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NONLINEAR QUANTUM FIELDS IN ≥ 4 DIMENSIONS AND COHOMOLOGY OF THE INFINITE HEISENBERG GROUP

J. PEDERSEN, I. E. SEGAL, AND Z. ZHOU

ABSTRACT. Aspects of the cohomology of the infinite-dimensional Heisenberg group as represented on the free boson field over a given Hilbert space are treated. The 1-cohomology is shown to be trivial in certain spaces of generalized vectors. From this derives a canonical quantization mapping from classical (unquantized) forms to generalized operators on the boson field. An example, applied here to scalar relativistic fields, is the quantization of a given classical interaction Lagrangian or Hamiltonian, i.e., the establishment and characterization of corresponding boson field operators. For example, if ϕ denotes the free massless scalar field in d-dimensional Minkowski space ($d \geq 4$, even) and if q is an even integer greater than or equal to 4, then $\int_{\mathbf{M}_0} : \phi(X)^q : dX$ exists as a nonvanishing, Poincaré invariant, hermitian, selfadjointly extendable operator, where $: \phi(X)^q :$ denotes the Wick power. Applications are also made to the rigorous establishment of basic symbolic operators in heuristic quantum field theory, including certain massive field theories; to a class of pseudo-interacting fields obtained by substituting the free field into desingularized expressions for the total Hamiltonian in the conformally invariant case $d = q = 4$ and to corresponding scattering theory.

1. INTRODUCTION

The quantum action and interaction energy integrals of heuristic quantum field theory, as well as the perturbative S-matrix coefficients, are symbolic operators that have never been established in a clear-cut or rigorous sense in the physical case of four-dimensional space-time. There is nevertheless a certain formal and quasi-rigorous property that these symbolic operators share. Namely, they are all formally of finite "order" in the sense that their successive commutators with the quantized field vanish after a finite number of bracket operations. The last nonvanishing commutator is a (classical, i.e., unquantized) multilinear form on a dense domain in the underlying space of classical fields (or "single-particle" space) and mathematically well-defined modulo technical details, such as the specification of the domains, or of associated test function spaces. Moreover, this classical form represents a type of cocycle for the infinite Heisenberg group, as represented by the boson field operators on the quantized field for the underlying noninteracting (or "free") context.

By exploiting this connection between symbolic operators and Heisenberg group cocycles, rigorous versions of these operators can be established and

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exploited, for certain classes of quantum fields. The method described here applies to general types of fields, by virtue of a conformal mapping method, which is described below. However, in order to provide explicit examples and avoid undue abstraction in the presentation of the basic theory, we give detailed treatments for certain scalar fields in Minkowski space. The basic cohomology result is a kind of generalized Poincaré lemma that establishes a unique quantization map from classical to quantized expressions having cogent regularity and covariance properties.

Global cohomology of the finite-dimensional Heisenberg group was treated in [15], in a small 'hard' space (that of all Hilbert-Schmidt operators). For applications to the singular operators of heuristic field theory, a large 'soft' space is more useful; and when a suitable such space is designated, the infinitesimal cohomology becomes tractable. We treat both global and infinitesimal cohomology, but only the latter is required for the application to relativistic fields that are made here.

Regularity properties of the quantum action for conformally covariant quantum fields were first treated in [8]. Physical aspects of this and some aspects of the present work are treated in [10].

2. Technical preliminaries

We use the notation and terminology of [9]. For convenience, we summarize the basic features as follows. $\mathbf{H}$ will denote a complex Hilbert space and $(\mathbf{K}, W, \Gamma, v)$ the free boson field over $\mathbf{H}$. Thus, $\mathbf{K}$ is a complex Hilbert space; W is a map from $\mathbf{H}$ to unitary operators on $\mathbf{K}$ satisfying the Weyl relations; Γ is a unitary representation on $\mathbf{K}$ of the unitary group $U(\mathbf{H})$ on $\mathbf{H}$, which intertwines appropriately with W; and v is a unit vector in $\mathbf{K}$ that is invariant under all $\Gamma(U)$ and is cyclic for the $\{W(z) : z \in \mathbf{H}\}$. Infinite-dimensional integrals will be with respect to Gaussian measure ν of variance parameter $\frac{1}{2}$ on a complex Hilbert space, the integrands being entire or anti-entire functions, and are defined as the limit of such integrals in the usual Lebesgue sense over finite-dimensional subspaces, as the subspaces tend to the full space. A domain $\mathbf{D}$ in a Hilbert space that is given an intrinsic topology (stronger than that in the ambient space) will be denoted as $[\mathbf{D}]$. The continuous sesquilinear forms on $[\mathbf{D}]$ will be denoted as $\mathscr{F}[\mathbf{D}]$. We use especially the domains of differentiable, analytic, and entire vectors for the basic Hamiltonians, denoted as B in the underlying single-particle space $\mathbf{H}$ and as H in the field space $\mathbf{K}$. For any selfadjoint operator A in $\mathbf{H}$, the selfadjoint generator of the one-parameter unitary group $\Gamma(e^{itA})$ will be denoted as $\partial\Gamma(A)$, e.g., $H = \partial\Gamma(B)$. In the basic theory, e^{-tB} will be a trace class operator for all $t > 0$ (implying the same for e^{-tH}); relativistic Hamiltonians will be treated by reduction to this case (exemplified for example by the generator of temporal displacement in the Einstein Universe in conformally covariant theories).

The term *form* will mean continuous sesquilinear form on the specified domain $[\mathbf{D}]$. A form F on the space $\mathbf{E}(H)$ of all entire vectors for H can be represented by a kernel $K(z, z')$ satisfying the inequality

$$|K(u, u')| \leq ce^{\|e^{tB}u\|^2+\|e^{tB}u'\|^2}$$

for some positive constants c and t, where z, z' are in $\mathbf{E}(B)$, that is an anti-entire function of z and an entire function of z', relative to the complex wave representation of the boson field over $\mathbf{H}$, in which $\mathbf{K}$ is represented as the space $\mathbf{H}^{-}L_2(\mathbf{H})$ of all square-integrable anti-entire functions on $\mathbf{H}$ (see [14]). Notationally, forms may be expressed as operators so that the form F evaluated on the pair (f, g) is expressible as

$$\langle Ff, g\rangle = \int_{\mathbf{H}\times\mathbf{H}} K(z, z')f(z')\overline{g(z)}\,d\nu(z)\,d\nu(z'),$$

in terms of the kernel for F. Here and elsewhere as indicated by context, vectors f and g in $\mathbf{K}$ are identified with anti-entire functions on $\mathbf{H}$. The standard kernel for the identity operator I is $e^{\langle z', z\rangle}$; we use also the *reduced kernel*, defined as $e^{-\langle z', z\rangle}K(z, z')$, where $K(z, z')$ is the standard kernel. Thus, the reduced kernel for I is 1.

Further notation is similar (e.g., $\mathbf{H}^{+}L_2(\mathbf{H})$ for the space of square-integrable entire vectors on $\mathbf{H}$ and $\mathbf{A}(B)$ for the topological domain of all analytic vectors for the selfadjoint operator B). We refer to loc. cit. for details.

3. Infinitesimal cohomology

Let $\mathbf{D}$ be a dense linear subset of $\mathbf{H}$, having a given topology such that the injection map is continuous; and let this topological space be denoted as $[\mathbf{D}]$ when this is necessary to avoid confusion. Let $\mathbf{D}'$ be similarly a dense linear subset of $\mathbf{K}$. Two main cases of this configuration will be involved below, called the *entire* and *analytic* cases. In the entire case, $\mathbf{D}$ is the space of entire vectors for B and $\mathbf{D}'$ is the space of entire vectors for H. The analytic case is the same with the substitution of analytic for entire.

Definition 3.1. An (infinitesimal) n-cocycle is a continuous multilinear map F from $[\mathbf{D}]^n$ to the space $\mathscr{F}[\mathbf{D}']$ of all continuous sesquilinear forms on $[\mathbf{D}']$ satisfying the following conditions:

(i) $F(z_1, \dots, z_n)$ is a symmetric function of the vectors $z_1, \dots, z_n$.

(ii) For arbitrary $z_1, \dots, z_n$ and z' in $\mathbf{D}$,

$$[F(z_1, \dots, z_n), \phi(z')] = [F(z', \dots, z_n), \phi(z_1)].$$

Example. Let F be arbitrary in $\mathscr{F}[\mathbf{D}]$, and set

$$F(z_1, \dots, z_n) = \partial_{z_1}\cdots\partial_{z_n}F,$$

where ∂_z denotes the operator $F \to [F, \phi(z)]$; we assume that ∂_z is a continuous linear operator on $\mathbf{F}(\mathbf{D}')$. Then $F(z_1, \dots, z_n)$ is an *exact* n-cocycle and will be called the *n-derivative* of F.

Remark. As indicated above, the requisite continuity on $F(z_1, \dots, z_n)$ is joint, but in the entire and analytic cases this is implied by separate continuity, by virtue of the fact that the spaces $\mathbf{E}(S)$ and $\mathbf{A}(S)$ are metrizable and barreled for an arbitrary selfadjoint operator S.

Theorem 3.1. *In the entire and analytic cases, every infinitesimal n-cocycle is exact.*

Proof. The following proof for the entire case applies also to the analytic case, with technical changes similar to those indicated in [9].

We note first that for any real-linear function R from $\mathbf{E}(B)$ to a complex vector space V, there exist unique complex-linear and antilinear functions $R^{\pm}$

from $\mathbf{E}(B)$ to V such that $R(z) = R^{+}(z) + R^{-}(z)$. We define $R^{\#}$ as the function on $\mathbf{E}(B) \times \mathbf{E}(B)$ given by the equation

$$R^{\#}(z, z') = R^{+}(z) + R^{-}(z').$$

Lemma 3.1.1. *Let $F(u, u')$ be a function on $\mathbf{H} \oplus \mathbf{H}$ that is analytic as a function of u' and antianalytic as a function of u. If $F(u, u) = 0$ and $F(\cdot, \cdot) \in L_2(\mathbf{H} \oplus \mathbf{H}, d\nu)$, then $F(\cdot, \cdot) \equiv 0$.*

Proof. If $\mathbf{H}$ is finite dimensional, the power series expansion that $F(u, u')$ enjoys in the components of u' and the complex conjugate of the components of u restricts to a power series on $\mathbf{H}$ on setting $u = u'$. The unicity of the coefficients of such a power series then implies that if $F(u, u) = 0$ identically in u, then $f(u, u') = 0$ identically in u and u'. If $\mathbf{H}$ is infinite dimensional, then for any finite-dimensional subspace $\mathbf{M}$, the restriction of F to $\mathbf{M} \oplus \mathbf{M}$ vanishes by the case just considered and hence, vanishes on all of $\mathbf{H} \oplus \mathbf{H}$ (e.g., as the L_2 limit of the corresponding function on $\mathbf{M} \oplus \mathbf{M}$, obtained by setting components orthogonal to $\mathbf{M}$ equal to 0).

