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NOTES ON RULED SYMPLECTIC 4-MANIFOLDS

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ABSTRACT. A symplectic 4-manifold (V, ω) is said to be ruled if it is the total space of a fibration whose fibers are 2-spheres on which the symplectic form does not vanish. This paper develops geometric methods for analysing the symplectic structure of these manifolds, and shows how this structure is related to that of a generic complex structure on V. It is shown that each V admits a unique ruled symplectic form up to pseudo-isotopy (or deformation). Moreover, if the base is a sphere or if V is the trivial bundle over the torus, all ruled cohomologous forms are isotopic. For base manfolds of higher genus this remains true provided that a cohomological conditon on the form is satisfied: one needs the fiber to be "small" relative to the base. These results correct the statement of Theorem 1.3 in *The structure of rational and ruled symplectic manifolds*, J. Amer. Math. Soc. 3 (1990), 679–712, and give more details of some of the proofs.

1. INTRODUCTION

A ruled symplectic 4-manifold is a compact manifold V which is a 2-sphere bundle $\pi : V \to M$ over a Riemann surface M, together with a symplectic form ω on V whose restriction to each fiber never vanishes. (π is then called a symplectic fibering or ruling.) An attempt was made in Theorem 1.3 of [8] to classify these manifolds up to isotopy. Lalonde recently pointed out that this theorem is not correct without extra hypotheses. In this note, we set the record straight, and push the methods of [8] as far as possible. Our results do not give complete information on the structure of these ruled manifolds, except when the base is a sphere or V is the product $T^2 \times S^2$. It would be very interesting to know what happens in the other cases.

Recall from [8] that a symplectic 4-manifold (V, ω) is said to be *minimal* if it contains no symplectically embedded 2-sphere with self-intersection -1. (Any symplectic 4-manifold can be made minimal by blowing down all such spheres.) Moreover, if V is minimal and contains a symplectically embedded 2-sphere S with trivial normal bundle then S forms one of the fibers of a symplectic fibering of V, so that V is ruled. Each ruled manifold has many symplectic fiberings which are inequivalent, in the sense that there is no fiber-preserving symplectomorphism between them: see the Note after [8, Lemma

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4.4]. Thus, the equivalence relation provided by fiber-preserving symplectomorphisms is too fine, and it is more interesting to try to classify these forms up to isotopy.

We remind the reader that two forms ω_0 and ω_1 on V are isotopic if they may be joined by a family of cohomologous symplectic forms, or equivalently, if there is a family of diffeomorphisms g_t of V with $g_0 = \mathrm{id}$ and such that $g_1^*(\omega_1) = \omega_0$. Another weaker relation which one can consider is that of pseudo-isotopy:[1] ω_0 and ω_1 are pseudo-isotopic if they may be joined by a family of not necessarily cohomologous symplectic forms. However, the geometric meaning of pseudo-isotopy is not clear: it is possible to have cohomologous symplectic forms which are pseudo-isotopic but not diffeomorphic (see [6]), though no such examples are known in dimension 4.

Up to diffeomorphism, there are only two orientable S^2-bundles over a Riemann surface M, the product $M \times S^2$ and the nontrivial bundle V_M, which is the pull-back by a map of degree 1 of the nontrivial S^2-bundle over S^2. The product bundle admits sections M_{2k} with even self-intersection number $2k$, and V_M admits sections M_{2k+1} with odd self-intersection number $2k+1$. We will choose the basis b, f for $H^2(M \times S^2, \mathbf{R})$ which is dual to the homology basis $[M_0] = [M \times pt]$, $F = [pt \times S^2]$, and will use the basis b_+, b_- for $H^2(V_M, \mathbf{R})$ which is dual to the homology basis $[M_1], [M_{-1}]$. Thus $b_+(M_1) = b_-(M_{-1}) = 1$. Observe that because $[M_1] \cdot [M_{-1}] = 0$ the following relations hold:

$$b_+ \cup b_+(V_M) = 1, \quad b_- \cup b_-(V_M) = -1, \quad b_+ \cup b_- = 0.$$

We will suppose in both cases that the fibration $\pi\ V \to M$ is fixed, and that *compatible* orientations of V, M, and the fiber F are prescribed, and then will consider only those symplectic forms which are *compatible* with this data, in the sense that they do not vanish on the fibers and are compatible with the given orientations. Such forms were called *π-compatible* in [8]. As we will see, the structure of (V, ω) depends largely on which homology classes may be represented by a symplectic section, that is, a section of π on which ω does not vanish.

Obviously, the only cohomology classes on $M \times S^2$ which are represented by a compatible symplectic form are the classes $\mu_1 m + \mu_2 f$ where $\mu_i > 0$ for $i = 1, 2$. Moreover, each such class contains a standard split form $\mu_1\sigma_1 \oplus \mu_2\sigma_2$. Thus in this case the question is whether there are any symplectic forms on $M \times S^2$ not isotopic to a split form. The best answer we have so far is the following.

Theorem 1.1. *Let ω be a symplectic form on $M \times S^2$ which is compatible with the projection onto M and the obvious orientations.*

(i) *If M is S^2 or T^2, ω is isotopic to a standard split form.*

(ii) *If M has genus $g > 1$, the statement in* (i) *holds provided that* $[\omega] = \mu_1 b + \mu_2 f$ *where $\mu_1 > q\mu_2$, where $q = [g/2]$.*

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[1]This terminology was suggested by Lalonde.

(iii) *ω is always pseudo-isotopic to a split form. Moreover, it is isotopic to a split form iff it has a symplectic section in class* $[M_0]$.

In the case of the nontrivial bundle V_M, if $a = \mu_+ b_+ + \mu_- b_-$ is the class of a symplectic form we must have $0 < a^2(V_M) = (\mu_+)^2 - (\mu_-)^2$, so that $|\mu_+| > |\mu_-|$. Since we are also assuming that a is positive on the class $[F] = [M_1] - [M_{-1}]$ of the fiber, we in fact need $\mu_+ > |\mu_-|$. However, note that μ_- need not be positive here. It turns out that when $M = S^2$, these classes a with $a^2 > 0$ do not all have compatible symplectic representatives, though they do for all other M.

Theorem 1.2. (i) *When $M = S^2$, the class $a = \mu_+ b_+ + \mu_- b_-$ may be represented by a compatible symplectic form on V_M only if $\mu_+ > \mu_- > 0$. Moreover, up to isotopy there is a unique such form in each class.*

(ii) *If M has genus $g > 0$, every class with $\mu_+ > |\mu_-| > 0$ has a compatible symplectic representative. There is a unique form up to isotopy in each class such that $q\mu_- > (q-1)\mu_+$, where $q = [(g+1)/2]$.*

(iii) *All these forms are pseudo-isotopic.*

Remark 1.3. (i) When $g = 0$ the second condition in (ii) is vacuous, and when $g = 1$ it says simply that $\mu_- > 0$. Since it may be rewritten as $\mu_- > (q-1)(\mu_+ - \mu_-)$, we may interpret it as limiting the size, $\mu_+ - \mu_-$, of the fiber relative to that, μ_-, of the cross section M_-.

(ii) As we shall see, there are "standard" forms on V_M in each class with $\mu_- > 0$. Our methods show that the form ω is isotopic to one of these standard forms iff $\mu_- > 0$ and ω admits a symplectic section in the class $[M_1]$: see Corollary 3.6. They yield no information about the isotopy class of a general form with $\mu_- \leq 0$.

