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KINEMATIC FORMULAS FOR MEAN CURVATURE POWERS OF HYPERSURFACES AND HADWIGER'S THEOREM IN $\mathbb{R}^{2n}$

JIAZU ZHOU

ABSTRACT. We first discuss the theory of hypersurfaces and submanifolds in the m-dimensional Euclidean space leading up to high dimensional analogues of the classical Euler's and Meusnier's theorems. Then we deduce the kinematic formulas for powers of mean curvature of the $(m-2)$-dimensional intersection submanifold $S_0 \cap gS_1$ of two C^2-smooth hypersurfaces S_0, S_1, i.e., $\int_G(\int_{S_0 \cap gS_1} H^{2k} d\sigma)\, dg$. Many well-known results, for example, the C-S. Chen kinematic formula and Crofton type formulas are easy consequences of our kinematic formulas. As direct applications of our formulas, we obtain analogues of Hadwiger's theorem in $\mathbb{R}^{2n}$, i.e., sufficient conditions for one domain K_β to contain, or to be contained in, another domain K_α.

0. INTRODUCTION

Let M^p, N^q be submanifolds of dimensions p, q, respectively, in a homogeneous space G/H and let I be an integral invariant (e.g., volume, surface area, etc.) of the submanifold $M^p \cap gN^q$. Then many works in integral geometry have been concerned with computing integrals of the following type

$$\int_{\{g \in G:\ M^p \cap gN^q \neq \varnothing\}} I(M^p \cap gN^q)\, dg, \tag{0.1}$$

where dg is the normalized kinematic density of G. For example in the case that G is the group of motions in an n-dimensional Euclidean space $\mathbb{R}^n$, M^p and N^q are submanifolds of $\mathbb{R}^n$ and

$$I(M^p \cap gN^q) = \mathrm{Vol}(M^p \cap gN^q)$$

evaluation of (0.1) leads to the formulas of Poincaré, Blaschke, Santaló and others (see [1]). R. Howard [10] obtained a kinematic formula for $I(M^p \cap gN^q) = \mathrm{Vol}(M^p \cap gN^q)$ in a homogeneous space. If $I(M \cap gN) = \chi(M \cap gN)$, where $\chi(\cdot)$ is the Euler-Poincaré characteristic of the intersection $M \cap gN$ of domains M and N in $\mathbb{R}^n$ with smooth boundaries, then (0.1) leads to S. S. Chern's kinematic fundamental formula [2]. Next, assume that $I(M^p \cap gN^q) = \mu(M^p \cap gN^q)$ is one of the integral invariants from the Weyl tube formula. Then (0.1) leads to the Chern-Federer kinematic formula [22]

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for submanifolds of $\mathbb{R}^n$. This integral also leads to the C-S. Chen kinematic formula [15] if we take $I(M\cap gN) = \int_{M\cap gN} \kappa^2(M\cap gN)\,d\sigma$, the total square of curvature of the intersection curve $M\cap gN$ of two compact closed surfaces M, N in $\mathbb{R}^3$. T. Shifrin [24] also obtained a kinematic formula of type (0.1) by letting $I(M^p\cap gN^q)$ be the integral of a Chern class. R. Howard [14] achieved more general kinematic formula in the case that $I(M^p\cap gN^q)$ is a homogeneous polynomial of the second fundamental form of $M^p\cap gN^q$ in a homogeneous space.

As is evident by the above examples, many known kinematic formulas are intrinsic. Only a few of them deal with the extrinsic invariants, for example, C-S. Chen's formula. In fact, Howard [14] gives some extrinsic kinematic formulas. In this paper, we will give kinematic formulas for powers of mean curvature in the case that M, N are hypersurfaces of Euclidean space $\mathbb{R}^n$. One of our results is the following.

Main Theorem. *Let S_i $(i = 0, 1)$ be compact smooth hypersurfaces of class C^2 in $\mathbb{R}^n$. Denote by H the mean curvature of the $(n-2)$-dimensional intersection submanifolds $S_0\cap gS_1$ and by $\tilde{\kappa}_n^r(S_i)$ (defined in equation (4.15) below) the rth integral of the principal curvatures of S_i, respectively, where $g\in G$, the rigid motions in $\mathbb{R}^n$. Then for any integer k with $0\le 2k\le n-1$ we have the kinematic formula*

$$\int_G\left(\int_{S_0\cap gS_1} H^{2k}d\sigma\right)dg = \sum_{\substack{i,j,l\\ i+j+l=k\\ l\ (even)}} c_{ijkln}\,\tilde{\kappa}_n^{l+2j}(S_0)\,\tilde{\kappa}_n^{l+2i}(S_1), \tag{0.2}$$

where dg is the kinematic density for $\mathbb{R}^n$, $d\sigma$ is the volume element and the c's are constants depending on the indices.

In the first part of this paper we seek formulas for submanifolds of $\mathbb{R}^n$ in which the integrand in (0.1) can be expressed as a homogeneous polynomial in the components of the curvature tensor of the submanifolds. If (M, N) is a pair of hypersurfaces in $\mathbb{R}^n$, we obtain kinematic formulas in which the right-hand side of (0.1) is expressed explicitly in terms of the total principal curvatures of the given hypersurfaces. Many well-known results (for example, the C-S. Chen formula [15] and Crofton type formulas) are easy consequences of our theorems. It is possible to use the ideas of this paper to obtain the general kinematic formula for p-dimensional submanifolds M^p and q-dimensional submanifolds N^q, i.e., for integrals of type $\int_G\left(\int_{M^p\cap gN^q} H^k\,d\sigma\right)dg$, $(0\le k\le p+q-n)$, in which the right-hand side of (0.1) can be expressed as a polynomial of the curvatures of the submanifolds.

In section 1 we give some basic concepts of differential geometry of hypersurfaces in $\mathbb{R}^n$ which are generalizations of the results of classical differential geometry (for example, Meusnier's and Euler's theorem). Some preliminaries of integral geometry are introduced in section 2. The basic concepts are in Santaló's book [1]. We obtain our kinematic formula (Theorem 1) and its consequences (for example, C-S. Chen's formula, Crofton type formulas) in section 3 and the proof of Main Theorem is given in detail in section 4. The methods used in this part are mostly based on those of Chern [26] and Howard [14].

In the second part of this paper we derive the analogues of Hadwiger's containment theorem. The general principle underlying this investigation can be briefly described as follows:

Containment problem. Let K_α, K_β be two suitable domains in the Euclidean space $\mathbb{R}^n$, for example, two convex bodies with interior points. Let G be the group of rigid motions of $\mathbb{R}^n$ and let m be its (suitably normalized) kinematic measure. Then

$$\begin{aligned} m\{g \in G : gK_\beta \subset K_\alpha \text{ or } gK_\beta \supset K_\alpha\} \\ = m\{g \in G : K_\alpha \cap gK_\beta \neq \varnothing\} - m\{g \in G : \partial K_\alpha \cap g\partial K_\beta \neq \varnothing\}. \end{aligned} \tag{0.3}$$

(If K_α, K_β are not convex, one assumes that their boundaries are connected.) By integral geometric methods it is possible to estimate the measure $m\{g \in G : K_\alpha \cap gK_\beta \neq \varnothing\}$ from below and the measure $m\{g \in G : \partial K_\alpha \cap g\partial K_\beta \neq \varnothing\}$ from above in terms of geometric invariants of K_α and K_β. This will result in an inequality of the form

$$m\{g \in G : gK_\beta \subset K_\alpha \text{ or } gK_\beta \supset K_\alpha\} \geq f(A_\alpha^1, \cdots, A_\alpha^k; A_\beta^1, \cdots, A_\beta^k), \tag{0.4}$$

where each A_γ^j is an integral geometric invariant of K_γ $(\gamma = \alpha, \beta)$. One can then state the following conclusion: If $f(A_\alpha^1, \cdots, A_\alpha^k; A_\beta^1, \cdots, A_\beta^k) > 0$, then there is a rigid motion g such that either gK_β is contained in K_α or gK_β contains K_α.