Lemma 3.1.2. *Let $G(u, u')$ be the reduced kernel of the form F. Let z be arbitrary in $\mathbf{E}(B)$. Then the reduced kernel for $\partial_z F$ is*

$$F(z, u, u') = -\frac{i}{\sqrt{2}}[d_{u'}G(u, u')z + d_u G(u, u')z].$$

Proof. Let f and g be arbitrary antianalytic functions in $\mathbf{E}(H)$. Then by direct calculation,

$$\langle [F, C(z)]f, g\rangle = \iint P(z, u, u')f(u')\overline{g(u)}\, d\nu(u)\, d\nu(u'),$$

where

$$iP(z, u, u') = -\langle z, u\rangle e^{\langle u', u\rangle} G(u, u') + e^{\langle u', u\rangle} G(u, u')\langle z, v'\rangle.$$

The expression given for $P(z, u, u')$ is not the standard kernel for $[F, C(z)]$, since the factor $\langle z, u'\rangle$ in the second term is antianalytic in u'. To obtain the standard kernel, recall that $(C(z)f)(u') = -i\langle z, u'\rangle f(u')$ and $(C(z)^{*}h)(u') = i(h)z$, whence

$$\begin{aligned} i\int & P(z, u, u')f(u')\, d\nu(u') \\ &= \int [-e^{\langle u', u\rangle} G(u, u')\langle z, u\rangle + d_{u'}[e^{\langle u', u\rangle} G(u, u')]z]f(u')\, d\nu(u') \\ &= \int e^{\langle u', u\rangle}[d_{u'}G(u, u')z]f(u')\, dg(u') \end{aligned}$$

from which the standard kernel for $[F, C(z)]$ can be read off.

Similarly,

$$\langle [F, C^{*}(z)]f, g\rangle = \iint Q(z, u, u')f(u')\overline{g(u)}\, d\nu(u)\, d\nu(u'),$$

where $Q(z, u, u') = e^{\langle u', u\rangle} d_u G(u, u')z$. Lemma 3.1.2 now follows from the equation $\phi(z) = \frac{1}{\sqrt{2}}[C(z) + C(z)^{*}]$ and the unicity of the standard kernel. □

Lemma 3.1.3. *Let* $R(z)$ *be a* 1*-cocycle, and let the reduced kernel for* $R(z)$ *be* $R(z, u, u')$. *Then*

$$d_{u'}R^{\#}(w, w', u, u')z' + d_uR^{\#}(w, w', u, u')z = d_{u'}R^{\#}(z, z', u, u')w' + d_uR^{\#}(z, z', u, u')w. \tag{3.1}$$

Proof. It follows from Lemma 3.1.2 and $[R(w), \phi(z)] = [R(z), \phi(w)]$ that

$$d_{u'}R(z, u, u')w + d_uR(z, u, u')w = d_{u'}R(w, u, u')z + d_uR(w, u, u')z,$$

which is equivalent to, by the definition of $R^{\#}$,

$$d_{u'}R^{\#}(w, w, u, u')z + d_uR^{\#}(w, w, u, u')z = d_{u'}R^{\#}(z, z, u, u')w + d_uR^{\#}(z, z, u, u')w.$$

The last equality is to the effect that equation (3.1) is valid on the diagonal $w = w'$, $z = z'$. The equation now follows everywhere off the diagonal from Lemma 3.1.1 on the unicity for a function on $\mathbf{H}\oplus\mathbf{H}$ that is respectively analytic and antianalytic in the two variables.

Lemma 3.1.4. *The special case of Theorem* 3.1 *in which* $n = 1$ *is valid.*

Proof. Let $R(z)$ be a given 1-cocycle and $R(z, u, u')$ be the reduced kernel for $R(z)$. Let

$$G(u, u') = i\sqrt{2}\int_0^1 R^{\#}(+u, u', su, su')\,ds, \tag{3.2}$$

and let G denote the form whose reduced kernel is $G(u, u')$. It will be shown that $R(z) = \partial_z G$. To this end it suffices to show that $G(u, u')$ satisfies the condition given in Lemma 3.1.2. By Lemma 3.1.3 and the linearity/antilinearity of $R^{\#}(z, z', u, u')$ as a function of z resp. z', it follows (using d_j to denote the partial differential with respect to the jth variable, e.g., $d_3R^{\#}(z, z', u, u')w = d/dt|_{t=0}R^{\#}(z, z', u + tw, u')$) that

$$\begin{aligned}
&= -\frac{1}{\sqrt{2}}[d_{u'}G(u, u')z + d_uG(u, u')z] \\
&= \int_0^1 [R^{\#}(0, z, su, su') + d_4R^{\#}(+u, u', su, su')(sz) \\
&\qquad + R^{\#}(z, 0, su, su') + d_3R^{\#}(+u, u', su, su')(sz)]\,ds \\
&= \int_0^1 [R^{\#}(z, z, su, su') + d_4R^{\#}(sz, sz, su, su')u' \\
&\qquad\qquad + d_3R^{\#}(sz, sz, su, su')u]\,ds \\
&= \int_0^1 \frac{d}{ds}R^{\#}(sz, sz, su, su')\,ds \\
&= R^{\#}(z, z, u, u') = R(z, u, u'). \quad \square
\end{aligned}$$

Completion of Proof of Theorem 3.1. Let ∂^{-1} denote the map from 1-cocycles to forms given by Lemma 3.1.4. More specifically, we write $\partial_z^{-1}R(z) = F$ where F is the form given by equation (3.2). If now $R(z_1, \dots, z_n)$ is an n-cocycle, then $\partial_{z_n}^{-1}R(z_1, \dots, z_{n-1}, z_n)$ is easily seen to be a cocycle as a function of

$z_1, \ldots, z_{n-1}$. It follows that $\partial_{z_1}^{-1} \cdots \partial_{z_n}^{-1} R(z_1, \ldots, z_n)$ is a form whose n-derivative is $R(z_1, \ldots, z_n)$. □

(Formula (3.2) can be extended to all n.)

The form whose derivative is a given cocycle is not at all unique, since in the case of a 1-cocycle, for example, the cocycle is unchanged if an arbitrary constant scalar multiple of I is added to the form.

Corollary 3.1.1. *Two forms whose derivatives are equal differ only by a constant.*

Proof. Corollary 3.1.1 is equivalent to the claim that a form F that commutes with all $\phi(z)$, $z \in \mathbf{E}(B)$, is a multiple of identity. This follows from the irreducibility of the $\phi(z)$, $z \in \mathbf{E}(B)$ [9]. □

We call ∂^{-1} in the specific form given by equation (3.2) the *quantization map* and denote it as $\mathscr{Q}$. We extend $\mathscr{Q}$ by linearity to finite sums of forms of varying degree.

Corollary 3.1.2. $\mathscr{Q}$ *is continuous from n-cocycles to forms, in the* $\mathbf{E}(B)$ *and* $\mathbf{E}(H)$ *topologies (or in corresponding analytic topologies).*

Proof. This follows from straightforward estimation using successively the integral formula for $\mathscr{Q}$ given above. □

Example. Consider the quantization of the conformal wave equation

$$(\partial_t^2 - \Delta + 1)\phi = 0$$

on $\mathbb{R}^1 \times S^3$, where Δ denotes the Laplacian on S^3. The Hilbert space consists of solutions of the equation of finite conformally invariant norm. All such solutions are periodic with period 2π in the time variable and may accordingly be identified with functions on $S^1 \times S^3$, which is conformally equivalent to the 2-fold cover $\overline{\overline{\mathbf{M}}}$ of the conformal compactification of Minkowski space $\mathbf{M}_0$; cf. [7]. Let B denote the Einstein energy (i.e., selfadjoint operator generating time evolution) in $\mathbf{H}$, and for arbitrary $\phi_1, \ldots, \phi_n$ in $\mathbf{E}(B)$ let

$$F(\phi_1, \ldots, \phi_n) = \int_{\overline{\overline{\mathbf{M}}}} \phi_1(u) \cdots \phi_n(u)\, du.$$

This is continuous on $\mathbf{E}(B)$ by a simple estimate that follows from [7]. Accordingly, F is an n-cocycle. Its antiderivative F on $\mathbf{E}(H)$, where H is the Einstein energy in the quantized field $\mathbf{K}$, is the integrated Wick product

$$\int_{\overline{\overline{\mathbf{M}}}} : \phi^n(u) : du.$$

It is straightforward to extend the analysis to arbitrary even space-time dimensions $(n \geq 4)$. More generally, the analysis may be extended to show the existence of forms corresponding to the m-cocycles

$$\int_{\overline{\overline{\mathbf{M}}}^m} : D_1\phi(u_1) \cdots D_n\phi(u_m) : T(u_1, \ldots, u_m)\, du_1 \cdots du_m,$$

where the D_j are differential operators on $\overline{\overline{\mathbf{M}}}$ that commute with the isometry group K of $S^1 \times S^{n-1}$ and $T(u_1, \ldots, u_m)$ is an arbitrary distribution on $\overline{\overline{\mathbf{M}}}^m$. Theorem 3.1 essentially establishes Wick product theory on the domains of entire and analytic vectors of the field Hamiltonian, from which extension to the domain of differentiable vectors may be made.

In the case $n = 4$, conformal covariance and other considerations detailed below show the corresponding existence and essential selfadjointness of the antiderivative of the 4-cocycle

$$\int_{\mathbf{M}_0} \phi_1(x) \cdots \phi_4(x)\, dx$$

on $\mathbf{E}(B_0)$, where the ϕ's are solutions of the wave equation on $\mathbf{M}_0$ and B_0 is an operator unitarily equivalent to B. This result is interpreted physically as the existence and selfadjointness of the quantized $\int_{\mathbf{M}_0} :\phi(x)^4:\, d^4x$ for the so-called massless ϕ_4^4 theory, ϕ representing the quantized free field for the wave equation.

4. Regularity, selfadjointness, and covariance

Temporal smoothing enhances regularity in many contexts, and we here require its manifestation in the entire and analytic cases.