When k is small relative to the genus g of M, the following reformulation of Corollary 3.6 gives some more information.

Corollary 1.4. *Let (V, ω) be a symplectic ruled 4-manifold which admits a symplectic section in class $[M_k]$ for some $k \geq 0$. Then, if $\omega(M_{-k}) > 0$, the symplectic form ω is isotopic to a standard form.*

These results are proved using the methods of [8, §4]. Most of what is written there is correct. However, [8, Lemma 4.15] is wrong as stated: it needs an extra hypothesis concerning the cohomology class of ω and the homology class of the section Γ. Also, the proof (though not the statement) of [8, Lemma 4.16] is incorrect, since the complex structure J_1' considered there has too many cusp curves: see the discussion in Proposition 3.12 below. Essential to a correct proof of [8, Lemma 4.16] is an understanding of the possible complex structures on ruled surfaces. In §2 we pick out relevant results from Atiyah's beautiful papers [1, 2] on this subject.

The proofs of the theorems are given in §3.

I wish to thank Francois Lalonde for reading my paper [8] with such a critical eye, and for his useful suggestions about a preliminary version of the present paper. I also wish to thank Gompf for an illuminating conversation on symplectic fibrations.

2. Complex structures on ruled surfaces

It is well known that every complex structure on a ruled surface V_M arises by considering V_M as the projectivization of a holomorphic bundle W of rank 2. The easiest structures to understand arise from bundles which split into the sum of line bundles. However, as we shall see, most of these are nongeneric and have too many holomorphic sections. It turns out that the most interesting complex structures are those with the fewest sections. In fact, when $g > 1$ the structures which interest us are exactly the stable rank 2 bundles. We will write $d(L)$ for the *degree* (i.e., the Chern number) of the line bundle L. All the results of this section are due to Atiyah [1, 2]. Another good general reference is [4].

(i) **Split complex structures.** If W splits as $L_1 \oplus L_2$, its projectivization V may be written as $P(L \oplus \mathbf{C})$, where $L = L_1 \otimes L_2^*$. We will suppose as we may that $d(L) \geq 0$. It is well known that V is diffeomorphic to the trivial bundle $M \times S^2$ when d is even, and to the nontrivial bundle V_M when d is odd. In all cases, there are two obvious sections, $M_d = P(0 \oplus \mathbf{C})$ and $M_{-d} = P(L \oplus 0)$, where, as above, the section

M_q has self-intersection q. These sections are the fixed points sets of the action of S^1 induced by multiplication by $e^{2\pi i t}$ in L. If $d > 0$ and $\lambda_1 > \lambda_2 > 0$, it is not hard to construct an S^1-invariant Kähler form on V whose integral over M_d (resp. M_{-d}) is λ_1 (resp. λ_2): see [7] for example. In particular, when $d = 1$, the numbers λ_1 and λ_2 coincide with the μ_+ and μ_- mentioned above, and so we get representatives of all classes with $\mu_+ > \mu_- > 0$.

When $d = 0$, a similar construction exists when $\lambda_1 = \lambda_2$, but then we can also make the integral over the fiber an arbitrary positive number, so that we still have a two-parameter family of choices.

Our main concern is to understand the holomorphic curves in these complex surfaces, in particular, how many holomorphic sections there are and in what homology classes A they lie. If possible, we would like to find complex structures J which are *regular* in the sense of Fredholm theory for a suitable set of homology classes $A \in H_2(V, \mathbf{Z})$. By Lemma 2.8 of [8], this will be the case iff the normal bundle ν_C of each J-holomorphic curve C in the class A is such that $H^1(C, \nu_C) = H^0(C, \nu_C^* \otimes K_M) = 0$, where K_M is the canonical bundle. In other words, we need the divisor corresponding to the bundle $\nu_C^* \otimes K_M$ to be noneffective. Since we are interested only in sections, we may assume that C is embedded. Then $H^1(C, \nu_C) = 0$ if $c_\nu = c_1(\nu_C) = C \cdot C$ is sufficiently positive ($> 2(g-1)$, to be exact, where g is the genus of M), since, then, $\nu_C^* \otimes K_M$ is negative. It may or may not be zero in the range $g - 1 \leq c_\nu \leq 2(g-1)$, and, if $c_\nu < g - 1$, it is never zero. In particular, sections with negative self-intersection number are never regular (unless $g = 0$).

When $g = 0$, the two split structures with $d = 0$ and 1 are regular for the classes $[M_{\pm d}]$. However, when $g > 0$ all the complex structures described above contain the nonregular sections M_{-d} except for one case which we now describe.

Lemma 2.1. *If $g = 1$ and L is a nontrivial bundle of degree 0, then $V = P(L \oplus \mathbf{C})$ is diffeomorphic to the product $T^2 \times S^2$ and contains exactly two* *holomorphic sections in the class* $[M_0] = [M \times pt]$. *Moreover, these are regular.*

Proof. Atiyah shows in [2, I.4] that sections of V correspond bijectively to line subbundles of $W = L \oplus \mathbf{C}$. There are two obvious sections which correspond to the summands $L \oplus 0$ and $0 \oplus \mathbf{C}$.

It is easy to see that their normal bundles are the nontrivial bundles L^* and L respectively. (In general, if L' is a line subbundle of W, the normal bundle of the corresponding section is $\mathrm{Hom}(L', W/L') \cong (L')^* \otimes W/L'$.) Hence, these sections are regular iff the bundles $L \otimes K_M$ and $L^* \otimes K_M$ have no nontrivial sections. But K_M is trivial for an elliptic curve, and a bundle over an elliptic curve of degree 0 has a nontrivial section iff it is the trivial bundle. Therefore, because L is nontrivial, these sections are regular.

If there were another holomorphic section of V in the class $[M_0]$, there would be a subbundle L' of degree $d = 0$, different from these two summands. Then L' would project onto the two factors L and $\mathbf{C}$ of W, which means that the line bundles $L'^* \otimes L$ and $L'^* \otimes \mathbf{C} = L'^*$ would both have nontrivial sections. Hence, if such L' existed, L would be trivial, contrary to hypothesis. □

(ii) **Complex structures on V_M for $g = 1$.** In order to find regular complex structures when $g > 0$, we must look at bundles W which do not split.

When $M = T^2$, there is just one corresponding complex structure on V_M, which we now describe.

Proposition 2.2. *For each complex structure on $M = T^2$, there is exactly one complex structure on V_M which comes from a nonsplit bundle. Each of these ruled surfaces contains a compact family of regular sections in class $[M_1]$, with exactly two curves through a generic point of V_M.*

Proof. Atiyah showed in [2] that, when $g = 1$, there is exactly one complex structure on V coming from a nonsplit bundle W. We may consider W to be the nonsplit extension $0 \to \mathbf{C} \to W \to L \to 0$ of a holomorphic line bundle L of degree 1 by the trivial bundle $\mathbf{C}$. Note that the space of holomorphic sections of L has dimension 1: thus, up to a multiple, there is only one section σ, which is zero at exactly one point, x_0 say. (In fact, this gives the isomorphism between the moduli space of holomorphic line bundles over T^2 and the Picard variety Pic^1.)