The first classical result is due to Hadwiger (see [1, 5, 6], 1941) who was the first to use the method of integral geometry to obtain some sufficient conditions for containment problem in Euclidean plane $\mathbb{R}^2$ and later in projective plane $\mathbb{R}P^2$ and hyperbolic plane H^2. Since then many mathematicians have been interested in getting a version of the containment problem in space $\mathbb{R}^n$, that is, getting sufficient conditions to insure that a given domain K_β of surface area F_β, bounded by a piecewise smooth boundary ∂D_β, of volume V_β may be moved 'inside' another domain K_α of surface area F_α, bounded by a piecewise smooth boundary ∂K_α, of volume V_α. Grinberg, Ren and Zhou (see [4, 28]) obtained a variation of Hadwiger's theorem in the plane of constant curvature ϵ and reinterpreted those sufficient conditions by the isoperimetric deficits of domains involved. But there was no general result or analogue of Hadwiger's theorem in space $\mathbb{R}^n$ $(n \geq 3)$ until works [7, 9, 10, 11, 12, 17] appeared, even if some very strong restrictions are put onto the domains involved (for example, the domains are supposed to be convex bodies and some topological conditions are put onto their boundaries and intersection). The situation of n-dimensional space $\mathbb{R}^n$ $(n \geq 3)$ is much more complex and difficult than that of 2-dimensional plane $\mathbb{R}^2$. None of the formulas and methods in $\mathbb{R}^2$ can be parallelly carried out in higher dimensions. One also hopes that analogous results may be achieved for hyperbolic and projective spaces of higher dimension. We have some results in a later paper [29].

By restricting the domains K_γ to the convex category, the author [9, 10, 11] obtained some sufficient conditions in $\mathbb{R}^n$ $(n \geq 3)$ which are the generalizations of Hadwiger's theorem to $\mathbb{R}^n$. Zhang [7] derived a sufficient condition in $\mathbb{R}^3$ for the domains belonging to the convex category. Goodey [13] obtained some related results by putting topological restrictions on the convex domains involved and their intersection $\partial K_\alpha \cap g\mathring{K}_\beta$. Later we removed the convexity restrictions and obtained analogues of Hadwiger's theorem in $\mathbb{R}^3$ [12, 17].

In section 5 of the second part of this paper we will give direct applications of our kinematic formulas and obtain the $\mathbb{R}^{2n}$ analogues (Theorem 4 and Theorem 5) of Hadwiger's theorem. It is surely possible to obtain analogues of Hadwiger's theorem in R^{2n+1}. But for this purpose we must seek new kinematic formulas. In fact, we can only obtain kinematic "inequalities".

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1. Differential geometry of the hypersurface in $\mathbb{R}^m$

Let M be an n-dimensional submanifold of $\mathbb{R}^m$. The second fundamental form of M at $x \in M$ is a symmetric bilinear mapping

$$h_x : M_x \times M_x \longrightarrow M_x^\perp, \tag{1.1}$$

where M_x is the tangent bundle of M and $M_x^\perp$ is the normal bundle of M at x. If $e_1, \cdots, e_m$ is a orthonormal basis of $\mathbb{R}^m$ such that $e_1, \cdots, e_n$ is a basis of M_x and $e_{n+1}, \cdots, e_m$ is a basis of $M_x^\perp$, then the components of h_x in this basis are the numbers $(h_x)_{ij}^\alpha = \langle h_x(e_i, e_j), e_\alpha\rangle$, $1 \le i, j \le n$, $n+1 \le \alpha \le m$, where $\langle\ ,\ \rangle$ is the usual inner product in $\mathbb{R}^n$. Associated to the bundle of orthonormal frames $\{e_A\}$ over M we have the coframes, which consist of m linearly independent linear differential forms $\{\omega_A\}$, such that the metric in M is

$$ds^2 = \sum_i \omega_i^2. \tag{1.2}$$

Restricted to M, we have $\omega_\alpha = 0$ and $0 = d\omega_\alpha = \sum_i \omega_{\alpha i} \wedge \omega_i$. By Cartan's lemma, we get

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \tag{1.3}$$

the second fundamental form of M with respect to frames $\{e_\alpha\}$ and coframes $\{\omega_i\}$ is

$$II = \sum_{i,j,\alpha} (h_{ij}^\alpha \omega_i \omega_j) e_\alpha. \tag{1.4}$$

The mean curvature vector is defined by

$$\vec{H} = \sum_\alpha \left(\frac{1}{n}\sum_i h_{ii}^\alpha\right) e_\alpha. \tag{1.5}$$

The square of mean curvature of M is

$$H^2 = \frac{1}{n^2}\sum_{\alpha}\left(\sum_i h_{ii}^{\alpha}\right)^2. \tag{1.6}$$

Let $X^n \subseteq Y^m \subseteq \mathbb{R}^{n+p}$ be two submanifolds. If we choose the frame

$$(x; e_1, \cdots, e_n, e_{n+1}, \cdots, e_m, e_{m+1}, \cdots, e_{n+p}) \tag{1.7}$$

such that $e_1, \cdots, e_n \in T(X^n)$ and $e_1, \cdots, e_m \in T(Y^m)$, then we have the mean curvature vector $\vec{H}_X$ of X^n, the mean curvature vector $\vec{H}_Y$ of Y^m, respectively, are

$$\vec{H}_X = \frac{1}{n}\sum_{\alpha=n+1}^{m}\left(\sum_i h_{ii}^{\alpha}\right)e_\alpha + \frac{1}{n}\sum_{\beta=m+1}^{n+p}\left(\sum_i h_{ii}^{\beta}\right)e_\beta = \vec{G}_{Y^m} + \vec{N}_{Y^m}, \tag{1.8}$$

$$\vec{H}_Y = \frac{1}{m}\sum_{\beta=m+1}^{n+p}\left(\sum_i h_{ii}^{\beta}\right)e_\beta = \frac{n}{m}\vec{N}_{Y^m}, \tag{1.9}$$

where $\vec{G}_{Y^m}$ is defined as the *mean geodesic curvature* of Y^m at $x \in X^n$ and $\vec{N}_{Y^m}$ is the *mean normal curvature* of Y^m at $x \in X^n$. Their lengths, i.e., $|\vec{G}_{Y^m}| = \kappa_g(Y^m)$, $|\vec{N}_{Y^m}| = \kappa_n(Y^m)$, are called, respectively, the geodesic curvature, normal curvature of Y^m (at $x \in Y^m$ in the direction X^n). Of course they are invariants. It is obvious that *two submanifolds X and X' of same dimension which are tangent at x have the same normal mean curvatures.* This actually is the classical Meusnier's theorem when X is a smooth curve in a surface Y^2 of 3-dimensional Euclidean space $\mathbb{R}^3$.

Let Σ be a hypersurface in an m-dimensional Euclidean space $\mathbb{R}^m$. For each point $x \in \Sigma$, choose the frames $\{x; e_1, e_2, \cdots, e_{m-1}, e_m\}$ such that $e_1, \cdots, e_{m-1} \in T_x(\Sigma)$, and $e_m \in N_x(\Sigma)$, the normal vector at $x \in \Sigma$. Then we have the fundamental equations

$$\begin{aligned} dx &= \sum_{i=1}^{m-1}\omega_i e_i, \\ de_i &= \sum_{j=1}^{m-1}\omega_{ij}e_j + \omega_{im}e_m, \quad \omega_{ij} = \omega_{ji}, \\ de_m &= -\sum_{i=1}^{m-1}\omega_{im}e_i, \end{aligned} \tag{1.10}$$

and the integrability conditions

$$\begin{aligned} &\sum_{i=1}^{m-1}\omega_i \wedge \omega_{im} = 0, \quad d\omega_i = \sum_{j=1}^{m-1}\omega_j \wedge \omega_{ij}, \\ &d\omega_{ij} = \sum_{k=1}^{m-1}\omega_{ik}\wedge\omega_{kj} + \omega_{im}\wedge\omega_{mj}, \\ &d\omega_{im} = \sum_{j=1}^{m}\omega_{ij}\wedge\omega_{jm}. \end{aligned} \tag{1.11}$$

From the above equations, there are functions h_{ij} such that

$$\omega_{im} = \sum_{j=1}^{m-1} h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{1.12}$$

Now we get

$$\mathrm{I} = \langle dx, dx\rangle = \sum_{i=1}^{m-1} (\omega_i)^2, \tag{1.13}$$

$$\mathrm{II} = -\langle dx, de_m\rangle = \sum_{j=1}^{m-1} \omega_j\omega_{jm} = \sum_{i,j=1}^{m-1} h_{ij}\omega_i\omega_j, \tag{1.14}$$

which are called, respectively, the *first* and *second fundamental form* of Σ (II is only defined up to a sign). The *principal directions* are the directions which diagonalize (h_{ij}), i.e., the eigenvalue directions of (h_{ij}). It is known that at each point of a hypersurface Σ in $\mathbb{R}^m$ there are $m-1$ principal directions and $m-1$ principal curvatures $\kappa_1, \cdots, \kappa_{m-1}$.