Theorem 4.1. *Let A be a selfadjoint operator in the Hilbert space $\mathbf{H}$ that is bounded below, and let F be a form on $\mathbf{E}(A)$ (resp. $\mathbf{A}(A)$). Let f be in $L_2(\mathbb{R})$ and such that $|\hat{f}(x)| < Ce^{-\kappa|x|}$ for all real κ (resp. $|\hat{f}(x)| < Ce^{-\kappa|x|}$ for some $\kappa > 0$). Then the form*

$$G(u, u') = \int F(\exp(itA)u, \exp(itA)u')f(t)\, dt$$

is a continuous operator from $\mathbf{E}(A)$ to itself (resp. $\mathbf{A}(A)$ to itself).

Proof. The proof for the analytic case is parallel to that for the entire case, and we give details only for the latter. To say that F is a form means that there exist a bounded linear operator B on $\mathbf{H}$ and real number a such that

$$F(u, u') = \langle Be^{aA}u, e^{aA}u'\rangle$$

for arbitrary u, $u' \in \mathbf{E}(A)$. It is no essential loss of generality to assume that A is bounded below by I. Writing $E(r)$ for the spectral resolution of A and m for the regular measure on $\mathbb{R}^2$ such that

$$\langle BE(r)u, E(s)u'\rangle = m([(x, y) : x < r, y < s],$$

then for arbitrary u, $u' \in E(A)$,

$$G(u, u') = \int \hat{f}(r - s)\exp(ar + as)\, dm(r, s).$$

To say that G is a continuous operator from $\mathbf{E}(A)$ to itself means that for arbitrary real b,

$$|G(u, \exp(bA)u')| < C(u, b)\|u'\|$$

for arbitrary u, $u' \in \mathbf{E}(A)$. Since u is in the domain of $\exp[(a + k)A]$ for arbitrary k, it suffices to show that this inequality holds when u is replaced by $\exp(-kA)u$. Note also that the total variation of m is bounded by $\|B\| dn(r)dp(s)$, where $n(r) = \langle E(r)u, u\rangle$ and $p(s) = \langle E(s)u', u'\rangle$.

Accordingly, it suffices to show that $|\hat{f}(r - s)|\exp(ar - kr)\exp(as + bs)$ is bounded as a function of s and r in $[1, \infty)$. Noting that this expression is bounded by

$$C\exp[-\kappa|r - s|]\exp[(a - k)(r - s)]\exp[(2a + b - k)s],$$

it is clear that this is the case for sufficiently large k and larger κ. □

We next establish a criterion for essential selfadjointness, of a type first treated by Poulsen [11], that will be useful below. A selfadjoint operator A in a Hilbert space $\mathbf{H}$ will be called *regular* in case it has pure point spectrum, and only a finite number of eigenvalues are in any bounded interval. A hermitian operator (or form) F (that is defined on the domain $\mathbf{B}$ of all finite combinations of eigenvectors for A is called *invariant* under e^{itA} in case $e^{-itA}Fe^{itA} = F$ for all $t \in \mathbb{R}$.

Theorem 4.2. *Let A be a regular operator in a Hilbert space $\mathbf{H}$, and let the hermitian operator F defined on $\mathbf{B}(A)$ be invariant under e^{itA}, $t \in \mathbb{R}$. Then F is essentially selfadjoint on $\mathbf{B}(A)$.*

Proof. Essential selfadjointness for F is equivalent to denseness for the ranges of $F \pm iI$. The argument for $F - iI$ is parallel to that for $F + iI$ we treat only the latter case in detail.

The orthocomplement M of the range of $F + iI$ is invariant under the e^{itA}, which implies that the projection P with range M commutes with the e^{itA}, $t \in \mathbb{R}$. The general such operator is a direct sum of projections P_j, each of which has range in an eigenspace for A. Thus $\langle (F - iI)x, P_n y\rangle = 0$ for all $x \in \mathbf{B}(A)$ and $y \in \mathbf{H}$. But $P_n y$ is itself in $\mathbf{B}(A)$ and taking $x = P_n y$, it follows that $P_n y = 0$ for all n. Since y is arbitrary, this means $P_n = 0$, implying that $P = 0$, and completing the proof. □

Corollary 4.2.1. *An invariant and hermitian form F under e^{itA} on the domain $\mathbf{E}(A)$ (resp. $\mathbf{A}(A)$) is essentially selfadjoint on the domain $\mathbf{B}(A)$.*

Proof. Theorem 4.1 implies that F is a continuous operator on $\mathbf{E}(A)$ (resp. $\mathbf{A}(A)$) (by taking f to be an arbitrary nonvanishing nonnegative function in the indicated space) and in particular defined on $\mathbf{B}(A)$. It follows from Theorem 4.2 that F is essentially selfadjoint on the domain $\mathbf{B}(A)$. □

We next consider covariance aspects of the quantization map $\mathscr{Q}$.

Theorem 4.3. *Let U be a unitary operator on $\mathbf{H}$ that is also an isomorphism of $[\mathbf{E}(B)]$ (resp. $[\mathbf{A}(B)]$) into itself. Then $\Gamma(U)$ is an isomorphism of $\mathbf{E}(H)$ (resp. $\mathbf{A}(H)$) into itself.*

Proof. The proof for the analytic case is similar to that for the entire case and we detail only the latter. Using the complex wave representation for $\mathbf{K}$, if $f \in \mathbf{E}(H)$, then $f = e^{-tH}g$ for arbitrarily large t and t-dependent $g \in \mathbf{K}$. The claim that $\Gamma(U)f \in \mathbf{E}(H)$ is equivalent to the claim that for arbitrary real s, $e^{sH}\Gamma(U)e^{-tH}g$ is in $\mathbf{K}$. This means that the function of z, $g(e^{-tB}Ue^{sB}z)$ is essentially in $\mathbf{K}$ (cf. [9]). Since U acts continuously on $[\mathbf{E}(B)]$, on this space $Ue^{sB} = e^{s'B}V$, where V is a bounded linear operator on $\mathbf{H}$. Taking $t > s'$, the question becomes whether $g(e^{-\varepsilon B}Vz)$ is square-integrable if $\varepsilon > 0$. The operator $S = e^{-\varepsilon B}V$ is Hilbert-Schmidt, so it suffices to show that $g(Sz) \in \mathbf{K}$ for arbitrary such S. Now since $|g(z)| \leq Ce^{\|z\|^2/2}$, $|g(Sz)| \leq Ce^{\|Sz\|^2/2}$, so that the hypothesis of Lemma 1.6 of [9] is satisfied, apart from the condition that $\|S\| < 1$, which is seen to be redundant by a finite-dimensional adjustment. This shows that $\Gamma(U)$ leaves $\mathbf{E}(H)$ invariant, and continuity of the action of $\Gamma(U)$ on $[\mathbf{E}(H)]$ follows by similar estimates. □

Corollary 4.3.1. *Let U be as in Theorem 4.3, and let F be an n-cocycle on $\mathbf{E}(B)$ (resp. $\mathbf{A}(B)$). Let $F_U(z_1, \ldots, z_n) = \Gamma(U)^{-1}F(Uz_1, \ldots, Uz_n)\Gamma(U)$. Then F_U is a cocycle and $\mathscr{Q}(F_U) = \Gamma(U)^{-1}\mathscr{Q}(F)\Gamma(U)$.*

Proof. The two cases (entire and analytic vectors) are similar, and we treat in detail only the entire case.

We show that $\partial_{z_1} \cdots \partial_{z_n}$ maps both $\Gamma(U)^{-1}\mathscr{Q}(F)\Gamma(U)$ and $\mathscr{Q}(F_U)$ into the same cocycle, whence the corollary follows from the unicity part of Theorem 3.1. On the one hand,

$$\partial_{z_1 \cdots z_n}\mathscr{Q}(F_U) = F_U(z_1, \ldots, z_n) = \Gamma(U)^{-1}F(Uz_1, \ldots, Uz_n)\Gamma(U).$$

On the other hand, note that in the case $n = 1$, for arbitrary $z \in \mathbf{E}(B)$,

$$\begin{aligned}\partial_z\Gamma(U)^{-1}Q(F)\Gamma(U) &= [\Gamma(U)^{-1}\mathscr{Q}(F)\Gamma(U), \phi(z)] \\ &= \Gamma(U)^{-1}[\mathscr{Q}(F), \Gamma(U)\phi(z)\Gamma(U)^{-1}]\Gamma(U) \\ &= \Gamma(U)^{-1}[\mathscr{Q}(F), \phi(Uz)]\Gamma(U).\end{aligned}$$

It follows by a finite induction that

$$\partial_{z_1 \cdots z_n}\Gamma(U)^{-1}\mathscr{Q}(F)\Gamma(U) = \Gamma(U)^{-1}F(Uz_1, \ldots, Uz_n)\Gamma(U),$$

which agrees with the n-derivative of the other side of the putative equality. □

These results imply the essential selfadjointness of the quantizations of invariant hermitian cocycles, where a cocycle $F(z_1, \ldots, z_n)$ is *hermitian* in case $F(z_1, \ldots, z_n)^* = (-1)^n F(z_1, \ldots, z_n)$.

Corollary 4.3.2. *Suppose that F is a given hermitian n-cocycle on $\mathbf{E}(H)$ (resp. $\mathbf{A}(H)$) such that*

$$F(e^{itB}z_1, \ldots, e^{itB}z_n)$$

Then the $\mathscr{Q}(F)$ is essentially selfadjoint on $\mathbf{E}(H)$ (resp. $\mathbf{A}(H)$).

Proof. Corollary 4.3.1 shows that $\mathscr{Q}(F)$ is an invariant form under e^{itH}. It is accordingly a continuous linear operator on $[\mathbf{E}(H)]$ (resp. $[\mathbf{A}(H)]$) by Theorem 4.1. $\mathscr{Q}(F)$ is a hermitian operator since F is a hermitian form. Finally it follows from Corollary 4.2.1 that it is essentially selfadjoint. □

Symmetries other than time evolution commute with the quantization map in appropriate cases by virtue of

Corollary 4.3.3. *Let $U(\cdot)$ be a continuous unitary representation of a Lie group G on the Hilbert space $\mathbf{H}$, which includes a one-parameter subgroup whose action on $\mathbf{H}$ is generated by the operator B above. Suppose that B dominates the other generators of G in this representation as regards entire vectors (resp. as regards analytic vectors). If the cocycle F is $U(g)$-invariant, then $\mathscr{Q}(F)$ and its closure are $\Gamma(U(g))$-invariant, for $g \in G$.*

Proof. This follows from Theorem 4.3 in conjunction with Corollary 4.3.1. □

Example. If $G = SO(2, n)$, U is irreducible, and B generates the $O(2)$ subgroup, it follows from the scalar character of $dU(C)$ for the Casimir of G that B analytically dominates all of G. For $n = 4$ and U the wave representation of G, the quantized action integral for the ϕ_4^4 theory is a conformally invariant essentially selfadjoint operator (cf. below).