Let us take this point x_0 to be the zero element $0 \in T^2$, so that we can define addition on the elliptic curve T^2. Further, for any $y \in T^2$, let L_y be the line bundle of degree 1 with sections vanishing at y. (So, in this notation, $L = L_0$.) If L' is any line bundle of degree 0, then L' may be identified with an element y of T^2 in such a way that the sections of the tensor product $L' \otimes L_0$ vanish at y. Thus, if we write L'_y for this bundle of degree 0, we have $L'_y \otimes L_0 = L_y$. Since this holomorphic map $y \mapsto L'_y$ preserves the zero elements, the proposition on [3, p. 326] implies that it is a group homomorphism, i.e., $L'_x \otimes L'_y = L'_{x+y}$. Note that if L'_y is a subbundle of W of degree 0, then we must have

$$\Lambda^2(W) = W \wedge W = L'_y \otimes (W/L'_y) = \mathbf{C} \otimes L_0 = L_0.$$

Therefore, the quotient W/L'_y equals L_{-y}.

Hence the homomorphism $L'_y \to L_0$ (induced by the quotient map $W \to L_0$) is a section of the bundle $(L'_y)^* \otimes L_0 = L_{-y}$ and so vanishes at $-y$. In other words L'_y has one fiber in common with $\mathbf{C}$, namely that over $-y$.

As in Lemma 2.1, sections of V_M in the class $[M_1]$ correspond to line subbundles L' of W of degree 0. Let us denote the section corresponding to the trivial subbundle $\mathbf{C}$ by Z_0 and identify it with T^2 in the obvious way. Then the section Z_y given by L'_y meets Z_0 in the point $-y$.

We claim that for each $y \neq x_0$ there is exactly one section of V_M (other than Z_0) which meets Z_0 at $-y$. (The point x_0 is not a regular point of the evaluation map.) Existence is proved by Atiyah: see [2, Corollary to Theorem 7], and Theorem 2.4 below. To prove uniqueness, observe that if there were more than one, there would be two distinct subbundles L'_1 and L'_2 isomorphic to L'_y.

The induced map $L'_1 \to W/L'_2$ would be a nontrivial section of $(L'_1)^* \otimes W/L_2 = L_{-2y}$ and so would vanish at $-2y$. But our previous argument shows that $L'_1 = L'_2 = \mathbf{C}$ at $-y$, so this is impossible if $y \neq 0$. Similarly, one can show that there is only one trivial line subbundle in W. Thus the sections of V_M in class $[M_1]$ are in bijective correspondence with the line bundles $L' \in \mathrm{Pic}^0$. These sections are all regular, since $c_\nu = 1 > 2(g-1)$.

To complete the proof that these sections form a compact family it would suffice to show that they vary continuously with $L' \in \mathrm{Pic}^0$. Alternatively, one can use the fact that if the moduli space of these curves were not compact, it would follow from the compactness theorem that there must be degenerate curves (or cusp curves) in class $[M_1]$. These degenerations would have to be the union of a section of negative self-intersection with some number of fibers. However, if there were a section in class $[M_{-q}]$, one would have to have $q = 1$, since we need $[M_{-q}] \cdot [M_1] = 1 - q \geq 0$, by positivity of intersections. Thus it would correspond to a subbundle L' which did not meet $\mathbf{C}$, and so would split W. □

Remark 2.3. (i) The above proof shows that the moduli space $\mathcal{M}$ of curves in V_T in class $[M_1]$ is diffeomorphic to a 2-torus. Observe that all these curves project bijectively onto the base and so have a fixed complex structure and a natural parametrization which is given by a parametrization of the base. Thus there is a holomorphic evaluation map $\mathcal{M} \times T^2 \to V_T$ (for some appropriate complex structure on $\mathcal{M} \times T^2$) whose composite with π is projection onto the second factor. This means that V_T is a quotient of $T^2 \times T^2$ by a holomorphic equivalence relation of the form $(y, u) \sim (y', u)$. If we write $y \in \mathcal{M}$ for the section Z_y corresponding to the line subbundle L_y, then it is easy to check that Z_y meets Z_z at a point in the fiber over $-y - z$. Thus $(y, -y-z) \sim (z, -y-z)$, which means that the equivalence relation is $(y, u) \sim (-y-u, u)$. Note that the map $\mathcal{M} \times T^2 \to V_T$ is 2 to 1 except over the branching locus $\{(y, u) : 2y = -u\}$, which meets each fiber $\{(y, u) : u = u_0\}$ at 4 distinct points.

(ii) There is a similar unique complex structure on the product $T^2 \times S^2$ coming from the nontrivial extension of $\mathbf{C}$ by $\mathbf{C}$. We will not discuss this in detail, since we already have found a complex structure on $T^2 \times S^2$ which is regular for the class $[M_0]$.

Besides, one can readily check that this new one is not regular for the class $[M_0]$.

(iii) **Complex structures when** $g > 1$. When M has genus $g > 1$ the moduli space of ruled surfaces over M is very complicated. We are interested in the most generic structures, i.e., the ones with the fewest sections. By [8, Lemma 2.8], the formal dimension (over $\mathbf{C}$) of the family of holomorphic sections in the class $[M_p]$ is $p - g + 1$, and we would like to find complex structures which have no sections in classes with $p - g + 1 < 0$ and have a family of regular sections for p such that $p - g + 1 = 0$ or 1.

Theorem 2.4. *When M has genus $g \geq 1$, there are complex stuctures on the ruled surfaces over M with the following properties.*

(i) *V_M has a finite number of regular holomorphic sections in class M_{2k-1} when $g = 2k$, and has a 1-parameter family of regular holomorphic sections in class M_{2k-1} when $g = 2k - 1$, and*

(ii) *$M \times S^2$ has a finite number of regular holomorphic sections in class M_{2k} when $g = 2k + 1$, and has a 1-parameter family of regular holomorphic sections in class M_{2k} when $g = 2k$.*

Moreover, all other holomorphic sections have larger self-intersection numbers.

Proof. We have already proved this when $g = 1$, and so will suppose that $g > 1$. The proof that these complex structures exist is contained in [1]. It will be convenient to identify line bundles with the corresponding divisors, and so we will use additive notation for the group of line bundles. Thus $K_M - L$ denotes the line bundle $K_M \otimes L^*$.

Following [1], we will consider our ruled surface to be the projectivization of a bundle W which is an extension of the form $0 \to L \to W \to \mathbf{C} \to 0$.

These extensions are classified by elements of $H^1(M, L) \cong H^0(M, K_M - L)$. The zero element corresponds to the trivial extension, and nonzero elements ζ, ζ' give rise to the same (nontrivial) extension iff $\zeta = k\zeta'$ for some $k \in \mathbf{C}$.

Therefore, the nontrivial extensions are classified by a projective space $Y(L)$ of dimension $h^1(L) - 1$, which we can identify with the projectivization of the space of sections $H^0(M, K_M - L)$.

Let us first consider the case when $g = 2k$ is even and the degree d of L is odd. Then our ruled surface is V_M. Recall that the normal bundle of the section Z_L given by L is $-L$. Therefore, in order for this section to be one of those mentioned in condition (i) we will take $d = 1 - 2k$. If we suppose further that the bundle $K_M + L$ of degree $g - 1$ is not effective, Z_L is regular, that is, $H^1(M, L^{-1}) = 0$. From now on, we will fix a bundle L satisfying these conditions, and will consider how to choose the extension ζ.

