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WELL-POSEDNESS AND STABILIZABILITY OF A VISCOELASTIC EQUATION IN ENERGY SPACE

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ABSTRACT. We consider the well-posedness and exponential stabilizability of the abstract Volterra integrodifferential system

$$v'(t) = -D^*\sigma(t) + f(t),$$
$$\sigma(t) = \nu Dv(t) + \int_{-\infty}^{t} a(t-s)Dv(s)ds, \quad t \geq 0,$$

in a Hilbert space. In a typical viscoelastic interpretation of this equation one lets v represent velocity, v' acceleration, σ stress, $-D^*\sigma$ the divergence of the stress, $\nu \geq 0$ pure viscosity (usually equal to zero), Dv the time derivative of the strain, and a the linear stress relaxation modulus of the material. The problems that we treat are one-dimensional in the sense that we require a to be scalar. First we prove well-posedness in a new semigroup setting, where the history component of the state space describes the absorbed energy of the system rather than the history of the function v. To get the well-posedness we need extremely weak assumptions on the kernel; it suffices if the system is "passive", i.e., a is of positive type; it may even be a distribution. The system is exponentially stabilizable with a finite dimensional continuous feedback if and only if the essential growth rate of the original system is negative. Under additional assumptions on the kernel we prove that this is indeed the case. The final part of the treatment is based on a new class of kernels. These kernels are of positive type, but they need not be completely monotone. Still, they have many properties similar to those of completely monotone kernels, and a number of results that have been proved earlier for completely monotone kernels can be extended to the new class.

1. INTRODUCTION

In this paper we treat the well-posedness and stabilizability problems for abstract Volterra integrodifferential systems of the type

$$\begin{aligned} &v'(t) = -D^*\sigma(t) + f(t), \\ &\sigma(t) = \nu Dv(t) + \int_{-\infty}^{t} a(t-s)Dv(s)ds, \quad t \geq 0, \end{aligned} \tag{1}$$

in a Hilbert space. These equations arise, for example, in the theory of linear viscoelasticity, in which case v represents velocity, v' acceleration, σ stress,

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$-D^*\sigma$ the divergence of the stress, $\nu \geq 0$ pure viscosity, Dv the time derivative of the strain, and a the linear (shear or tensile) stress relaxation modulus of the material. A typical case is a mechanical system consisting of a number of rigid bodies with flexible members attached to them. To speed up the damping of vibrations of the system control forces are applied to the rigid parts.

Exponential stabilization of a system of the type described above is impossible unless there is some internal damping. This damping must be due to the deformations of the flexible members. To model the behavior of these flexible members one uses standard viscoelastic models. If we let σ denote the stress and ϵ the strain, then a typical constitutive assumption is that the stress can be computed from the strain in the form

$$\sigma(t) = \frac{d}{dt}\left(\nu\epsilon(t) + \int_{-\infty}^{t} a(t-s)\epsilon(s)\,ds\right),$$

where $\nu \geq 0$ (usually equal to zero) represents pure viscosity, and a is the linear stress relaxation modulus. (For the purpose of this discussion, let us assume that $\epsilon(t)$ vanishes for all sufficiently large negative t, so that there is no problem with the convergence of the integral.) The stress relaxation modulus is a material function that is rather difficult to measure. It is widely taken to be completely monotone, but there is no direct physical evidence supporting this assumption. We shall argue below that in some cases it is more appropriate to allow a not to be completely monotone.

In the case where the stress relaxation modulus is completely monotone, the question that we address below has been completely resolved in [4]. Our purpose is to extend that result to a larger class of relaxation moduli. Indeed, we succeed to get well-posedness for the equation in a setting similar to the one in [4] under the minimal assumption that the relaxation modulus is of positive type. This is a necessary physical condition, related to the fact that the system must be passive, i.e., it cannot emit more energy than what it has previously absorbed (see (12) with the lower bound 0 in the integral replaced by $-\infty$). To get exponential stabilizability we have to restrict the class of moduli further. In our more general setting several new phenomena show up, not present in [4], and although the skeleton remains the same, the final theory is much richer than in the completely monotone case.

