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STABLE VECTOR BUNDLES ON ALGEBRAIC SURFACES

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ABSTRACT. We prove an existence result for stable vector bundles with arbitrary rank on an algebraic surface, and determine the birational structure of a certain moduli space of stable bundles on a rational ruled surface.

1. INTRODUCTION

Let $\mathcal{M}_L(r; c_1, c_2)$ be the moduli space of L-stable (in the sense of Mumford-Takemoto) rank-r vector bundles with Chern classes c_1 and c_2 on an algebraic surface X. The nonemptiness of $\mathcal{M}_L(2; 0, c_2)$ has been studied by Taubes [22], Gieseker [9], Artamkin [1], Friedman [8], Jun Li, etc. The generic smoothness of $\mathcal{M}_L(2; c_1, c_2)$ has been proved by Donaldson [6], Friedman [8], and Zuo [23]. For an arbitrary r and c_1, Maruyama [17] proved that for any integer s, there exists an integer c_2 with $c_2 \geq s$ such that $\mathcal{M}_L(r; c_1, c_2)$ is nonempty; however, no explicit formula for the lower bound of c_2 was given. Using deformation theory on torsionfree sheaves, Artamkin [1] showed that if $c_2 > (r+1) \cdot \max(1, p_g)$, then the moduli space $\mathcal{M}_L(r; 0, c_2)$ is nonempty and contains a vector bundle V with $h^2(X, \mathrm{ad}(V)) = 0$ where $\mathrm{ad}(V)$ is the tracefree subvector bundle of $\mathrm{End}(V)$. Based on certain degeneration theory, Gieseker and J. Li [10] announced the generic smoothness of the moduli space $\mathcal{M}_L(r; c_1, c_2)$.

In the first part of this paper, we determine the nonemptiness of $\mathcal{M}_L(r; c_1, c_2)$ in the most general form and show that at least one of the components of moduli space is generically smooth. Using an explicit construction, we show the following.

Theorem 1.1. *For any ample divisor L on X, there exists a constant α depending only on X, r, c_1, and L such that for any $c_2 \geq \alpha$ there exists an L-stable rank-r bundle V with Chern classes c_1 and c_2. Moreover, $h^2(X, \mathrm{ad}(V)) = 0$.*

This is proved in §2. Our starting point is the classical Cayley-Bacharach property. A well-known result (see [11, p. 731]) says that there exists a rank-2 bundle given by an extension of $\mathcal{O}_X(L'') \otimes I_Z$ by $\mathcal{O}_X(L')$ if and only if the 0-cycle Z satisfies the Cayley-Bacharach property with respect to the complete linear system $|(L'' - L' + K_X)|$, that is, any curve in $|(L'' - L' + K_X)|$ containing all but one point in Z must contain the remaining point. It follows that to

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construct a rank-r bundle V as an extension of

$$\bigoplus_{i=1}^{(r-1)}[\mathcal{O}_X(L_i)\otimes I_{z_i}]$$

by $\mathcal{O}_X(L')$, we need only make sure that Z_i satisfies the Cayley-Bacharach property with respect to $|(L_i - L' + K_X)|$ for each i. Now, let L be an ample divisor and normalize c_1 such that $-rL^2 < c_1 \cdot L \leq 0$. Let $L' = c_1 - (r-1)L$ and $L_i = L$. Our main argument is that if the length of Z_i is sufficiently large and if Z_i is generic in the Hilbert scheme $\mathrm{Hilb}^{l(Z_i)}(X)$ for each i, then the vector bundle V is L-stable and

$$h^2(X, \mathrm{ad}(V)) = 0.$$

A similar construction for stable rank-2 bundles is well known [20].

We notice that there have been extensive studies for stable rank-2 bundles on $\mathbf{P}^2$ and on a ruled surface [3, 14, 13, 4, 5, 8,16, 21] and for stable bundles with arbitrary rank on $\mathbf{P}^2$ [15, 18, 7, 1]. In the rest of this paper, we study the structure of $\mathcal{M}_L(r; c_1, c_2)$ for a suitable ample divisor L on a ruled surface X. In §3, we prove that $\mathcal{M}_L(r; c_1, c_2)$ is empty if $(c_1 \cdot f)$ is not divisible by r and that $\mathcal{M}_L(r; \mathrm{t}f, c_2)$ is nonempty if $-r < t \leq 0$ and $c_2 \geq 2(r-1)$; moreover, we show that the restriction of any bundle in $\mathcal{M}_L(r; \mathrm{t}f, c_2)$ to the generic fiber of the ruling π must be trivial.

In §4, we assume that X is a rational ruled surface and verify that a generic bundle V in $\mathcal{M}_L(r, tf, c_2)$ sits in an exact sequence of the form:

$$0 \to \bigoplus_{i=1}^{r}\mathcal{O}_X(-n_i f) \to V \to \bigoplus_{i=1}^{c_2}(\tau_i)_*\mathcal{O}_{f_i}(-1) \to 0 \tag{1.2}$$

where $\{f_1, \dots, f_{c_2}\}$ are distinct fibers with τ_i being the natural embedding $f_i \hookrightarrow X$ and the integer n_i is defined inductively by (4.20). The idea is a natural generalization of those in [4, 5, 8]. Since the restriction of V to the generic fiber is trivial, $\pi_* V$ is a rank-r bundle on $\mathbf{P}^1$; thus, we can construct $(r-1)$ exact sequences:

$$0 \to \mathcal{O}_X(-n_i f) \to V_i^{**} \to V_{i-1} \to 0$$

where $i = r, \dots, 2$, $V_r = V$, and V_i is a torsionfree rank-i sheaf. By estimating the numbers of moduli of V_i and V_i^{**}, we conclude that for a generic V, the sheaves $V_2, \dots, V_r$ are all locally free and $V_1 = \mathcal{O}_X((c_2 - n_1)f) \otimes I_Z$ where Z consists of c_2 points lying on distinct fibers. Then the exact sequence (1.2) follows.

In §5, based on (1.2), we define a rational map Φ from $\mathcal{M}_L(r; tf, c_2)$ to $\mathbf{P}^{c_2}$ and show that the fiber is unirational. We thus obtain our second main result.

Theorem 1.3. *Let X be a rational ruled surface. Assume that the moduli space $\mathcal{M}_L(r; tf, c_2)$ is nonempty where $r \geq 2$, $-r < t \leq 0$, and L satisfies condition* (3.3). *Then $\mathcal{M}_L(r; tf, c_2)$ is irreducible and unirational.*

One consequence of Theorem 1.3 is that the moduli space $\mathcal{M}_L(r; 0, c_2)$ on $\mathbf{P}^2$ which is known to be irreducible [15, 7] is unirational. In fact, we shall show that any irreducible component of a nonempty moduli space on a rational surface is unirational and determine the irreducibility and rationality in the rank-3 case. Details will appear elsewhere

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Notation and conventions

X stands for an algebraic surface over the complex number field $\mathbf{C}$. The stability of a vector bundle is in the sense of Mumford-Takemoto. Furthermore, we make no distinction between a vector bundle and its associated locally free sheaf.

$K_X =:$ the canonical divisor of X.

$p_g =: h^0(X, \mathcal{O}_X(K_X))$, the geometric genus of X.

$l(Z) =:$ the length of the 0-cycle Z on X.

$\mathrm{Hilb}^l(X) :=$ the Hilbert scheme parametrizing all 0-cycles of length-l on X;

$r =:$ an integer larger than one;

$\mu_L(V) =: c_1(V)\cdot L/\mathrm{rank}(V)$ where L is an ample divisor on X and V is a torsionfree sheaf on X.

$\mathrm{ad}(V) =: \ker(\mathrm{Tr}: \mathrm{End}(V) \to \mathcal{O}_X)$. Then, $\mathrm{End}(V) = \mathrm{ad}(V) \oplus \mathcal{O}_X$.

$[x] =:$ the integer part of the number x.

When X is a ruled surface, we also fix the following notation.

$\pi =:$ a ruling from X to an algebraic curve C.

$f =:$ a fiber to the ruling π.

$\sigma =:$ a section to π such that σ^2 is the least.

$e =: -\sigma^2$.

$r_L =: b/a$ where $L \equiv (a\sigma + bf)$ and $a \neq 0$.

$\mathbf{d}f =: \pi^*(\mathbf{d})$ where $\mathbf{d}$ is a divisor on C; in this case, d stands for degree $(\mathbf{d})$.

$P^1_K =:$ the generic fiber of the ruling π.

