X \in g \,/\, \mathrm{ad}(H) = \alpha(H)X \quad \forall H \in h\}.$$

Choosing appropriately $X_\alpha \in g_\alpha$, Ω is given by

$$\Omega = \sum H_i^2 + \sum_{\alpha \in \Phi^+} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) = \sum H_i^2 + \sum_{\alpha \in \Phi^+} (H_\alpha + 2X_{-\alpha} X_\alpha).$$

3. Eigenvalues of $\mathbf{D}$

If we fix a minimal left ideal in the Clifford algebra of p_0, the resulting representation of $so(p_0)$ breakes into two irreducible representations. Composed with the adjoint action of k_0 on p_0, this lifts to a representation S of K, called the spin representation. Let $\{X_i\}_{i=1}^{2n}$ be an orthonormal base of p_0, let c be the operation of left Clifford multiplication and let V_σ be an irreducible representation of K of maximal weight σ (Φ_k^+-dominant). The Dirac operator

$$\mathbf{D} : L^2(G/K\,, V_\sigma \otimes S) \quad \rightarrow \quad L^2(G/K\,, V_\sigma \otimes S)$$

is defined by

$$\mathbf{D} = \sum_{i=1}^{2n} (1 \otimes c(X_i))\, X_i \tag{3.1}$$

where the X_i act as left-invariant differential operators for all i. The spin representation S decomposes into a sum of two subrepresentations $S = S^+ \oplus S^-$. If $X \in p_0$, then $c(X)S^\pm = S^\mp$, so

$$\mathbf{D}^\pm : L^2\left(G/K\,, V_\sigma \otimes S^\pm\right) \quad \rightarrow \quad L^2\left(G/K\,, V_\sigma \otimes S^\mp\right) \tag{3.2}$$

are also well defined.

We list some properties of the Dirac operator $\mathbf{D}$. $\mathbf{D}$ is an elliptic G-invariant differential operator, and as the riemannian metric of G/K is complete, $\mathbf{D}$ and $\mathbf{D}^2$ are essentially selfadjoint in $L^2(G/K\,, V_\sigma \otimes S)$ [W]; that is, the minimal extension is the unique selfadjoint closed extension starting from the set of smooth compactly supported functions. So, we consider $\mathbf{D}$ densely defined by this extension, which coincides with the maximal one [A]. The eigenvalues of $\mathbf{D}$ are defined as the eigenvalues of the unique selfadjoint extension.

Let L_d^2 be the closure of the sum of all irreducible G-invariant closed subspaces of $L^2(G/K\,, V_\sigma \otimes S)$; Harish-Chandra has proved that L_d^2 is the direct sum of a finite number of square integrable G-irreducible closed subspaces, that is a finite sum of discrete series

$$L_d^2 \simeq \bigoplus_{\lambda \in F} n_\lambda H_\lambda \tag{3.3}$$

with F a finite set and n_λ the multiplicity of the discrete series H_λ with parameter λ.

A theorem of Connes and Moscovici [C-M] ensures that if

$$D\colon L^2(G/K\,, V_\sigma \otimes S) \quad \rightarrow \quad L^2(G/K\,, V_\sigma \otimes S)$$

is an elliptic G-invariant operator, each eigenspace of D is a finite sum of discrete series and D has a finite number of eigenvalues.

Take Φ^+ such that σ is a Φ^+-dominant weight. If Ω is the Casimir element of the universal enveloping algebra $\mathscr{U}(g)$ of g, the Parthasarathy equality for the square of the operator $\mathbf{D}$ [A-S] is

$$\mathbf{D}^2 = -\Omega + (\sigma - \rho_n\,, \sigma - \rho_n + 2\rho)I.$$

This equality restricted to an immersion of a discrete series H_λ (with infinitesimal character χ_λ) in L_d^2 is

$$\mathbf{D}^2|_{H_\lambda} = \left(-\|\lambda\|^2 + \|\rho\|^2 + (\sigma - \rho_n\,, \sigma - \rho_n + 2\rho)\right) I \tag{3.4}$$

because the Casimir acts on H_λ by the constant $\|\lambda\|^2 - \|\rho\|^2$ (see (2.5)).

Recall that n_λ denotes the multiplicity of the discrete series with parameter λ which occur in $L^2(G/K, V_\sigma \otimes S)$, that is

$$n_\lambda = \dim \mathrm{Hom}_G\left(H_\lambda, L^2(G/K, V_\sigma \otimes S)\right) = \dim \mathrm{Hom}_K(H_\lambda, V_\sigma \otimes S)$$

by Frobenius reciprocity. If the maximal weight σ of V_σ is sufficiently far from the walls of the Weyl chambers of K, or more precisely, if

$$(\sigma + \gamma, \alpha) > 0 \quad \forall \gamma \in P(S), \ \forall \alpha \in \Phi_k^+ \tag{3.5}$$

with $P(S)$ the set of weight of S, then,

$$V_\sigma \otimes S = \bigoplus_{\gamma \in P(S)} V_{\sigma+\gamma} \tag{3.6}$$

where $V_{\sigma+\gamma}$ is the irreducible K-module with maximal weight $\sigma + \gamma$. This happens because the multiplicity of each weight of S is one, and

$$\begin{aligned}
\chi_{V\otimes S} = \chi_V \cdot \chi_S &= \Delta_K^{-1} \sum_{w \in W_K} \det w \quad e^{w(\sigma+\rho_k)} \sum_{\gamma \in P(S)} e^\gamma \\
&= \Delta_K^{-1} \sum_{w \in W_K} \sum_{\gamma \in P(S)} \det w \quad e^{w(\sigma+\rho_k)+\gamma} = \Delta_K^{-1} \sum_{w \in W_K} \sum_{\gamma \in P(S)} \det w \quad e^{w(\sigma+\gamma+\rho_k)} \\
&= \sum_{\gamma \in P(S)} \chi_{\sigma+\gamma} \qquad \text{(by (3.5))}
\end{aligned}$$

where χ_W denotes the character of the K-module W. By (3.6), we have that

$$n_\lambda = \sum_{\gamma \in P(S)} \dim \mathrm{Hom}_K(H_\lambda, V_{\sigma+\gamma}). \tag{3.7}$$

So, we only have to analyse when the isotypic component ($H_\lambda[\sigma+\gamma]$), of the representation H_λ restricted to K of maximal weight $\sigma+\gamma$, is not zero. In the cases $G = SU(n,1)$ and $G = Spin(2n,1)$ it is known that if $H_\lambda[\sigma+\gamma] \neq 0$, then $H_\lambda[\sigma+\gamma]$ is irreducible because each K-type of any principal series has this property; that is,

$$n_\lambda = |\{\gamma \in P(S) \colon H_\lambda[\sigma+\gamma] \neq 0\}|. \tag{3.8}$$

Denote by $\mathrm{Eig}(\mathbf{D})$ the set of eigenvalues of $\mathbf{D}$, and by $\mathbf{W}_\alpha(\mathbf{D})$ the eigenspace of the operator $\mathbf{D}$ associated to the eigenvalue α.