The first five sections of this paper treat the well-posedness problem under minimal assumptions on the kernel, and relates our results to those previously known for completely monotone kernels. Our new class of relaxation moduli is introduced in Section 6. The next two sections discuss the spectrum of the generator, and investigates the exponential stabilizability and spectrum determined growth property of the semigroup. In addition we point out that the same approach applies to the case where the kernel is a distribution rather than a function. The following section contains examples illustrating the theory. Finally, the last two sections give some time and frequency domain estimates for the new class of relaxation moduli.

We shall refer the reader to [4] for further background information on the problem, and for explicit viscoelastic examples. The book [16] contains a nice short discussion of linear viscoelasticity. Additional relevant references are, among others, [1], [2], [3], [8], [10], [11], [12], [14], and [17].

2. THE ABSORBED ENERGY SPACE SETUP FOR WELL-POSEDNESS

The setting that we use is very similar to the one in [4], apart from the fact that we use a different class of kernels and a different norm. The functions v and f in (1) take their values in an infinite-dimensional Hilbert space X, and σ takes its values in another infinite-dimensional Hilbert space Y. The unbounded, closed, and densely defined operator D maps $\operatorname{dom} D \subset X$ into Y, and its adjoint D^* maps $\operatorname{dom} D^* \subset Y$ into X. We use the graph norms $(\|x\|_X^2 + \|Dx\|_Y^2)^{1/2}$ and $(\|x\|_Y^2 + \|D^*x\|_X^2)^{1/2}$ in $\operatorname{dom} D$ and $\operatorname{dom} D^*$, and assume that the imbeddings $\operatorname{dom} D \subset X$ and $\operatorname{dom} D^* \subset Y$ are compact. In particular, DD^* and D^*D are selfadjoint, with compact resolvents. These operators have the same spectra, except that zero may belong to the spectrum of one of the two, but not to the spectrum of the other. As we shall see below in Remark 6, we lose no generality by assuming that D^* is one-to-one, i.e., by assuming that zero does not belong to the spectrum of DD^* (but it may belong to the spectrum of D^*D). (These notations differ slightly from those in [4]. To convert the notation used in [4] to our notation one should exchange X and Y, and also D and D^*.)

The kernel a in (1) is supposed to be (scalar) real-valued, of positive type, and locally integrable. Moreover, it must not vanish identically. This means that, by one version of Bochner's theorem (see., e.g., [6, Theorem 2.5, p. 495]), a is the restriction to $\mathbf{R}^+$ of the inverse distribution Fourier transform of a positive locally finite measure $2\pi\mu$ (that does not vanish identically). Usually this inverse transform has to be interpreted in the distribution sense, but if a is in addition continuous, then we may write

$$a(t) = \int_{-\infty}^{\infty} e^{i\omega t} \mu(d\omega), \quad t > 0. \tag{2}$$

(Here we have for convenience dropped the factor $1/(2\pi)$ that is commonly used in front of the integral.) If a is not continuous, then we still occasionally use the same equation, but regard it as an informal way of writing that a is the restriction to $\mathbf{R}^+$ of the inverse distribution Fourier transform of $2\pi\mu$. Since we assume a to be real, the measure μ is symmetric, i.e., $\mu(-E) = \mu(E)$ for all sets E. It need not be finite, and it need not even satisfy $\int_{-\infty}^{\infty}(1+|\omega|)^{-1}\mu(d\omega) < \infty$, but it does satisfy

$$\lim_{|\omega|\to\infty} \mu([\omega, \omega+1]) = 0;$$

to see this one may, e.g., use the facts that the function a_1 defined by $a_1(t) = (1-|t|)a(|t|)$ for $|t| \le 1$, $a_1(t) = 0$ otherwise, is integrable, and that its Fourier transform, which is the convolution of μ with the nonnegative Fejér kernel, tends to zero at infinity. In particular, we always have

$$\int_{-\infty}^{\infty} \frac{\mu(d\omega)}{1+\omega^2} < \infty. \tag{3}$$

The distribution Laplace transform $\hat{a}$ of a exists in the open half-plane $\Re\lambda > 0$. If μ is finite, i.e., if a is bounded at zero, then a short computation shows that $\hat{a}(\lambda)$ will be given by an integral of Cauchy type, i.e.,

$$\hat{a}(\lambda) = \int_{-\infty}^{\infty} \frac{\mu(d\omega)}{\lambda - i\omega}, \quad \Re\lambda > 0. \tag{4}$$