2. Existence of stable bundles on algebraic surfaces

2.1. **The Cayley-Bacharach property.** Fix divisors $L', L_1, \dots, L_{r-1}$ and reduced 0-cycles $Z_1, \dots, Z_{r-1}$ on the algebraic surface X such that $Z_i \cap Z_j = \varnothing$ for $i \neq j$. Put $Z = \bigcup Z_i$ and

$$W = \bigoplus_{i=1}^{(r-1)} [\mathcal{O}_X(L_i) \otimes I_{Z_i}].$$

Let W_i be the obvious quotient $W/[\mathcal{O}_X(L_i) \otimes I_{Z_i}]$. It is well known that there exists an extension e_i in $\mathrm{Ext}^1(\mathcal{O}_X(L_i) \otimes I_{Z_i}, \mathcal{O}_X(L'))$ whose corresponding exact sequence

$$0 \to \mathcal{O}_X(L') \to V_i \to \mathcal{O}_X(L_i) \otimes I_{Z_i} \to 0$$

gives a bundle V_i if and only if Z_i satisfies the Cayley-Bacharach property with respect to the complete linear system $|(L_i - L' + K_X)|$; i.e., if a curve D in $|(L_i - L' + K_X)|$ contains all but one point of Z_i, then D contains the remaining point. Note that

$$\mathrm{Ext}^1(W, \mathcal{O}_X(L')) = \bigoplus_{i=1}^{(r-1)} \mathrm{Ext}^1(\mathcal{O}_X(L_i) \otimes I_{Z_i}, \mathcal{O}_X(L')).$$

In the following, we study the existence of a bundle V sitting in an extension

$$0 \to \mathcal{O}_X(L') \to V \xrightarrow{\varphi} W \to 0. \tag{2.1}$$

Proposition 2.2. *There exists an extension $e \in \mathrm{Ext}^1(W, \mathcal{O}_X(L'))$ whose corresponding exact sequence* (2.1) *gives a bundle V if and only if for each $i = 1, \dots, (r-1)$ the 0-cycle Z_i satisfies the Cayley-Bacharach property with respect to $|(L_i - L' + K_X)|$.*

Proof. Put $e = (e_1, \dots, e_{r-1})$ where $e_i \in \mathrm{Ext}^1(\mathcal{O}_X(L_i) \otimes I_{Z_i}, \mathcal{O}_X(L'))$. Let V_i be the subsheaf $\varphi^{-1}(\mathcal{O}_X(L_i) \otimes I_{Z_i})$ of V. Then V_i is given by the extension e_i:

$$0 \to \mathcal{O}_X(L') \to V_i \to \mathcal{O}_X(L_i) \otimes I_{Z_i} \to 0.$$

Note that V is locally free outside the 0-cycle Z and sits in an exact sequence

$$0 \to V_i \to V \to W_i \to 0.$$

Since W_i is locally free at the points in Z_i, we see that V is locally free at the points in Z_i if and only if V_i is locally free at the points in Z_i, that is, Z_i satisfies the Cayley-Bacharach property with respect to $|(L_i - L' + K_X)|$. Hence, our result follows. □

Corollary 2.3. *If $h^0(X, \mathcal{O}_X(L_i - L' + K_X) \otimes I_{Z_i - \{x\}}) = 0$ for every i and for every $x \in Z_i$, then there exists a bundle V sitting in the exact sequence* (2.1).

2.2. **Construction of rank-r bundle** V. Let L be a very ample divisor on X, and let V be a rank-r bundle. Note that

$$c_1(V \otimes \mathcal{O}_X(nL)) = c_1(V) + nrL.$$

Thus, by tensoring some line bundle to V, we may assume that $-rL^2 < c_1(V) \cdot L \leq 0$. Without loss of generality, from now on, we fix a divisor c_1 with $-rL^2 < c_1 \cdot L \leq 0$.

We start with three lemmas. In these lemmas, we prove certain properties satisfied by a generic 0-cycle in the Hilbert scheme $\mathrm{Hilb}^l(X)$ when l is sufficiently large.

Lemma 2.4. *Let Z be a generic 0-cycle Z in the Hilbert scheme* $\mathrm{Hilb}^l(X)$.

(i) *If $l \geq h^0(X, \mathcal{O}_X(rL - c_1 + K_X))$, then $h^0(X, \mathcal{O}_X(rL - c_1 + K_X) \otimes I_Z) = 0$.*

(ii) *If $l \geq p_g$, then $h^0(X, \mathcal{O}_X(K_X) \otimes I_Z) = 0$.*

Proof. This is straightforward. □

Lemma 2.5. *Let $l \geq \max(p_g, h^0(X, \mathcal{O}_X(rL - c_1 + K_X)))$. Then a generic 0-cycle Z' in the Hilbert scheme $\mathrm{Hilb}^{l+1}(X)$ satisfies the Cayley-Bacharach property with respect to $|rL - c_1 + K_X|$; moreover, $h^0(X, \mathcal{O}_X(K_X) \otimes I_{Z'}) = 0$.*

Proof. In view of Lemma 2.4(ii), we need only to prove the first statement. Define an open dense subset U_l of $\mathrm{Hilb}^l(X)$ such that if $Z \in U_l$, then Z is reduced and

$$h^0(X, \mathcal{O}_X(rL + K_X - c_1) \otimes I_Z) = 0.$$

By Lemma 2.4(i), this can be done. Define V_l to be the open subset of $\mathrm{Hilb}^l(X)$ consisting of reduced 0-cycles. Hence U_l is an open dense subset of V_l. Define Z^{l+1} to be the universal family in $V_{l+1} \times X$:

$$Z^{l+1} = \{([Z], x) \in V_{l+1} \times X | x \in Z\}.$$

Then there is a surjective morphism $\pi : Z^{l+1} \to V_l$ given by $\pi([Z], x) = (Z - x)$. Hence, $Z^{l+1} - \pi^{-1}(U_l)$ is a proper closed subset of Z^{l+1}. Define the natural projection:

$$Z^{l+1} \subset V_{l+1} \times X \xrightarrow{\rho} V_{l+1}.$$

Then ρ is a flat surjection and $\rho(Z^{l+1} - \pi^{-1}(U_l))$ is a proper closed subset of V_{l+1}. So we can choose an element $Z' \in V_{l+1} - \rho(Z^{l+1} - \pi^{-1}(U_l))$. Hence, $\rho^{-1}([Z']) \subset \pi^{-1}(U_l)$; this means that for any point x in Z', $Z' - x \in U_l$, that is, we have

$$h^0(X, \mathcal{O}_X(rL + K_X - c_1) \otimes I_{Z'-x}) = 0 \quad \text{for any } x \in Z'.$$

So Z' satisfies the Cayley-Bacharach property with respect to $|rL+K_X-c_1|$. □

The above two lemmas will be used to construct a rank-r bundle, while the following lemma will be used to show the L-stability of that bundle.

Lemma 2.6. *There exists a reduced* 0*-cycle* Z'' *of length* $l(Z'') \geq 4(r-1)^2 \cdot L^2$ *such that if* $h^0(X, \mathcal{O}_X(F) \otimes I_{Z''}) > 0$, *then we have* $F \cdot L \geq 2(r-1) \cdot L^2$.

Proof. Choose $2(r-1)$ distinct smooth curves $L_1, \dots, L_{2(r-1)}$ in the complete linear system $|L|$. Choose a set Z_i'' of $2(r-1) \cdot L^2$ many distinct points in the open subset

$$L_i - \left(\bigcup_{j \neq i} L_j \right)$$

of L_i. Let $Z'' = \bigcup_{i=1}^{2(r-1)} Z_i''$. Suppose that $h^0(X, \mathcal{O}_X(F) \otimes I_{Z''}) > 0$. Then F is effective. If F contains all the curves L_i as its irreducible components, then

$$F \cdot L \geq 2(r-1) \cdot L^2.$$

If F does not have L_i as its irreducible component for some i, then $F \cap L_i \supset Z_i''$ and

$$F \cdot L = F \cdot L_i \geq l(Z_i'') = 2(r-1) \cdot L^2. \quad \square$$

Now, for $i = 1, \dots, (r-1)$, we can choose a reduced 0-cycle $Z_i = Z_i' \cup Z_i''$ such that Z_i' is chosen as in Lemma 2.5 and Z_i'' is chosen as in Lemma 2.6; moreover, we may assume that $Z_1, \dots, Z_{r-1}$ are disjoint. Put $Z = \bigcup_{i=1}^{r-1} Z_i$ and

$$W = \bigoplus_{i=1}^{(r-1)} [\mathcal{O}_X(L) \otimes I_{Z_i}].$$

Since $h^0(X, \mathcal{O}_X(rL + K_X - c_1) \otimes I_{Z_i'-x}) = 0$ for any $x \in Z_i'$,

$$h^0(X, \mathcal{O}_X(rL + K_X - c_1) \otimes I_{Z_i' \cup Z_i'' - x}) = 0$$

for any $x \in Z_i = Z_i' \cup Z_i''$. Hence Z_i satisfies the Cayley-Bacharach property with respect to $|rL + K_X - c_1|$. By Corollary 2.3, there is a bundle V sitting in an extension:

$$0 \to \mathcal{O}_X(c_1 + (1-r)L) \to V \to W \to 0. \tag{2.7}$$

Note that $c_1(V) = c_1$ and that, since Z is nonempty, the extension (2.7) is nontrivial.

2.3. **L-stability of the vector bundle V.** In the following, we show the L-stability of the bundle V constructed above.