As in Proposition 2.2, any section of V_M is given by a line subbundle L' of W, and has normal bundle $W/L' - L' = L - 2L'$. Since there is a homomorphism from L' to $\mathbf{C}$, $-L'$ must be effective, and the divisor Q which it represents is exactly the set of points on M at which the subbundles L and L' coincide. Note that if two different subbundles L'_1 and L'_2 correspond to Q, there is a nontrivial homomorphism $L'_1 \to W/L'_2$ and so the divisor $L + 2Q$ is effective. Thus, when $L + 2Q$ is not effective, there is at most one corresponding section, which, moreover, is regular. Given a divisor Q which appears in this way, we will write L_Q and Z_Q for the corresponding subbundle and section.

Atiyah showed that if $K_M - L$ has no basepoints, then the canonical map $\sigma : M \to Y(L)$ has the following property.

Lemma 2.5. *The extension W_ζ given by $\zeta \in Y(L)$ has a subbundle with divisor $\subset Q$ iff ζ lies in the linear subspace P_Q spanned by the points in $\sigma(Q)$. (Such a subspace is called an r-chordal where $r = |Q|$.)*

Proof. This is [1, Corollary 5.2]. □

Now, in our case $K - L$ has degree $3(g-1)$ and, by Riemann-Roch, $Y(L)$ has dimension $2g-3$. Since the image of σ is a complex curve of dimension 1, the set of points of $Y(L)$ which lie on some r-chordal P_Q has dimension at most $2r-1$. Therefore, there is $\zeta \in Y(L)$ which lies on no such set for $r < g-1$ and which lies on a finite number of such sets when $r = g-1$. By choosing a generic point ζ, we may suppose that each of the corresponding divisors Q of degree $g-1$ gives rise to a unique regular section Z_Q since, as explained above, the conditions for this are that the divisors $L + 2Q$ and $K_M - L - 2Q$ of degree $g-1$ are not effective. (Observe that, since each effective divisor of degree $g-i$ is a set of $g-i$ points, the subvariety of effective divisors of degree $g-i$ has dimension $g-i$, and hence has codimension $\geq i$ in the g-dimensional space Pic^{g-i} of all equivalence classes of divisors of degree $g-i$.) Note that $Z_Q \cdot Z_L = |Q| = g-1$, hence the homology class of Z_Q is $[M_{2k-1}]$ as required. By construction, there are no holomorphic sections in classes $[M_p]$ with $p < 2k-1$.

In the case $V = M \times S^2$ where $g = 2k+1$, we must again start with a bundle L of degree $d = 1-g$. The above argument then applies word for word. If either $V = V_M$ with $g = 2k-1$ or $V = M \times S^2$ with $g = 2k$, we should start with a bundle L of degree $-g$. Then $Y(L)$ has dimension $2g-2$, and when $r = g$ a generic point ζ lies on a 1-parameter family of subspaces P_Q with $|Q| = g$. As before, in order to ensure that the sections are regular, we need the divisors $K - L - 2Q$ of degree $g-2$ to be noneffective. This can be arranged since the set of effective divisors which we must avoid has complex codimension at least 2. □

3. Symplectic structures on ruled surfaces

We will begin by proving the existence part of Theorem 1.2. We give two proofs: the first uses methods from complex geometry.

Proposition 3.1. *When $g > 0$ there is a Kähler structure on V_M in every class $a = \mu_+ b_+ + \mu_- b_-$ with $\mu_+ > |\mu_-|$.*

Proof 1. We will suppose that V_M has the complex structure constructed in Theorem 2.4. Nakai's criterion implies that a cohomology class $a \in H^2(V_M, \mathbf{Z})$ with $a^2 > 0$ has a Kähler representative iff $a(C) > 0$ for every holomorphic curve C in V_M: see [4, V, Theorem 1.10]. (Note that $H^2(V_M, \mathbf{C}) = H^{1,1}(V_M, \mathbf{C})$, so that every integral class a is the Chern class of some holomorphic line bundle.) If $a = \mu_+ b_+ + \mu_- b_-$ then the condition that $a^2 > 0$ is exactly $\mu_+ > |\mu_-|$. We will show that this condition is sufficient, i.e. that if $C = p[M_1] + q[F]$ has a holomorphic representative, then $p\mu_+ + q(\mu_+ - \mu_-) > 0$. Here $[F] = [M_1] - [M_{-1}]$ is the class of the fiber. Since the class $[F]$ has a holomorphic representative, we must have $(p[M_1] + q[F]) \cdot [F] = p \geq 0$ by positivity of intersections. Hence the only case which might cause problems is when $q < 0$, and here, because $2\mu_+ > \mu_+ - \mu_-$, it suffices to show that $p + 2q \geq 0$. The curve C is a p-fold branched cover of the base, and so, by the Riemann-Hurwitz formula, its genus g_C satisfies the inequality $g_C \geq p(g-1) + 1$, where g is the genus of M. Using the adjunction formula

$$g_C \leq 1 + \tfrac{1}{2}(C \cdot C + K_V \cdot C),$$

where K_V is the canonical class of V_M, one finds that

$$1 + p(g-1) \leq 1 + \tfrac{1}{2}(p^2 + 2pq - p - 2q).$$

Thus $(p+2q)(p-1) \geq 0$, and so, either $p = 1$ and C is a section, or $p > 1$ and $p + 2q \geq 0$. Since the only sections represented have $q > 0$, we have $p + 2q \geq 0$ in all cases, as required.

This shows that every cohomology class with $\mu_+ > |\mu_-|$ and μ_-/μ_+ rational is represented by a Kähler form. But, if ω is a Kähler form on (V_M, J) and if σ is an area form on M, the form $\omega + \epsilon\pi^*\sigma$ is also Kähler for all $\epsilon > 0$, because π is J-holomorphic. The desired result now follows easily. □

Proof 2. Here is a different procedure (suggested to me by Hitchin) for constructing these symplectic forms. First consider the case when M is the torus T^2. Identify T^2 with the quotient $\mathbf{R}^2/\mathbf{Z}^2$, and consider the action of $\mathbf{Z}^2$ on S^2 in which the two generators τ_1, τ_2 of $\mathbf{Z}^2$ act by half-turns h_1 and h_2 around two mutually orthogonal axes. The quotient V_T of $\mathbf{R}^2 \times S^2$ by the diagonal action is the nontrivial S^2-bundle over T^2. (This holds because the given representation of $\mathbf{Z}^2$ on $SO(3)$ does not lift to $SU(2)$.) Because $\mathbf{Z}^2$ acts by elements of order 2, the square

$$I_z = [0, 2] \times [0, 2] \times \{z\} \subset \mathbf{R}^2 \times S^2, \qquad z \in S^2,$$

descends to a closed submanifold in V_T which lies in the class $4[M_+] - 2[F] = 2[M_+] + 2[M_-]$. Thus, if σ is an area form on S^2 with total area 1, the symplectic form $\lambda_1 ds \wedge dt \oplus \lambda_2 \sigma$ on $\mathbf{R}^2 \times S^2$ descends to a form on V_T in the class $a = \mu_+ b_+ + \mu_- b_-$ where $\mu_+ + \mu_- = 2\lambda_1$ and $\mu_+ - \mu_- = \lambda_2$. Thus by a correct choice of the parameters λ_1 and λ_2 one can get a form in any class a.