Assume that $e_1, \cdots, e_{m-1}$ are $m-1$ principal directions, for each unit vector $v \in T_x(\Sigma)$, $v = \sum v_i e_i$. Then we have the *normal mean curvature* at $x \in \Sigma$ along the direction v,

$$\kappa_\Sigma(v) = \frac{\mathrm{II}(v)}{\mathrm{I}(v)} = \frac{\sum \kappa_i\omega_i^2(v)}{\sum \omega_i^2(v)} = \sum_{i=1}^{m-1} \kappa_i v_i^2. \tag{1.15}$$

This is *Euler's formula* for higher dimensional hypersurface. In particular, the normal curvatures along the directions of frames e_j are

$$\kappa_\Sigma(e_j) = \frac{\mathrm{II}(e_j)}{\mathrm{I}(e_j)} = h_{jj} = \kappa_j, \qquad j = 1, \cdots, m-1. \tag{1.16}$$

If $v \in T_x(\Sigma)$, a unit vector, then v can be expressed by the spherical coordinates

(1.17)
$$v = (\cos\phi_1\cos\phi_2\cdots\cos\phi_{m-3}\cos\phi_{m-2}, \cos\phi_1\cos\phi_2\cdots\cos\phi_{m-3}\sin\phi_{m-2}, \\ \cos\phi_1\cos\phi_2\cdots\cos\phi_{m-4}\sin\phi_{m-3}, \cdots, \cos\phi_1\sin\phi_2, \sin\phi_1),$$

with

$$0 \le \phi_1 < 2\pi; \quad 0 \le \phi_2, \cdots, \phi_{m-2} \le \pi, \tag{1.18}$$

where ϕ_i are the angles between v and the principal directions e_i. Then we have

$$\begin{aligned} \kappa_\Sigma(v) = {} & \kappa_1\cos^2\phi_1\cos^2\phi_2\cdots\cos^2\phi_{m-3}\cos^2\phi_{m-2} \\ & + \kappa_2\cos^2\phi_1\cos^2\phi_2\cdots\cos^2\phi_{m-3}\sin^2\phi_{m-2} \\ & + \kappa_3\cos^2\phi_1\cos^2\phi_2\cdots\cos^2\phi_{m-4}\sin^2\phi_{m-3} + \cdots \\ & + \kappa_{m-2}\cos^2\phi_1\sin^2\phi_2 + \kappa_{m-1}\sin^2\phi_1. \end{aligned} \tag{1.19}$$

This is *Euler's formula* in classical differential geometry when $m = 3$. The mean curvature of Σ is

$$H_\Sigma = \frac{1}{m-1}\operatorname{Tr}(\mathrm{II}) = \frac{1}{m-1}\sum_{i=1}^{m-1} h_{ii} = \frac{1}{m-1}\sum_{i=1}^{m-1}\kappa_i. \tag{1.20}$$

Now for the frame $e_1, \cdots, e_n$ and $e_1', \cdots, e_n'$ attached to S_0 and S_1 the mean curvature vectors of S_0, S_1, respectively, are

$$\vec{H}_0 = \frac{1}{n-1}\sum_{i=1}^{n-1} h_{ii}e_n, \tag{1.21}$$

$$\vec{H}_1 = \frac{1}{n-1}\sum_{i=1}^{n-1} h_{ii}'e_n'. \tag{1.22}$$

The mean curvature vector of $S_0 \cap gS_1$ is

$$\begin{aligned}\vec{H} &= \frac{1}{n-2}\left[\sum_{i=1}^{n-2} h_{ii}^{n-1}e_{n-1} + \sum_{i=1}^{n-2} h_{ii}e_n\right]\\ &= \frac{1}{n-2}\left[\sum_{i=1}^{n-2} h'^{n-1}_{ii}e_{n-1}' + \sum_{i=1}^{n-2} h_{ii}'e_n'\right].\end{aligned} \tag{1.23}$$

Let U, V be an orthonormal basis of the normal space of $S_0 \cap gS_1$ and H_0, H_1 be the mean curvatures of S_0, S_1, respectively. We choose U in the direction of $\vec{H}$, the direction of the mean curvature of $S_0 \cap gS_1$. Then

$$\operatorname{Tr}\langle U, d^2x\rangle = (n-2)H_0, \quad \operatorname{Tr}\langle V, d^2x\rangle = 0. \tag{1.24}$$

Let

$$e_n = \cos\theta\, U + \sin\theta\, V, \tag{1.25}$$

then

$$\langle e_n, d^2x\rangle = \cos\theta\langle U, d^2x\rangle + \sin\theta\langle V, d^2x\rangle. \tag{1.26}$$

Hence

$$\begin{aligned}\operatorname{Tr}\langle e_n, d^2x\rangle &= h_{11} + \cdots + h_{n-2\,n-2}\\ &= \kappa(e_1) + \cdots + \kappa(e_{n-2}) = (n-1)H_0 - \kappa(e_{n-1}).\end{aligned} \tag{1.27}$$

Thus

$$\langle \vec{H}, e_n\rangle = (\vec{H}\cdot e_n) = \frac{1}{n-2}[(n-1)H_0 - \kappa(e_{n-1})] \tag{1.28}$$

is an invariant, where $\kappa(e_{n-1})$ is the principal curvature of S_0 in the direction e_{n-1}. Similarly,

$$\langle \vec{H}, e_n'\rangle = (\vec{H}\cdot e_n') = \frac{1}{n-2}[(n-1)H_1 - \kappa'(e_{n-1}')] \tag{1.29}$$

is also an invariant and where $\kappa'(e_{n-1}')$ is the principal curvature of S_1 in the direction e_{n-1}'.

By the following formula

$$(1.30)\quad \begin{aligned}(a_1+\cdots+a_{m-1})^P &= \sum_{i_1+\cdots+i_{m-1}=P} \frac{P!}{i_1!\cdots i_{m-1}!} a_1^{i_1}\cdots a_{m-2}^{i_{m-2}} a_{m-1}^{i_{m-1}} \\ &= \sum_{i_1,\cdots,i_{m-1}} c_{i_1\cdots i_{m-1}} a_1^{i_1}\cdots a_{m-2}^{i_{m-2}} a_{m-1}^{i_{m-1}},\end{aligned}$$

we have

$$(1.31)\qquad \kappa_\Sigma^P(v) = \sum_{i_1,\cdots,i_{m-1}} c_{i_1\cdots i_{m-1}} v_1^{2i_1}\cdots v_{m-1}^{2i_{m-1}} \kappa_1^{i_1}\kappa_2^{i_2}\cdots\kappa_{m-1}^{i_{m-1}},$$

where

$$(1.32)\qquad c_{i_1\cdots i_{m-1}} = \frac{P!}{i_1!\cdots i_{m-1}!}, \qquad i_1+\cdots+i_{m-1}=P.$$

If $d\sigma$ denotes the area element of Σ, we define the *rth homogeneous mean curvature* as

$$(1.33)\qquad \tilde{M}^r_{i_1\cdots i_{m-1}}(\Sigma) = \binom{m-1}{r}^{-1} \int_\Sigma (\kappa_1^{i_1}\cdots\kappa_{m-1}^{i_{m-1}})\, d\sigma,$$

where $(\kappa_1^{i_1}\cdots\kappa_{m-1}^{i_{m-1}})$ denotes the rth homogeneous polynomial of the principal curvatures. In particular, $\tilde{M}^0$ is the area and $\tilde{M}^1$ is the total mean curvature of Σ.

When $m=3$, all the results here agree with those in classical differential geometry.