Lemma 2.8. *The rank-r bundle V in* (2.7) *is L-stable.*

Proof. Let U be a proper subvector bundle of V such that the quotient V/U is torsion-free. Let U_2 be the image of U in W, and let U_1 be the kernel of the surjection $U \to U_2 \to 0$. Then we have a commutative diagram of morphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(c_1 + (1-r)L) & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & U_1 & \longrightarrow & U & \longrightarrow & U_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Case (a): $U_1 \neq 0$. Then $c_1(U_1) = (c_1 + (1-r)L) - E_1$ for some effective divisor E_1. From $U_2 \hookrightarrow W$, we have $U_2^{**} \hookrightarrow W^{**} = \mathcal{O}_X(L)^{\oplus(r-1)}$; thus,

$$\bigwedge^{r_2}(U_2^{**}) \hookrightarrow \bigwedge^{r_2}(\mathcal{O}_X(L)^{\oplus(r-1)}) = \mathcal{O}_X(r_2L)^{\oplus\binom{r-1}{r_2}}$$

where r_2 is the rank of U_2. Thus, $c_1(U_2) = r_2L - E_2$ for some effective divisor E_2 and

$$c_1(U) = (c_1 + (1 + r_2 - r)L) - (E_1 + E_2).$$

It follows that $c_1(U) \cdot L \leq (c_1 + (1 + r_2 - r)L) \cdot L$. Therefore,

$$\mu_L(U) = \frac{c_1(U) \cdot L}{(1 + r_2)} \leq \frac{(c_1 + (1 + r_2 - r)L) \cdot L}{(1 + r_2)} < \frac{c_1 \cdot L}{r} = \mu_L(V).$$

Case (b): $U_1 = 0$. Then $U \hookrightarrow W$; thus, we see that

$$\bigwedge^{\bar{r}}(U) \hookrightarrow \bigwedge^{\bar{r}}(W) = \bigoplus_{\beta}[\mathcal{O}_X(\bar{r}L) \otimes I_{\bigcup_{i\in\beta} Z_i}]$$

where $\bar{r}$ denotes the rank of U and β runs over the set of $\bar{r}$ choices from $(r-1)$ letters. It follows that for some β and for some $i \in \beta$,

$$h^0(X, \mathcal{O}_X(\bar{r}L - c_1(U)) \otimes I_{Z_i}) > 0.$$

In particular, $h^0(X, \mathcal{O}_X(\bar{r}L - c_1(U)) \otimes I_{Z_i''}) > 0$. In view of Lemma 2.6, we have

$$(\bar{r}L - c_1(U)) \cdot L \geq 2(r-1)L^2 \geq 2\bar{r}L^2.$$

So $c_1(U) \cdot L \leq -\bar{r}L^2 < \bar{r} \cdot (c_1 \cdot L)/r$ and $\mu_L(U) < \mu_L(V)$.

Thus, in both cases, $\mu_L(U) < \mu_L(V)$. Therefore, V is L-stable. □

In the next lemma, we are going to prove that $h^2(X, \mathrm{ad}(V)) = 0$, that is, the irreducible component of $\mathcal{M}_L(r; c_1, c_2)$ containing V is generically smooth (equivalently, this means that the versal deformation space of V is smooth).

Lemma 2.9. *Let V be the rank-r bundle in* (2.7). *If $rL^2 > K_X \cdot L$, then*
(i) $\mathrm{Hom}(W, V \otimes \mathcal{O}_X(K_X)) = 0$;
(ii) $h^2(X, \mathrm{ad}(V)) = 0$.

Proof. (i) Let $\beta \in \mathrm{Hom}(W, V \otimes \mathcal{O}_X(K_X))$. Then β induces a map β' from W^{**} to $V \otimes \mathcal{O}_X(K_X)$ such that we have commutative diagram of maps:

$$\begin{array}{ccc} W & \hookrightarrow & W^{**} = \mathcal{O}_X(L)^{\oplus(r-1)} \\ \downarrow \beta & \swarrow \beta' & \\ V \otimes \mathcal{O}_X(K_X) & & \end{array}$$

To show that $\beta = 0$, it suffices to show that $H^0(X, V \otimes \mathcal{O}_X(K_X - L)) = 0$.

Since $c_1 \cdot L \le 0$ and $K_X \cdot L < rL^2$, $(c_1 - rL + K_X) \cdot L < 0$. Thus,

$$H^0(X, \mathcal{O}_X(c_1 - rL + K_X)) = 0.$$

By our choice of the 0-cycles Z_i', $H^0(X, \mathcal{O}_X(K_X) \otimes I_{Z_i'}) = 0$. Thus,

$$H^0(X, W \otimes \mathcal{O}_X(K_X - L)) = 0.$$

Now, tensoring (2.7) by $\mathcal{O}_X(K_X - L)$ and taking cohomology, we see that

$$H^0(X, V \otimes \mathcal{O}_X(K_X - L)) = 0.$$

(ii) We follow the argument as in the proof of Lemma 4.5.4 in [19]. By the Serre duality, we have $H^2(X, \mathrm{ad}(V) \cong H^0(X, \mathrm{ad}(V) \otimes \mathcal{O}_X(K_X))$. Let

$$\varphi \in H^0(X, \mathrm{ad}(V) \otimes \mathcal{O}_X(K_X)) \subseteq H^0(X, \mathrm{End}(V) \otimes \mathcal{O}_X(K_X)).$$

Then we obtain a map φ from V to $V \otimes \mathcal{O}_X(K_X)$. Consider the diagram:

$$(2.10) \qquad 0 \to \mathcal{O}_X(c_1 + (1-r)L) \xrightarrow{\theta} V \xrightarrow{\rho} W \to 0$$

$$\downarrow \varphi$$

$$(2.11) \qquad 0 \to \mathcal{O}_X(c_1 + (1-r)L + K_X) \xrightarrow{\theta'} V \otimes \mathcal{O}_X(K_X) \xrightarrow{\rho'} W \otimes \mathcal{O}_X(K_X) \to 0.$$

By our choice of the 0-cycles Z_i', $H^0(X, \mathcal{O}_X(rL - c_1 + K_X) \otimes I_{Z_i'}) = 0$. Thus,

$$\mathrm{Hom}(\mathcal{O}_X(c_1 + (1-r)L), W \otimes \mathcal{O}_X(K_X)) = 0,$$

so $\rho' \circ \varphi \circ \theta = 0$. Applying $\mathrm{Hom}(\mathcal{O}_X(c_1 + (1-r)L), \cdot)$ to (2.11), we obtain

$$0 \to H^0(X, \mathcal{O}_X(K_X)) \xrightarrow{\lambda} \mathrm{Hom}(\mathcal{O}_X(c_1 + (1-r)L), V \otimes \mathcal{O}_X(K_X))$$
$$\xrightarrow{\rho' \circ} \mathrm{Hom}(\mathcal{O}_X(c_1 + (1-r)L), W \otimes \mathcal{O}_X(K_X)) = 0.$$

It follows that there exists $\tau \in H^0(X, \mathcal{O}_X(K_X))$ such that

$$\varphi \circ \theta = \lambda(\tau) = (\tau \otimes \mathrm{Id}_V) \circ \theta$$

where Id_V is the identity morphism in $\mathrm{End}(V)$. Thus, $(\varphi - \tau \otimes \mathrm{Id}_V) \circ \theta = 0$. Applying $\mathrm{Hom}(\cdot\,, V \otimes \mathcal{O}_X(K_X))$ to (2.10), we get an exact sequence:

$$\mathrm{Hom}(W, V \otimes \mathcal{O}_X(K_X)) \to H^0(X, \mathrm{End}(V) \otimes \mathcal{O}_X(K_X))$$
$$\stackrel{\circ\theta}{\to} \mathrm{Hom}(\mathcal{O}_X(c_1 + (1-r)L), V \otimes \mathcal{O}_X(K_X)).$$

From (i), we conclude that $(\varphi - \tau \otimes \mathrm{Id}_V) = 0$. Since $0 = \mathrm{Tr}(\varphi) = \tau$, $\varphi = 0$. Hence,

$$h^2(X, \mathrm{ad}(V)) = 0. \quad \square$$

Finally, we state and prove the main result in this section.

Theorem 2.12. *For any ample divisor L on X, there exists a constant α depending only on X, r, c_1, and L such that for any $c_2 \geq \alpha$, there exists an L-stable rank-r bundle V with Chern classes c_1 and c_2. Moreover, $h^2(X, \mathrm{ad}(V)) = 0$.*

Proof. We may rescale the ample divisor L such that L is very ample and that $rL^2 > K_X \cdot L$. Note that $c_1(W) = (r-1)L$ and $c_2(W) = l(Z) + (r-1)(r-2)/2 \cdot L^2$.

From the exact sequence (2.7), we see that $c_1(V) = c_1$ and

$$c_2(V) = l(Z) + (r-1)(c_1 \cdot L) - r(r-1)/2 \cdot L^2.$$

By the construction of the 0-cycle Z, we get

$$l(Z) = \sum_{i=1}^{(r-1)} [l(Z_i') + l(Z_i'')]$$
$$\geq (r-1)[1 + \max(p_g, h^0(X, \mathcal{O}_X(rL - c_1 + K_X))) + 4(r-1)^2 \cdot L^2].$$

Let α be the integer:

$$(r-1)[1 + \max(p_g, h^0(X, \mathcal{O}_X(rL - c_1 + K_X))) + 4(r-1)^2 \cdot L^2]$$
$$+ (r-1)(c_1 \cdot L) - r(r-1)/2 \cdot L^2.$$

Then α depends only on X, r, c_1, and L. By Lemma 2.8, for any $c_2 \geq \alpha$, there exists an L-stable rank-r bundle V with Chern classes c_1 and c_2.

Moreover, since $rL^2 > K_X \cdot L$, $h^2(X, \mathrm{ad}(V)) = 0$ by Lemma 2.9(ii). $\square$

Remark 2.13. In [2], Artamkin showed that $\mathcal{M}_L(r; 0, c_2)$ is nonempty whenever

$$c_2 > (r+1) \cdot \max(1, p_g);$$

in particular, when we only consider the case of $c_1 = 0$, the lower bound of the integer c_2 does not depend on the ample divisor L. By contrast, the constant α in Theorem 2.12 depends on L. In fact, if we want a universal lower bound of c_2 for all c_1, this bound must depend on the ample divisor L. We shall see this fact from Theorem 3.1 in the next section that on a ruled surface there exists a divisor c_1 such that for any integer c_2 we can find an ample divisor L with $\mathcal{M}_L(r; c_1, c_2)$ being empty.