This construction may be adapted to the case when M has genus > 1. Here, think of M as $\mathscr{H}/\Gamma$, where $\mathscr{H}$ is the Siegel upper half-plane, and consider the action of Γ on S^2 in which a pair of conjugate generators act by the half-turns h_1 and h_2, and the others act trivially. The bundle so obtained is the pull-back by a map of degree 1 of the above bundle on T^2, and all the above remarks apply. □

Remark 3.2. Because the space of conjugacy classes of representations of $\pi_1(M)$ into $SO(3)$ is connected, the isotopy class of a form obtained by the second construction is determined by its cohomology class. A similar conclusion holds for forms obtained by the first construction, this time because the space of relevant complex structures is connected.

When discussing the question of uniqueness it is useful to have *standard forms* with which to compare an arbitrary form. The standard forms on the product $M \times S^2$ are of course the split forms. In the case of V_M, we will in this paper discuss this question only for forms in classes with $\mu_- > 0$. Therefore, we can use the same standard forms as in [8]. Namely, we consider S^1-invariant forms which are Kähler with respect to the complex structure induced by the bundle $\mathbf{C} \oplus L$ where $c_1(L) = 1$. The corresponding moment map has a minimum at the section $\mathbf{P}(0 \oplus L)$ of self-intersection -1 and a maximum at the section $\mathbf{P}(\mathbf{C} \oplus 0)$ of self-intersection $+1$. Moreover, it is easy to check that, up to equivariant symplectomorphism, there is a unique such form in each cohomology class: see [7, Lemma 4] for example.

Let us now consider the structure of an arbitrary compatible symplectic form ω on the ruled surface V over M. The arguments given in [8, §4] which prove the structure theorems of §1 in the case $g = 0$ are correct. The higher genus case is deduced from this by a procedure of cutting and pasting, which we now describe. We will suppose that M has been assembled from a $4g$-sided polygon P by identifying its sides in the standard way $a_1, a_2, a_1^{-1}, a_2^{-1}, \dots, a_{2g}^{-1}$.

The vertices of P are identified to a point x_0 in M and the sides form embedded loops $\lambda_1, \dots, \lambda_{2g}$ which intersect only at x_0.

Let $\Lambda = \lambda_1 \cup \cdots \cup \lambda_{2g}$ and let F be the fiber at x_0. For each loop $\lambda_i \in \Lambda$ there is an area-preserving holonomy map $h_i\, F \to F$, which is induced by the characteristic flow round the hypersurface $\pi^{-1}(\lambda_i) \cong S^1 \times F$ in (V, ω). (Note that the characteristic flow is tangent to the null directions of ω along the hypersurface, and so is always transverse to the fibers of π.)

Proposition 3.3. *Suppose that all these holonomy maps are the identity. Then, if V is the nontrivial bundle, $[\omega] = \mu_+ b_+ + \mu_- b_-$ where $\mu_- > 0$. Moreover, in all cases, ω is isotopic to a standard form.*

Proof. The hypothesis implies that we can cut V open along $\pi^{-1}(\Lambda)$ to get a ruled surface over P equipped with a symplectic form which splits over some neighbourhood U of its boundary, i.e., we can identify $\pi^{-1}(U)$ with the product $U \times S^2$ in such a way that the restriction of ω to $U \times S^2$ is a split form $\sigma_1 \oplus \sigma_2$. Therefore, by identifying the boundary of P to a point, we obtain a symplectic form ω' on a ruled surface over S^2. The class a of ω' may still be written as $\mu_+ b_+ + \mu_- b_-$, and so $\mu_- > 0$ by the known result for $M = S^2$. Moreover, ω is standard because ω' is. For further details, see [8, Lemma 4.13]. □

The next step is to show that there is a pseudo-isotopy from ω to a form ω' which has trivial holonomy round Λ. In [8, Lemma 4.14], we show that there is a suitable pseudo-isotopy which decomposes into two parts: first we increase the size of the base by the pseudo-isotopy

$$\omega_t = \omega + t\pi^*(\tau), \qquad 0 \le t \le 1, \tag{1}$$

where τ is a 2-form on M supported in a small neighbourhood D of x_0, and second we isotop ω_1 to ω' by an isotopy with support in $\pi^{-1}D$. This proves that ω is always pseudo-isotopic to a standard form as is claimed in part (iii) of Theorems 1.1 and 1.2.

The final step is to show that when π admits a suitable symplectic section, one can replace the above pseudo-isotopy ω_t, $0 \leq t \leq 1$, by an isotopy. Contrary to what is claimed in [8, Lemma 4.15], our method works only under a compatibility condition between the homology class of the section and the cohomology class of ω. Here is a corrected version of this lemma. I am indebted to Lalonde for pointing out the original mistake and the need for Lemmas 3.9 and 3.11. He adopts a slightly different approach to these questions in [5].

Proposition 3.4. *Suppose that* $\omega(M_{-k}) > 0$ *and that* ω *is nondegenerate on some section* Z *of* π *in the class* $[M_k]$, *where* $k > 0$. *Then* ω *is isotopic to a form* ω' *which is* π*-compatible on* $\pi^{-1}(\Lambda)$ *and has trivial holonomy round* Λ.

Remark 3.5. It is easy to check that the condition $\omega(M_{-k}) > 0$ is equivalent to the condition $2\mu_1 > k\mu_2$ when $[\omega] = \mu_1 b + \mu_2 f$ on $M \times S^2$, and to $(k+1)\mu_- > (k-1)\mu_+$ when $[\omega] = \mu_+ b_+ + \mu_- b_-$ on V_M. For, in the first case $[M_{-k}] = [M_0] - (k/2)[F]$, and in the second, $[M_{-k}] = [M_1] - ((k+1)/2)([M_1] - [M_{-1}])$.

Corollary 3.6. *Under the above hypotheses, the form* ω *is standard.*

Proof. This follows immediately from the two previous propositions. □

In the proof of Proposition 3.4, we will need the following results about symplectic structures on vector bundles.

Lemma 3.7. *Let* $\pi : E \to M$ *be a complex line bundle of degree* $k \geq 0$, *and let* ω *be a* π*-compatible symplectic form which is defined in some open neighbourhood* U *of the zero-section* Z_M *and does not vanish on* Z_M. *Then there is a closed 2-form* ρ *supported in* U *which represents the Thom class of* E *and is such that the form* $\omega + s\rho$ *is symplectic on* U *for each* $s > 0$. *Moreover, given a point* $x_0 \in M$, *and a trivialization* $\pi^{-1}D = D \times \mathbf{C}$ *of* E *in some neighbourhood* D *of* x_0 *such that* ω *splits in* $\pi^{-1}D \cap U$, *we may assume that the forms* $\omega + s\rho$ *also split in* $\pi^{-1}D \cap U$.