2. Preliminaries on integral geometry

Let S_0 and S_1 be two piecewise smooth hypersurfaces in $\mathbb{R}^n$ of class C^2. We assume S_0 fixed and S_1 moving under the rigid motion g of $\mathbb{R}^n$ with the kinematic density dg. Consider generic positions gS_1 so that the intersection $S_0\cap gS_1$ is an $(n-2)$-dimensional manifold. Let $x\in S_0\cap gS_1$ and consider the orthonormal frames $(x; e_1,\cdots,e_n)$ such that $\operatorname{span}\{e_1,\cdots,e_{n-2}\} = T_x(S_0\cap gS_1)$, the tangent $(n-2)$-space to $S_0\cap gS_1$ at x. Let e'_{n-1}, e'_n be the unit vectors that are, respectively, tangent and normal to S_1 at x, so that $(x; e_1,\cdots,e_{n-2}, e'_{n-1}, e'_n)$ is a second orthonormal frames of origin x. We denote the kinematic density dT_0 on S_0 (i.e., the density for sets of frames $(x; e_1,\cdots,e_n)$ such that e_n is the unit normal to S_0) by

$$(2.1)\quad \begin{aligned} dT_0 = \bigwedge_i (dx\cdot e_i) \bigwedge_{h<j} (e_h\cdot de_j) \wedge (dx\cdot e'_{n-1}) \wedge (de_1\cdot e'_{n-1}) \\ \wedge\cdots\wedge (de_{n-2}\cdot e'_{n-1}), \qquad i,h,j=1,2,\cdots,n-2.\end{aligned}$$

Similarly, the kinematic density on S_1 (density for frames $(x; e_1,\cdots,e_{n-2}, e'_{n-1}, e'_n)$ such that e'_n is the unit normal to S_1) is denoted by

$$(2.2)\qquad dT_1 = \bigwedge_i (dx\cdot e_i) \bigwedge_{h<j} (e_h\cdot de_j), \quad i,h,j=1,2,\cdots,n-1,$$

so that the kinematic density for $\mathbb{R}^n$ is

$$(2.3)\qquad dg = dT_1\wedge(dx\cdot e_n)\wedge(de_1\cdot e_n)\wedge\cdots\wedge(de_{n-1}\cdot e_n).$$

The kinematic density on $S_0 \cap gS_1$ (i.e., density for frames $(x; e_1, \cdots, e_{n-2})$ attached to $S_0 \cap gS_1$) is defined by

$$dT_{01} = \bigwedge_i (dx \cdot e_i) \bigwedge_{h<j} (e_h \cdot de_j), \quad i, h, j = 1, 2, \cdots, n-2. \tag{2.4}$$

Then we have the basic formula (see [1, pp. 262, (15.35)])

$$dT_{01} \wedge dg = \sin^{n-1} \phi \, d\phi \wedge dT_0 \wedge dT_1, \tag{2.5}$$

where ϕ is the angle between S_0 and S_1 (i.e., between e_n and e'_n).

If L_1 is a hyperplane, assume that $dL_{n-1[x]}$ is the density of the $(n-1)$-plane about x and that dL_{n-1} is the density for sets of $(n-1)$-plane in $\mathbb{R}^n$. Then we have (see [1, pp. 244, (14.64)])

$$d\sigma(x) \wedge dL_{n-1} = \sin^{n-1}\phi \, d\phi \wedge d\sigma^{(0)}(x) \wedge dL_{n-1[x]}, \tag{2.6}$$

where $d\sigma(x)$ is the volume element of the intersection submanifolds $S_0 \cap gL_1$ and $d\sigma^{(0)}(x)$ is the volume element of S_0.

Denote by

$$\begin{aligned} \kappa_n(S_0) &= \langle \overrightarrow{H}, e_n \rangle = \frac{1}{n-2}[(n-1)H_0 - \kappa(e_{n-1})], \\ \kappa_n(S_1) &= \langle \overrightarrow{H}, e'_n \rangle = \frac{1}{n-2}[(n-1)H_1 - \kappa'(e'_{n-1})], \end{aligned} \tag{2.7}$$

the normal mean curvatures of S_0, S_1, respectively. Then by the laws of sines and cosines we have

Lemma 1. *Let H be the mean curvature of the intersection manifold $S_0 \cap gS_1$ of two compact hypersurfaces S_0 and S_1. Denoted by ϕ the angle between S_0 and S_1, i.e., $\cos\phi = (e_n, e'_n)$. Then we have*

$$\begin{aligned} H^2 \sin^2 \phi &= (\overrightarrow{H} \cdot e_n)^2 + (\overrightarrow{H} \cdot e'_n)^2 - 2(\overrightarrow{H} \cdot e_n)(\overrightarrow{H} \cdot e'_n)\cos\phi \\ &= \kappa_n^2(S_0) + \kappa_n^2(S_1) - 2\kappa_n(S_0)\,\kappa_n(S_1)\cos\phi, \end{aligned} \tag{2.8}$$

or

$$H \sin\phi = (\kappa_n^2(S_0) + \kappa_n^2(S_1) - 2\kappa_n(S_0)\,\kappa_n(S_1)\cos\phi)^{1/2}; \quad 0 \le \phi < \pi. \tag{2.9}$$

When $n = 3$, (2.8) becomes a formula of classical differential geometry in $\mathbb{R}^3$.

Lemma 2. *Let $\kappa_n^{(i)}$ $(i = 0, 1)$ be the normal curvatures of surfaces S_i in 3-dimensional Euclidean space $\mathbb{R}^3$. Denoted by κ the curvature of the intersection curve $S_0 \cap gS_1$ of two compact surfaces S_0 and S_1 and by ϕ the angle between S_0 and S_1, i.e., $\cos\phi = (e_3, e'_3)$. Then we have*

$$\kappa \sin\phi = [(\kappa_n^{(0)})^2 + (\kappa_n^{(1)})^2 - 2\kappa_n^{(0)}\kappa_n^{(1)}\cos\phi]^{1/2}. \tag{2.10}$$

From the formula

$$(a_1 + a_2 + a_3)^k = \sum_{\substack{j_1, j_2, j_3 \\ j_1+j_2+j_3=k}} \frac{k!}{j_1! j_2! j_3!} a_1^{j_1} a_2^{j_2} a_3^{j_3}, \tag{2.11}$$

for $0 \le 2k \le n-1$ we have

$$(2.12)\quad \begin{aligned} H^{2k}\sin^{2k}\phi &= \left(\kappa_n^2(S_0)+\kappa_n^2(S_1)-2\kappa_n(S_0)\,\kappa_n(S_1)\cos\phi\right)^k \\ &= \sum_{\substack{i,j,l\\ i+j+l=k}} \frac{(-2)^l k!}{i!j!l!}\,\kappa_n^{l+2i}(S_0)\,\kappa_n^{l+2j}(S_1)\cos^l\phi. \end{aligned}$$

From (2.5) and (2.12), we come to

(2.13)

$$\begin{aligned} &H^{2k}dT_{01}\wedge dg \\ &\quad = \sum_{\substack{i,j,l\\ i+j+l=k}} \frac{(-2)^l k!}{i!j!(l)!}\,\kappa_n^{l+2i}(S_0)\,\kappa_n^{l+2j}(S_1)\cos^l\phi\,\sin^{n-2k-1}\phi\,d\phi\wedge dT_0\wedge dT_1, \end{aligned}$$

with

$$(2.14)\qquad 0\le\phi<\pi,$$

the angle between S_0 and S_1 and dg the density for $\mathbb{R}^n$. Also

(2.15)

$$\int_0^\pi \cos^l\phi\,\sin^{n-2k-1}\phi\,d\phi = \begin{cases} \dfrac{(l-1)!!(n-2k-2)!!}{(n-k-i-j-1)!!}\cdot\pi, & l \text{ even},\ n \text{ odd};\\[2mm] \dfrac{(l-1)!!(n-2k-2)!!}{(n-k-i-j-1)!!}\cdot 2, & l \text{ even},\ n \text{ even};\\[2mm] 0, & l \text{ odd}.\end{cases}$$

Let

$$(2.16)\qquad C_{ijlkn} = \begin{cases} \dfrac{2^l k!(l-1)!!(n-2k-2)!!}{i!j!l!(n-k-i-j-1)!!}\cdot\pi, & n \text{ odd};\\[2mm] \dfrac{2^{l+1} k!(l-1)!!(n-2k-2)!!}{i!j!l!(n-k-i-j-1)!!}, & n \text{ even}.\end{cases}$$