3. Restriction of a stable bundle on a ruled surface to the generic fiber

From now on, we study stable bundles on a ruled surface X. Our first goal in this section is to show that if $0 < (c_1 \cdot f) < r$ and if $r_L \gg 0$, then $\mathcal{M}_L(r; c_1, c_2)$ is empty.

Theorem 3.1. *Let* $0 < (c_1 \cdot f) < r$. *Then there exists a constant* r_0 *depending only on* X, r, c_1, *and* c_2 *such that* $\mathcal{M}_L(r; c_1, c_2)$ *is empty whenever* $r_L > r_0$.

Proof. Assume that $V \in \mathcal{M}_L(r; c_1, c_2)$. Let $c_1 = (a\sigma + \mathbf{b}f)$; then $0 < a < r$. For any divisor $\mathbf{k}$ on C, we see that $c_1(V \otimes \mathcal{O}_X(-\sigma + \mathbf{k}f)) = (a-r)\sigma + (\mathbf{b} + r\mathbf{k})f$ and that

$$c_2(V \otimes \mathcal{O}_X(-\sigma + \mathbf{k}f)) = c_2 + (r-1)(a\sigma + \mathbf{b}f) \cdot (-\sigma + \mathbf{k}f) + \frac{r(r-1)}{2} \cdot (-\sigma + \mathbf{k}f)^2.$$

By the Riemann-Roch formula, we conclude the following:

$$\chi(V \otimes \mathcal{O}_X(-\sigma + \mathbf{k}f)) = a \cdot k + a \cdot (b + 1 - g_C) - c_2 - \frac{e(a^2 - a)}{2}.$$

Let $k = g_C - b + [c_2/a + e(a-1)/2] + 1$. Then $\chi(V \otimes \mathcal{O}_X(-\sigma + \mathbf{k}f)) > 0$. Thus, $h^i(X, V \otimes \mathcal{O}_X(-\sigma + \mathbf{k}f)) > 0$ where $i = 0$ or 2. On the other hand, put

$$r_0 = \max\left\{e + \frac{kr + b}{r - a}, e - \frac{2r\chi(\mathcal{O}_X) + er + kr + b}{r + a}\right\}.$$

Then r_0 is a number depending only on X, r, c_1, and c_2. If

$$h^0(X, V \otimes \mathcal{O}_X(-\sigma + \mathbf{k}f)) > 0,$$

then there exists an injective map $\mathcal{O}_C(\sigma - \mathbf{k}f) \hookrightarrow V$. By the stability of V, we see that $(\sigma - \mathbf{k}f) \cdot L < (a\sigma + bf) \cdot L/r$. By direct calculations, we get

$$r_L < e + \frac{kr + b}{r - a};$$

but this contradicts with the choice of the numbers r_0 and r_L.

If $h^2(X, V \otimes \mathcal{O}_X(-\sigma + \mathbf{k}f)) > 0$, then $h^0(X, V^* \otimes \mathcal{O}_X(K_X + \sigma - \mathbf{k}f)) > 0$. Hence, there is a nonzero map $V \to \mathcal{O}_X(K_X - \sigma + \mathbf{k}f)$ which can be extended to

$$V \to \mathcal{O}_X(K_X + \sigma - \mathbf{k}f) \otimes \mathcal{O}_X(-E) \otimes I_Z \to 0$$

for some effective divisor E. By the stability of V, we must have

$$c_1(V) \cdot L/r < K_X \cdot L + (\sigma - \mathbf{k}f) \cdot L - E \cdot L \leq K_X \cdot L + (\sigma - \mathbf{k}f) \cdot L.$$

By a straightforward calculation, we obtain that

$$r_L \leq e - \frac{2r\chi(\mathcal{O}_X) + er + kr + b}{r + a};$$

again, this contradicts our choices of r_0 and r_L.

Therefore, if $r_L > r_0$, the moduli space $\mathcal{M}_L(r; c_1, c_2)$ is empty. □

Remark 3.2. Theorem 3.1 only says that for a fixed c_1 with $0 < c_1 \cdot f < r$ and for a fixed c_2, the moduli space $\mathcal{M}_L(r; c_1, c_2)$ is empty for some special ample divisor L (e.g., when $r_L > r_0$). For another ample divisor L', $\mathcal{M}_{L'}(r; c_1, c_2)$ can be nonempty (see [21] when $r = 2$); we will discuss this issue in other places.

In view of Theorem 3.1, our next goal is to study the moduli space $\mathcal{M}_L(r; \mathbf{t}f, c_2)$ where $-r < t \leq 0$. Let $V \in \mathcal{M}_L(r; \mathbf{t}f, c_2)$ where L is of the form $(\sigma + \mathbf{r}_L f)$ with

$$r_L \geq \max\{e/2 - \chi(\mathcal{O}_X) + r(g_C + |c_2|) + 1, 2|e| + r(g_C + |c_2|)\}. \tag{3.3}$$

We want to show that the restriction of the stable bundle V to the generic fiber is trivial. To start with, we prove the following technical lemma.

Lemma 3.4. *Let U be a rank-s bundle with an injection $U \hookrightarrow V$.*

(i) *For any divisor* $\mathbf{d}$ *with* $d \geq -r(g_C - |c_2|) - 1$, $h^2(X, U^* \otimes \mathcal{O}_X(\mathbf{d}f)) = 0$.

(ii) *If* $c_1(U) = -\mathbf{a}f$ *with* $0 < a \leq (r-s)(g_C + |c_2|)$ *and* $c_2(U) \leq c_2$, *then* U *sits in*

$$0 \to U_1 \to U \to \mathcal{O}_X(\mathbf{n}f) \otimes I_Z \to 0$$

where U_1 *is a rank-*$(s-1)$ *bundle with an injection* $U_1 \hookrightarrow V$; *moreover,* $c_1(U_1) = -(\mathbf{a}+\mathbf{n})f$ *with* $0 < (a+n) \leq (r-s+1)(g_C + |c_2|)$ *and* $c_2(U_1) \leq c_2$.

Proof. (i) By the Serre duality,

$$h^2(X, U^* \otimes \mathcal{O}_X(\mathbf{d}f)) = h^0(X, U \otimes \mathcal{O}_X(K_X - \mathbf{d}f)).$$

If

$$h^0(X, U \otimes \mathcal{O}_X(K_X - \mathbf{d}f)) > 0,$$

then we have $\mathcal{O}_X(\mathbf{d}f - K_X) \hookrightarrow U \hookrightarrow V$; by the stability of V, we obtain that

$$(\mathbf{d}f - K_X) \cdot L < \frac{tf \cdot L}{r} \leq 0.$$

On the other hand, we have $(\mathbf{d}f - K_X) \cdot L = d - 2(e/2 - \chi(\mathcal{O}_X)) + 2r_L \geq 0$ in view of the assumption (3.3); but this is a contradiction.

(ii) By the Riemann-Roch formula, one checks that

$$\chi(U^* \otimes \mathcal{O}_X(\mathbf{k}f)) = s \cdot k + s \cdot \chi(\mathcal{O}_X) + a - c_2(U) \geq s \cdot k + s \cdot \chi(\mathcal{O}_X) + a - c_2.$$

Let $k = g_C + [(c_2 - a)/s]$. Then $\chi(U^* \otimes \mathcal{O}_X(\mathbf{k}f)) > 0$. Since

$$k \geq g_C + \frac{c_2 - (r-s)(g_C + |c_2|)}{s} - 1 \geq -r(g_C + |c_2|) - 1,$$

$h^0(X, U^* \otimes \mathcal{O}_X(\mathbf{k}f)) > 0$ by (i); thus, there is an exact sequence:

$$0 \to U_1 \to U \to \mathcal{O}_X(\mathbf{k}f - E) \otimes I_Z \to 0$$

where E is effective and Z is a 0-cycle. Since U/U_1 is torsion-free, U_1 is a bundle. Let $E \equiv (\lambda\sigma + \mu f)$. Then $\lambda \geq 0$; moreover, $\mu \geq 0$ when $e \geq 0$ and $\mu \geq \lambda e/2$ when $e < 0$. We claim that $\lambda = 0$; otherwise, $\lambda \geq 1$; then

$$\begin{aligned} c_1(U_1) \cdot L &= (\lambda\sigma + (\mu - a - k)f) \cdot L \\ &= \lambda(r_L - e) + \mu - a - k \\ &\geq (r_L - e) - |e| - a - k. \end{aligned}$$

But

$$\begin{aligned} a + k &\leq (r-s)(g_C + |c_2|) + g_C + [(c_2 - a)/s] \\ &\leq (r-s)(g_C + |c_2|) + g_C + |c_2| \\ &= (r-s+1)(g_C + |c_2|) \\ &\leq r(g_C + |c_2|). \end{aligned}$$

So $c_1(U_1) \cdot L \geq r_L - 2|e| - r(g_C + |c_2|) \geq 0$ by our assumption about r_L; but this contradicts the stability of V. Therefore, E is supported in the fibers of the ruling and U sits in the desired exact sequence; moreover, $c_2(U_1) \leq c_2(U) \leq c_2$. Note that $c_1(U_1) = -(\mathbf{a}+\mathbf{n})f$ and that $(a+n) \leq (a+k) \leq (r-s+1)(g_C + |c_2|)$. By the stability of V, $-(a+n)/(s-1) < -t/r \leq 0$. Thus, $(a+n) > 0$. □

Theorem 3.5. *Let* $V \in \mathcal{M}_L(r; \mathbf{t}f, c_2)$ *where* $-r < t \leq 0$ *and* L *satisfies* (3.3). *Then*

$$V|_{\mathbf{P}^1_K} = \mathcal{O}^{\oplus r}_{\mathbf{P}^1_K}.$$

Proof. By Lemma 3.4(ii) and induction on the rank of subbundles of V, we conclude that there exists a flag of subbundles of V : $V_1 \subset V_2 \subset \cdots \subset V_{r-1} \subset V_r = V$ such that $\operatorname{rank}(V_i) = i$, $c_2(V_i) \leq c_2$, $c_1(V_i) = -\mathbf{b}_i f$ with $0 < b_i \leq r(g_C + |c_2|)$ for $i < r$, and $V_i/V_{i-1} = \mathcal{O}_X((\mathbf{b}_{i-1} - \mathbf{b}_i)f) \otimes I_{Z_i}$ where Z_i is an 0-cycle. Hence $V|_{\mathbf{P}^1_K} = \mathcal{O}^{\oplus r}_{\mathbf{P}^1_K}$. □

Next, we prove the following simple observation.