5. Integrals of local products of quantized fields

The quantum action and energy integrals of heuristic quantum field theorem are symbolic operators that are formally of finite order, where a form T is said to be of order n if $\partial_{z_1} \cdots \partial_{z_m} T = 0$ for $m = n+1$ but is not identically zero (for the $z_j \in \mathbf{E}(\mathbf{B})$) for $m = n$. The derived cocycle

$$F(z_1, \ldots, z_n) = \partial_{z_1} \cdots \partial_{z_n} T$$

is in practice a well-defined multilinear form, and the theory above permits rigorous versions of the cited heuristic operators to be established and explored.

The method presented here is general, but in order to avoid undue abstraction and provide concrete examples, we treat explicitly only certain scalar fields in Minkowski space. The criterion for essential selfadjointness and the conformal mapping method that we use apply to fields of arbitrary spin.

For the reader's convenience we summarize the most basic notation and results used below which are drawn from Paneitz and Segal [7] and Branson [4]. Minkowski space is denoted as $\mathbf{M}_0$, and it will be assumed in the following that $\mathbf{M}_0$ has even dimension $d \geq 4$. The usual coordinates on $\mathbf{M}_0$ are denoted as x_j $(j = 0, 1, \ldots, d-1)$ and the fundamental Lorentzian form as $x_0^2 - x_1^2 - \cdots - x_{d-1}^2$. $\overline{\mathbf{M}}$ denotes the conformal compactification of $\mathbf{M}_0$, $\overline{\overline{\mathbf{M}}}$ the 2-fold cover of $\overline{\mathbf{M}}$, and $\widetilde{\mathbf{M}}$ the universal cover of $\overline{\mathbf{M}}$. The universal cover of the conformal group on $\widetilde{\mathbf{M}}$ locally isomorphic to $O(2, d)$ is denoted $\widetilde{G}$. P denotes the Poincaré group on $\mathbf{M}_0$, as extended by the scaling transformations $x_j \to l x_j$ $(l > 0)$, and $\widetilde{P}$ its universal cover. The isotropy subgroup in the action of $\widetilde{G}$ on $\widetilde{\mathbf{M}}$ is isomorphic to $\widetilde{P}$, and the isomorphism will be denoted as γ. There is a canonical embedding $\imath$ of $\mathbf{M}_0$ into $\widetilde{\mathbf{M}}$, in which a point at infinity on $\mathbf{M}_0$ corresponds to a fixed point for $\gamma(\widetilde{P})$ and which is P-covariant: $\imath(gx) = \gamma(g)\imath(x)$ for arbitrary $g \in P$ and $x \in \mathbf{M}_0$.

$\widetilde{\mathbf{M}}$ is conformally equivalent to the 'Einstein Universe' $\mathbf{E}$, the Lorentzian manifold $\mathbb{R} \times S^{d-1}$ of metric $dt^2 - ds^2$, where t is the 'Einstein time' or component in $\mathbb{R}$ and ds is the element of arc length on S^{d-1} in radians. We represent S^{d-1} as the unit sphere in $\mathbb{R}^d$, $\{(u_1, \ldots, u_d) : u_1^2 + \cdots + u_d^2 = 1\}$. We define $u_{-1} = \cos t$, $u_0 = \sin t$, and note that $\overline{\overline{\mathbf{M}}}$ is conformally equivalent to $\overline{\overline{\mathbf{E}}}$, which is covered by $\mathbf{E}$ by the conformally invariant map $(t, \mathbf{u}) \to (e^{it}, \mathbf{u})$, where $\overline{\overline{\mathbf{E}}}$ is represented as $S^1 \times S^{d-1}$; the coordinates t, u_{-1}, and u_0 are adapted to $\overline{\overline{\mathbf{E}}}$ in the obvious way. We set ρ for the distance on S^{d-1} from the base point $(0, 0, \ldots, 1)$ and set p for the function on $\mathbf{E}$ or $\overline{\overline{\mathbf{E}}}$, $p(u) = \frac{1}{2}(\cos t + \cos \rho)$. Relative to the standard embedding indicated above, the Lorentzian and Einstein metrics on $\mathbf{M}_0$ and $\mathbf{E}$ are related by the equation $dt^2 - ds^2 = p^2(dx_0^2 - dx_1^2 - \cdots - dx_{d-1}^2)$. We write dx for $dx_0 dx_1 \cdots dx_{d-1}$ and du for $dt d\mathbf{u}$, where $d\mathbf{u}$ denotes the element of volume on S^{d-1}. The relations $du = p^d dx$ and $d\mathbf{u} = p^{d-1} d\mathbf{x}$ will be used below. For further details, see loc. cit.

We denote as Δ_0 and Δ the usual Laplacians on $\mathbb{R}^{d-1}$ and S^{d-1} and by $\Box_0$ and $\Box$ the usual wave operators on $\mathbf{M}_0$ or $\mathbf{E}$ (or on $\overline{\overline{\mathbf{E}}}$, where it will be clear from the context, it not entirely immaterial, whether $\mathbf{E}$ or $\overline{\overline{\mathbf{E}}}$ is meant). We also set $C_0 = (-\Delta_0)^{1/4}$ and $C = (c_d - \Delta)^{1/4}$, where $c_d = ((d-2)/2)^2$

and Δ_0 and Δ are formulated in the usual way as selfadjoint operators in $L_2(\mathbb{R}^{d-1})$ and $L_2(S^{d-1})$, which Hilbert spaces we denote as $\mathbf{F}_0$ and $\mathbf{F}$. We denote as $\mathbf{H}_0$ the complex Hilbert space of real normalizable solutions of the wave equation $\Box_0\phi_0 = 0$, where the complex structure and the inner product are as follows, in terms of Cauchy data, which may be at an arbitrary time and in an arbitrary Lorentz frame. These data $f = \phi_0(t_0, \cdot)$ and $g = \dot{\phi}_0(t_0, \cdot)$, at time t_0, are in the Hilbert spaces denoted $\mathbf{F}_{0,1/2}$ and $\mathbf{F}_{0,-1/2}$ consisting of the completion of the domains of C_0 and C_0^{-1} in their natural Hilbert metrics. The complex structure j is given in matrix form on the direct sum $\mathbf{F}_{0,1/2} \oplus \mathbf{F}_{0,-1/2}$, with which $\mathbf{H}_0$ is identified via the solution of the Cauchy problem, as $\begin{pmatrix} 0 & C_0^{-2} \\ -C_0^2 & 0 \end{pmatrix}$. The real part of the inner product in $\mathbf{H}_0$ is the real form obtained by polarization from the norm

$$\|\phi\|^2 = \|C_0 f\|_{L_2}^2 + \|C_0^{-1} g\|_{L_2}^2;$$

the imaginary part is determined by the given complex structure and the real part. We note that vectors in $\mathbf{H}_0$ are determined by the restrictions of their Fourier transformations to the positive-frequency region, in terms of which j is represented by complex multiplication by i; the present formalism, however, has certain advantages in symmetry and generality over the more familiar formulation of $\mathbf{H}_0$ in relativistic theory. Thus, we denote by $\mathbf{H}$ the complex Hilbert space of all real normalizable solutions of the conformal wave equation $(\Box + c_d)\phi = 0$, with the complex structure and inner product defined in the same way except for the replacement of C_0 by C and $L_2(\mathbb{R}^{d-1})$ by $L_2(S^{d-1})$. We recall that if $\phi \in \mathbf{H}$, then $\phi_0 \in \mathbf{H}_0$, where $\phi_0(x) = p(\imath(x))^{(d-2)/2}\phi(\imath(x))$, and that the mapping $T: \phi \to \phi_0$ is unitary and intertwines the respective actions of G on $\mathbf{H}$ and $\mathbf{H}_0$ that derive from and express the conformal invariance of the equations. We refer to loc. cit. for further details.

It follows from the unicity of the free boson field over a given Hilbert space that the fields over $\mathbf{H}_0$ and $\mathbf{H}$ are unitarily equivalent via a unique unitary operator $\Gamma(T)$ from $\mathbf{K}$ onto $\mathbf{K}_0$ such that $\Gamma(T)W(\phi) = W_0(T\phi)$ and $\Gamma(T)v = v_0$, where the subscript 0 distinguishes the field over $\mathbf{H}_0$ from that over $\mathbf{H}$. The notation $\Gamma(T)$ is convenient notwithstanding the fact that T is not in $U(\mathbf{H})$, by virtue of covariance features.

The single-particle Hilbert spaces of heuristic quantum field theory are usually concretely formulated as function spaces, but they are more invariantly expressible as sub- or subquotient spaces of the section spaces of induced bundles, and the use of different presentations for the relevant bundles is in essence the main difference between the spaces $\mathbf{H}_0$ and $\mathbf{H}$. We next show that the local products of field operators that are involved in the heuristic theory on $\mathbf{M}_0$ and on $\mathbf{E}$ for the quantized wave equations are similarly essentially the same except for their presentations. The treatment here exemplifies a general induced bundle method and will be applied below to deal with massive fields.

The quantization of a general class of fields representable by induced bundles including the wave equation fields on $\mathbf{M}_0$ and $\widetilde{\mathbf{M}}$ may be given an invariant form as follows. Let P be a Lie subgroup of the Lie group G, and consider the bundle over G/P induced from a given finite-dimensional real representation of P. Let $\mathbf{R}$ denote the space of C^∞ sections, and suppose that $\mathbf{R}$ has an invariant subspace $\mathbf{S}$ on which the action of G is unitarizable. Let $\mathbf{H}$ denote the Hilbert space completion of $\mathbf{S}$. Consider now the quantization of the fields represented by $\mathbf{S}$ in terms of the free boson field over $\mathbf{H}$ as underlying single-particle space.