Proposition 3.1. *Let* $\mathbf{D}$ *be the Dirac operator defined in* $L^2(G/K, V_\sigma \otimes S)$. *Then,*

(i) *If* $\beta \in \mathrm{Eig}(\mathbf{D}^2)$, $\beta \neq 0$, *and* α *is the positive square root of* β,

$$\mathbf{W}_{\alpha^2}(\mathbf{D}^2) = \mathbf{W}_\alpha(\mathbf{D}) \oplus \mathbf{W}_{-\alpha}(\mathbf{D}) \qquad \textit{and} \qquad \mathbf{W}_0(\mathbf{D}^2) = \mathbf{W}_0(\mathbf{D}).$$

(ii) *If* α *is a nonzero eigenvalue of* $\mathbf{D}$, $\mathbf{W}_\alpha(\mathbf{D})$ *is equivalent to* $\mathbf{W}_{-\alpha}(\mathbf{D})$ *as a* G*-module, so that each discrete series which occurs in* $\mathbf{W}_{\alpha^2}(\mathbf{D}^2)$ *has even multiplicity.*

(iii) $L_d^2 = \bigoplus_{\alpha \in \mathbf{Eig}(\mathbf{D})} \mathbf{W}_\alpha(\mathbf{D})$.

(iv) *The set of the eigenvalues of* $\mathbf{D}^2$ *is*

$$\mathrm{Eig}(\mathbf{D}^2) = \{-\|\lambda\|^2 + \|\sigma + \rho_k\|^2 \mid \lambda \textit{ is a } \Phi_k^+\textit{-dominant Harish-Chandra parameter and } H_\lambda[\sigma+\gamma] \neq 0 \textit{ for some } \gamma \in P(S)\}$$

and the set of the eigenvalues of $\mathbf{D}$ *is*

$$\operatorname{Eig}(\mathbf{D}) = \left\{\alpha:\ \alpha^2 \in \operatorname{Eig}(\mathbf{D}^2)\right\}.$$

Note. Using the Atiyah-Schmid result, which ensures that the kernel of $\mathbf{D}$ is equivalent to $H_{\sigma+\rho_k}$, this proposition says that the multiplicity of each discrete series which occurs in L^2_d is even except for $H_{\sigma+\rho_k}$.

Proof. Since $\beta = \|\mathbf{D}f\|^2/\|f\|^2 > 0$, it makes sense to take the positive square root α.

(i) Since $\mathbf{D}^2$ is an essentially selfadjoint operator its eigenvalues are real. If $\beta \neq 0$, let $f \in \mathrm{W}_\beta(\mathbf{D}^2)$, then $f \pm \alpha^{-1}\mathbf{D}f \in \mathrm{W}_{\pm\alpha}(\mathbf{D})$, with α the positive square root of β, because

$$\mathbf{D}(f \pm \alpha^{-1}\mathbf{D}f) = \mathbf{D}f \pm \alpha^{-1}\mathbf{D}^2 f = \mathbf{D}f \pm \alpha f = \pm\alpha(\pm\alpha^{-1}\mathbf{D}f + f).$$

Then, since

$$f = \tfrac{1}{2}(f + \alpha^{-1}\mathbf{D}f) + \tfrac{1}{2}(f - \alpha^{-1}\mathbf{D}f)$$

we have that $\mathrm{W}_{\alpha^2}(\mathbf{D}^2) \subset \mathrm{W}_\alpha(\mathbf{D}) \oplus \mathrm{W}_{-\alpha}(\mathbf{D})$.

$\mathbf{D}^2$ is essentialy selfadjoint, so if f is in the domain of $\mathbf{D}^2$, then

$$(\mathbf{D}^2 f, f) = (\mathbf{D}f, \mathbf{D}f).$$

If f also is in the kernel of $\mathbf{D}^2$, $\|\mathbf{D}f\| = 0$, that is $\mathbf{D}f = 0$; and as the kernel of $\mathbf{D}^2$ is closed, $\mathrm{W}_0(\mathbf{D}^2) = \mathrm{W}_0(\mathbf{D})$.

(ii) If $f \in L^2(G/K, V_\sigma \otimes S) = L^2(G/K, V_\sigma \otimes S^+) \oplus L^2(G/K, V_\sigma \otimes S^-)$, then $f = (f^+, f^-)$ and $\mathbf{D}f = (\mathbf{D}^- f^-, \mathbf{D}^+ f^+)$ because of (3.2). The map

$$\mathrm{W}_\alpha(\mathbf{D}) \to \mathrm{W}_{-\alpha}(\mathbf{D}), \quad (f^+, f^-) \to (f^+, -f^-)$$

is really an isomorphism between $\mathrm{W}_\alpha(\mathbf{D})$ and $\mathrm{W}_{-\alpha}(\mathbf{D})$. In fact,

$$\mathbf{D}(f^+, -f^-) = (-\mathbf{D}^- f^-, \mathbf{D}^+ f^+) = (-\alpha f^+, \alpha f^-) = -\alpha(f^+, -f^-).$$

(iii) The equality (3.4) implies that each discrete series in L^2_d is in an eigenspace of $\mathbf{D}^2$, the eigenvalue depends on the norm of the parameter λ. Then L^2_d is the sum of eigenspaces of $\mathbf{D}^2$, and by (i), we have

$$L^2_d \simeq \bigoplus_{\beta \in \operatorname{Eig}(\mathbf{D}^2)} \mathrm{W}_\beta(\mathbf{D}^2) \simeq \bigoplus_{\alpha \in \operatorname{Eig}(\mathbf{D})} \mathrm{W}_\alpha(\mathbf{D}).$$

(iv) The equality (3.7) ensures that $n_\lambda \neq 0$ if and only if $H_\lambda[\sigma + \gamma] \neq 0$ for some $\gamma \in P(S)$. Then by the equality (3.4) and (iii) if $H_\lambda[\sigma + \gamma] \neq 0$ for some $\gamma \in P(S)$, one has that $H_\lambda \in \operatorname{Eig}(\mathbf{D}^2)$. But

$$\begin{aligned}\|\rho\|^2 + (\sigma - \rho_n, \sigma - \rho_n + 2\rho) &= (\rho, \rho) + 2(\sigma - \rho_n, \rho) + (\sigma - \rho_n, \sigma - \rho_n) \\ &= (\sigma - \rho_n + \rho, \sigma - \rho_n + \rho) = \|\sigma + \rho_k\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{Eig}(\mathbf{D}^2) = \{-\|\lambda\|^2 + \|\sigma + \rho_k\|^2 \mid \lambda \text{ is a } \Phi_k^+\text{-dominant Harish-Chandra} \\ \text{parameter, and } H_\lambda[\sigma+\gamma] \neq 0 \text{ for any } \gamma \in P(S)v\}.\end{aligned}$$ □

4. $G = SU(n, 1)$

Let K be the usual immersion of $S(U(n) \times U(1))$ in G, so K is a maximal compact subgroup of G. Let T be the torus of diagonal matrices of K, so T is also a compact Cartan subgroup of G. Let g_0, k_0, h_0 be their Lie algebras and g, k, h the complexifications. Choose an orthonormal base $\{H_1, \dots, H_n\}$ of the real Lie algebra ih_0 with respect to $-B(\ ,\ )$, where B is the Killing form of g $(B(X, Y) = \frac{1}{n}\mathrm{tr}(XY))$.

If $H = \sum ih_jE_{jj} \in ih_0$, let $e_j \in (ih_0)'$ be given by

$$e_j(H) = h_j, \qquad j = 1, \dots, n+1.$$

Denote by $(\ ,\ )$ the dual symmetric form to the Killing form of g.