Lemma 3.6. *If the moduli space* $\mathcal{M}_L(r; \mathbf{t}f, c_2)$ *is nonempty, then it is smooth with dimension* $2rc_2 - (r^2-1)(1-g_C)$; *in particular,* $c_2 \geq (1-g_C)(r^2-1)/(2r)$.

Proof. Since L satisfies (3.3), $K_X \cdot L \leq 0$. By a well-known result of Maruyama, $\mathcal{M}_L(r; \mathbf{t}f, c_2)$ is smooth with the expected dimension $2rc_2 - (r^2-1)(1-g_C)$. □

We notice that the ample divisor L in Theorem 3.5 depends on the integer c_2 (that is, condition (3.3)). However, in our existence result Theorem 2.12, the integer c_2 has to be bigger than some constant depending on L. Thus, Theorem 2.12 cannot apply to the present situation to guarantee the nonemptiness of the moduli space $\mathcal{M}_L(r; \mathbf{t}f, c_2)$. The following result deals with this problem.

Proposition 3.7. *Let* $r \geq 2$, $-r < t \leq 0$, *and* $L = (\sigma + \mathbf{r}_L f)$ *with* $r_L \geq (|e| + 2r - 2)$. *If* $c_2 \geq 2(r-1)$, *then the moduli space* $\mathcal{M}_L(r; \mathbf{t}f, c_2)$ *is nonempty.*

We omit the proof since it is a slight modification of the proof of Theorem 2.12 (replacing the L in W by f). It seems to us that a stronger result should hold; that is, if $c_2 \geq (r+t)$, then $\mathcal{M}_L(r; \mathbf{t}f, c_2)$ is nonempty (see Theorem 5.4(iii)).

4. Generic bundles in $\mathcal{M}_L(r; tf, c_2)$ on a rational ruled surface

From now on, X will be a rational ruled surface. In this section, we will study the structure of a generic bundle in $\mathcal{M}_L(r; tf, c_2)$ where L satisfies (3.3) and $-r < t \leq 0$.

4.1. **Exact sequences associated to a bundle** V **in** $\mathcal{M}_L(r; tf, c_2)$. In this subsection, we will construct $(r-1)$ exact sequences for each vector bundle in the moduli space $\mathcal{M}_L(r; tf, c_2)$. We begin with two lemmas.

Lemma 4.1. *Let* U *be a rank-*i *bundle with* $c_1(U) = af$ *and* $U|_{\mathbf{P}^1_K} = \mathcal{O}^{\oplus i}_{\mathbf{P}^1_K}$. *Then*

(i) $\pi_* U$ *is a rank-*i *bundle on* $\mathbf{P}^1$;

(ii) $\deg c_1(\pi_* U) \geq (a - c_2(U))$.

Proof. (i) Note that $\pi_* U$ is always torsion-free. Thus, $\pi_* U$ is a vector bundle. Since $U|_{\mathbf{P}^1_K}$ is equal to $\mathcal{O}^{\oplus i}_{\mathbf{P}^1_K}$, the rank of $\pi_* U$ is equal to i.

(ii) Since $U|_{\mathbf{P}^1_K} = \mathcal{O}^{\oplus i}_{\mathbf{P}^1_K}$, $R^1\pi_* U$ is a torsion sheaf supported in some points; thus, $\deg c_1(R^1\pi_* U) \geq 0$. By the Grothendieck-Riemann-Roch formula (see [12, p. 436]),

$$\operatorname{ch}(\pi_* U) - \operatorname{ch}(R^1\pi_* U) = \pi_*(\operatorname{ch}(U) \cdot \operatorname{td}(T_\pi)) = i + (a - c_2(U)) \cdot [pt]$$

where T_π is the relative tangent bundle, $\mathrm{td}(T_\pi) = 1 + (\sigma - e/2 \cdot f)$ and $[pt]$ stands for the class determined by a point. Therefore,

$$\deg c_1(\pi_* U) = \deg c_1(R^1\pi_* U) + (a - c_2(U)) \geq (a - c_2(U)). \quad \square$$

Lemma 4.2. *Let U be a rank-i bundle with $c_1(U) = af$ and $U|_{\mathbf{P}^1_K} = \mathcal{O}^{\oplus i}_{\mathbf{P}^1_K}$. If*

$$\pi_* U = \mathcal{O}_{\mathbf{P}^1}(-n)^{\oplus j} \oplus \mathcal{O}_{\mathbf{P}^1}(-n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(-n_{i-j}) \tag{4.3}$$

where $1 \leq j \leq i$ and $n < n_1 \leq \cdots \leq n_{i-j}$, then

(i) *$in + (i - j) \leq (c_2(U) - a)$;*

(ii) *$in - h^0(\mathbf{P}^1, \pi_* U \otimes \mathcal{O}_{\mathbf{P}^1}(n)) \leq (c_2(U) - a) - i$;*

(iii) *the bundle U sits in an exact sequence of the form:*

$$0 \to \mathcal{O}_X(-nf) \to U \to W \to 0 \tag{4.4}$$

where W is a torsion-free rank-$(i-1)$ sheaf with $W|_{\mathbf{P}^1_K} = (W^{**})|_{\mathbf{P}^1_K} = \mathcal{O}^{\oplus(i-1)}_{\mathbf{P}^1_K}$.

Proof. (i) Since $n < n_1 \leq \cdots \leq n_{i-j}$, by Lemma 4.1(ii), we have

$$(c_2(U) - a) \geq -\deg c_1(\pi_* U) = jn + \sum_{k=1}^{i-j} n_k \geq in + (i - j).$$

(ii) Note that $h^0(\mathbf{P}^1, \pi_* U \otimes \mathcal{O}_{\mathbf{P}^1}(n)) = j$. Therefore, by (i),

$$in - h^0(\mathbf{P}^1, \pi_* U \otimes \mathcal{O}_{\mathbf{P}^1}(n)) \leq [(c_2(U) - a) - (i - j)] - j = (c_2(U) - a) - i.$$

(iii) Since there is a natural injection $\pi^*(\pi_* U) \hookrightarrow U$, we have

$$\mathcal{O}_X(-nf) \hookrightarrow U.$$

We claim that the quotient $W = U/\mathcal{O}_X(-nf)$ is torsion-free: otherwise, we have

$$\mathcal{O}_X(-nf) \hookrightarrow \mathcal{O}_X(-nf + D) \hookrightarrow U \tag{4.5}$$

where D is some nontrivial effective divisor. Since $U|_{\mathbf{P}^1_K}$ is equal to $\mathcal{O}^{\oplus i}_{\mathbf{P}^1_K}$, D is supported in the fibers of π. Put $D = df$ where $d > 0$. Applying π_* to (4.5), we obtain

$$\mathcal{O}_{\mathbf{P}^1}(-n) \hookrightarrow \mathcal{O}_{\mathbf{P}^1}(-n + d) \hookrightarrow \pi_* U;$$

but this is impossible in view of the assumption (4.3).

Thus, we have the exact sequence (4.4). Since W is torsion-free, W is locally free outside possibly finite many points. Restricting (4.4) to $\mathbf{P}^1_K$, we see that

$$0 \to \mathcal{O}_{\mathbf{P}^1_K} \to \mathcal{O}^{\oplus i}_{\mathbf{P}^1_K} \to W|_{\mathbf{P}^1_K} = W^{**}|_{\mathbf{P}^1_K} \to 0.$$

Since $c_1(W) = (a + n)f$, we conclude that $W|_{\mathbf{P}^1_K} = W^{**}|_{\mathbf{P}^1_K} = \mathcal{O}^{\oplus(i-1)}_{\mathbf{P}^1_K}$. $\square$

Proposition 4.6. *Let $V \in \mathcal{M}_L(r; tf, c_2)$ where L satisfies condition (3.3) and $-r < t \leq 0$. Then there exist $(r - 1)$ exact sequences:*

$$0 \to \mathcal{O}_X(-n_i f) \to V_i^{**} \to V_{i-1} \to 0 \tag{4.7}$$

where $i = r, \ldots, 2$, $V_r = V$, and V_i is a torsion-free rank-i sheaf such that

(i) *$\pi_*(V_i^{**}) = \mathcal{O}_{\mathbf{P}^1}(-n_i)^{\oplus j_i} \oplus \mathcal{O}_{\mathbf{P}^1}(-n_{i,1}) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(-n_{i,i-j_i})$ with $n_i < n_{i,k}$;*

(ii) $(V_i)|_{\mathbf{P}^1_K} = (V_i^{**})|_{\mathbf{P}^1_K} = \mathscr{O}_{\mathbf{P}^1_K}^{\oplus i}$;
(iii) $in_i + (i - j_i) \le (c_2(V_i^{**}) - t - \sum_{k=i+1}^r n_k)$;
(iv) $in_i - h^0(\mathbf{P}^1, \pi_*(V_i^{**}) \otimes \mathscr{O}_{\mathbf{P}^1}(n_i)) \le (c_2(V_i^{**}) - t - \sum_{k=i+1}^r n_k) - i$.