Then we have

Lemma 3. *Let S_i $(i=0,1)$ be smooth compact hypersurfaces of class C^2 in an n-dimensional Euclidean space $\mathbb{R}^n$. Then for any k with $0\le 2k\le n-1$ we have*

$$(2.17)\quad \begin{aligned} &\int_{\{g\,:\,S_0\cap gS_1\neq\varnothing\}}\left(\int H^{2k}dT_{01}\right)dg \\ &\qquad = \sum_{\substack{i,j,l\\ i+j+l=k\\ l \text{ even}}} C_{ijlkn}\int\kappa_n^{l+2i}(S_0)\,dT_0\int\kappa_n^{l+2j}(S_1)\,dT_1. \end{aligned}$$

3. Kinematic formulas for hypersurfaces

Let S be a smooth hypersurface of class C^2 in an n-dimensional Euclidean space $\mathbb{R}^n$ and $\kappa_n(S)$ be the normal mean curvature of S. Denoted by $\tilde{\kappa}_n^r(S)$ the rth total normal mean curvature of S, i.e.,

$$(3.1)\qquad \tilde{\kappa}_n^r(S) = \int_S \kappa_n^r(S)\,d\sigma,$$

where $d\sigma$ is the volume element. Let

$$C(n) = O_{n-2}^2 O_{n-3} \cdots O_1 ,$$

$$J^{(1)}_{i_1\cdots i_{n-1}} = \binom{n-1}{l+2i} c_{i_1\cdots i_{n-1}} I(i_1, \cdots, i_{n-1}), \tag{3.2}$$

$$J^{(0)}_{j_1\cdots j_{n-1}} = \binom{n-1}{l+2j} c_{j_1\cdots j_{n-1}} I(j_1, \cdots, j_{n-1}),$$

$$i_1 + \cdots + i_{n-1} = l + 2i, \quad j_1 + \cdots + j_{n-1} = l + 2j, \tag{3.3}$$

$$I(i_1, \cdots, i_{n-1}) = O_{n-2} \frac{(2i_1-1)!! \cdots (2i_{n-1}-1)!!}{(n-1)(n+1)\cdots(n+2p-3)}, \tag{3.4}$$

where $p = \sum_{k=1}^{n-1} i_k$. O_m is the surface area of the m-dimensional unit sphere and its value is

$$O_m = \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)}, \tag{3.5}$$

where Γ denotes the gamma function. Then we have the following kinematic formula (3.6) which can be explicitly expressed as integrals of the principal curvatures of S_0 and S_1, i.e.,

Theorem 1. *Let S_i $(i = 0, 1)$ be two compact smooth hypersurfaces of class C^2 in an n dimensional Euclidean space $\mathbb{R}^n$. Then for any integer k with $0 \le 2k \le n-1$ we have the following kinematic formula*

$$\begin{aligned} &\int_G \left(\int_{S_0 \cap gS_1} H^{2k}\, d\sigma \right) dg \\ &\quad = C(n) \sum_{i,j,l} C_{ijkln} \sum_{i_1, \cdots, i_{n-1}} J^{(1)}_{i_1\cdots i_{n-1}} \tilde{M}^{l+2i}_{i_1\cdots i_{n-1}}(S_1) \\ &\quad\quad \times \sum_{j_1, \cdots, j_{n-1}} J^{(0)}_{j_1\cdots j_{n-1}} \tilde{M}^{l+2j}_{j_1\cdots j_{n-1}}(S_0). \end{aligned} \tag{3.6}$$

Let Σ be a C^2-smooth hypersurface in $\mathbb{R}^n$, and let R_Σ, H_Σ be, respectively, the scalar curvature, the mean curvature of Σ. Denote by $\tilde{R}_\Sigma$ and $\tilde{H}_\Sigma^{(2)}$ the total scalar curvature and the total square of mean curvature of Σ, i.e.,

$$\tilde{R}_\Sigma = \int_\Sigma R_\Sigma\, d\sigma, \quad \tilde{H}_\Sigma^{(2)} = \int_\Sigma H_\Sigma^2\, d\sigma. \tag{3.7}$$

If $k = 1$ then we have

Corollary 1. *Let S_i $(i = 0, 1)$ be C^2-smooth closed hypersurfaces in $\mathbb{R}^n$. Denote by F_i the surface area of S_i. Then we have*

$$\begin{aligned} \int_G \left(\int_{S_0 \cap gS_1} H^2\, d\sigma \right) dg &= \frac{\pi\, O_{n-2}^2 O_{n-3} \cdots O_1}{(n+1)(n-1)} \\ &\quad \times \left\{ 3(n-1)^2 (F_0 \tilde{H}_1^{(2)} + F_1 \tilde{H}_0^{(2)}) - 4(F_0 \tilde{R}_1 + F_1 \tilde{R}_0) \right\}. \end{aligned} \tag{3.8}$$

When $k = 1$ and $n = 3$, the scalar curvature is Gaussian curvature. Then C-S. Chen's kinematic formula [15] is an easy consequence of our Corollary 1.

Corollary 2. *Let S_i $(i = 0, 1)$ be compact C^2-smooth surfaces in $\mathbb{R}^3$. Denote by H_i, R_i, $\tilde{H}_i^{(2)}$ and $\tilde{R}_i$, respectively, the mean curvature, Gaussian curvature, the total square of mean curvature and the total Gaussian curvature of S_i. Then we have*[1]

$$\int_G \left(\int_{S_0 \cap gS_1} \kappa^2 \, ds \right) dg = 2\pi^3 (3\tilde{H}_0^{(2)} - \tilde{R}_0) F_1 + 2\pi^3 (3\tilde{H}_1^{(2)} - \tilde{R}_1) F_0, \tag{3.9}$$

where κ is the curvature of the intersection curves $S_0 \cap gS_1$, ds is the arc element of $S_0 \cap gS_1$ and F_i are the surface areas of S_i.

Let

$$\begin{aligned} C_{nk} &= \int_0^\pi \sin^{n-2k-1} \phi \, d\phi \\ &= \begin{cases} \dfrac{n-2k-2}{n-2k-1} \cdot \dfrac{n-2k-4}{n-2k-3} \cdots \dfrac{4}{5} \cdot \dfrac{3}{2} \cdot 2, & n \text{ even}, \\ \dfrac{n-2k-2}{n-2k-1} \cdot \dfrac{n-2k-4}{n-2k-3} \cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \pi, & n \text{ odd}, \end{cases} \end{aligned} \tag{3.10}$$

$$c'_{i_1 \cdots i_{n-1}} = \binom{n-1}{2k} \frac{(2k)!}{i_1! \cdots i_{n-1}!} \frac{(2i_1 - 1)!! \cdots (2i_{n-1} - 1)!!}{(n-1)(n+1) \cdots (n+4k-3)}, \tag{3.11}$$

we also have

Theorem 2. *Let L_1 be a hyperplane and let S_0 be a closed hypersurface in $\mathbb{R}^n$. Denote by $\tilde{M}^r_{i_1 \cdots i_{n-1}}$ the rth homogeneous mean curvature of S_0 and H the mean curvature of submanifold of $S_0 \cap gL_1$. Then for any k $(0 \le 2k \le n-1)$, we have*

$$\begin{aligned} &\int_G \left(\int_{S_0 \cap gL_1} H^{2k} \, d\sigma \right) dg \\ &\quad = O_{n-2}^3 O_{n-3} \cdots O_1 \, C_{nk} \sum_i \sum_{\substack{i_1, \cdots, i_{n-1} \\ i_1 + \cdots + i_{n-1} = i}} c'_{i_1 \cdots i_{n-1}} \tilde{M}^i_{i_1 \cdots i_{n-1}} (S_0). \end{aligned} \tag{3.12}$$

Theorem 2 can be restated as follows:

Corollary 3. *Let $AG(n, n-1)$ be the Grassmannian manifold of all affine hyperplanes in $\mathbb{R}^n$. Then for any compact hypersurface S in $\mathbb{R}^n$ and for k $(0 \le 2k \le n-1)$, we have the following Crofton type formula*

$$\begin{aligned} &\int_{AG(n, n-1)} \left(\int_{S \cap L} H^{2k} \, d\sigma \right) dL \\ &\quad = C_{nk} \, O_{n-2}^3 O_{n-3} \cdots O_1 \sum_i \sum_{\substack{i_1, \cdots, i_{n-1} \\ i_1 + \cdots + i_{n-1} = i}} c'_{i_1 \cdots i_{n-1}} \tilde{M}^i_{i_1 \cdots i_{n-1}} (S). \end{aligned} \tag{3.13}$$

[1] The formula was first proved by C-S. Chen in 1972 (see [15]) and reproved by this author in 1992 (see [12]) by a different method.