The root set of (g, h) is

$$\Phi(h, g) = \{e_i - e_j : i \neq j\ , i, j = 1, \dots, n+1\}$$

and

$$\Phi_k = \{e_i - e_j : i \neq j\ , i, j = 1, \dots, n\}, \qquad \Phi_n = \{\pm(e_i - e_{n+1}) : i = 1, \dots, n\}.$$

Fix

$$\Phi_k^+ = \{e_i - e_j : i < j < n+1\}. \tag{4.1}$$

The number of choices of Φ_n^+ such that $\Phi_k^+ \cup \Phi_n^+$ is a positive root system of $\Phi(h, g)$ is $n+1 = |W_G|/|W_K|$, because W_G is the set of permutations of $n+1$ elements and W_K that of n elements. The different Φ_n^+ are

$$\Psi^r = \{e_i - e_{n+1} : 1 \le i \le r-1\} \cup \{-e_i + e_{n+1} : r \le i \le n\} \tag{4.2}$$

with $1 \le r \le n+1$.

From now on fix r such that $\Phi_n^+ = \Psi^r$, then

$$\rho_k = \frac{1}{2}\sum_{i<j<n+1}(e_i - e_j) = \frac{1}{2}\sum_{i=1}^{n}(n - 2i + 1)e_i,$$

$$\rho_n = \frac{1}{2}\left(\sum_{i=1}^{r-1} e_i - \sum_{i=r}^{n} e_i + (n - 2r + 2)e_{n+1}\right), \tag{4.3}$$

$$\rho = \frac{1}{2}\left(\sum_{i=1}^{r-1}(n - 2i + 2)e_i + \sum_{i=r}^{n}(n - 2i)e_i + (n - 2r + 2)e_{n+1}\right).$$

Let $\lambda \in (ih_0)'$ be an integral weight. Then λ satisfies $\lambda = \sum_{i=1}^{n+1}\lambda_i e_i$ with $\sum_{i=1}^{n+1}\lambda_i = 0$ because the element $H^\lambda = \sum_{j=1}^{n+1} i\lambda_j E_{jj} \in ih_0$ such that $\lambda = -B(\ , H^\lambda)$ has Trace$(H^\lambda) = 0$. Moreover, $\|e_j - e_{j+1}\| = 2$ gives

$$\frac{2(\lambda, e_j - e_{j+1})}{\|e_j - e_{j+1}\|^2} = (\lambda, e_j - e_{j+1}) = \lambda_j - \lambda_{j+1} \in \mathbb{Z} \quad \forall j = 1, \dots, n.$$

This implies that for some $s \in \mathbb{Z}$, $0 \le s < n+1$,

$$\lambda_i = m_i + \frac{s}{n+1}, \qquad m_i, s \in \mathbb{Z} \quad \forall i = 1, \dots, n+1. \tag{4.4}$$

Also note that λ is a Φ_k^+-dominant weight if and only if

$$\lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_1 \tag{4.5}$$

and it is Ψ^r-dominant if and only if

$$\lambda_r \leq \lambda_{n+1} \leq \lambda_{r-1}. \tag{4.6}$$

Suppose λ is a Φ^+-dominant Harish-Chandra parameter. Then as $\lambda+\rho$ and ρ are integral (as $SU(n, 1)$ is simply connected, ρ is integral for any positive root system), λ satisfies (4.4), and since λ also is nonsingular, at (4.5) and (4.6) the strict inequalities hold.

To determine when a K-type occurs at a discrete series of G, fix $\Phi^+ = \Phi_k^+ \cup \Psi^r$. Denote by $m_\lambda(\tau)$ the multiplicity of the irreducible representation of highest weight τ in H_λ.

Proposition 4.1. *Let $\lambda = \sum_{i=1}^{n+1} \lambda_i e_i$ be a Harish-Chandra parameter of a discrete series of the group $SU(n, 1)$ which is $(\Phi_k^+ \cup \Psi^r)$-dominant, and let $\tau = \sum_{i=1}^{n+1} \tau_i e_i$ be a Φ_k^+-dominant weight. If $\mu = \lambda + \rho_n - \rho_k = \sum_{i=1}^{n+1} \mu_i e_i$, then*

$$m_\lambda(\tau) = 1 \;\Leftrightarrow \begin{cases} \tau_n \leq \mu_n \leq \tau_{n-1} \leq \cdots \leq \tau_r \leq \mu_r < \mu_{r-1} \leq \tau_{r-1} \leq \cdots \leq \mu_1 \leq \tau_1, \\ \tau_i - \mu_i \in \mathbb{Z} \qquad \forall i = 1, \ldots, n. \end{cases}$$

Proof. If $\tau' = \tau + \rho_k$ and $\mu' = \mu + \rho_k$, then the inequality of the proposition is equivalent to

$$\tau'_n \leq \mu'_n < \tau'_{n-1} \leq \cdots < \tau'_r \leq \mu'_r < \mu'_{r-1} \leq \tau'_{r-1} < \mu'_{r-2} \leq \cdots < \mu'_1 \leq \tau'_1 \tag{4.7}$$

because $(\rho_k)_{i+1} = (\rho_k)_i + 1$ for each i.

The Blattner formula is

$$m_\lambda(\tau) = \sum \det s \; Q(s^{-1}\tau' - \mu')$$

where $Q(\sigma)$ is the number of expressions of the weight σ as a sum of positive noncompact roots.

Suppose $m_\lambda(\tau) \neq 0$, so $Q_s = Q(s^{-1}\tau' - \mu') \neq 0$ for some $s \in W_K$. Since $\Phi^+ = \Phi_k^+ \cup \Psi^r$, from (4.2) we get $(s^{-1}\tau' - \mu', e_i) \in \mathbb{Z}$ and

$$(s^{-1}\tau' - \mu', e_i) \begin{cases} \geq 0, & 1 \leq i \leq r-1, \\ \leq 0, & r \leq i \leq n, \end{cases} \tag{4.8}$$

because $s^{-1}\tau' - \mu' = \sum_{i=1}^{n} n_i(e_i - e_{n+1})$ with $n_i \geq 0$ for $i < r$ and $n_i \leq 0$ for $r \leq i < n+1$. Now W_K is the permutation set of the elements $\{e_1, \ldots, e_n\}$, so if π is a permutation of n elements, then

$$(s^{-1}\tau' - \mu')_i = \begin{cases} \tau'_{\pi(i)} - \mu'_i \geq 0, & 1 \leq i \leq r-1, \\ \tau'_{\pi(i)} - \mu'_i \leq 0, & r \leq i \leq n. \end{cases} \tag{4.9}$$

Since $\mu'_n < \mu'_{n-1} < \cdots < \mu'_1$, (4.8) ensures that π leaves invariant the sets $\{1, \ldots, r-1\}$ and $\{r, \ldots, n\}$, because if $1 \leq i < r$ and $r \leq j \leq n$ (because τ is dominant), then $\tau'_{\pi(j)} \leq \mu'_j < \mu'_i \leq \tau'_{\pi(i)}$, implies $\pi(j) > \pi(i)$ $\forall i, j$ in the given intervals.