Proof. By Theorem 3.5, $V|_{\mathbf{P}^1_K} = \mathscr{O}_{\mathbf{P}^1_K}^{\oplus r}$. Now the exact sequences (4.7) and the properties (i) and (ii) follow from induction and Lemma 4.2(iii). Note that

$$c_1(V_i^{**}) = c_1(V_i) = c_1(V_{i+1}^{**}) + n_{i+1}f = \left(t + \sum_{k=i+1}^{r} n_k\right) f.$$

Therefore, the properties (iii) and (iv) follow from Lemma 4.2(i) and (ii). □

4.2. **The number of moduli of V_i and V_i^{**}**. In this subsection, we estimate the number of moduli of V_i and V_i^{**}. These estimations will be used in the next subsection to study generic bundles in the moduli space $\mathscr{M}_L(r; tf, c_2)$ where L satisfies condition (3.3) and $-r < t \le 0$. To begin with, we collect some properties satisfied by the sheaf V_i.

Lemma 4.8. (i) *For each i, there exists a canonical exact sequence*

$$0 \to V_i \to V_i^{**} \to Q_i \to 0 \tag{4.9}$$

where Q_i is a torsion sheaf supported on finitely many points in X;
(ii) $\dim \mathrm{Hom}(V_i, \mathscr{O}_X(-n_{i+1}f)) + 1 \le \dim \mathrm{Aut}(V_{i+1}^{**})$;
(iii) $\mathrm{Ext}^2(V_i, \mathscr{O}_X(-n_{i+1}f)) = 0$;
(iv) $-\chi(V_i, \mathscr{O}_X(-n_{i+1}f)) = c_2(V_i) + (t + \sum_{k=i+1}^r n_k) + i \cdot n_{i+1} - i$.

Proof. (i) This is a standard fact. The torsion sheaf Q_i is supported on those points where V_i is not locally free.

(ii) Applying the functor $\mathrm{Hom}(V_{i+1}^{**}, \cdot)$ to the exact sequence (4.10), we have

$$0 \to \mathrm{Hom}(V_{i+1}^{**}, \mathscr{O}_X(-n_{i+1}f)) \to \mathrm{End}(V_{i+1}^{**}) \xrightarrow{\psi_{i+1}} \mathrm{Hom}(V_{i+1}^{**}, V_i)$$

where $\psi_{i+1}(\mathrm{Id}) = p_i$ for the identity endomorphism Id in $\mathrm{End}(V_{i+1}^{**})$. Thus,

$$\dim \mathrm{Aut}(V_{i+1}^{**}) = \dim \mathrm{End}(V_{i+1}^{**}) \ge 1 + \dim \mathrm{Hom}(V_{i+1}^{**}, \mathscr{O}_X(-n_{i+1}f)).$$

Similarly, applying the functor $\mathrm{Hom}(\cdot, \mathscr{O}_X(-n_{i+1}f))$ to (4.10), we obtain

$$0 \to \mathrm{Hom}(V_i, \mathscr{O}_X(-n_{i+1}f)) \to \mathrm{Hom}(V_{i+1}^{**}, \mathscr{O}_X(-n_{i+1}f));$$

thus, $\dim \mathrm{Hom}(V_{i+1}^{**}, \mathscr{O}_X(-n_{i+1}f)) \ge \dim \mathrm{Hom}(V_i, \mathscr{O}_X(-n_{i+1}f))$. Hence,

$$\dim \mathrm{Aut}(V_{i+1}^{**}) \ge 1 + \dim \mathrm{Hom}(V_i, \mathscr{O}_X(-n_{i+1}f)).$$

(iii) Since $\mathscr{O}_X(K_X + n_{i+1}f)|_{\mathbf{P}^1_K} = \mathscr{O}_{\mathbf{P}^1_K}(-2)$ and $(V_i)|_{\mathbf{P}^1_K} = \mathscr{O}_{\mathbf{P}^1_K}^{\oplus i}$, we see that $H^0(X, V_i \otimes \mathscr{O}_X(K_X + n_{i+1}f)) = 0$. By the Serre duality,

$$\mathrm{Ext}^2(V_i, \mathscr{O}_X(-n_{i+1}f)) \cong H^0(X, V_i \otimes \mathscr{O}_X(K_X + n_{i+1}f)) = 0.$$

(iv) Recall that by definition, $\chi(\mathscr{F}_1, \mathscr{F}_2) = \sum_{i=0}^2 (-1)^i \dim \mathrm{Ext}^i(\mathscr{F}_1, \mathscr{F}_2)$ for two sheaves $\mathscr{F}_1$ and $\mathscr{F}_2$ on X. Let $\mathrm{td}(X)$ be the Todd class of X, and let $\mathrm{ch}(\mathscr{F})$ be the Chern character of a sheaf $\mathscr{F}$. Then we have the formula:

$$\chi(\mathscr{F}_1, \mathscr{F}_2) = (\mathrm{ch}(\mathscr{F}_1)^* \cdot \mathrm{ch}(\mathscr{F}_2) \cdot \mathrm{td}(X))_4$$

where $*$ acts on $H^{2i}(X, \mathbf{Z})$ by multiplication of $(-1)^i$. Thus, we obtain

$$-\chi(V_i, \mathcal{O}_X(-n_{i+1}f)) = c_2(V_i) - \frac{1}{2}(K_X \cdot c_1(V_i)) + i \cdot n_{i+1} - i.$$

Since $c_1(V_i) = (t + \sum_{k=i+1}^r n_k)f$, the conclusion follows immediately. □

Next, for convenience, we introduce some notation.

Notation 4.10. (i) Let $l_i = h^0(X, Q_i)$ for $i = 1, \dots, r-1$.
(ii) Let $\delta_i = [\#(\text{moduli of } V_i) - \dim \operatorname{Aut}(V_i)]$ for $i = 1, \dots, r-1$.
(iii) Let $\delta_i^{**} = [\#(\text{moduli of } V_i^{**}) - \dim \operatorname{Aut}(V_i^{**})]$ for $i = 1, \dots, r$.
Now we estimate the number of moduli of Q_i, V_i, and V_i^{**}.

Lemma 4.11. (i) $\#(\textit{moduli of } Q_i) - \dim \operatorname{Aut}(Q_i) \le l_i$.
(ii) $\delta_i \le \delta_i^{**} + (i+1)l_i$.
(iii) $\delta_i^{**} \le \delta_{i-1} - \chi(V_{i-1}, \mathcal{O}_X(-n_i f)) - h^0(X, V_i^{**} \otimes \mathcal{O}_X(n_i f))$.

Proof. (i) From (4.7), we have an exact sequence

$$0 \to \mathcal{O}_X(-n_{i+1}f) \to V_{i+1}^{**} \to V_i \to 0. \tag{4.12}$$

Applying $\operatorname{Hom}(\cdot, Q_i)$ to (4.12), we obtain

$$\dim \operatorname{Hom}(V_i, Q_i) \le \dim \operatorname{Hom}(V_{i+1}^{**}, Q_i) = (i+1)l_i.$$

Applying $\operatorname{Hom}(\cdot, Q_i)$ to (4.9), we get

$$0 \to \operatorname{Hom}(Q_i, Q_i) \to \operatorname{Hom}(V_i^{**}, Q_i) \to \operatorname{Hom}(V_i, Q_i)$$
$$\to \operatorname{Ext}^1(Q_i, Q_i) \to \operatorname{Ext}^1(V_i^{**}, Q_i) = 0.$$

It follows that

$$\dim \operatorname{Ext}^1(Q_i, Q_i) - \dim \operatorname{Hom}(Q_i, Q_i)$$
$$= \dim \operatorname{Hom}(V_i, Q_i) - \dim \operatorname{Hom}(V_i^{**}, Q_i) \le (i+1)l_i - il_i = l_i.$$

Since

$$\#(\text{moduli of } Q_i) \le \dim \operatorname{Ext}^1(Q_i, Q_i)$$

and

$$\dim \operatorname{Aut}(Q_i) = \dim \operatorname{Hom}(Q_i, Q_i),$$

$$\#(\text{moduli of } Q_i) - \dim \operatorname{Aut}(Q_i) \le l_i. \tag{4.13}$$

(ii) From the exact of (4.9), we see that

$$\#(\text{moduli of } V_i) \le \#(\text{moduli of } V_i^{**}) + \#(\text{moduli of } Q_i) + \dim \operatorname{Hom}(V_i^{**}, Q_i)$$
$$- \dim \operatorname{Aut}(V_i^{**}) - \dim \operatorname{Aut}(Q_i) + 1$$
$$= \delta_i^{**} + [\#(\text{moduli of } Q_i) - \dim \operatorname{Aut}(Q_i)] + \dim \operatorname{Hom}(V_i^{**}, Q_i) + 1$$
$$\le \delta_i^{**} + l_i + il_i + 1 \le \delta_i^{**} + (i+1)l_i + 1.$$

Since $\dim \operatorname{Aut}(V_i) \ge 1$, we obtain that $\delta_i \le \delta_i^{**} + (i+1)l_i$.