Proof. Let ω_M be the restriction of ω to the zero section $Z_M \equiv M$. Choose an inner product on E, and a connection 1-form β on the unit circle bundle S of E so that $d\beta = -\pi^*(\phi\,\omega_M)$, for some function ϕ on M which $= 0$ on D and is ≥ 0 everywhere. We may further suppose that this connection is compatible with the given trivialization of E over D. Let r be the radial distance function in the fibers of E, and extend β to $E - Z_M$ by pulling it back by the obvious radial projection $E - Z_M \to S$. Then, for every $\epsilon > 0$, there is a representative of the Thom class of E with the formula

$$\rho_\epsilon = -d(\psi_\epsilon(r^2)\beta),$$

where $\psi_\epsilon(t)$ is a smooth nonincreasing function which equals $1/2\pi - t$ near $t = 0$ and 0 for $t > \epsilon$. (Note that the 2-form $d(\psi_\epsilon(r^2)\beta)$ extends over Z_M although the 1-form $\psi_\epsilon(r^2)\beta$ does not.)

It is easy to check that the 2-form

$$\omega_0 = \pi^*\omega_M + d(r^2\beta) = (1 - r^2\pi^*\phi)\pi^*(\omega_M) + 2r dr \wedge \beta$$

is nondegenerate near Z_M and restricts to ω_M on Z_M. Therefore, by the symplectic neighbourhood theorem, there is an isotopy g_t which is the identity on Z_M and has support in U such that $g_1^*\omega = \omega_0$ in some open neighbourhood $U' \subset U$ of Z_M. (Note that g_t is the composite of two isotopies: the first isotops ω so that for each $x \in Z_M$, $\omega(x) = \omega_0(x)$, and the second (which is C^1-small) makes the two forms agree near Z_M.)

Since $\phi = 0$ in D, the connection form β is flat over D and so may be used to identify $\pi^{-1}D$ with $D \times \mathbf{C}$. Then, ω_0 splits with respect to this product structure, and so, because ω also splits with respect to this product structure, we may assume that g_t is the identity in $\pi^{-1}D$.

Choose $\epsilon > 0$, so that $\{x : r(x) \leq 2\epsilon\} \subset U'$. Then

$$\omega_s = \omega_0 + s\rho_\epsilon = f_s \pi^* \omega_M + g_s r dr \wedge \beta$$

in U', where the functions f_s, g_s are strictly positive. Thus ω_s is nondegenerate in U' for all $s > 0$, and we may take $\rho = (g_1{}^{-1})^* \rho_\epsilon$. □

Remark 3.8. It is easy to see that, given any loop λ in M based at x_0, the characteristic foliation induced by $\omega_s = \omega_0 + s\rho_\epsilon$ on $\pi^{-1}(\lambda)$ is tangent to the horizontal distribution $\ker\beta$. In particular the intersection $\lambda_Z = Z_M \cap \pi^{-1}(\lambda)$ is a closed characteristic. Thus, if F_0 is the fiber $\pi^{-1}x_0$ and x_Z is the base point $Z_M \cap F_0$, the holonomy of ω_s round λ is a germ of a self-map of F_0 at x_Z. Moreover, it coincides with the holonomy of the connection β, and so is independent of s.

We now adapt this construction so that we can keep control of the holonomy of $\omega + s\rho$ round the loops of Λ. Observe that, because the isotopy g_t need not be C^1-small, the forms $\omega + s\rho$ constructed above may not be compatible with either π or $\pi \circ g_1$ on the whole of E, although it is compatible with both fibrations near Z_M.

Lemma 3.9. *Suppose that the loop λ_Z is a closed characteristic for ω. Then ω is isotopic (by a C^1-small isotopy) to a π-compatible form $\tilde{\omega}$ whose holonomy round λ is trivial near the base point x_Z. Further, we may choose ρ in the above lemma so that all the forms $\tilde{\omega} + s\rho$ are π-compatible on $\pi^{-1}(\lambda)$ and have the same holonomy round λ.*

Proof. Our hypothesis is equivalent to requiring that at each point $x \in \lambda_Z$ the fiber $F_x = \pi^{-1}(x)$ is ω-perpendicular to the tangent space $T_x Z_M$. Thus, we may choose β and r in Lemma 3.7 above so that $\omega = \omega_0$ on $T_x E$ for each $x \in Z_M$. (This is just a matter of choosing an appropriate inner product so that the form ω_0 is correct in the fiber direction.) We may also choose β to be flat near λ, so that $\phi = 0$ near λ.

It follows that we may choose the isotopy g_t so that $dg_t(x) = \mathrm{id}$ for $x \in \lambda_Z$ and all t. Thus g_1 is C^1-close to the identity near λ_Z. Hence, if U is sufficiently small, $g_t|_{\mathrm{nbhd}\lambda_Z}$ may be extended to a C^1-small isotopy h_t such that $h_1^*(\omega)$ is π-compatible. Since $h_1^*(\omega)$ equals ω_0 near $Z_M \cap \pi^{-1}(\lambda)$, its holonomy round λ is trivial near the base point, and so we may take $\tilde{\omega} = h_1^*(\omega)$.

Now apply the argument of Lemma 3.7 to $\tilde{\omega}$, taking U so small that $\tilde{\omega} = \omega_0$ near $U \cap \pi^{-1}(\lambda)$. Then we may suppose that g_t (which has support in U) is the identity near $\pi^{-1}(\lambda)$, and so we obtain a family of forms $\tilde{\omega}_s = \tilde{\omega} + s\rho_\epsilon$

which equal ω_s on $U \cap \pi^{-1}(\lambda)$ and equal $\tilde{\omega}$ elsewhere on $\pi^{-1}(\lambda)$. The result follows. □

Remark 3.10. If ω has trivial holonomy round λ, it is not hard to arrange that all the forms $\tilde{\omega} + s\rho$ also have trivial holonomy round λ.

We want to apply these lemmas to the symplectic section Z of $\pi : V \to M$. We first have to arrange that the hypothesis of the previous lemma is satisfied. Given a loop λ in M, we will say that its lift $\lambda_Z = Z \cap \pi^{-1}(\lambda)$ to Z is a closed characteristic if it is a closed leaf of the characteristic foliation on $\pi^{-1}(\lambda)$.

Lemma 3.11. *Suppose that a π-compatible form ω is nondegenerate on some section Z of π. Then, given any loop λ in M, ω is isotopic to a π-compatible form ω' which is nondegenerate on Z and has λ_Z as a closed characteristic.*

Proof. We need to arrange that the fibers of π are ω'-orthogonal to Z at all points $x \in \lambda_Z$. To do this, we will find an isotopy g_t of V supported near λ_Z such that

(i) $g_t(x) = x$ for $x \in Z$;
(ii) ω is nondegenerate on the fibers of $\pi' = \pi \circ g_1$;
(iii) for each $x \in \lambda_Z$ the fiber of π' through x is tangent to the subspace

$$T_xZ^{\perp} = \{v : \omega(v, w) = 0 \text{ for all } w \in T_xZ\}.$$

Clearly, we may then take $\omega' = (g_1^{-1})^*\omega$.

By the symplectic neighbourhood theorem we may identify a neighbourhood of λ_Z in (V, ω) with a neighbourhood of the circle $S^1 \times \{0, 0, 0\}$ in the product $(S^1 \times \mathbf{R}^3, ds \wedge dt + du \wedge dv)$, where (s, t, u, v) are the obvious coordinates. We may also suppose that Z corresponds to the surface $u = v = 0$.