Example 1. G is the connected Poincaré group on $\mathbf{M}_0$, and P is its Lorentz subgroup. G/P is identifiable with $\mathbf{M}_0$, and the usual relativistic fields are obtained. The massless scalar fields correspond to the invariant subspace defined by the wave equation, and the massive scalar fields are those defined by the Klein-Gordon equation. These subspaces are suitably unitarizable under G.

Example 2. Let $d = 4$, and consider the one-dimensional representation of P in which the scaling transformation, which acts in $\mathbf{M}_0$ as $x_j \to lx_j$, acts as l. $\mathbf{R}$ is then the space of scalar fields of conformal weight 1 and has a unique irreducibly invariant subspace $\mathbf{S}$. In a suitable parallelization, $\mathbf{R}$ may be represented by the space of all C^∞ fields on $\overline{\overline{\mathbf{M}}}$, and $\mathbf{S}$ then becomes the subspace of solutions of the conformal wave equation, which as noted is unitarizable.

The sections $\phi \in \mathbf{S}$ in these examples are representable by point functions, but it is only in the simplest case, of the usual relativistic fields, that there is an entirely canonical way to do so (e.g., in Example 2, the parallelization involved is not unique). In general, the notion of the value $\phi(x)$ of the section ϕ at the point $x \in G/P$ is well defined only relative to choices of parallelization or local trivializations. In order to treat integrals given symbolically as $\int_{G/P} \phi(x)^q \, dx$, some preliminaries are therefore needed.

The concept of the quantized fields at a point, say $\boldsymbol{\phi}(x)$, where boldface letters distinguish quantized from classical fields, involves in its simplest rigorous form, a linear mapping from a space of test functions f into operators, given in symbolic form as $f \to \int \boldsymbol{\phi}(x) f(x) \, dx$. However, in order for this mapping to have an invariant significance, it is in general necessary to formulate what appears as test functions in the simplest cases as sections of the bundle dual to that of which the corresponding classical fields, whose quantization is in question, are sections. To clarify this matter, we use underlined letters to distinguish abstract sections, defined by induction as above, from concrete representatives, obtained by divers parallelizations or trivializations.

Assuming now that G/P is compact, or alternatively using sections of compact support, the real bilinear pairing $\langle \underline{\phi}, \underline{f} \rangle$ is invariantly defined. This bilinear expression defines a linear functional on $\mathbf{S}$, which we assume is continuous in the topology on $\mathbf{H}$, as is often the case in practice, and so may be represented in the form $\operatorname{Im}\langle \underline{\phi}, P\underline{f} \rangle$, where P is a linear map from the section space $\mathbf{S}^*$ of the dual bundle into $\mathbf{H}$. If $(\mathbf{K}, W, \Gamma, v)$ denotes the free boson fields over $\mathbf{H}$ and $\boldsymbol{\phi}$ denotes the map ∂W from $\mathbf{H}$ into selfadjoint operators on $\mathbf{K}$, the map $\underline{f} \to \boldsymbol{\Phi}(\underline{f}) = \boldsymbol{\phi}(P\underline{f})$ defines the rigorous and invariant form of the symbolic map $f \to \int \boldsymbol{\phi}(x) f(x) \, dx$; for quasi-bookkeeping purposes, it may be helpful to denote $\boldsymbol{\Phi}(f)$ as "$\int \boldsymbol{\phi}(x) f(x) \, dx$".

In order to treat the nonlinear operations involved in local quantum field theory, we must now similarly invariantly formulate products such as $\boldsymbol{\phi}(x_1)\boldsymbol{\phi}(x_2) \cdots \boldsymbol{\phi}(x_n)$. This product will be realized as a form on a suitable domain, whose formulation requires some additional assumptions. We assume there is given an infinitesimal generator X of G (or similarity class of such) called infinitesimal time evolution (e.g., $\partial/\partial x_0$ for the Poincaré group on $\mathbf{M}_0$ or $\partial/\partial t$ for the conformal group on $\mathbf{E}$). We set U for the unitary action of G on $\mathbf{H}$, A for $\partial U(X)$, and H for $\partial\Gamma(A)$; colloquially, A is the single-particle Hamiltonian and H is the quantized field Hamiltonian. *We assume that A is bounded below by a positive scalar and that e^{-tA} is of trace class for arbitrary $t > 0$.* This condition is satisfied by the Einstein Hamiltonian but not by the Minkowski Hamiltonian, which nevertheless may be effectively treated in terms of the Einstein Hamiltonian (e.g., every C^∞ vector for the latter is also a C^∞ vector for the former). Let $[\mathbf{D}_\infty(H)]$ denote the space of all C^∞ vectors for H, in it usual topology. The concept of the quantized field as a point function may now be given invariant formulation by

Scholium 5.1. *There exists a unique continuous map $u \to \underline{\phi}(u)$ from $\mathbf{E}$ to forms on $[\mathbf{D}_\infty(H)]$ such that for arbitrary $\underline{f} \in \mathbf{S}^*$, and w, $w' \in \mathbf{D}_\infty(H)$,*

$$\langle \Phi(\underline{f})w, w'\rangle = \langle\langle\langle \underline{\phi}(\cdot)w, w'\rangle, \underline{f}(\cdot)\rangle\rangle$$

where $\langle \underline{\phi}(u)w, w'\rangle$ denotes the inner product in the fiber at the point u.

The proof is by arguments similar to those given in Paneitz and Segal [7] and is omitted. □

The case of products of fields may now be treated in the same format as follows. Since the quantized field at a point is merely a form and not a bona fide operator, a product such as $\underline{\phi}(u_1)\cdots\underline{\phi}(u_n)$ is a priori undefined; a presumptive 'finite part', known as the Wick product, and denoted by enclosing the symbolic product in : ... : is used in the heuristic literature. It has been treated in various contexts in the rigorous literature (e.g., [2] or references therein), and the following formula adapts these treatments.

Scholium 5.2. *There exist unique continuous maps Φ_n $(n = 1, 2, \ldots)$ from $\mathbf{S}^{*^n}$ to the space $\mathbf{L}([\mathbf{D}_\infty(H)]$ of all continuous linear operators on $[\mathbf{D}_\infty(H)]$ (in the weak operator topology) such that the following relations hold:*

(i) *If f is a direct product section, $f = f_1 \times \cdots \times f_n$ $(f_j \in \mathbf{S}^*$ and $g \in \mathbf{S}^*$, then setting $h_j = f_1 \times \hat{f}_j \times \cdots \times f_n$ (i.e., the factor f_j is omitted),*

$$[\Phi_n(f), \Phi(g)] = -i\sum_{j=1}^{n} \Phi_{n-1}(h_j)\,\mathrm{Im}\langle Pf_j, Pg_j\rangle_{\mathbf{H}},$$

where for $n = 1$, $h_1 = 1$ and $\Phi_0(1) = 1$.

(ii) $\langle \Phi_n(f)v, v\rangle = 0$ $(n > 0)$.

Proof. Cf. loc. cit. for similar arguments. □

The mappings Φ_n extend from smooth sections to distributions, by arguments similar to those treating the case of functions in loc. cit., as stated finally in

Theorem 5.1. *There exist unique continuous maps from distribution sections of $\mathbf{S}^{*^n}$ to forms on $[\mathbf{D}_\infty(H)]$ (in weak operator topology) that extend the maps given by Lemma 5.2 and satisfy the corresponding extensions of* (i) *and* (ii). *In particular, for arbitrary $u_1, \ldots, u_n$ in $\mathbf{E}^n$, there exists a form denoted $:\underline{\phi}(u_1)\cdots\underline{\phi}(u_n):$ such that for any distribution $f(u_1, \ldots, u_n)$ in $\mathbf{S}^{*^n}$,*

$$\Phi(f) = \int_{M^n} :\underline{\phi}(u_1)\cdots\underline{\phi}(u_n): f(u_1, \ldots, u_n)\, d^n u.$$

And if $f \in C^\infty$, *then* $\Phi(f)$ *is a continuous operator on* $[\mathbf{D}_\infty(H)]$.

Proof. We refer to loc. cit. for details in quite representative special cases.

6. Applications to scalar fields

We now specialize to the case of the scalar bundles over $\mathbf{M}_0$ or $\widetilde{\mathbf{M}}$ of certain weights, corresponding to the cases of massless and massive relativistic fields. The massless case corresponds to the weight $w = (d-2)/2$, and we denote by $\mathscr{S}$ the space of abstract sections Φ that are C^∞ and transform according to the generator ζ of the infinite cyclic center of $\widetilde{G}$ as $\Phi \to (-1)^w \Phi$. $\mathscr{S}$ has a unique irreducibly invariant subspace under $\widetilde{G}$ that is stably unitarizable, giving rise to a corresponding Hilbert space $\mathscr{H}$ and a unitary representation $\mathscr{U}$ of $\widetilde{G}$ on $\mathscr{H}$.

$(\mathscr{H}, \mathscr{U})$ may be concretely represented as the solution manifolds of wave equations in $\widetilde{\mathbf{M}}$ or $\mathbf{M}_0$, with their transformation properties and spatio-temporal localizations, using appropriate concrete functional representations for the sections Φ; cf. [4] and [7]. More specifically, $\mathscr{H}$ is equivalent in the indicated aspects to the space $\mathbf{H}$ earlier treated and, with restriction to $\mathbf{M}_0$ (which by virtue of the action of ζ is an isomorphism), by $\mathbf{H}_0$. The isomorphisms $\Phi \to \phi_0$ give rise to the following relation between ϕ_0 and ϕ:

$$\phi_0(x) = p_0(x)^w \phi(\iota x), \qquad p_0(x) = p((\iota^{-1}x)) = [(1 - x^2/4)^2 + x_0^2]^{-1}.$$

The quantization theory earlier presented then implies

$$\Phi_0(x) = p_0(x)^w \widetilde{\Phi}(\iota x),$$

and similarly for the local Wick products. Noting that the designated measures du in $\mathbf{M}$ and dx in $\mathbf{M}_0$ are connected by the equation $du = p^d dx$, relativistic and similar quantized action integrals in $\mathbf{M}_0$ are seen to be unitarily equivalent to well-defined action integrals in $\mathbf{M}$ and therefore finite.

Scholium 6.1. *If* b *is bounded and in* $C^\infty(\mathbf{M}_0)$ *and* $q \geq 4$ *when* $d \equiv 0 \bmod 4$ *or* $q \geq 3$ *when* $d = 2 \bmod 4$, *then*

$$L_0 = \int_{\mathbf{M}_0} b(x) \colon \Phi_0(x)^q \colon dx$$

exists as a continuous linear operator on $[\mathbf{D}_\infty(\widetilde{H}')]$, *where* $\widetilde{H}' = \Gamma(T)^{-1} \widetilde{H} \Gamma(T)$.