(iii) Similarly, from the exact sequence (4.7), we have

$$\#(\text{moduli of } V_i^{**}) \le \#(\text{moduli of } V_{i-1}) + \dim \operatorname{Ext}^1(V_{i-1}, \mathcal{O}_X(-n_i f))$$
$$- \dim \operatorname{Hom}(\mathcal{O}_X(-n_i f), V_i^{**}) - \dim \operatorname{Aut}(V_{i-1}) + 1$$
$$= \delta_{i-1} + \dim \operatorname{Ext}^1(V_{i-1}, \mathcal{O}_X(-n_i f)) - h^0(X, V_i^{**} \otimes \mathcal{O}_X(n_i f)) + 1$$
$$= \delta_{i-1} - \chi(V_{i-1}, \mathcal{O}_X(-n_i f)) - h^0(X, V_i^{**} \otimes \mathcal{O}_X(n_i f))$$
$$+ 1 + \dim \operatorname{Hom}(V_{i-1}, \mathcal{O}_X(-n_i f))$$

where we have used Lemma 4.8(iii) in the last equality. By Lemma 4.8(ii),

$$\delta_i^{**} \le \delta_{i-1} - \chi(V_{i-1}, \mathcal{O}_X(-n_i f)) - h^0(X, V_i^{**} \otimes \mathcal{O}_X(n_i f)). \quad \square$$

Proposition 4.14. $\delta_i^{**} \le \delta_{i-1}^{**} + 2(c_2 - \sum_{k=i}^{r-1} l_k) - (2i-1) + i l_{i-1}$.

Proof. By Lemma 4.8(iv) and Proposition 4.6(iv), we have

$$\begin{aligned}
-\chi(V_{i-1}, \mathcal{O}_X(-n_i f)) &= c_2(V_{i-1}) + \left(t + \sum_{k=i}^{r} n_k\right) + (i-1)n_i - (i-1) \\
&= c_2(V_i^{**}) + \left(t + \sum_{k=i+1}^{r} n_k + i n_i\right) - (i-1) \\
&\le c_2(V_i^{**}) + [c_2(V_i^{**}) + h^0(\mathbf{P}^1, \pi_*(V_i^{**}) \otimes \mathcal{O}_{\mathbf{P}^1}(n_i))] - (2i-1) \\
&= 2c_2(V_i^{**}) + h^0(X, V_i^{**} \otimes \mathcal{O}_X(n_i f)) - (2i-1) \\
&= 2\left(c_2 - \sum_{k=i}^{r-1} l_k\right) + h^0(X, V_i^{**} \otimes \mathcal{O}_X(n_i f)) - (2i-1).
\end{aligned}$$

Therefore, by Lemma 4.11(ii) and (iii), we conclude that

$$\begin{aligned}
\delta_i^{**} &\le \delta_{i-1}^{**} - \chi(V_{i-1}, \mathcal{O}_X(-n_i f)) - h^0(X, V_i^{**} \otimes \mathcal{O}_X(n_i f)) + i l_{i-1} \\
&\le \delta_{i-1}^{**} + \left[2\left(c_2 - \sum_{k=i}^{r-1} l_k\right) - (2i-1)\right] + i l_{i-1}. \quad \square
\end{aligned}$$

4.3. **Generic bundles in the moduli space $\mathcal{M}_L(r; tf, c_2)$.** Our purpose is to determine the structure of a generic bundle in $\mathcal{M}_L(r; tf, c_2)$.

Lemma 4.15. *Assume $\mathcal{M}_L(r; tf, c_2)$ is nonempty where $-r < t \le 0$ and L satisfies* (3.3). *Then for a generic bundle V in $\mathcal{M}_L(r; tf, c_2)$, there are $(r-1)$ exact sequences:*

$$0 \to \mathcal{O}_X(-n_i f) \to V_i \to V_{i-1} \to 0 \tag{4.16}$$

for $r \ge i \ge 2$ with the following properties:

(i) $V_r = V$, V_i *is a rank-i bundle for* $i = r-1, \dots, 2$, *and*

$$V_1 = \mathcal{O}_X\left(\left(t + \sum_{i=2}^{r} n_i\right) f\right) \otimes I_{Z_1};$$

(ii) $l(Z_1) = c_2$, *and Z_1 is supported in c_2 distinct fibers;*

(iii) $n_r = [\frac{c_2 - t}{r}]$, *and* $n_i = [\frac{(c_2-t) - \sum_{k=i+1}^{r} n_i}{i}]$ *for* $i = r-1, \dots, 2$.

Proof. Note that $\delta_1^{**} = \#(\text{moduli of } V_1^{**}) - \dim \operatorname{Aut}(V_1^{**}) = -1$. By Proposition 4.14,

$$\begin{aligned}
\delta_r^{**} &\le \delta_1^{**} + \sum_{i=2}^{r} \left[2c_2 - 2\sum_{k=i}^{r-1} l_k - (2i-1) + i l_{i-1}\right] \\
&= -1 + \left[2(r-1)c_2 + (1 - r^2) + \sum_{i=1}^{r-1} (3-i) l_i\right].
\end{aligned}$$

Since $\delta_r^{**} = \#(\text{moduli of } V) - 1$ and $\sum_{i=1}^{r-1} l_i = c_2$, we have

$$(4.17)\qquad \#(\text{moduli of } V) \le 2rc_2 + (1 - r^2) + \sum_{i=1}^{r-1}(1-i)l_i \le 2rc_2 + (1-r^2).$$

By Lemma 3.6, since $\mathcal{M}_L(r; tf, c_2)$ is nonempty, we always have

$$\#(\text{moduli of } V) = 2rc_2 + (1 - r^2);$$

thus, in particular, all the inequalities in (4.17), (4.13), and Proposition 4.6(iii) become equalities. Hence, for a generic bundle V in $\mathcal{M}_L(r; tf, c_2)$, we conclude that:

(a) Since (4.17) is an equality, $l_2 = \cdots = l_{r-1} = 0$; so $l_1 = c_2$. It follows that $V_2, \ldots, V_{r-1}$ are bundles, and (4.16) comes from (4.7). Since V_1 is of rank-1,

$$V_1 = \mathcal{O}_X\left(\left(t + \sum_{i=2}^{r} n_i\right) f\right) \otimes I_{Z_1}$$

for some 0-cycle Z_1 on X. Thus, $Q_1 = \mathcal{O}_{Z_1}$ and $l(Z_1) = l_1 = c_2$. This proves (i).

(b) Since (4.13) is an equality and $Q_1 = \mathcal{O}_{Z_1}$,

$$\#(\text{moduli of } Z_1) = \#(\text{moduli of } Q_1) = 2l_1 = 2c_2.$$

Thus, for a generic bundle V, Z_1 is reduced and supported in c_2 distinct fibers. This proves (ii).

(c) Since Proposition 4.6(iii) is an equality, for $i = 2, \ldots, r$, we have

$$i \cdot n_i + (i - j_i) = c_2(V_i^{**}) - t - \sum_{k=i+1}^{r} n_k = c_2 - t - \sum_{k=i+1}^{r} n_k;$$

note that $0 \le (i - j_i) < i$; thus, $n_r = [\frac{c_2-t}{r}]$, and

$$n_i = \left[\frac{(c_2 - t) - \sum_{k=i+1}^{r} n_i}{i}\right]$$

for $i = r-1, \ldots, 2$. This proves (iii) and completes the proof. □

Proposition 4.18. *Assume that* $\mathcal{M}_L(r; tf, c_2)$ *is nonempty where* $-r < t \le 0$ *and* L *satisfies* (3.3). *Then a generic bundle* V *in* $\mathcal{M}_L(r; tf, c_2)$ *sits in an exact sequence:*

$$(4.19)\qquad 0 \to \bigoplus_{i=1}^{r} \mathcal{O}_X(-n_i f) \to V \to \bigoplus_{i=1}^{c_2} (\tau_i)_* \mathcal{O}_{f_i}(-1) \to 0$$

where the integer n_i *is defined by induction as follows:*

$$(4.20)\qquad n_i = \left[\frac{(c_2 - t) - \sum_{k=i+1}^{r} n_i}{i}\right] \text{ for } i < r \quad \text{with } n_r = \left[\frac{c_2 - t}{r}\right],$$

and $\{f_1, \ldots, f_{c_2}\}$ *are distinct fibers with* τ_i *being the natural embedding* $f_i \hookrightarrow X$.