4. Proofs of theorems

Proof of Theorem 1. Equation (2.2) may be written as

$$dT_1 = \bigwedge_i (dx \cdot e_i) \bigwedge_{h<j} (e_h \cdot de_j) = d\sigma^{(1)} \wedge dg_1 , \qquad i, h, j = 1, 2, \cdots, n-1, \tag{4.1}$$

where $d\sigma^{(1)}$ is the volume element of S_1 at x and dg_1 is the kinematic density for S_1, i.e., for $SO(n-1)$. Then for any positive integer p we have

(4.2)

$$\begin{aligned}
\int \kappa_n^p(S_1(v))\, dT_1 &= \int \kappa_n^p(S_1(v))\, d\sigma^{(1)} \wedge dg_1 \\
&= \sum_{i_1+\cdots+i_{n-1}=p} c_{i_1\cdots i_{n-1}} \int_{S_1} \left(\kappa_1^{i_1}\cdots\kappa_{n-1}^{i_{n-1}}\right) d\sigma^{(1)} \int_{SO(n-1)} v_1^{2i_1}\cdots v_{n-1}^{2i_{n-1}}\, dg_1 \\
&= \mathrm{Vol}\,(SO(n-2)) \sum_{i_1, \cdots, i_{n-1}} \binom{n-1}{p} c_{i_1\cdots i_{n-1}} \tilde{M}^p_{i_1\cdots i_{n-1}}(S_1)\, I(i_1, \cdots, i_{n-1}),
\end{aligned}$$

where

$$\tilde{M}^p_{i_1\cdots i_{n-1}}(S_1) = \binom{n-1}{p}^{-1} \int_{S_1} \left(\kappa_1^{i_1}\cdots\kappa_{n-1}^{i_{n-1}}\right) d\sigma^{(1)}$$

are the pth homogeneous mean curvature of S_1,

$$c_{i_1\cdots i_{n-1}} = \frac{p!}{i_1!\cdots i_{n-1}!}, \quad i_1+\cdots i_{n-1} = p; \tag{4.3}$$

$$\begin{aligned}
\int_{SO(n-1)} v_1^{2i_1}\cdots v_{n-1}^{2i_{n-1}}\, dg_1 &= \mathrm{Vol}\,(SO(n-2)) \int_{S^{n-2}} v_1^{2i_1}\cdots v_{n-1}^{2i_{n-1}}\, d\sigma \\
&= \mathrm{Vol}\,(SO(n-2))\, I(i_1, \cdots, i_{n-1}).
\end{aligned} \tag{4.4}$$

Equality (4.4) follows from the fibering of $SO(n-1)$ over S^{n-2} with the fiber $SO(n-2)$, and the values of the integrals are known (see, for example, H. Weyl [18]),

$$\begin{aligned}
I(i_1, \cdots, i_{n-1}) &= \int_{S^{n-2}} v_1^{2i_1}\cdots v_{n-1}^{2i_{n-1}}\, d\sigma \\
&= O_{n-2} \frac{(2i_1-1)!!\cdots(2i_{n-1}-1)!!}{(n-1)(n+1)\cdots(n+2p-3)},
\end{aligned} \tag{4.5}$$

where $p = \sum_{k=1}^{n-1} i_k$.

So we have

$$\int \kappa_n^p(S_1(v))\, dT_1 = \mathrm{Vol}\,(SO(n-2)) \sum_{i_1, \cdots, i_{n-1}} J^{(1)}_{i_1\cdots i_{n-1}} \tilde{M}^p_{i_1\cdots i_{n-1}}(S_1), \tag{4.6}$$

where

$$J^{(1)}_{i_1\cdots i_{n-1}} = \binom{n-1}{p} c_{i_1\cdots i_{n-1}} I(i_1, \cdots, i_{n-1}). \tag{4.7}$$

Similarly, for any q we have

$$\int \kappa_n^q(S_0(v))\, dT_0 = \mathrm{Vol}\,(SO(n-2)) \sum_{j_1, \cdots, j_{n-1}} J^{(0)}_{j_1, \cdots, j_{n-1}} \tilde{M}^q_{j_1\cdots j_{n-1}}(S_0), \tag{4.8}$$

where

$$J^{(0)}_{j_1\cdots j_{n-1}} = \binom{n-1}{q} c_{j_1\cdots j_{n-1}} I(j_1, \cdots, j_{n-1}); \tag{4.9}$$

$$c_{j_1\cdots j_{n-1}} = \frac{q!}{j_1!\cdots j_{n-1}!}, \quad j_1+\cdots+j_{n-1} = q; \tag{4.10}$$

and the $I(j_1, \cdots, j_{n-1})$ are computed as in (4.5). Also we have

$$\mathrm{Vol}\,(SO(n-2)) = O_{n-2}\cdots O_1. \tag{4.11}$$

Putting formula (4.6), (4.8) and (4.11) into (2.17) we obtain

$$\begin{aligned} &O_{n-3}\cdots O_1 \int_G \left(\int_{S_0\cap gS_1} H^{2k}\, d\sigma\right) dg \\ &= (O_{n-2}\cdots O_1)^2 \sum_{i,j,l} C_{ijkln} \sum_{i_1,\cdots,i_{n-1}} J^{(1)}_{i_1\cdots i_{n-1}} \tilde{M}^{l+2i}_{i_1\cdots i_{n-1}}(S_1) \\ &\quad \times \sum_{j_1,\cdots,j_{n-1}} J^{(0)}_{j_1\cdots j_{n-1}} \tilde{M}^{l+2j}_{j_1\cdots j_{n-1}}(S_0), \end{aligned} \tag{4.12}$$

i.e.,

$$\begin{aligned} &\int_G \left(\int_{S_0\cap gS_1} H^{2k}\, d\sigma\right) dg \\ &= O^2_{n-2}\cdot O_{n-3}\cdots O_1 \sum_{i,j,l} C_{ijkln} \sum_{i_1,\cdots,j_{n-1}} J^{(1)}_{i_1\cdots j_{n-1}} \tilde{M}^{l+2i}_{i_1\cdots i_{n-1}}(S_1) \\ &\quad \times \sum_{j_1,\cdots,j_{n-1}} J^{(0)}_{j_1\cdots j_{n-1}} \tilde{M}^{l+2j}_{j_1\cdots j_{n-1}}(S_0), \end{aligned} \tag{4.13}$$

with

$$i_1+\cdots+i_{n-1} = l+2i, \quad j_1+\cdots+j_{n-1} = l+2j. \tag{4.14}$$

We complete the proof of Theorem 1.

Let

$$\begin{aligned} c_{ijkln} &= C(n)\, C_{ijkln}, \\ \tilde{\kappa}^{l+2i}_n(S_1) &= \sum_{i_1,\cdots,i_{n-1}} J^{(1)}_{i_1\cdots i_{n-1}} \tilde{M}^{l+2i}_{i_1,\cdots,i_{n-1}}(S_1), \\ \tilde{\kappa}^{l+2j}_n(S_0) &= \sum_{j_1,\cdots,j_{n-1}} J^{(0)}_{j_1\cdots j_{n-1}} \tilde{M}^{l+2j}_{j_1,\cdots,j_{n-1}}(S_0), \end{aligned} \tag{4.15}$$

and we obtain our Main Theorem.