Proof. This is an immediate consequence of Scholium 5.2 together with the considerations regarding the conformal mapping of Wick products. □

L_0 may also be expressed as $L_0 = \Gamma(T)^{-1} L \Gamma(T)$, where

$$L = \frac{1}{2} \int_{\overline{\overline{\mathbf{M}}}} b(\iota^{-1}u) p(u)^{qw-d} \colon \widetilde{\Phi}^q(u) \colon du$$

is a continuous linear operator on $[\mathbf{D}_\infty(\widetilde{H})]$.

A similar analysis applies to integrals only over space, such as those representing putative interaction energy operators, but without the strongly regularizing effect of time integration, the result is only a form.

Scholium 6.2. *If* b *is a bounded* C^∞ *function on* S^{d-1} *and if* $q \geq 2$, *then*

$$\widetilde{H}_i = \int_{S^{d-1}} b(x) : \Phi(0, \mathbf{x})^q : d\mathbf{x}$$

exists as a form on $[\mathbf{D}_\infty(\widetilde{H})]$.

Proof. The same argument applies in conjunction with the observation that $d\mathbf{x} = p^{d-1} d\mathbf{u}$ at $t = x_0 = 0$. □

Remark. It is not difficult to show that $\int_{\mathbf{M}_0} : \Phi^q : dx$ has a selfadjoint extension, for $q \geq 4$. However, the integral is in general not invariant under Einstein temporal evolution, so its essential selfadjointness does not follow. The integral is invariant under Minkowski temporal evolution, and [11] would imply essential selfadjointness *if* the domain were $[\mathbf{D}_\infty(H)]$, but it is doubtful whether the integral can be extended to a form on this domain, due to the softness of the Minkowski Hamiltonian.

We now turn to the consideration of massive fields. Whereas in Minkowski space the conformal weight w does not affect the relativistic transformation properties, the value of w is quite material to considerations in $\widetilde{\mathbf{M}}$. In the scalar bundle of weight $d/2 - 1$ earlier treated, the positive-energy section subspace is indecomposable under the conformal group, so the massive subspace is at best only invariant under K or is conformally invariant only as a quotient space. The quantization of an indecomposable section space as a whole presents problems theoretically distinct from those of the present paper. Accordingly we treat here rather the scalar bundle of weight $d/2$, whose positive energy subspace is unitarily equivalent to the massive positive energy quotient of the section space of the scalar bundle with $w = 1$, modulo the wave equation subspace when $d = 4$. Moreover, the usual invariant inner product in the Klein-Gordon single-particle subspace results from the direct integral decomposition of the inner product in the $w = d/2$ bundle positive-energy subspace.

For applicability, we describe the present massive fields primarily in terms of relativistic theoretical usage and make the

Definition 6.1. The general massive field in $\mathbf{M}_0$ is that consisting of all L_2 real functions Φ whose Fourier transforms vanish in the 'tachyonic' region of the dual ('momentum') space, consisting of energy-momenta K for which K^2 is negative.

We note that, as earlier, the real Hilbert space $\mathbf{H}_{\text{real}}$ of all Φ becomes a complex space $\mathbf{H}$ in which energy is positive on defining the action of i as the Hilbert transform with respect to the time x_0. This is clearly Poincaré invariant. In order to formulate the conformal transformation properties of the Φ, it is convenient to map conformally as earlier into the Einstein universe. The singularity of the global action of the conformal group in $\mathbf{M}_0$ produces an awkwardness whose resolution is tedious without the use of conformal mapping.

Accordingly we note that the positive-energy subspace of the $w = \frac{d}{2}$ bundle section space on $\widetilde{\mathbf{M}}$ transforming under ζ as $(-1)^{d/2-1}$ is unitarily equivalent to $\mathbf{H}$, via the canonical imbedding of $\mathbf{M}_0$ into $\widetilde{\mathbf{M}}$ together with corresponding unitarization by multiplication by the square root of the Jacobian. Thus the action of G is unitary, taking the form for $f \in L_2(\widetilde{\mathbf{M}})$:

$$f(u) \to f(g^{-1}u) J(g^{-1})^{1/2} \qquad (g \in G),$$

where $J(g)$ is the Jacobian of g. It is known that $L_2(\overline{\mathbf{M}})$ splits into irreducible positive and negative energy subspaces and a tachyonic two-sided energy spectral subspace under the action of G (when $d = 4$, [7]; for the general case, [4]; we thank T. P. Branson for reference to the relevant computations).

Now taking account of the differences in weight, we may apply a similar analysis to that above. Observing that δ_x is a well-defined element of $\mathbf{F}(\mathbf{E}(B))$ (i.e., for $\phi \in \mathbf{E}(B)$ and $x \in \mathbf{M}_0$, $\phi \to \Phi(x)$ is continuous), the quantized boson field $\Phi(x)$ is correspondingly defined.

Scholium 6.3. *Let $\Phi(x)$ denote the quantized massive scalar field (of weight $d/2$) in $\mathbf{M}_0$. Let b be a bounded function in $C^\infty(\mathbf{M}_0)$. Then for all integrals $q > 1$, $\int : \phi(x)^q : b(x)dx$ is a continuous linear operator on $\mathbf{E}(\widetilde{H}')$.*

Proof. The argument is parallel to that for the case $w = 1$ treated above. □

Remark. The proceeding results apply without essential change to multicomponent fields involving similar polynomial interactions between different fields, both massless and massive. The Lagrangian $\phi^2\psi$, where ϕ is massless and ψ is massive, provides a conformally invariant, essentially selfadjoint operator on the tensor product of the respective free field Hilbert spaces. The results also apply to K-invariant subspaces of the massive field without essential change in the argument. These cases include Einstein Universe analogs of the generalized free fields in Minkowski space [6] and provide models for particles of finite width.

7. Pseudo-interacting fields and scattering

By a pseudo-interacting field we mean one whose Hamiltonian is obtained by substituting the free field in an expression for the classical Hamiltonian of the interacting theory. The pseudo-interacting field itself is then obtained by prescribing Cauchy or temporally asymptotic (here equivalent to Goursat [1]) data for it, consisting in practice of its coincidence with the corresponding data for the free field. In two-dimensional constructive quantum field theory, the Cauchy problem at an arbitrary fixed finite time was used; and this approach has been fruitful. However, physically it appears unjustified to assume that the free and the putative interacting fields are unitarily equivalent at any finite time. In physical theory it is rather the asymptotic coincidence between the free and interacting fields at infinite Minkowski times (within unitary equivalence) that is assumed and appears conceptually appropriate. The analysis here uses a rigorous form of this assumed coincidence.

We treat here the prototypical case of the ϕ_4^4 theory, which has been of topical interest in connection with particle theory [3]. It will be seen that the method is a general one for conformally invariant interactions and is adaptable to interactions that are merely K-invariant. We begin by deriving expressions for the classical Hamiltonian of ϕ_4^4 theory that are less singular than the fixed-time formulation of the Hamiltonian, using temporal smoothing to desingularize.

The Hamiltonian naturally depends on a particular choice of temporal evolution group. All Minkowski evolution groups are conjugate within the scaling-extended Poincaré group and all Einstein evolution groups are conjugate within the conformal group, but these two classes of evolution group are not conjugate to each other (as is clear from the differing spectra of the corresponding

Hamiltonians). From a classical viewpoint there is ultimately a correspondence between the results using the one Hamiltonian or the other (e.g., [1]), the issue being essentially a choice of one-parameter family of spacelike surfaces for a given hyperbolic equation and thus, basically a matter of technical convenience. In the quantum case there is only a formal presumption of basic equivalence between the Minkowski and Einstein temporal evolution formats; in rigorous terms (including its precise formulation) the question is open. The key different between the classical and quantum cases is the concept of the vacuum in the latter case; and as the lowest eigenstate of the Hamiltonian, the vacuum is not clearly independent of the choice of conjugacy class of the temporal evolution group. The theory above suggests a possibly more fundamental role for the Einstein format, which will be used here (cf., e.g., [10] for comment on the Minkowski format).

Scholium 7.1. *Let* $\mathbf{S}$ *denote the class of* C^∞ *finite-Einstein energy solutions of the nonlinear wave equation*

$$(\Box + 1)\tilde{\phi} + g\tilde{\phi}^3 = 0$$

in $\mathbf{E}$. *Let* f *be a continuous nonnegative function on* $\mathbb{R}^1$ *that is periodic of period* 2π *and* $\int_0^{2\pi} f(t)\,dt = 1$. *For an arbitrary* C^∞ *function* $\tilde{\phi}$ *let*

$$E_f(\tilde{\phi}) = \int_{-\pi}^{\pi}\int_{\mathbf{S}^3}\left[\frac{1}{2}(\nabla\tilde{\phi})^2 + \frac{1}{2}(\partial_t\tilde{\phi})^2 + \frac{1}{2}\tilde{\phi}^2 + \frac{1}{4}g\tilde{\phi}^4\right] d\mathbf{u} f(t)\,dt.$$

Then

(i) *If* $\tilde{\phi} \in \mathbf{S}$, *then* $E_f(\tilde{\phi})$ *is the Einstein energy of* $\tilde{\phi}$.

(ii) *The restriction of* E_f *to the space* $\mathbf{E}(B)$ *of solutions of the free wave equations is the sum of quadratic ('free') and biquadratic ('interacting') cocycles.*

(iii) *If* $\mathscr{Q}(E_f)$ *is essentially selfadjoint on* $\mathbf{E}(H)$, *then its closure is a continuous univalent function of the coupling constant* g *(in the strong topology on not necessarily bounded selfadjoint operators in* $\mathbf{K}$).