Then, the fiber of π through x is taken to a submanifold $F_{s,t}$ which is tangent at $x \equiv (s, t, 0, 0)$ to an affine subspace $W_{s,t}$ of $S^1 \times \mathbf{R}^3 \equiv \mathbf{R}^4/\mathbf{Z}$ through $(s, t, 0, 0)$. Thus we may alter π by a C^1-small isotopy so that its fibers coincide with the $W_{s,t}$ near the circle. By rescaling both ω and the coordinates in $\mathbf{R}^3$ as necessary, we may suppose that this is true in the set $\{(s, t, u, v) : |t| \le 1, \ r^2 = u^2 + v^2 \le 1\}$. Each $W_{s,t}$ is a graph over the u, v-plane. Thus there are coefficients a, b, c, d which vary smoothly with s, t, such that, for each s, t, $W_{s,t}$ is the image of the linear map

$$(u, v) \mapsto (s + au + bv, t + cu + dv, u, v).$$

Since $W_{s,t}$ is symplectic, the determinant $D = ad - bc$ must always be > -1. Given a bijective increasing smooth map $\alpha : [0, 1] \to [0, 1]$, let $W^{\alpha}_{s,t}$ be the image of the map

$$(u, v) \mapsto (s + \alpha(r)(au + bv), \ t + \alpha(r)(cu + dv), \ u, \ v), \qquad \text{for } (u, v) : r \le 1,$$

where $r^2 = u^2 + v^2$. Then $W^{\alpha}_{s,t}$ will be smooth if all the derivatives of α vanish at $r = 0$, and it will fit together smoothly with $W_{s,t}$ for $r \ge 1$ if $\alpha(r) = 1$ for r near 1. Further, it is easy to check that the restriction of the symplectic form $ds \wedge dt + du \wedge dv$ to $W^{\alpha}_{s,t}$ is $(1 + (r\alpha\alpha' + \alpha^2)D)du \wedge dv$, and so $W^{\alpha}_{s,t}$ is symplectic iff

$$r\alpha\alpha' + \alpha^2 \le 1 + \varepsilon, \tag{2}$$

where $\varepsilon > 0$ is chosen so that $(1+\varepsilon)D > -1$.

We claim that, for each $\varepsilon > 0$ there is a function α which satisfies all the above requirements. To see this, note that, because $\alpha(r) \le 1$, condition (2) will be satisfied if $\alpha'(r) \le \varepsilon/r$. The other conditions can then be satisfied because $\int_0^1 1/r\, dr = \infty$.

Fix such an α. We want the fibration π' to have fibers of the form $W^{\alpha}_{s,0}$, at all points $t = 0$. It is not hard to check that these manifolds $W^{\alpha}_{s,0}$ are disjoint. As $|t|$ increases to 1, the fibers should interpolate between the old and the new: thus the fiber should be the graph of

$$(u, v) \mapsto (s + \beta(|t|, r)(au + bv),\ t + \beta(|t|, r)(cu + dv),\ u,\ v),$$

for a suitable function $\beta(t, \cdot)$ which equals $\alpha(r)$ when $t = 0$ and 1 when $|t| = 1$. These fibers would also be symplectic and disjoint if we could make $\partial\beta/\partial t$ sufficiently small, i.e., if we could extend the range of t relative to the range of r. Evidently, this may be achieved by replacing the function α by the function α_δ, where $\alpha_\delta(r) = \alpha(r/\delta)$ for some small constant $\delta > 0$. (This does not affect the validity of (2).) This constructs a fibration π' satisfying condition (iii). It is obvious from the construction that a suitable isotopy g_t also exists. □

Proof of Proposition 3.4. By the symplectic neighbourhood theorem, we may suppose that ω splits in some neighbourhood $\pi^{-1}D = D \times F$ of the fiber $F = \pi^{-1}(x_0)$, and then may isotop Z (and shrink D if necessary) so that $Z \cap \pi^{-1}D = D \times \{z_0\}$ for some point z_0. By Lemmas 3.11 and 3.9 we may isotop ω (without changing it near $Z \cap \pi^{-1}D$) first so that all the loops in Λ are closed characteristics, and then so that, for suitable ρ, all the forms $\omega + s\rho$ are π-compatible on $\pi^{-1}(\Lambda \cup D)$ and have the same holonomy round the loops of Λ.

Now choose the form τ to have support in D as in (1), and consider the forms

$$\omega_{t,s} = \omega + t\pi^*\tau + s\rho, \qquad s, t \ge 0.$$

By construction, they are all symplectic, and all have the same holonomy round Λ. Moreover, because this holonomy is trivial near the base point $z_0 \in Z$, the construction given in [8, Lemma 4.14], which isotops $\omega_{1,0}$ in $\pi^{-1}D$ to a form ω' which has trivial holonomy round Λ, works just as well for the forms $\omega_{1,s}$, $s > 0$. Thus each $\omega_{1,s}$ is isotopic, by an isotopy with support in $\pi^{-1}D$, to a form with trivial holonomy.

Since multiplying a symplectic form by a scalar κ does not change its holonomy, it now suffices to check that if $\omega(M_{-k}) > 0$ we can choose $s(t)$ and $\kappa(t)$ so that the forms $\kappa(t)\omega_{t,s(t)}$, $0 \le t \le 1$, are cohomologous. To see this, let $T = \int_M \tau$. When $V = M \times S^2$, because $\rho(M_{-k}) = 0$, we find $[\rho] = kb/2 + f$. Thus

$$[\omega_{t,s}] = \mu_1 b + \mu_2 f + tTb + s\left(\frac{k}{2}b + f\right) = \left(\mu_1 + tT + \frac{sk}{2}\right)b + (\mu_2 + s)f,$$

which is a positive multiple of $\mu_1 b + \mu_2 f$ iff $2\mu_1 - k\mu_2 > 0$. When $V = V_M$, one has

$$[\omega_{t,s}] = \mu_+ b_+ + \mu_- b_- + tT(b_+ + b_-) + s\left(\frac{k+1}{2}b_+ + \frac{k-1}{2}b_-\right),$$

and the result follows as before. □

It remains to establish which homology classes in V may be represented by symplectic sections, i.e., sections Z on which ω does not vanish. One might try to construct these by cutting and pasting, but there seems to be no way of doing this which allows one to keep control of the homology class of Z.

Another approach constructs Z as a J-holomorphic curve.

Proposition 3.12. *Let (V, ω_0) be a ruled surface over a Riemann surface of genus $g > 0$ and let $\omega_t, 0 \le t \le 1$, be a pseudo-isotopy such that ω_1 is a Kähler form for one of the complex structures J constructed in Theorem 2.4. Then the regular J-holomorphic sections whose existence is established in this theorem persist under the deformation ω_t. In particular, there are sections in the given classes which are ω_0-symplectic.*

Remark 3.13. We attempted to prove a similar result in Lemma 4.16 of [8]. There we considered sections in class M_{2g-1} on V_M, and tried to show that such a section would persist under a pseudo-isotopy. The proof is correct, except for one point. The integrable complex structure J_1' which we used to establish the degree of the evaluation map is nongeneric and has too many cusp curves. In fact, because there is a J_1'-holomorphic section M_{1-2g}, there is a cusp curve through any set of $2g-1$ points, namely the union of M_{1-2g} with $2g-1$ fibers. Since we are only at liberty to choose g points, it is impossible to avoid these cusp curves, and indeed, when we move J_1' to a nearby generic J some (but not all) of these cusp curves will join up to form J-holomorphic curves in class M_{2g-1}. One suspects that the end result will always be to increase the degree of the evaluation map e: this is certainly true if we may assume that our generic J is integrable. (An instructive example to consider is the case when $M = T^2$, when we know from Proposition 2.2 that e has degree 2.) However, if J is not integrable, it is not yet known whether evaluation maps are always locally orientation preserving for curves of genus > 0 (see [9]), and so it is conceivable that that the curves which one knows about are cancelled out by some others where the evaluation map is negatively oriented. Thus, in order to establish that the evaluation map has positive degree, we must construct a generic (i.e., regular) integrable J as we have done here.