Proof. First of all, we notice that if $(c_2 - t) = ar + \varepsilon$ with $0 \le \varepsilon < r$, then

$$(4.21)\qquad n_i = \begin{cases} a & \text{if } i = \varepsilon + 1, \ldots, r, \\ a + 1 & \text{if } i = 1, \ldots, \varepsilon. \end{cases}$$

In particular, $n_i \le n_j$ if $i > j$. By Lemma 4.15, for a generic bundle V in $\mathcal{M}_L(r; tf, c_2)$, we have $(r-1)$ exact sequences (4.16). Consider the first two exact sequences:

$$0 \to \mathcal{O}_X(-n_r f) \to V \xrightarrow{p_{r-1}} V_{r-1} \to 0,$$
$$0 \to \mathcal{O}_X(-n_{r-1} f) \to V_{r-1} \to V_{r-2} \to 0.$$

Then the subsheaf $p_{r-1}^{-1}(\mathcal{O}_X(-n_{r-1}f))$ of V sits in an exact sequence:

$$0 \to \mathcal{O}_X(-n_r f) \to p_{r-1}^{-1}(\mathcal{O}_X(-n_{r-1}f)) \to \mathcal{O}_X(-n_{r-1}f) \to 0.$$

Since $n_r \le n_{r-1}$, $\mathrm{Ext}^1(\mathcal{O}_X(-n_{r-1}f), \mathcal{O}_X(-n_r f)) = 0$; thus,

$$p_{r-1}^{-1}(\mathcal{O}_X(-n_{r-1}f)) = \bigoplus_{i=r-1}^{r} \mathcal{O}_X(-n_i f).$$

We check that $V/\bigoplus_{i=r-1}^{r} \mathcal{O}_X(-n_i f) = V_{i-1}/\mathcal{O}_X(-n_{r-1}f) = V_{i-2}$. Thus, V sits in

$$0 \to \bigoplus_{i=r-1}^{r} \mathcal{O}_X(-n_i f) \to V \to V_{r-2} \to 0.$$

By induction and the fact that $\mathrm{Hom}(\mathcal{O}_X(-n_1 f), V_1) \cong H^0(X, \mathcal{O}_X(c_2 f) \otimes I_{Z_1}) \ne 0$, we conclude that V sits in an exact sequence:

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_X(-n_i f) \to V \to V_1/\mathcal{O}_X(-n_1 f) \to 0.$$

Now the exact sequence (4.19) follows from the observation that

$$V_1/\mathcal{O}_X(-n_1 f) = I_{Z_1}/\mathcal{O}_X(-c_2 f) = \bigoplus_{i=1}^{c_2} (\tau_i)_* \mathcal{O}_{f_i}(-1)$$

where $f_1, \dots, f_{c_2}$ are the c_2 distinct fibers supporting the 0-cycle Z_1. □

Remark 4.22. (i) By Theorem 3.5, for any stable bundle V in $\mathcal{M}_L(r; tf, c_2)$, $\pi^*(\pi_* V)$ is a locally free rank-r subsheaf of V with the quotient Q being supported on the fibers of the ruling π over which the restriction of V is nontrivial. Another possible approach to prove Proposition 4.18 is to study the exact sequence

$$0 \to \pi^*(\pi_* V) \to V \to Q \to 0$$

and to estimate the number of moduli of these V's in terms of the data of Q and the rank-r bundle $\pi_* V$ on $\mathbf{P}^1$. In fact, this approach has been used very successfully by Friedman [8] to study stable rank-2 bundles on an arbitrary ruled surface. However, for $r > 2$, the difficulty of this approach lies in the observation that the deformation of Q is quite complicated.

(ii) From the exact sequence (4.19), we conclude that

$$\pi^*(\pi_* V) = \bigoplus_{i=1}^{r} \mathcal{O}_X(-n_i f)$$

for a generic bundle V in the moduli space $\mathcal{M}_L(r; tf, c_2)$.

5. The moduli space $\mathfrak{M}_L(r; tf, c_2)$ on a rational ruled surface

In this section, based on the results from the previous section, we determine the birational structure of the moduli space $\mathfrak{M}_L(r; tf, c_2)$ on a rational ruled surface where L satisfies (3.3) and $-r < t \le 0$. First of all, we introduce the following notation.

Notation 5.1. (i) Let n_i, f_i, and τ_i be as in Proposition 4.18. Put

$$W_0 = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{P}^1}(-n_i), \; W = \pi^*(W_0) = \bigoplus_{i=1}^{r} \mathcal{O}_X(-n_i f), \text{ and } Q = \bigoplus_{i=1}^{c_2} (\tau_i)_* \mathcal{O}_{f_i}(-1).$$

(ii) Let $\mathfrak{M}$ be the Zariski open and dense subset in $\mathfrak{M}_L(r; tf, c_2)$ parametrizing all bundles sitting in exact sequences of the form (4.19).

(iii) Let $\Phi : \mathfrak{M} \to U$ be the morphism defined by

$$\Phi(V) = \sum_{i=1}^{c_2} \pi(f_i)$$

where U is a Zariski open and dense subset in $\mathrm{Sym}^{c_2}(\mathbf{P}^1) \cong \mathbf{P}^{c_2}$.

Next, we want to determine the fiber $\Phi^{-1}(u)$ for $u \in U$. We start with a lemma.

Lemma 5.2. (i) $\mathrm{Hom}(W, V) \cong \mathrm{End}(W)$;

(ii) $\dim \mathrm{Aut}(W) = r^2$ *and* $\dim \mathrm{Aut}(Q) = c_2$;

(iii) $\dim \mathrm{Ext}^1(Q, W) = 2rc_2$.

Proof. (ii) and (iii) follow from (4.21) and the definitions of W and Q. In the following, we prove (i). Since $W = \pi^* W_0$, $\mathrm{End}(W) \cong \mathrm{End}(W_0)$. Since $\pi_* Q$ is torsion and

$$H^0(\mathbf{P}^1, \pi_* Q) = H^0(X, Q) = 0,$$

$\pi_* Q$ must be zero. Applying π_* to (4.19), we have $\pi_* V \cong \pi_* W = W_0$. Thus,

$$\begin{aligned}\mathrm{Hom}(W, V) &\cong H^0(X, V \otimes W^*) = H^0(\mathbf{P}^1, \pi_*(V \otimes \pi^*(W_0^*))) \\ &\cong H^0(\mathbf{P}^1, W_0 \otimes W_0^*) \cong \mathrm{End}(W_0) \cong \mathrm{End}(W). \quad \square\end{aligned}$$

Proposition 5.3. *Let $u \in U$. Then the fiber $\Phi^{-1}(u)$ is birational to* $\mathrm{Ext}^1(Q, W)$ *modulo the $(c_2 + r^2 - 1)$-dimensional group actions from* $\mathrm{Aut}(W)/\mathbf{C}^*$ *and* $\mathrm{Aut}(Q)$.

Proof. By Lemma 5.2(i), $\mathrm{Hom}(W, V) \cong \mathrm{End}(W)$.

From the proof of Lemma 4.15, we see that generic extensions in $\mathrm{Ext}^1(Q, W)$ must correspond to bundles in the Zariski open and dense subset $\mathfrak{M}$. It follows that $\Phi^{-1}(u)$ is birational to $\mathrm{Ext}^1(Q, W)$ modulo the group actions from $\mathrm{Aut}(W)/\mathbf{C}^*$ and $\mathrm{Aut}(Q)$. By Lemma 5.2(ii),

$$\dim \mathrm{Aut}(W) = r^2 \quad \text{and} \quad \dim \mathrm{Aut}(Q) = c_2.$$

Therefore, the group actions are $(c_2 + r^2 - 1)$-dimensional. $\square$

Now, we prove the second main result in this paper.

Theorem 5.4. *Assume that the moduli space* $\mathcal{M}_L(r; tf, c_2)$ *is nonempty where* $r \geq 2$, $-r < t \leq 0$, *and the ample divisor* L *satisfies condition* (3.3). *Then*

(i) $\mathcal{M}_L(r; tf, c_2)$ *is irreducible and unirational*;

(ii) *a generic bundle* V *in* $\mathcal{M}_L(r; tf, c_2)$ *sits in an exact sequence*

$$(5.5) \qquad 0 \to \bigoplus_{i=1}^{r} \mathcal{O}_X(-n_i f) \to V \to \bigoplus_{i=1}^{c_2} (\tau_i)_* \mathcal{O}_{f_i}(-1) \to 0$$

where the integer n_i *is defined by induction as follows*:

$$(5.6) \qquad n_i = \left[\frac{(c_2 - t) - \sum_{k=i+1}^{r} n_i}{i}\right] \quad \text{for } i < r \text{ with } n_r = \left[\frac{c_2 - t}{r}\right],$$

and $\{f_1, \dots, f_{c_2}\}$ *are distinct fibers with* τ_i *being the natural embedding* $f_i \hookrightarrow X$;

(iii) $(c_2 - t) \geq r$.

Proof. (i) By Lemma 5.2(iii), the extension group $\mathrm{Ext}^1(Q, W)$ has dimension $2rc_2$. By Proposition 4.24, we have a rational map Φ from the moduli space $\mathcal{M}_L(r; tf, c_2)$ to $\mathbf{P}^{c_2}$ such that a generic fiber $\Phi(u)$ is birational to

$$[\mathrm{Aut}(W)/\mathbf{C}^*] \backslash \mathbf{C}^{\oplus 2rc_2} / \mathrm{Aut}(Q).$$

Therefore, $\mathcal{M}_L(r; tf, c_2)$ is irreducible and unirational.

(ii) This is the same as Proposition 4.18.

(iii) Since $\mathcal{O}_X(-n_r f) \hookrightarrow V$ and V is L-stable, $-n_r f \cdot L < tf \cdot L/r \leq 0$; thus, $n_r \geq 1$. Since $n_r = [(c_2 - t)/r] \leq (c_2 - t)/r$, we get $(c_2 - t) \geq r$. □

Remark 5.7. In Theorem 1.9 of [2], Artamkin showed that if $c_2 \geq r \geq 2$, then $\mathcal{M}_L(r; 0, c_2)$ is nonempty and irreducible. Therefore, by Theorem 5.4(iii), we conclude that $\mathcal{M}_L(r; 0, c_2)$ is nonempty if and only if $c_2 \geq r$.

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