Proof. The coincidence on $\mathbf{S}$ of $E_f(\tilde{\phi})$ with the energy of $\tilde{\phi}$ is immediate from energy conservation. The continuity of the total Hamiltonian

$$H(f) = \text{ closure of } (\widetilde{H}_0 + gM(f)),$$

where $M(f) = \mathscr{Q}(C)$, C being the cocycle

$$C(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4) = \int_{-\pi}^{\pi}\int_{S^3} f(t)\tilde{\phi}_1\tilde{\phi}_2\tilde{\phi}_3\tilde{\phi}_4\,dt\,du,$$

follows from the continuity of $H(f)w$ as a function of g, for arbitrary $w \in \mathbf{E}(\widetilde{H}_0)$. The univalence of $H(f)$ as a function of g follows from the non-identical vanishing of C and the unicity part of Theorem 3.1. □

We are now in a position to define the pseudo-interacting fields ϕ to be treated here. These fields depend on f, which will be held fixed unless otherwise indicated. It is convenient here to define ϕ as an operator-valued function on test functions in a *lightcone* format; regarding classical aspects (cf. e.g., [1]). The "reference lightcone" in $\mathbf{E}$ may be defined as the manifold (with singularities, which however have measure 0 in the present L_2 context and thus are immaterial) of all points (t, u) in $\mathbf{E}$ such that $t = \rho(u)$, where $\rho(u)$ is the distance on S^3 from u to the origin $(0, 0, 0, 1)$, which may be interpreted as the point of observation. Other lightcones are obtained as conformal transforms of the reference one, and we use especially the cones $C(\tau)$ obtained from the reference cone by Einstein temporal translation through τ. Thus $C(\tau) = \{(t, u) | t - \rho(u) = \tau\}$ [1]. If $\tilde{\phi}$ is any vector in $\mathbf{A}(B)$, its values (or Goursat data) on $C(\tau)$ are in L_2 with respect to the measure on $C(\tau)$ obtained by its identification with S^3 given in [1] (but this association of S^3 with lightcones is distinct from its association with the spacelike surfaces obtained by fixing the Einstein time). Thus instead of describing the free field as an operator-valued function at any fixed time on test functions defined on space (corresponding to the Cauchy data for the field at that time and thus involving also the first time derivative of the field in an irreducible set of field variables), we describe it as an operator-valued function at any fixed 'Goursat' time τ on test functions on $C(\tau)$, whose test functions form a dense subset $\mathbf{D}$ of $L_2(C)$.

The map $V(\tau)$ from $\phi \in \mathbf{H}$ to its Goursat data on $C(\tau)$ is unitary, in an appropriate metric, on the class $\mathbf{D}$ of functions on $C(\tau)$ corresponding to vectors in $\mathbf{A}(B)$ via the Goursat problem. Moreover, $V(s) = \exp(isB)V(0)\exp(-isB)$. If Y is any rotation (in $SO(4)$) on physical space S^3 in $\mathbf{E}$, $U(Y)$ commutes with B, and there is therefore a corresponding unitary transformation $V(Y)$ on the Hilbert space completion of $\mathbf{D}$ with respect to the conformally invariant metric in this space corresponding to that in $\mathbf{H}$. In summary, it follows that for any fixed τ, $\mathbf{H}$ is unitarily equivalent to a Hilbert space of Goursat data on $C(\tau)$ and that this unitary equivalence is K-covariant. We denote by $\psi_0(\tau, h)$ the free field operator corresponding to the Goursat datum h on $C(\tau)$ and note explicitly the K-covariance in the form

$$\exp(-itH_0)\psi_0(\tau, h)\exp(itH_0) = \psi_0(\tau + t, h),$$
$$\Gamma(U(g))^{-1}\psi(\tau, h)\Gamma(U(g)) = \psi(\tau, V(g)h),$$

where g is arbitrary in the group K_S of purely spatial transformations in $\mathbf{K}$, representable as $SU(2) \times SU(2)$, which act as rotations on space, but whose action on Goursat data consists of point transformations only on the diagonal subgroup (geometrically, rotations leaving fixed the base point in space).

The pseudo-interacting field will be defined by giving its weighted averages on the cones $C(\tau)$ with respect to test functions h in the space of functions on the cone corresponding to $\mathbf{E}(B)$:

$$\psi(\tau, h) = e^{i(\tau+\pi)H}\psi_0(-\pi, h)e^{-i(\tau+\pi)H}.$$

We now take up the properties of the pseudo-interacting field. To treat the scattering we use its formulation in $\mathbf{E}$, according to which the S-operator is the unitary operator that implements the action of the conformally invariant element ζ on the interacting field, modulo the free action of ζ. ζ is the generator of the infinite cyclic center of $\widetilde{G}$ and takes the form

$$(t, u) \to (t + \pi, Au),$$

where A is the antipodal map on S^3. It maps the infinite past in Minkowski as imbedded in $\mathbf{E}$ into the infinite future in Minkowski space, Poincaré-covariantly.

Theorem 7.1. *If $f \equiv 1$ or, more generally, if the closure of $\mathscr{Q}(E_f)$ is selfadjoint, the pseudo-interacting field has the properties*:

(i) *on $C(-\pi)$ (i.e., in the infinite Minkowski past) it is coincident with the free field*;

(ii) *it is K-covariant: for a suitable (unique, within possible constant phase factors) unitary representation Γ_{int} of K on $\mathbf{K}$, and test pair $T = (t, h)$*

$$\Gamma_{\text{int}}^{-1}(g)\phi(T)\Gamma_{\text{int}}(g) = \phi(g(T)),$$

where $g = s \times g_1$ sends T into $(t+s, V(g_1)h)$;

(iii) *the field operators are selfadjoint and satisfy the canonical relations on lightcones (in the Weyl form)*;

(iv) *the S operator exists, is a continuous function of $g \geq 0$, and is univalent as a function of g for sufficiently small g*;

(v) *Sw is for $w \in \mathbf{E}(\widetilde{H}_0)$ a differentiable function of g at $g = 0$, with derivative $\frac{1}{2}\int_{S^1\times S^3} :\phi(t, \mathbf{u})^4: dt d\mathbf{u}$ (independent of f!)*;

(vi) *the Hamiltonian $H(f)$ as a form on $\mathbf{E}(\widetilde{H}_0)$ converges as $f \to \delta(t)$ to the Hamiltonian for the putative interacting field.*

Proof. When $f \equiv 1$, the interaction Hamiltonian is essentially selfadjoint as seen earlier and, being invariant under Einstein temporal displacement, commutes (strongly) with the free Hamiltonian; it then follows from spectral theory that their sum has selfadjoint closure. Now assuming that $\mathscr{Q}(E_f)$ has this property, we verify the conclusions of Theorem 7.1 seriatim. Regarding (i), on setting $\tau = -\pi$, ϕ and ϕ_0 agree as functions of the test function h. Thus the fields coincide on $C(-\pi)$, and the infinite past in Minkowski space is represented precisely by $C(-\pi)$.

Regarding (ii), the general element g of K takes the form $g_1 g_2$ where g_1 is an Einstein time translation, say $t \to t+s$, and g_2 is a purely spatial transformation on S^3 as the Einstein space. K-covariance means that there exists a unitary representation Γ_{int} of K with the property that

$$\Gamma_{\text{int}}(g)^{-1}\phi(\tau, h)\Gamma_{\text{int}}(g) = \phi(g((\tau, h)),$$

where $g(\tau, h) = (g_1(t), g_2(h))$. Here $g_1(t) = t+s$ and $g_2(h) = V(g_2)h$, this being the action of K on the present test functions, irrespective of the fields being 'tested'. Now setting

$$\Gamma_{\text{int}}(g) = e^{isH}\Gamma(U(g_2)),$$

it is straightforward to check, noting the invariance of $\mathbf{M}$ under $\Gamma_{\text{int}}(A)$, that this covariance relation is satisfied; and Γ_{int} is evidently a continuous unitary representation of K.

Assertion (iii) is immediate from the Weyl relations for the free field.

Regarding (iv), ζ is carried by the action of K on the interacting field into $\exp(i\pi H(f))\Gamma_{\text{int}}(A)$, while on the free field its action is $\exp(i\pi\widetilde{H}_0)\Gamma_{\text{int}}(A)$, whence the S-operator is

$$S = e^{i\pi H(f)}e^{-i\pi\widetilde{H}_0}.$$

This is a continuous function of g by definition of the strong topology in the space of selfadjoint (not necessarily bounded) operators. To show its differentiability as a function of g, note that Duhamel's formula is applicable to

$\exp(itH(f))$ in the form

$$e^{itH(f)}e^{-it\widetilde{H}_0}w = w + \int_{-\pi}^{\pi} e^{isH(f)}gMe^{-is\widetilde{H}_0}w\,ds\,,$$

if $w \in \mathbf{A}(\widetilde{H}_0)$. This follows from the observation that $e^{itH(f)}e^{-it\widetilde{H}_0}$ is a differentiable function of f with derivative $e^{isH(f)}(H(f) - \widetilde{H}_0)e^{-is\widetilde{H}_0}w$, followed by integration with respect to t. It results that

$$Sw = w + \int_{-\pi}^{\pi} e^{isH(f)}gMe^{-is\widetilde{H}_0}w\,ds\,,$$

whence

$$(Sw - w)/g = \int_{-\pi}^{\pi} e^{isH(f)}Me^{-is\widetilde{H}_0}w\,ds\,.$$

As $g \to 0$, the right side converges by strong continuity to

$$\int_{-\pi}^{\pi} e^{isH_0}Me^{-isH_0}w\,ds\,.$$

But as a function of f, $e^{isH_0}M(f)e^{-isH_0} = M(f_s)$, where $f_s(t) = f(t+s)$. Accordingly the integral over s is $\int_{-\pi}^{\pi} M(f_s)\,ds = M(1)$, showing that $S'(0)$ exists and has the value $iM(1)$.

Finally, the convergence of $H(f)$ as a form on $\mathbf{A}(H_0)$ to $H_0 + H(\delta(t))$ follows by the same argument as the establishment of $H(\delta(t))$ as a form (cf. Scholium 6.2). □

Remark. Heuristic perturbation theory gives a formally similar determination of $S'(0)$, modulo an infinite 'coupling constant renormalization'. The higher terms in the symbolic power series expansion of $S(g)$ as a function of g on which heuristic theory is based correspond to higher derivatives of $S(g)$ at $g = 0$. These higher derivatives may be computed as above but are materially dependent on f. The third author has shown in work to be presented elsewhere that as $f \to \delta$, the vacuum expectation value of $S''(g)$ approaches $+\infty$. However, the question of whether $S(g)$ converges in the space of unitary operators as $f \to \delta$ remains open.

It is not expected that $H(f)$ itself converges as $f \to \delta$ but possibly that the expectation value functionals for ground states of the $H(f)$ converge on bounded functions of a finite number of field operators of the form $\phi(z)$, where z is a K-finite vector in $\mathbf{H}$. It is presently an open question whether $H(f)$ is bounded below. We note finally that $H(f)$ and all of the other hermitian operators treated here have selfadjoint extensions by a reality argument similar to that given in [13].