Proof. We begin by recalling some facts from [8].

Let A be the homology class of the sections under consideration, let $\mathcal{J}$ denote the space of all almost complex structures on V which are ω_t-tame for some $0 \le t \le 1$, and let $\mathcal{T}_g$ be the Teichmüller space for surfaces of genus g. Given $J \in \mathcal{J}$, consider the space $\mathcal{M}(J, \mathcal{T}_g, A)$ of all holomorphic maps $(\Sigma_g, j_\tau) \to (V, J)$, where j_τ is the complex structure on the (fixed) surface Σ_g which corresponds to $\tau \in \mathcal{T}_g$. For regular J, this is a smooth manifold which has (real) dimension 2 if $g > 1$ and dimension 4 when $g = 1$.

However, the unparametrized moduli space $\mathcal{M}(J, \mathcal{T}_g, A)/G_g$ is a manifold of dimension 2 for all g. Here G_g is the modular group for $g > 1$ and is an extension of $PSL(2, \mathbf{Z})$ by the rotation group T^2 when $g = 1$, acting in the obvious way. Thus, if we denote its action on $\mathcal{T}_g$ by $\gamma \in G_g : \tau \mapsto \gamma \cdot \tau$, then there is an injective homomorphism $\gamma \mapsto \phi_\gamma$ from G_g into the group of diffeomorphisms of Σ_g such that $(\phi_\gamma)_*(j_\tau) = j_{\gamma\cdot\tau}$, and the action on $\mathcal{M}(J, \mathcal{T}_g, A)$ is given by

$$\gamma(u, j_\tau) = (u \circ \phi_\gamma, j_{\gamma\cdot\tau}).$$

Since G_g acts freely on $\mathcal{M}(J, \mathcal{T}_g, A)$, the quotient $\mathcal{M}(J, \mathcal{T}_g, A)/G_g$ is a manifold. Further, if $\dim\mathcal{M}(J, \mathcal{T}_g, A)$ is positive, the argument of [6, §4] shows that both $\mathcal{M}(J, \mathcal{T}_g, A)$ and $\mathcal{M}(J, \mathcal{T}_g, A)/G_g$ have a natural orientation, which is induced by the obvious complex structure when J is integrable.

The fibration $\mathcal{M}_\Sigma(J) \to \mathcal{M}(J, \mathcal{T}_g, A)$ with fiber (Σ_g, j_τ) at (u, τ) descends to a fibration $\Sigma_g \to W(J, A) \to \mathcal{M}(J, \mathcal{T}_g, A)/G_g$. Moreover, there is an evaluation map $e : W(J, A) \to V$, which takes the point z in the fiber over (u, τ) to the point $u(z) \in V$. When J is integrable, $W(J, A)$ has a complex structure which extends the obvious complex structures on its fibers, and both maps $W(J, A) \to \mathcal{M}(J, \mathcal{T}_g, A)/G_g$ and $W(J, A) \to V$ are holomorphic.

By [8, (2.4)], there is an open, dense and path-connected subset $\mathcal{U}$ of $\mathcal{J}$ consisting of J which admit no holomorphic sections with self-intersection number less than $A \cdot A$. It follows easily from positivity of intersections that every A-cusp curve must contain such a section. Thus the moduli space $\mathcal{M}(J, \mathcal{T}_g, A)/G_g$ of unparametrized J-holomorphic A-curves, and hence also $W(A, J)$, is compact for $J \in \mathcal{U}$.

Let us now get back to the problem in hand. There are two cases to consider, one when the curves in question form a 2-(real)-dimensional family, and one in which they are isolated. In Theorem 2.4, we established in both cases the existence of regular elements $J_1 \in \mathcal{U}$ which are integrable and are ω_1-tame. Thus, in the first case, $e(J_1) : W(A, J) \to V$ is a holomorphic map between compact 4-manifolds, and so it has positive degree, d say. Since $\mathcal{U}$ is path-connected, it follows that $e(J)$ has degree d for every regular element J of $\mathcal{U}$. In particular, $W(J, A)$ is nonempty for all such J. But, by [8, (2.4)], there is a regular element J_0 of $\mathcal{U}$ which is ω_0-tame. Moreover, by [8, Lemma 4.4], we may assume that the fibers of π are J_0-holomorphic.

Hence, there is a J_0-holomorphic A-curve, which must be a section of π because $A \cdot F = 1$. This gives us the required ω_0-symplectic section.

In the second case, the sections of interest are isolated and the moduli space $\mathcal{M}(J, \mathcal{T}_g, A)/G_g$ consists of isolated oriented points. The orientation procedure shows that when J is integrable each of these points is positively oriented.

Therefore, because the moduli space is nonempty for an integrable $J \in \mathcal{U}$ it must be nonempty for every $J \in \mathcal{U}$. Thus, as above, there is an ω_0-symplectic section.

We end with a direct geometric argument which shows, in this second case, that $\mathcal{M}(J, \mathcal{T}_g, A)$ can never be empty.

Let $A' = A + F$. Because $c(F) = 2$, the moduli space $\mathcal{M}(J, \mathcal{T}_g, A')/G$ has dimension 4. Hence, for a generic point $x \in V$ the moduli space $\mathcal{M}(J, \mathcal{T}_g, x, A')/G$ of A'-curves which go through x has dimension 2. Observe that, for $J \in \mathscr{U}$, the only possible A'-cusp curves are unions of an F-curve (i.e., fiber of π_J) with an A-curve. Since A-curves are isolated, we may always assume that x is not on an A-curve, but there still may be a finite number of A'-cusp curves through x since there is always a fiber through x. Observe that, if $\mathscr{M}(J, \mathscr{T}_g, A)$ is empty for some $J \in \mathscr{U}$, the space $\mathscr{M}(J, \mathscr{T}_g, x, A')/G$ must be compact.

We now complete the proof by showing that $\mathscr{M}(J, \mathscr{T}_g, x, A')/G$ cannot be compact. Consider the map

$$e_P = \mathscr{M}(J, \mathscr{T}_g, x, A')/G \to \mathbf{C}P^1 = P_{\mathbf{C}}(T_x V)$$

which takes a curve to its complex tangent plane at x. The above remarks show that when $J \in \mathscr{U}$, e_P is a proper map onto $\mathbf{C}P^1 - \{T_F\}$, where T_F is the tangent space to the fiber of π_J though x. It therefore has a well-defined degree, which is independent of $J \in \mathscr{U}$.

Taking J to be one of the generic integrable ω_1-tame complex structures of Theorem 2.4, one sees that this degree is positive. Thus, if $\mathscr{M}(J, \mathscr{T}_g, x, A')/G$ is compact, e_P must be surjective.

This means that there is an A'-curve which is tangent to an F-curve. Thus, by positivity of intersections, $A' \cdot F \geq 2$. But this is impossible because $A' \cdot F = 1$. □

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