Let H be the permutation set that permute the τ'_j's in each interval $[\mu'_i, \mu'_{i-1})$ with $1 \leq i < r$ $(\mu'_0 = \infty)$. For $s_1 \in H$, since $Q_s = Q_{ss_1}$,

$$m_\lambda(\tau) = \sum \det s \; Q_s = \sum \det s \; Q_{ss_1} = \sum \det s(s_1)^{-1} \; Q_s = \det(s_1)^{-1} m_\lambda(\tau).$$

H always contains a transposition unless $H = 1$, and the sign of a transposition (its determinant) is -1, so $H = 1$. Then, because of the decreasing order of τ'_j's $(j \neq n+1)$ and (4.8)

$$\mu'_{r-1} \leq \tau'_{r-1} < \mu'_{r-2} \leq \cdots < \mu'_1 \leq \tau'_1.$$

The same argument for the intervals $(\mu'_{i+1}, \mu'_i]$ with $r \leq i < n+1$ $(\mu'_{n+1} = -\infty)$ yields

$$\tau'_n \leq \mu'_n < \tau'_{n-1} \leq \cdots < \tau'_r \leq \mu'_r.$$

Thus, the unique s such that $Q_s \neq 0$ is $s = 1$, so $m_\lambda(\tau) = \det 1\, Q_1 = 1$. □

The proposition will be used for $\tau = \sigma + \gamma$ with σ a Φ_k^+-dominant weight and γ a weight of S. In this case

$$\begin{aligned} P(S) &= \left\{\tfrac{1}{2}(\pm\alpha_1 \pm \alpha_2 \pm \cdots \pm \alpha_n) \colon \alpha_i \in \Psi^r\right\} \\ &= \left\{\tfrac{1}{2}(\pm e_1 \pm \cdots \pm e_n + m e_{n+1}) \colon m = \text{number of } (-) - \text{number of } (+)\right\} \end{aligned}$$

$$\sigma = \sum_{i=1}^{n+1} \sigma_i e_i, \qquad \frac{\sigma_i = m_i + s}{n+1}, \quad s, m_i \in \mathbb{Z}, \quad 0 \leq s < n+1,$$

$$\sigma + \gamma = \sum_{i=1}^{n+1} (\sigma_i + \varepsilon_i) e_i, \qquad \varepsilon_i = (\gamma, e_i) = \begin{cases} \pm\frac{1}{2}, & i \neq n+1, \\ -\sum_{i=1}^{n} \varepsilon_i, & i = n+1. \end{cases}$$

We retain the notation of §3.

Proposition 4.2. *Let* $\lambda = \sum_{i=1}^{n+1} \lambda_i e_i$ *be a* Ψ^r*-dominant Harish-Chandra parameter, and let* L^2_d *be the discrete part of* $L^2(G/K, V_\sigma \otimes S)$ *as in* (3.3) *and* σ *be as in* §3. *Then*

(i)

$$n_\lambda \neq 0 \Leftrightarrow \begin{cases} (\sigma + \rho_k - \lambda)_i \in \mathbb{Z}, \quad i = 1, \ldots, n, \\ \lambda_i \in [\sigma_{i+1} + \frac{1}{2}(n - 2i - 1), \sigma_i + \frac{1}{2}(n - 2i + 1)], \quad 1 \leq i < r-1, \\ \lambda_{r-1} \in (\sigma_r + \frac{1}{2}(n - 2r + 1), \sigma_{r-1} + \frac{1}{2}(n - 2r + 3)], \\ \lambda_r \in [\sigma_r + \frac{1}{2}(n - 2r + 1), \lambda_{r-1}), \\ \lambda_i \in [\sigma_i + \frac{1}{2}(n - 2i + 1), \sigma_{i-1} + \frac{1}{2}(n - 2i + 3)], \quad r < i \leq n. \end{cases}$$

(ii) $n_\lambda \neq 0 \Rightarrow n_\lambda = 2^m$, $0 \leq m \leq n$.

(iii) $n_\lambda = 1 \Leftrightarrow \lambda = \sigma + \rho_k$.

Remark. If $\sigma + \rho_k$ is a Harish-Chandra parameter, then $\mathbf{W}_0(\mathbf{D}^2) = \mathbf{W}_0(\mathbf{D}) \supset H_{\sigma+\rho_k}$ by (iii) of the last proposition and (iv) of Proposition 3.1. Actually, the equality is true by the irreducibility of $\mathbf{W}_0(\mathbf{D})$ [A-S].

Proof. (i) Suppose that $n_\lambda \neq 0$, then $m_\lambda(\sigma + \gamma) \neq 0$ for some $\gamma \in P(S)$, so by Proposition 4.1 and (4.3)

$$\sigma_i + \varepsilon_i + (\rho_k)_i - \mu_i = \sigma_i + \varepsilon_i + (\rho_k)_i - (\lambda_i \pm \tfrac{1}{2}) \in \mathbb{Z} \quad \forall i$$

if and only if $\sigma_i + (\rho_k)_i - \lambda_i \in \mathbb{Z}$ $\forall i$ and

$$\begin{aligned} &\lambda_i \in [\sigma_{i+1} + \varepsilon_{i+1} + \tfrac{1}{2}(n - 2i), \sigma_i + \varepsilon_i + \tfrac{1}{2}(n - 2i)], \qquad 1 \leq i < r-1, \\ &\lambda_{r-1} \in (\sigma_r + \varepsilon_r + \tfrac{1}{2}(n - 2(r-1)), \sigma_{r-1} + \varepsilon_{r-1} + \tfrac{1}{2}(n - 2(r-1))], \\ &\lambda_r \in [\sigma_r + \varepsilon_r + \tfrac{1}{2}(n - 2(r-1)), \lambda_{r-1}), \\ &\lambda_i \in [\sigma_i + \varepsilon_i + \tfrac{1}{2}(n - 2(i-1)), \sigma_{i-1} + \varepsilon_{i-1} + \tfrac{1}{2}(n - 2(i-1))], \qquad r < i \leq n. \end{aligned}$$

As $\varepsilon = \pm\frac{1}{2}$ the components of λ are in the given intervals.

Conversely, we want to know when there exist $\gamma \in P(S)$ such that $m_\lambda(\sigma+\gamma) \neq 0$. Denote

for $i \leq r-1$

$$N_i = [\sigma_{i+1} + \tfrac{1}{2}(n-2i-1), \sigma_{i+1} + \tfrac{1}{2}(n-2i+1)),$$
$$B_i = [\sigma_{i+1} + \tfrac{1}{2}(n-2i+1), \sigma_i + \tfrac{1}{2}(n-2i-1)],$$
$$M_i = (\sigma_i + \tfrac{1}{2}(n-2i-1), \sigma_i + \tfrac{1}{2}(n-2i+1)];$$

for $i = r-1$

$$N_{r-1} = (\sigma_r + \tfrac{1}{2}(n-2(r-1)-1), \sigma_r + \tfrac{1}{2}(n-2(r-1)+1)),$$
$$B_{r-1} = [\sigma_r + \tfrac{1}{2}(n-2(r-1)+1), \sigma_{r-1} + \tfrac{1}{2}(n-2(r-1)-1)],$$
$$M_{r-1} = (\sigma_{r-1} + \tfrac{1}{2}(n-2(r-1)-1), \sigma_{r-1} + \tfrac{1}{2}(n-2(r-1)+1)];$$

for $i = r$

$$N_r = [\sigma_r + \tfrac{1}{2}(n-2(r-1)-1), \sigma_r + \tfrac{1}{2}(n-2(r-1)+1))),$$
$$B_r = [\sigma_r + \tfrac{1}{2}(n-2(r-1)+1), \lambda_{r-1}),$$
$$M_r = \varnothing;$$

for $r < i \leq n$

$$N_i = [\sigma_i + \tfrac{1}{2}(n-2(i-1)-1), \sigma_i + \tfrac{1}{2}(n-2(i-1)+1)),$$
$$B_i = [\sigma_i + \tfrac{1}{2}(n-2(i-1)+1), \sigma_{i-1} + \tfrac{1}{2}(n-2(i-1)-1)],$$
$$M_i = (\sigma_{i-1} + \tfrac{1}{2}(n-2(i-1)-1), \sigma_{i-1} + \tfrac{1}{2}(n-2(i-1)+1)].$$

Observe that the intervals N_i and M_i have length one, except when they are empty. Suppose $H_\lambda[\sigma+\gamma] \neq 0$. When $\lambda_i \in N_i$, for $i < r$, set $\varepsilon_{i+1}(\gamma) = -\frac{1}{2}$ and for $i \geq r$, set $\varepsilon_i(\gamma) = -\frac{1}{2}$. Similarly, for $\lambda_i \in M_i$, put $\varepsilon_i(\gamma) = \frac{1}{2}$, when $i < r$ and $\varepsilon_{i+1}(\gamma) = \frac{1}{2}$ when $i > r$. If λ is a Harish-Chandra parameter whose components satisfy the conditions on the right-hand side of (i), then two consecutive components λ_i and λ_{i+1} of λ cannot be at N_i and M_{i+1} respectively. So, either case determines the value of the corresponding component of γ. If $\lambda \in B_i$, $\varepsilon_i(\gamma)$ can take either value. So, there exist a γ such that $H_\lambda[\sigma+\gamma] \neq 0$.