8. Global cohomology

The following notion of cocycle applies generally to representations of the infinite Heisenberg group, but we treat here only the case of the free field representation. The simplest nontrivial example of the type of cohomology involved in the treatment of the Heisenberg group action on the Hilbert-Schmidt operators on $L_2(\mathbb{R}^n)$ in [15]. Here $\mathbb{R}^n$ is replaced effectively by Hilbert space and the Hilbert-Schmidt operators are replaced by operators of the much more singular type that arise in quantum field theory.

Definition 8.1. A global 1-cocycle is a map $z \to F(z)$ from $\mathbf{E}(B)$ to $\mathbf{F}(H)$ that satisfies the conditions

$$W(z')^{-1}F(z)W(z') + F(z') = F(z+z') \qquad (z, z' \in \mathbf{E}(B)). \tag{8.1}$$

A global n-cocycle is a map from $\mathbf{E}(B)^n$ to $\mathbf{F}(H)$, $F(z_1, \ldots, z_n)$, that is a symmetric function of $z_1, \ldots, z_n$, and is a 1-cocycle as a function of z_1 when $z_2, \ldots, z_n$ are fixed.

The structure of such cocycles can be characterized in terms of the holomorphic ('standard') kernels that provide representations for forms as integral operators on $\mathbf{K}$ in terms of its representation by anti-entire functions on $\mathbf{H}$ [9]. The case $n = 1$ is basic, as in §3, and is treated by

Theorem 8.1. *For the given form F on $\mathbf{E}(H)$, there exists a function $K_{z,z'}(u, u')$ defined for $z, z', u, u' \in \mathbf{E}(B)$, that is holomorphic in z', u' and anti-holomorphic in z, u, such that:*

(i) *for some $c, s > 0$ and arbitrary $u, u', z, z' \in \mathbf{E}(B)$,*

$$|K_{z,z'}(u, u')| \le c\exp(\|e^{sB}u\|^2 + \|e^{sB}u'\|^2 + \|e^{sB}z\|^2 + \|e^{sB}z'\|^2);$$

(ii) *the standard kernel for $\delta_z F$ is $K_{z,z}(u, u')$, where $\delta_z F$ denotes the 1-cocycle $W(z)^{-1}FW(z) - F$.*

Conversely, given a cocycle $F(z)$ whose kernels K_z have the form $K_z(u, u') = K_{z,z}(u, u')$ where $K_{z,z'}(u, u')$ is holomorphic in z', u' and antiholomorphic in z, u and which satisfies the estimate

$$|K_{z,z'}(0, 0)| \le c\exp(\|e^{sB}z\|^2 + \|e^{sB}z'\|^2)$$

for some $c, s > 0$, there exists a form F on $\mathbf{E}(H)$ such that

$$F(z) = \delta_z F = W(z)^{-1}FW(z) - F.$$

Proof. Note first that a given map $z \to F(z)$ from $\mathbf{E}(B) \to \mathbf{F}(H)$ forms a 1-cocycle if and only if the standard kernel K_z for $F(z)$ satisfies the equation (where $\sigma = \frac{1}{\sqrt{2}}$)

$$K_{z+z'}(u, u') = K_{z'}(u, u') + K_z(u+\sigma z', u'+\sigma z')e^{-\frac{1}{2}\|z'\|^2 - \sigma\langle z', u\rangle - \sigma\langle u', z'\rangle}. \tag{8.2}$$

This is an immediate deduction from Lemma 4.3 of [9] and the uniqueness of the standard kernel for a given form. Lemma 4.3 of [9] also implies that if F is a given form on $\mathbf{E}(H)$ and if $F(z) = \delta_z F$, then the standard kernel of $F(z)$ has the form

$$K_z(u, u') = K(u + \sigma z, u' + \sigma z)e^{-1/2\|z\|^2 - \sigma\langle z, u\rangle - \sigma\langle u', z\rangle} - K(u, u').$$

By an observation above, a cocycle $F(z)$ cannot be of the form $\delta_z F$ for some form F unless the kernels K_z of the $F(z)$ are obtainable as restrictions to the diagonal of the kernels $K_{z,z'}$ that are holomorphic in z' and anitholomorphic in z. If the form $F(z) = \delta_z F$, then kernel K_z for $F(z)$ is obtainable by setting $z = z'$ in

$$K_{z,z'}(u, u') = K(u+\sigma z, u'+\sigma z')\exp[-\tfrac{1}{2}\langle z', z\rangle - \sigma\langle u', z\rangle - \sigma\langle z', u\rangle] - K(u, u')$$

which is holomorphic in u' and z' and antiholomorphic in u and z. By Theorem 1 of [9], there exist, $c, s > 0$ such that

$$\begin{aligned} |K_{z,z'}(u, u')| &\le c\exp[\|e^{sB}(u + \sigma z)\|^2 + \|e^{sB}(u' + \sigma z')\|^2] \\ &\le c\exp[\|e^{tB}u\|^2 + \|e^{tB}u'\|^2 + \|e^{tB}z\|^2 + \|e^{sB}z\|^2] \end{aligned} \tag{8.3}$$

for some $t > 0$, completing the proof of the direct part of the theorem. To establish the converse, set

$$K(u, u') = K_{\sigma^{-1}u, \sigma^{-1}u'}(0, 0)e^{\langle u' u\rangle}$$

for u, $u' \in \mathbf{E}(B)$. The assumed bound ensures that K is the standard kernel of some form F on $\mathbf{E}(H)$. To show that $F(z) = \delta_z F$, recall that a function of two variables that is holomorphic in one variable and antiholomorphic in the other is determined by its values on the diagonal (Lemma 3.1.1). Using this in conjunction with the observation at the beginning of the proof, it follows that

$$\begin{aligned} &K_{z+z_1, z'+z_2}(u, u') \\ &\quad = K_{z_1, z_2}(u, u') + K_{z, z'}(u + \sigma z_1, u' + \sigma z_2)e^{-\langle z_2, z_1\rangle/2 - \sigma\langle z_2, u\rangle - \sigma\langle u', z_1\rangle}. \end{aligned}$$

It follows that the kernel of $W^{-1}(z)FW(z) - F$ is

$$\begin{aligned} &K(u + \sigma z, u' + \sigma z)e^{-\sigma\langle z, u\rangle - \sigma\langle u', z\rangle - \langle z, z\rangle/2} - K(u, u') \\ &\quad = K_{\sigma^{-1}u+z, \sigma^{-1}u'+z}(0, 0)e^{-\sigma\langle z, u\rangle - \sigma\langle u', z\rangle - \langle z, z\rangle/2 + \langle u'+\sigma z, u+\sigma z\rangle} \\ &\qquad - K_{\sigma^{-1}u, \sigma^{-1}u'}(0, 0)e^{\langle u', u\rangle} \\ &\quad = e^{\langle u', u\rangle}(K_{\sigma^{-1}u+z, \sigma^{-1}u'+z}(0, 0) - K_{\sigma^{-1}u, \sigma^{-1}u'}(0, 0)) \\ &\quad = e^{\langle u', u\rangle}\{K_{z, z}(u, u')e^{-\langle\sigma^{-1}u', \sigma^{-1}u\rangle/2}\} \\ &\quad = K_{z, z}(u, u'). \end{aligned}$$

Thus the given cocycle $F(z) = \delta_z F$, where F is the form whose kernel is K, completing the proof.

Corollary 8.1.1. *The form F in the converse part of the theorem is unique within addition of an additive multiple of I.*

Proof. This follows from the irreducibility of the Heisenberg group (or Weyl system) action on the space of entire vectors for H, in the generalized sense of the absence of any invariant sesquilinear forms other than multiples of the inner product in $\mathbf{K}$ (cf. [9]). □

Corollary 8.1.2. *Let U be a unitary representation of a topological group G on $\mathbf{H}$, and suppose that the cocycle $F(\cdot)$ is invariant under the corresponding action on $\mathbf{K}$; $\Gamma(U(g))F(z)\Gamma(U(g))^{-1} = F(U(g)z)$ for all $g \in G$ and $z \in \mathbf{E}(\mathbf{H})$. Let F be the unique form such that $\delta_z F = F(z)$ and $\langle Fv, v\rangle = 0$. Then F is invariant under the $\Gamma(U(g))$.*

Proof. It is immediate that for an arbitrary form F,

$$\Gamma(V)\delta_z F\Gamma(V)^{-1} = \delta_{V_z}(\Gamma(V)F\Gamma(V)^{-1})$$

for arbitary $V \in U(\mathbf{H})$. In particular, $\delta_z(\Gamma(U(g))F\Gamma(U(g))^{-1}) = F(z)$. By Corollary 8.1.1, $\Gamma(U(g))F\Gamma(U(g))^{-1} = F + c(g)I$, but since the $\Gamma(U(g))$ leave v fixed, $c(g) = 0$. □

The following corollary extends to the infinite-dimensional case the existence part of [15] (but not the inversion formula).

Corollary 8.1.3. *Let $F(z_1, \dots, z_n)$ be a bounded, continuous, and symmetric map from $\mathbf{E}(B)^n$ to the space $L_2(\mathbf{B}(\mathbf{K}))$ of all Hilbert-Schmidt operators on $\mathbf{K}$; and suppose that it is a cocycle as a function of z_1 for arbitrary fixed $z_2, \dots, z_n$.*

Then there exists a unique operator $F \in L_2(\mathbf{B}(\mathbf{K}))$ *such that* $F(z_1, \ldots, z_n) = \delta_{z_1} \cdots \delta_{z_n} F$ *for all* $z_1, \ldots, z_n$ *in* $\mathbf{E}(B)$.

Proof. The case $n = 1$ follows from [9] and Corollary 8.1.1, which establishes the bound $\|F\|_2 \leq \sup_z \|F(z)\|$ (where the subscript 2 indicate the Hilbert-Schmidt norm). The general case proceeds by induction on n. □

As earlier, the arguments for the entire vector case carry over without nontrivial change to the analytic case.

Corollary 8.1.4. *Theorem* 8.1 *and its Corollaries* 8.1.1 *and* 8.1.2 *apply equally if the spaces* $\mathbf{E}(B)$ *and* $\mathbf{E}(H)$ *are replaced by the spaces* $\mathbf{A}(B)$ *and* $\mathbf{A}(H)$ *and the bounds required for all* $s > 0$ *are replaced by the same bounds for some* $s > 0$, *together with the same replacement in the conclusion.*

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Department of Physics, University of Aalborg, Aalborg, Denmark

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824