Proof of Theorem 2. If $S_1 = L$ is a hyperplane, then all principal curvatures of L vanish. Then (2.9) becomes

$$H^2\sin^2\phi = \kappa^2_n(S), \tag{4.16}$$

where H is the mean curvature of submanifold $S\cap L$ and ϕ is the angle between S and L. From (2.6) we have

$$H^{2k} d\sigma\wedge dL_{n-1} = \kappa^{2k}_n(S)\,\sin^{n-2k-1}\phi\, d\phi\wedge d\sigma^{(0)}\wedge dL_{n-1[x]}, \tag{4.17}$$

and thus

$$\begin{aligned}\int_G\left(\int_{S\cap gL} H^{2k}\,d\sigma\right)dg \\ = C_{nk}O_{n-2}^3O_{n-3}\cdots O_1\sum_i \sum_{\substack{i_1,\cdots,i_{n-1}\\ i_1+\cdots+i_{n-1}=i}} c'_{i_1\cdots i_{n-1}}\tilde{M}^i_{i_1\cdots i_{n-1}}(S).\end{aligned}\tag{4.18}$$

This proves Theorem 2.

In particular, we have the following kinematic inequality.

Theorem 3. *Let* S_i $(i=0,1)$, F_i, R_i, H_i, $\tilde{R}_i$, *and* $\tilde{H}_i^{(2)}$ *be as in Corollary 1. Then we have*

(4.19)

$$\begin{aligned}\int_G\left(\int_{S_0\cap gS_1} H\,d\sigma\right)dg &\le \pi A_n\left\{B_nF_0F_1\left[\left(3(n-1)^2\tilde{H}_0^{(2)}-4\tilde{R}_0\right)F_1\right.\right.\\ &\qquad\left.\left.+\left(3(n-1)^2\tilde{H}_1^{(2)}-4\tilde{R}_1\right)F_0\right]\right\}^{\frac12}\\ &=\pi A_n\left\{B_nF_0F_1\left[3(n-1)^2\left(\tilde{H}_0^{(2)}F_1+\tilde{H}_1^{(2)}F_0\right)-4\left(\tilde{R}_0F_1+\tilde{R}_1F_0\right)\right]\right\}^{\frac12},\end{aligned}$$

where $A_n=O_{n-2}O_{n-3}\cdots O_1$ *and* $B_n=O_nO_{n-2}O_{n-1}^{-1}/(n^2-1)$.

Proof of Theorem 3. By Hölder's inequality we have

$$\begin{aligned}\int_{S_0\cap gS_1} H\,d\sigma &\le \left(\int_{S_0\cap gS_1} 1^2\cdot d\sigma\right)^{\frac12}\cdot\left(\int_{S_0\cap gS_1} H^2 d\sigma\right)^{\frac12}\\ &=(\mathrm{Vol}(S_0\cap gS_1))^{\frac12}\cdot\left(\int_{S_0\cap gS_1} H^2 d\sigma\right)^{\frac12}.\end{aligned}\tag{4.20}$$

The Santaló kinematic formula [1] for the volume of intersection $S_0\cap gS_1$ reads

$$\int_G \mathrm{Vol}(S_0\cap gS_1)\,dg=\frac{O_n\cdots O_1O_{n-2}}{O_{n-1}^2}F_0F_1.\tag{4.21}$$

Integrating (4.21) with respect to kinematic density dg and using Hölder's inequality, (3.8) and (4.22) we come to

(4.22)

$$\begin{aligned}&\int_G\left(\int_{S_0\cap gS_1} H\,d\sigma\right)dg\\ &\le\int_G(\mathrm{Vol}(S_0\cap gS_1))^{\frac12}\cdot\left(\int_{S_0\cap gS_1} H^2\,d\sigma\right)^{\frac12}dg\\ &\le\left(\int_G \mathrm{Vol}(S_0\cap gS_1)\,dg\right)^{\frac12}\cdot\left(\int_G\left(\int_{S_0\cap gS_1} H^2\,d\sigma\right)dg\right)^{\frac12}\\ &=\pi A_n\left\{B_nF_0F_1\left[3(n-1)^2(\tilde{H}_0^{(2)}F_1+\tilde{H}_1^{(2)}F_0)-4(\tilde{R}_0F_1+\tilde{R}_1F_0)\right]\right\}^{\frac12}.\end{aligned}$$

This proves Theorem 3.

When $n=3$, we have the following

Corollary 4. *Let* S_i $(i = 0, 1)$ *be two simply closed surfaces of class* C^2 *in* $\mathbb{R}^3$. *Assume that* F_i, R_i, H_i, $\tilde{R}_i$ *and* $\tilde{H}_i^{(2)}$ *are as in Corollary* 1. *Then we have*

$$\int_G \left(\int_{S_0 \cap gS_1} \kappa\, ds\right) dg \le 2\pi^3 \left\{F_0 F_1 \left[(3\tilde{H}_0^{(2)} - \tilde{R}_0)F_1 + (3\tilde{H}_1^{(2)} - \tilde{R}_1)F_0\right]\right\}^{\frac{1}{2}}. \tag{4.23}$$

5. Analogues of Hadwiger's theorem—sufficient conditions for one domain to contain another in $\mathbb{R}^{2n}$

It is known that at each point of a hypersurface Σ in $\mathbb{R}^n$ there are $n-1$ principal directions and $n-1$ principal curvatures $\kappa_1, \cdots, \kappa_{n-1}$. Let $d\sigma$ denote the volume element of Σ, then the *rth integral of mean curvature*

$$M_r(\Sigma) = \binom{n-1}{r}^{-1} \int_\Sigma \{\kappa_{i_1}, \cdots, \kappa_{i_r}\}\, d\sigma, \tag{5.1}$$

where $\{\kappa_{i_1}, \cdots, \kappa_{i_r}\}$ denotes the rth *elementary symmetric function of the principal curvatures.* In particular, M_0 is the area, M_{n-1} is a numerical multiple of the degree of mapping of Σ into the unit hypersphere defined by the normal field and M_1 is the total mean curvature of Σ.

In this section, we will use indices α and β for the two domains and their integral geometric invariants. We assume that K_α and K_β are bounded by the simple hypersurfaces ∂K_α and ∂K_β, respectively, which we assume to be of class C^2. One restriction we put on domains is that for all $g \in G$, the group of rigid motions in $\mathbb{R}^{2n}$, the Euler-Poincaré characteristic $\chi(K_\alpha \cap gK_\beta)$ of the intersection $K_\alpha \cap gK_\beta$ is at most n_0, a finite integer, i.e., $\chi(K_\alpha \cap gK_\beta) \le n_0$. Denote by M_i^α, M_i^β the ith integrals of mean curvatures of ∂K_α, ∂K_β, respectively. Then Chern's kinematic fundamental formula [1, 2] reads

$$\begin{aligned} &\int_G \chi(K_\alpha \cap gK_\beta)\, dg \\ &\quad = O_{n-2} \cdots O_1 \left[O_{n-1}(V_\alpha \chi(K_\beta) + V_\beta \chi(K_\alpha)) + \frac{1}{n}\sum_{h=0}^{n-2} \binom{n}{h+1} M_h^\alpha M_{n-2-h}^\beta\right], \end{aligned} \tag{5.2}$$

where $\chi(\cdot)$ is the Euler-Poincaré characteristic and V_γ $(\gamma = \alpha, \beta)$ are the volumes of K_γ.