(ii) Suppose that $\lambda_{i_j} \notin B_{i_j}$, $j = 1, \dots, m$, and $\lambda_k \in B_k$ for $k \neq i_j$. Then $\lambda_{i_j} \in N_{i_j} \cup M_{i_j}$, so this determines exactly m components values of the γ's such that $m_\lambda(\sigma+\gamma) \neq 0$. Thus there exist 2^{n-m} weight γ such that $m_\lambda(\sigma+\gamma) \neq 0$.

(iii) $n_\lambda = 1$ is equivalent to the existence of a unique $\gamma \in P(S)$ such that $m_l(\sigma+\gamma) \neq 0$, so the components of λ determine every components of γ, or equivalently $\lambda_i \in N_i \cup M_i \quad \forall i = 1, \dots, n$. Note that $M_r = \varnothing$, so $\lambda_r \in N_r$. This implies that $\lambda_i \in N_i \quad \forall i > r$. The component $\lambda_{r-1} \in M_{r-1}$, because

$\lambda_{r-1} \geq \lambda_r + 1 \geq \sigma_r + \frac{1}{2}(n-2(r-1)-1) + 1 =$ right extreme of the open set N_{r-1}.

So $\lambda_i \in M_i$ for $i < r$. Again, as the lengths of N_i and M_i are one,

$$(\sigma + \rho_k - \lambda)_i \in \mathbb{Z} \qquad \forall i = 1, \dots, n,$$
$$(\sigma + \rho_k)_i \in M_i, \qquad i < r,$$
$$(\sigma + \rho_k)_i \in N_i, \qquad i \geq r,$$

so the conclusion is $\lambda = \sigma + \rho_k$.

The converse is true because each component of λ is in $N_i \cup M_i$ and this determine exactly $\gamma = \rho_n^r$ by a similar argument to that used before. This γ satisfies $H_\lambda[\sigma + \gamma] \neq 0$, that is $n_\lambda = 1$. □

5. $G = Spin(2n, 1)$

In this case the maximal compact subgroup K is $Spin(2n)$. Fix T a maximal torus in K with Cartan subalgebra h_0, and an ordered orthonormal base $\{H_1, \dots, H_n\}$ of the real Lie algebra ih_0. Let $\{e_1, \dots, e_n\}$ be the dual base to $\{H_1, \dots, H_n\}$, so

(5.1) $$e_j(H_j) = \delta_{ij}.$$

The root system $\Phi(h, g)$ lies in $(ih_0)'$, the real dual of ih_0. It is known that

$$\Phi_k = \{e_i \pm e_j \colon i \neq j, i, j = 1, \dots, n\}, \qquad \Phi_n = \{\pm e_i : i = 1, \dots, n\}.$$

Fix

(5.2) $$\Phi_k^+ = \{e_i \pm e_j \colon i < j\}.$$

Now we have two choices of Φ_n^+ such that $\Phi^+ = \Phi_k^+ \cup \Phi_n^+$ is a positive root system, these are

(5.3) $$\Psi^1 = \{e_1, \dots, e_n\}, \qquad \Psi^2 = \{e_1, \dots, e_{n-1}, -e_n\}.$$

With (5.1) in mind

(5.4) $$\rho_k = \sum_{i=1}^{n}(n-i)e_i, \qquad \rho_n^1 = \frac{1}{2}\sum_{i=1}^{n} e_i, \qquad \rho_n^2 = \frac{1}{2}\left(\sum_{i=1}^{n-1} e_i - e_n\right)$$

where ρ_n^i correspond to choice of Ψ^i as positive noncompact root system. Let $\lambda \in (ih_0)'$ be an integral weight, so $\lambda = \sum \lambda_i e_i$ with $\lambda_i \in \mathbb{Z}$ $\forall i = 1, \dots, n$ or $\lambda_i = \frac{1}{2}(2k_i + 1)$ with $k_i \in \mathbb{Z}$ $\forall i = 1, \dots, n$. Note that λ is Φ_k^+-dominant, is equivalent to

(5.5) $$0 \leq |\lambda_n| \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$$

because $(\lambda, e_i - e_j) = \lambda_i - \lambda_j \geq 0$ if $i < j$, and $(\lambda, e_i + e_j) = \lambda_i + \lambda_j \geq 0$. λ is Φ_n^+-dominant is equivalent to $\lambda_n = \operatorname{sgn} e_n |\lambda_n|$ having in mind the choice made in (5.3). Recall that λ is a Harish-Chandra parameter of a discrete series if λ is nonsingular and $\lambda + \rho$ is integral. Thus, when λ is Φ^+-dominant, this is equivalent to having strict inequalities at (5.4) and λ being integral (because ρ is integral). The restriction that λ is Φ^+-dominant is equivalent to be Φ_n^+-dominant. From now on, λ shall be Φ_k^+-dominant.

The next proposition gives a necessary and sufficient condition for when a K-type occurs in a discrete series of $Spin(2n, 1)$ of parameter λ. Denote by $m_\lambda(\tau)$ the multiplicity of the irreducible component of maximal weight τ in this discrete series.

Proposition 5.1. *Let* $\lambda = \sum_{i=1}^{n} \lambda_i e_i$ *be a* Φ^+*-dominant Harish-Chandra parameter (for either of the two choices of* Φ_n^+ *). Let* $\tau = \sum_{i=1}^{n} \tau_i e_i$ *be a* Φ_k^+*-dominant weight and set* $\mu = \lambda + \rho_n - \rho_k = \sum_{i=1}^{n} \mu_i e_i$ *. Then,*

$$m_\lambda(\tau) = 1 \quad \Leftrightarrow \quad \begin{cases} \tau_i - \mu_i \in \mathbb{Z}, \\ |\lambda_n| + \frac{1}{2} \leq |\tau_n| \leq \mu_{n-1} \leq \tau_{n-1} \leq \cdots \leq \mu_1 \leq \tau_1, \\ \operatorname{sgn} \lambda_n = \operatorname{sgn} \tau_n. \end{cases}$$

Proof. Fix $\Phi_n^+ = \Psi^1$, and let λ be Ψ^1-dominant, or equivalently $\lambda_n > 0$. Let $\tau' = \tau + \rho_k$ and $\mu' = \mu + \rho_k = \lambda + \rho_n$, then we have to prove

$$m_\lambda(\tau) = 1 \quad \text{if and only if} \quad \mu'_j \leq \tau'_j < \mu'_{j-1}, \qquad j = 1, \dots, n \ (\mu_0 = \infty).$$