In fact, an approximation argument shows that the same formula must be valid whenever the integral above converges absolutely, i.e., whenever $\int_{-\infty}^{\infty}(1 + |\omega|)^{-1}\mu(d\omega) < \infty$. If it does not converge absolutely, then we may use the symmetry of μ to write this as

(5)
$$\hat{a}(\lambda) = \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{1}{\lambda - i\omega} + \frac{1}{\lambda + i\omega}\right)\mu(d\omega) = \int_{-\infty}^{\infty}\frac{\lambda}{\lambda^2 + \omega^2}\,\mu(d\omega), \quad \Re\lambda > 0,$$

where the integral does converge absolutely due to (3). Moreover, it is also possible to compute $\Re\hat{a}(\lambda)$ from the absolutely converging integral

(6)
$$\Re\hat{a}(\lambda) = \int_{-\infty}^{\infty}\Re\left\{\frac{1}{\lambda - i\omega}\right\}\mu(d\omega) = \int_{-\infty}^{\infty}\frac{\Re\lambda}{(\Re\lambda)^2 + (\Im\lambda - \omega)^2}\,\mu(d\omega), \quad \Re\lambda > 0.$$

We shall need to integrate various other functions, too, against μ. This is no problem if they belong to L^1 with respect to μ, but not all our functions will be of this type. Fortunately, those that are not have the same property as above, i.e., it is possible to define

$$\int_{-\infty}^{\infty}\varphi(i\omega)\,\mu(d\omega) = \frac{1}{2}\int_{-\infty}^{\infty}(\varphi(i\omega) + \varphi(-i\omega))\,\mu(d\omega), \tag{7}$$

where the integral on the right converges absolutely. In particular, if $\varphi(z) = \psi(z) + c/(\lambda - z)$, where $\psi \in L^1(\mu)$ and c and λ are constants with $\Re\lambda > 0$, then

$$\int_{-\infty}^{\infty}\varphi(i\omega)\,\mu(d\omega) = \int_{-\infty}^{\infty}\psi(i\omega)\,\mu(d\omega) + \hat{a}(\lambda)c. \tag{8}$$

The setup that we shall use in our semigroup formulation of (1) is similar to the one in [4]. There are two main differences: we replace the measure that generates a completely monotone kernel that was used in [4] by the measure μ introduced above, and instead of using the full space $L^2(\mu; Y)$ as a component of the state space we use the weighted Hardy space $H^2(\mu; Y)$ over the left half-plane. The details will follow.

To motivate our choice of state space, let us make the following computation. Assume for the moment that v is continuous with values in $\operatorname{dom} D$, and small enough as $t \to -\infty$ so that the convolution in (1) is well defined. Also suppose that a is continuous, so that $\int_{-\infty}^{\infty}\mu(d\omega) < \infty$ and (2) is valid. Then

$$\begin{aligned}\int_{-\infty}^{t} a(t-s)Dv(s)\,ds &\left(= \int_{-\infty}^{t}\int_{-\infty}^{\infty} e^{i\omega(t-s)}\,\mu(d\omega)Dv(s)\,ds\right)\\ &= \int_{-\infty}^{\infty}\varphi(t, i\omega)\,\mu(d\omega),\end{aligned} \tag{9}$$

where

$$\varphi(t, z) = \int_{-\infty}^{t} e^{z(t-s)} Dv(s)\,ds = \int_{-\infty}^{0} e^{-zs} Dv(s+t)\,ds. \tag{10}$$

Observe that, for each fixed t, the function $z \mapsto \varphi(t, z)$ is the bilateral Laplace transform of the function $\chi_{(-\infty,0]}\tau_t Dv$, where $\tau_t Dv(s) = Dv(s+t)$ is the function $Dv(s)$ translated to the left by the amount t. In particular, as a function of z, $\varphi(t, z)$ it is continuous in $\Pi^- \overset{\text{def}}{=} \{\Re z \le 0\}$, it is analytic in the interior of Π^-, and $z\varphi(t, z)$ tends to a finite limit (depending on t) as $|z| \to \infty$, $z \in \Pi^-$. Moreover, as a function of t,

$$\varphi'(t, z) = z\varphi(t, z) + Dv(t), \tag{11}$$

where we have used a prime to denote differentiation with respect to t. To solve (1) in the forward time direction, starting from some initial time t, with a given initial value of $v(t)$, one does not need to know the values of $v(s)$ for $s < t$, provided one knows the present and future values of the integral term. This term can be recovered from φ through the formula (9). The development of φ is fully determined by (11) together with an initial value of φ. Thus, instead of keeping track of the old values of v one may keep track of the present value of φ. This substitution of the history of v by the function φ turns the equation into an abstract ordinary differential equation (without memory).