Let

$$\tilde{C}_{ijkln} = \frac{2^{l+1} k!(l-1)!!(2n-2k-2)!!}{i!j!l!(2n-k-i-j-1)!!}, \tag{5.3}$$

$$\begin{aligned} f(i, j, k, l, n, \tilde{M}_\beta^{l+2i}, \tilde{M}_\alpha^{l+2j}) = \sum_{i,j,l} \tilde{C}_{ijkln} &\sum_{i_1, \cdots, i_{2n-1}} J^{(\beta)}_{i_1 \cdots i_{2n-1}} \tilde{M}^{l+2i}_{i_1 \cdots i_{2n-1}}(\partial K_\beta) \\ \times &\sum_{j_1, \cdots, j_{2n-1}} J^{(\alpha)}_{j_1 \cdots j_{2n-1}} \tilde{M}^{l+2j}_{j_1 \cdots j_{2n-1}}(\partial K_\alpha). \end{aligned}$$

Denoted by $\tilde{\kappa}_n^r(\partial K_\gamma)$ the rth integral of homogeneous mean curvature of ∂K_γ. As a direct application of our kinematic formula (0.2) or (3.6) we have an analogue of Hadwiger's theorem as follows

Theorem 4. *Let K_α, K_β be domains in $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ bounded by connected hypersurfaces ∂K_α, K_β, respectively, which we suppose to be C^2-smooth. We assume that for all $g \in G$, the group of rigid motions in $\mathbb{R}^{2n}$, the Euler-Poincaré characteristic satisfy $\chi(K_\alpha \cap gK_\beta) \leq n_0$. Then either of following inequalities gives a sufficient condition for K_α to enclose K_β or for K_β to enclose K_α:*

$$(5.4)\qquad \begin{aligned} &O_{2n-1}(V_\alpha\chi(K_\beta) + V_\beta\chi(K_\alpha)) + \frac{1}{2n}\sum_{h=0}^{2n-2}\binom{2n}{h+1}M_h^\alpha M_{2n-2-h}^\beta \\ &\qquad - n_0\sum_{i,j,l}\tilde{C}_{ijkln}\,\tilde{\kappa}_n^{l+2i}(\partial K_\alpha)\,\tilde{\kappa}_n^{l+2j}(\partial K_\beta) > 0; \end{aligned}$$

$$(5.5)\qquad \begin{aligned} &O_{2n-1}(V_\alpha\,\chi(K_\beta) + V_\beta\,\chi(K_\alpha)) + \frac{1}{2n}\sum_{h=0}^{2n-2}\binom{2n}{h+1}M_h^\alpha M_{2n-2-h}^\beta \\ &\qquad - n_0\,f(i, j, k, l, n, \tilde{M}_\beta^{l+2i}, \tilde{M}_\alpha^{l+2j}) > 0. \end{aligned}$$

Moreover, if $V_\alpha \geq V_\beta$, then K_α can enclose K_β.

Proof of Theorem 4. Let M be an n-dimensional closed submanifold in Euclidean space $\mathbb{R}^m$ and H be the mean curvature of M. Then B-Y. Chen's formula (i.e., the generalized Fenchel's theorem) [8] says

$$(5.6)\qquad \int_M |H|^n d\sigma \geq O_n,$$

the equality of (5.6) holds if and only if M is imbedded as an n-sphere of $\mathbb{R}^m$.

Now, we estimate integral

$$(5.7)\qquad \int_{\{g\,:\,\partial K_\alpha\cap g\partial K_\beta\neq\varnothing\}} dg.$$

For almost every rigid motion $g \in G$ in $\mathbb{R}^{2n}$, the intersection submanifold $\partial K_\alpha \cap g\partial K_\beta$ may be composed of several components, i.e., $\partial K_\alpha \cap g\partial K_\beta = \bigcup_{i=1}^{N_g}\Sigma_i$, where Σ_i is a connected imbedded closed $(2n-2)$-dimensional submanifold and N_g is always finite and only depends on g. By using B-Y. Chen's formula (5.6) to the generic $(2n-2)$-dimensional submanifolds $\partial K_\alpha \cap g\partial K_\beta$ we have

$$(5.8)\qquad \int_{\partial K_\alpha\cap g\partial K_\beta} H^{2n-2} d\sigma \geq O_{2n-2}.$$

The above equality holds if and only if $\partial K_\alpha \cap g\partial K_\beta$ is composed of only one component and is a $(2n-2)$-sphere in $\mathbb{R}^{2n}$. If the equality in (5.8) holds for almost all rigid motions $g \in G$, that is, the intersections $\partial K_\alpha \cap g\partial K_\beta$ of the boundaries of two domains K_α and K_β are always balls (or empty). Then the two domains K_α and K_β must be balls (a consequence of a result of P. Goodey [13, 25]). Therefore

$$(5.9)\qquad O_{2n-2}\int_{\{g\,:\,\partial K_\alpha\cap g\partial K_\beta\neq\varnothing\}} dg \leq \int_G\left(\int_{\partial K_\alpha\cap g\partial K_\beta} H^{2n-2}d\sigma\right)dg,$$

and thus by (0.2) we have

$$\int_{\{g:\ \partial K_\alpha \cap g\partial K_\beta \neq \varnothing\}} dg \le C_n \sum_{i,j,k} \tilde{C}_{ijkln}\, \tilde{\kappa}_n^{l+2i}(\partial K_\alpha)\, \tilde{\kappa}_n^{l+2j}(\partial K_\beta), \tag{5.10}$$

where $C_n = O_{2n-2} \cdots O_1$. From

$$\int_{\{g:\ K_\alpha \cap gK_\beta \neq \varnothing\}} dg \le n_0 \int_G \chi(K_\alpha \cap gK_\beta)\, dg, \tag{5.11}$$

(5.2), (5.10) and (5.11) we have the kinematic measure of one domain moving into another under the group G of the rigid motions in $\mathbb{R}^{2n}$, i.e.,

(5.12)

$$\begin{aligned}
&\tilde{m}\{g \in G:\ gK_\beta \subset K_\alpha \text{ or } gK_\beta \supset K_\alpha\} \\
&\quad = \int_{\{g:\ K_\alpha \cap gK_\beta \neq \varnothing\}} dg - \int_{\{g:\ \partial K_\alpha \cap g\partial K_\beta \neq \varnothing\}} dg \\
&\quad \ge \frac{1}{n_0} C_n \left[O_{2n-1}(V_\alpha \chi(K_\beta) + V_\beta \chi(K_\alpha) + \frac{1}{2n} \sum_{h=0}^{2n-2} \binom{2n}{h+1} M_h^\alpha M_{2n-2-h}^\beta \right] \\
&\qquad - C_n \sum_{i,j,l} \tilde{C}_{ijkln}\, \tilde{\kappa}_n^{l+2i}(\partial K_\alpha)\, \tilde{\kappa}_n^{l+2j}(\partial K_\beta).
\end{aligned}$$

Equality holds above if and only if the two domains are balls. Hence a sufficient condition for K_α to contained, or to be contain in, K_β is

$$\tilde{m}\{g \in G:\ gK_\beta \subset K_\alpha \text{ or } gK_\beta \supset K_\alpha\} \ge 0. \tag{5.13}$$

Therefore (5.2) or (5.3) is a sufficient condition for K_α to enclose, or to be enclosed in K_β. We have proved Theorem 4.

If K_α and K_β are convex bodies in $\mathbb{R}^{2n}$, then we have $\chi(K_\alpha) = \chi(K_\beta) = \chi(K_\alpha \cap gK_\beta) = n_0 = 1$ for almost all $g \in G$. So we have

Theorem 5. *Let K_α, K_β be two C^2-smooth convex bodies in $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$. Then either of the following inequalities is a sufficient condition for K_α to contain, or to be contained in, K_β:*

$$\begin{aligned}
&O_{2n-1}(V_\alpha + V_\beta) + \frac{1}{2n} \sum_{h=0}^{2n-2} \binom{2n}{h+1} M_h^\alpha M_{2n-2-h}^\beta \\
&\qquad - \sum_{i,j,l} \tilde{C}_{ijkln}\, \tilde{\kappa}_n^{l+2i}(\partial K_\alpha)\, \tilde{\kappa}_n^{l+2j}(\partial K_\beta) > 0;
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&O_{2n-1}(V_\alpha + V_\beta) + \frac{1}{2n} \sum_{h=0}^{2n-2} \binom{2n}{h+1} M_h^\alpha M_{2n-2-h}^\beta \\
&\qquad - f(i, j, k, l, n, \tilde{M}_\beta^{l+2i}, \tilde{M}_\alpha^{l+2j}) > 0.
\end{aligned} \tag{5.15}$$

Moreover, if $V_\alpha \ge V_\beta$, then K_α can contain K_β.

Remark. Of course, these conditions are not necessary. It would be very interesting to remove the 'smoothness' restriction to the convex bodies involved in this paper. All the notations except $\tilde{\kappa}_n^r(\partial K_\gamma)$ (or $\tilde{M}_\gamma^r$) are well-defined for nonsmooth convex bodies. If we could find substitutes for $\tilde{\kappa}_n^r(\partial K_\gamma)$ (or $\tilde{M}_\gamma^r$), the conditions in this paper can be interpreted for arbitrary convex bodies. This is definitely worth investigating.

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DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19122
E-mail address: zhou@euclid.math.temple.edu