In this case the Weyl group W_K of K is the set of maps

$$s\colon (e_1, \dots, e_n) \to (\pm e_{\pi(1)}, \dots, \pm e_{\pi(n)})$$

with an even number of minus signs where π is a permutation of a set of n elements; the determinant of s is the sign of π. The Blattner formula say that

$$m_\lambda(\tau) = \sum_{s \in W_K} \det s \; Q(s^{-1}\tau' - \mu')$$

where $Q(\sigma)$ is the number of expressions of σ as a sum of positive noncompact roots. If $s \in W_K$, one has that $Q_s = Q(s^{-1}\tau' - \mu') \neq 0$ if and only if $\pm\tau'_{\pi(k)} - \mu'_k$ is a nonnegative integer for all k. Since the number of minus sign is even, and μ'_n , $\tau'_j \geq 0$, except for τ'_n , then s cannot change signs, so $\tau'_n \geq 0$. Besides, since $\mu'_n \leq \mu'_j \ \ \forall j$, it follows that $\tau'_j \geq \mu'_n \ \ \forall j$ (otherwise $Q_s = 0 \ \ \forall s$). Suppose that $m_\lambda(\tau) \neq 0$, so $Q_s \neq 0$ for some s. Let H be the permutation subgroup which changes the elements τ'_j which are in the interval $[\mu'_k, \mu'_{k-1})$. Since the order of τ'_j in the interval is irrelevant, if $\pi \in H$ and $s_1 \in W_K$ corresponds to π, then $Q_{ss_1} = Q_s$.

$$m_\lambda(\tau) = \sum \det s \; Q_s = \sum \det s \; Q_{ss_1} = \sum \det s(s_1)^{-1} \; Q_s = \det(s_1)^{-1} m_\lambda(\tau).$$

But H always has a transposition, except when $H = \{1\}$, in which case there is only one τ'_j in each interval $[\mu'_k, \mu'_{k-1})$. This holds for $k = 1, \dots, n$ where $\mu_0 = \infty$. Since $\tau'_n \geq \mu'_n$ and the coefficients τ'_j are ordered, $m_\lambda(\tau) \neq 0$ only if the condition of the proposition holds.

Conversely if the condition of the proposition holds, $\tau'_{\pi(k)} - \mu'_k \geq 0$ if and only if $\pi = 1$, so $Q_1 = 1$ and $Q_s = 0$ if $s \neq 1$, that is $m_\lambda(\tau) = \det 1 \; Q_1 = 1$ (we know that in the case of $Spin(2n, 1)$ that $m_\lambda(\tau)$ is at the most 1).

Now consider $\lambda_n < 0$, or equivalently λ is Ψ^2-dominant. If we change the positive noncompact root set Ψ^1 to Ψ^2, then $\lambda = \sum_{i=1}^{n} \lambda_i e_i + (-\lambda_n)(-e_n)$ with $-\lambda_n > 0$, so the conditions are the same as in the first part of the proof. In this situation we must have

$$-\tau_n \geq |\lambda_n| + \tfrac{1}{2} > 0 \Rightarrow \tau_n < 0 \Rightarrow \operatorname{sgn} \lambda_n = \operatorname{sgn} \tau_n$$

and the proof is complete. □

We will use the last proposition in the case $\tau = \sigma + \gamma$ with σ a Φ_k^+-dominant weight and γ a weight of S, because that is what we need to obtain the set of elements of $\operatorname{Eig}(\mathbf{D}^2)$ (see Proposition 3.1(iv)). In this case

$$P(S) = \{\tfrac{1}{2}(\pm e_1 \pm \cdots \pm e_n)\}.$$

Let

$$\sigma = \sum \sigma_i e_i\,, \qquad \sigma_i \in \mathbb{Z} \quad \forall i\,, \quad \text{or} \quad 2\sigma_i \text{ is odd} \quad \forall i.$$

Thus,

$$\sigma + \gamma = \sum (\sigma_i + \varepsilon_i) e_i\,, \qquad \varepsilon_i = (\gamma\,, e_i) = \pm\frac{1}{2}.$$

Proposition 5.2. *Let $\lambda = \sum_{i=1}^n \lambda_i e_i$ be a Φ_k^+-dominant Harish-Chandra parameter, and let L_d^2 be the discrete part of $L^2(G/K\,, V_\sigma \otimes S)$ as in* (3.3), *and σ as in* (3.5). *Then,*

(i)

$$n_\lambda \neq 0 \Leftrightarrow \begin{cases} \sigma_i - \lambda_i \in \mathbb{Z} \quad \forall i\,, \\ \lambda_i \in [\sigma_{i+1} + n - i - 1\,, \sigma_i + n - i]\,, \qquad i < n\,, \\ |\lambda_n| \in (0\,, |\sigma_n|]\,, \\ \lambda \text{ and } \sigma \text{ are in the same Weyl chamber for } \Phi^+. \end{cases}$$

(ii) $n_\lambda \neq 0 \Rightarrow n_\lambda = 2^m\,,\ 0 \leq m \leq n\,.$

(iii) $n_\lambda = 1 \Leftrightarrow \lambda = \sigma + \rho_k\,.$

(iv) $\|\lambda\|^2 \leq \|\sigma + \rho_k\|$ *and* $\|\lambda\|^2 = \|\sigma + \rho_k\| \ \Leftrightarrow\ \lambda = \sigma + \rho_k\,.$

Remark. Using the notation of the Proposition 3.1, the equality $\mathrm{W}_0(\mathbf{D}^2) = \mathrm{W}_0(\mathbf{D}) = H_{\sigma+\rho_k}$ holds.

Proof. (i) Suppose that $n_\lambda \neq 0$, so $m_\lambda(\sigma + \gamma) \neq 0$ for some $\gamma \in P(S)$, so

$$\sigma_i + \varepsilon_i - \mu_i = \sigma_i + \varepsilon_i - (\lambda_i + \tfrac{1}{2}) \in \mathbb{Z} \quad \forall i \ \Leftrightarrow\ \sigma_i - \lambda_i \in \mathbb{Z} \quad \forall i\,,$$
$$\lambda_i \in [\sigma_{i+1} + \varepsilon_{i+1} + n - i - \tfrac{1}{2}\,, \sigma_i + \varepsilon_i + n - i - \tfrac{1}{2}] \quad \text{for } i < n\,,$$
$$|\lambda_n| \in (0\,, |\sigma_n + \varepsilon_n| - \tfrac{1}{2}]\,,$$
$$\operatorname{sgn} \lambda_n = \operatorname{sgn}(\sigma_n + \varepsilon_n) = \operatorname{sgn} \sigma_n$$

by the last proposition and (5.4). Note that $|\lambda_n| + \frac{1}{2} \leq |\sigma_n + \varepsilon_n|$, λ integral and nonsingular, ensures that $\operatorname{sgn}(\sigma_n + \varepsilon_n) = \operatorname{sgn} \sigma_n$.

Conversely, we want to find $\gamma \in P(S)$ such that $m_\lambda(\sigma + \gamma) \neq 0$. Denote

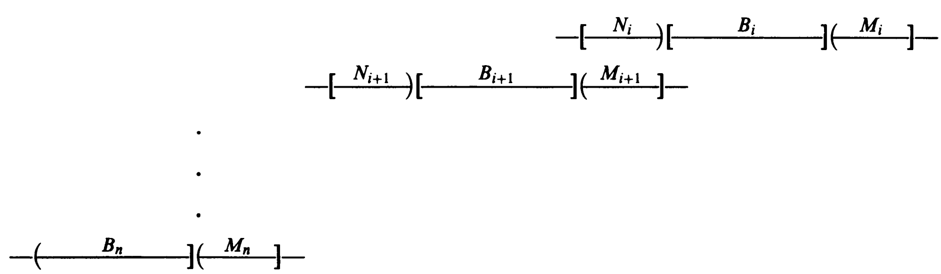
for $i < n$

$$N_i = [\sigma_{i+1} + n - i - 1\,, \sigma_{i+1} + n - i)\,,$$
$$B_i = [\sigma_{i+1} + n - i\,, \sigma_i + n - i - 1]\,,$$
$$M_i = (\sigma_i + n - i - 1\,, \sigma_i + n - i]\,;$$

for $i = n$

$$N_n = \varnothing\,,$$
$$B_n = (0\,, |\sigma_n| - 1]\,,$$
$$M_n = (|\sigma_n| - 1\,, |\sigma_n|].$$

This is the situation graphically:



If $\lambda_i \in N_i$, this fixes the value of $\varepsilon_{i+1}(\gamma) = -\frac{1}{2}$ for γ's such that $H_\lambda[\sigma+\gamma] \neq 0$. Similarly, $\lambda_{i+1} \in M_{i+1}$ ensures $H_\lambda[\sigma+\gamma] = 0$ for $\varepsilon_{i+1}(\gamma) = \frac{1}{2}$. But both cannot occur simultaneously, because N_i and M_{i+1} have both length one and equal extremes, and $\lambda_{i+1} - \lambda_i \in \mathbb{Z}$, that is that only one of the cases determines the value of $\varepsilon_{i+1}(\gamma)$. So there is a γ such that $m_\lambda(\sigma+\gamma) \neq 0$.

(ii) Suppose that $\lambda_{i_j} \notin B_{i_j}$, $j = 1, \dots, m$, and $\lambda_k \in B_k$ for $k \neq i_j$. Then $\lambda_{i_j} \in N_{i_j} \cup M_{i_j}$, this determines exactly m component values of the γ's for which $m_\lambda(\sigma+\gamma) \neq 0$. So there exist 2^{n-m} weights γ such that $m_\lambda(\sigma+\gamma) \neq 0$.

(iii) $n_\lambda = 1$ is equivalent to the existence of a unique $\gamma \in P(S)$ such that $m_\lambda(\sigma+\gamma) \neq 0$, so that the components of λ determine every component of γ, or equivalently $\lambda_i \in N_i \cup M_i \ \ \forall i$. Now note that $N_n = \varnothing$ and this ensures that $\lambda_n \in M_n$. But two consecutive components of λ cannot be in the same interval (M_i and N_{i-1} have the same extremes), so $\lambda_{n-1} \in M_{n-1}$. Repeating the same argument we obtain that $\lambda_i \in M_i \ \forall i$. Then, as $\lambda_i - \sigma_i \in \mathbb{Z}$, $\lambda = \sigma + \rho_k$.

(iv) By (i) $|\lambda_i| \leq |(\sigma+\rho_k)_i| \ \forall i$, so

$$\|\lambda\|^2 = \sum \lambda_i^2 \leq \sum (\sigma+\rho_k)_i^2 = \|\sigma+\rho_k\|^2$$

and the equality holds if and only if $\lambda = \sigma + \rho_k$. □

6. $G = Sp(2, \mathbb{R})$

In the cases $G = SU(n,1)$ and $G = Spin(2n,1)$ we proved that the multiplicity n_λ of the discrete series H_λ of parameter λ which occurs in $L^2(G/K, V_\sigma \otimes S)$ is a power of 2 with exponent less than or equal n. For the $G = Sp(2, \mathbb{R})$ we will show that there exist parameters λ's such that n_λ is nonzero and is not a power of 2. By (3.7) we know that

$$n_\lambda = \sum_{\gamma \in P(S)} \dim \operatorname{Hom}_K (H_\lambda, V_{\sigma+\gamma}).$$

We will give some examples where the number of elements $\gamma \in P(S)$ such that $H_\lambda[\sigma+\gamma] \neq 0$ is not a power of 2.

Let $G = Sp(2, \mathbb{R})$. The Lie algebra is

$$g_0 = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & {}^t\!-X_1 \end{pmatrix} : \quad X_1, X_2, X_3 \in \mathbb{R}^{2\times 2}, \ X_2, X_3 \text{ symmetric} \right\}.$$

Let $g_0 = k_0 + p_0$ be the Cartan decomposition of g_0, where

$$k_0 = \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} : \quad X_1 = -{}^tX_1 , \ X_2 = {}^t X_2 \right\},$$

$$p_0 = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} : \quad X_1 = {}^t X_1 , \ X_2 = {}^t X_2 \right\}.$$

There is an algebra isomorphism $k_0 = g_0 \cap u(4) \cong u(2)$ given by

$$k_0 \to u(2), \quad \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \to X_1 + iX_2.$$

A Cartan subalgebra of k_0 and g_0 is

$$h_0 = \mathbb{R}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

where the first summand is the center z_0 of k_0. Let g, k, p, h, z be the complexifications of g_0, k_0, p_0, h_0, z_0 respectively. The root system of (g, h) is

$$\Phi(h, g) = \{\pm e_1 \pm e_2\} \cup \{\pm 2e_1, \pm 2e_2\} \tag{6.1}$$

where

$$e_j \begin{pmatrix} 0 & 0 & ih_1 & 0 \\ 0 & 0 & 0 & ih_2 \\ -ih_1 & 0 & 0 & 0 \\ 0 & -ih_2 & 0 & 0 \end{pmatrix} = h_j, \qquad j = 1, 2.$$

Let

$$\Phi_k = \{\pm(e_1 - e_2)\}, \qquad \Phi_n = \{\pm(e_1 + e_2), \pm 2e_1, \pm 2e_2\}$$

and fix

$$\Phi_k^+ = \{e_1 - e_2\}, \qquad \Phi_n^+ = \{e_1 + e_2, 2e_1, 2e_2\}, \qquad \Phi^+ = \Phi_k^+ \cup \Phi_n^+. \tag{6.2}$$

Let E_α be the root vectors such that $B(E_\alpha, E_{-\alpha}) = 2\|\alpha\|^2$, where B is the Killing form. Define $H_\alpha = [E_\alpha, E_{-\alpha}]$, so H_α satisfies $\alpha(H_\alpha) = 2$. Thus

$$h = z \oplus \mathbb{C}H_{e_1 - e_2} = \mathbb{C}H_{e_1 + e_2} \oplus \mathbb{C}H_{e_1 - e_2}.$$

Let $(ih_0)'$ be the dual space of ih_0; if $\mu \in (ih_0)'$, then

$$\mu = \mu_1(e_1 + e_2) + \mu_2(e_1 - e_2).$$

Denote

$$p^+ = \sum_{\alpha \in \Phi_n^+} g_\alpha, \qquad p^- = \sum_{\alpha \in \Phi_n^+} g_{-\alpha}.$$

It is known that if λ is Φ^+-dominant with Φ^+ as in (6.2), H_λ is a holomorphic discrete series of $Sp(2, \mathbb{R})$. Then (see [S]) the restriction of the representation to K of the K-finite elements of H_λ is equivalent to the representation $S(p^+) \otimes V_\Lambda$, where $S(p^+)$ is the symmetric algebra of p^+ and $\Lambda = \lambda + \rho_n - \rho_k$. To obtain the irreducible representations of K that occur at $S(p^+)$ we will need the fact that $S(p^+)$ is the dual of $S(p^-)$ and the result of [S]. Select the maximal ordered subset $\Delta = \{\alpha_1, \dots, \alpha_r\}$ of p^- selected such that α_1 is the small root of p^-, and if $\alpha_1, \dots, \alpha_s$ has been chosen, α_{s+1} is the small root of p^- strongly orthogonal to $\alpha_1, \dots, \alpha_s$ $(\alpha_{s+1} \pm \alpha_i \notin \Phi,\ i = 1, \dots, s)$. Then, the results of [S] says any irreducible representation of K which occurs in $S(p^+)$ has multiplicity one and its maximal weight is $k_1\gamma_1 + \cdots + k_r\gamma_r$; $k_i \in \mathbb{Z}_{\geq 0}$; $\gamma_i = -\alpha_1 - \cdots - \alpha_i$. Moreover, this representation occurs in polynomials of degree at most $k_1 + 2k_2 + \cdots + rk_r$. In our case $\Delta = \{-2e_1, -2e_2\}$, so

$$\gamma_1 = 2e_1, \qquad \gamma_2 = 2e_1 + 2e_2$$

and the highest weight of the irreducible representations of $S(p^+)$ is

$$\begin{aligned} \mu &= k_1 2e_1 + k_2(2e_1 + 2e_2) \\ &= (k_1 + 2k_2)(e_1 + e_2) + k_1(e_1 - e_2), \qquad k_i \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

Therefore,

$$S(p^+) = \bigoplus_{k_1, k_2 \geq 0} \mathbb{C}_{(k_1+2k_2)(e_1+e_2)} \otimes V'_{k_1(e_1-e_2)}$$

where $V'_{k_1(e_1-e_2)}$ is an $SU(2)$-module of maximal weight $k_1(e_1 - e_2)$, and $\mathbb{C}_{(k_1+2k_2)(e_1+e_2)}$ is the one-dimensional representation of the center of $U(2)$ given by $\det(\ )^{k_1+2k_2}$. The $U(2)$-module V_Λ is equivalent to $\mathbf{C}_{a(e_1+e_2)} \otimes V'_{b(e_1-e_2)}$ if $\Lambda = a(e_1+e_2) + b(e_1 - e_2)$, so using the Clebsh-Gordon formula for the tensor product of two $SU(2)$-modules,

$$\begin{aligned} S(p^+) \otimes V_\Lambda &= \bigoplus_{k_1, k_2 \geq 0} \left(\mathbb{C}_{(k_1+2k_2)(e_1+e_2)} V'_{k_1(e_1-e_2)} \otimes \mathbb{C}_{a(e_1+e_2)} V'_{b(e_1-e_2)}\right) \\ &= \bigoplus_{k_1, k_2 \geq 0} \mathbb{C}_{(k_1+2k_2+a)(e_1+e_2)} \left(V'_{k_1(e_1-e_2)} \otimes V'_{b(e_1-e_2)}\right) \\ &= \bigoplus_{k_1, k_2 \geq 0} \left(\bigoplus_{t=0}^{\min(2k_1, 2b)} \mathbb{C}_{(k_1+2k_2+a)(e_1+e_2)} V'_{(k_1+b-t)(e_1-e_2)} \right). \end{aligned}$$

If the discrete series H_λ occurs in $L^2(G/K, V_\sigma \otimes S)$ where V_σ is the irreducible representation of K of maximal weight $\sigma = \sigma_1 e_1 + \sigma_2 e_2$, where σ is sufficiently far from the walls as in (3.5); then the K-type $H_\lambda[\sigma + \gamma]$ is nonzero for some $\gamma \in P(S)$.

Denote the noncompact roots by

$$\begin{aligned} \alpha_1 &= 2e_1 = (e_1 + e_2) + (e_1 - e_2), \\ \alpha_2 &= 2e_2 = (e_1 + e_2) - (e_1 - e_2), \\ \alpha_3 &= e_1 + e_2. \end{aligned}$$

Then $P(S) = \{\rho_n - \sum m_i \alpha_i \colon m_i = 0\ ,\ 1\}$.

We will give one example of a parameter λ such that n_λ is not a power of 2. In the cases of $Spin(2n, 1)$ and $SU(2n, 1)$ it happens that

$$n_\lambda = |\{\gamma \in P(S) \colon H_\lambda[\sigma + \gamma] \neq 0\}|$$

but for $Sp(2, \mathbb{R})$ this is not true.

Take $\lambda = \sigma + \rho_k - \alpha_1 - \alpha_2$ with σ chosen so that λ is Φ^+-dominant.

The highest weight of the minimal K-type of H_λ is

$$\Lambda = \lambda + \rho_n - \rho_k = \sigma + \rho_n - \alpha_1 - \alpha_2.$$

Since $\rho_n - \alpha_1 - \alpha_2 \in P(S)$, H_λ occurs in $L^2(G/K, V_\sigma \otimes S)$. The multiplicity of each K-type is equal to the number of expressions of its maximal weight in the form

$$(k_1 + 2k_2 + a)(e_1 + e_2) + (k_1 + b - t)(e_1 - e_2)$$

with $k_i \geq 0$ and $0 \leq t \leq \min(2k_1, 2b)$. Since σ is nonsingular and Φ^+-dominant, $b = \sigma_1 - \sigma_2 > 0$. To obtain n_λ we need the multiplicity of each K-type $\sigma + \gamma$ in H_λ with $\gamma \in P(S)$.

$$\begin{gathered} \sigma + \rho_n - \alpha_1 - \alpha_2 = a(e_1 + e_2) + b(e_1 - e_2), \\ k_1 = 0, \qquad k_2 = 0, \qquad t = 0, \\ \text{multiplicity} = 1, \end{gathered}$$

$$\sigma + \rho_n = (2+a)(e_1+e_2) + b(e_1-e_2)\,,$$
$$k_1 = 0\,, \qquad k_2 = 1\,, \qquad t = 0\,,$$
$$k_1 = 2\,, \qquad k_2 = 0\,, \qquad t = 2\,,$$
$$\text{multiplicity} = 2\,,$$
$$\sigma + \rho_n - \alpha_1 = (1+a)(e_1+e_2) + (-1+b)(e_1-e_2)\,,$$
$$k_1 = 1\,, \qquad k_2 = 0\,, \qquad t = 2\,,$$
$$\text{multiplicity} = 1\,,$$
$$\sigma + \rho_n - \alpha_2 = (1+a)(e_1+e_2) + (1+b)(e_1-e_2)\,,$$
$$k_1 = 1\,, \qquad k_2 = 0\,, \qquad t = 0\,,$$
$$\text{multiplicity} = 1\,,$$
$$\sigma + \rho_n - \alpha_3 = (1+a)(e_1+e_2) + b(e_1-e_2)\,,$$
$$k_1 = 1\,, \qquad k_2 = 0\,, \qquad t = 1\,,$$
$$\text{multiplicity} = 1\,,$$
$$\sigma + \rho_n - \alpha_2 - \alpha_3 = a(e_1+e_2) + (1+b)(e_1-e_2)\,,$$
$$\text{multiplicity} = 0\,,$$
$$\sigma + \rho_n - \alpha_1 - \alpha_3 = a(e_1+e_2) + (-1+b)(e_1-e_2)\,,$$
$$\text{multiplicity} = 0\,,$$

$$\sigma + \rho_n - 2\rho_n = (-1+a)(e_1+e_2) + b(e_1-e_2)\,,$$
$$\text{multiplicity} = 0\,,$$

Then $n_\lambda = 6 \neq 2^m$ and $|\{\gamma \in P(S) \colon H_\lambda[\sigma+\gamma] \neq 0\}| = 5 \neq 2^m$.

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