To find out what norm to use for φ we make another short computation. Let v and a satisfy the same assumptions as above, and, in addition, assume that σ is continuous with values in $\operatorname{dom} D^*$. Substitute the value of σ from the second equation in (1) into the first, take the real part of the X-inner product with v, and integrate over $[0, T]$. This leads to the equation

(12)
$$\begin{aligned}\frac{1}{2}\|v(T)\|_X^2 + \nu \int_0^T \|Dv(t)\|_X^2 dt + \int_0^T \Re\left\langle v(t), D^* \int_{-\infty}^{t} a(t-s) Dv(s)\,ds \right\rangle_X dt \\ = \frac{1}{2}\|v(0)\|_X^2 + \int_0^T \Re\langle v(t), f(t)\rangle_X.\end{aligned}$$

Several terms in this equation have obvious physical interpretations. The term $\frac{1}{2}\|v(T)\|_X^2 - \frac{1}{2}\|v(0)\|_X^2$ represents the total change in kinetic energy, the term $\nu \int_0^T \|Dv(t)\|_X^2 dt$ the total energy loss due to viscosity, and the term $\int_0^T \Re\langle v(t), f(t)\rangle_X$ is the work done by the body force f. Thus, the term containing the two integrals equals the total amount of work absorbed by the system, not counting the work needed to overcome the pure viscosity. Of course, this integral term is the one that contains most of the "interesting" information about the behavior of (1).

It is a very interesting and crucial fact that the term with the two integrals can be expressed in a simple way in terms of the transformed function φ. To see this, use (9) and (11) to get

$$\begin{aligned}
&\Re\left\langle v(t), D^* \int_{-\infty}^{t} a(t-s)Dv(s)\,ds \right\rangle_X = \Re\left\langle Dv(t), \int_{-\infty}^{t} a(t-s)Dv(s)\,ds \right\rangle_Y \\
&\quad = \Re\left\langle Dv(t), \int_{-\infty}^{\infty} \varphi(t, i\omega)\,\mu(d\omega) \right\rangle_Y \\
&\quad = \Re\left\langle \varphi'(t, i\omega) - i\omega\varphi(t, i\omega), \int_{-\infty}^{\infty} \varphi(t, i\omega)\,\mu(d\omega) \right\rangle_Y \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } \langle \varphi(t, i\omega), \varphi(t, i\omega)\rangle_Y \text{ is real)} \\
&\quad = \int_{-\infty}^{\infty} \Re\,\langle \varphi'(t, i\omega), \varphi(t, i\omega)\rangle_Y\, \mu(d\omega).
\end{aligned}$$

This is a perfect differential, which may be integrated into

(13)

$$\begin{aligned}
&\int_0^T \Re\left\langle v(t), D^* \int_{-\infty}^{t} a(t-s)Dv(s)\,ds \right\rangle_X dt \\
&\qquad = \frac{1}{2}\int_{-\infty}^{\infty} \|\varphi(T, i\omega)\|_Y^2\, \mu(d\omega) - \frac{1}{2}\int_{-\infty}^{\infty} \|\varphi(0, i\omega)\|_Y^2\, \mu(d\omega).
\end{aligned}$$

Thus, the μ-weighted L^2-norm of φ in Y can be interpreted as an internal energy, and this seems to be the obvious norm to use for φ. For obvious reasons we shall refer to this norm as the *absorbed energy norm.*

Actually, in a certain sense the full space $L^2(\mu; Y)$ suggested above is not the best possible one to use in the formulation of the problem, due to the fact that it may be bigger than what we really need. Recall that the functions φ that we are interested in are, at least in the case of smooth data, analytic in Π^-, and $z\varphi(z)$ has a finite limit as $|z| \to \infty$, $z \in \Pi^-$. Thus, as a state space for φ we shall use $H^2(\mu; Y)$, which we define to be the closure in $L^2(\mu; Y)$ of the space of functions φ that are Y-valued analytic in Π^- with $z\varphi(z)$ tending to a finite limit as $|z| \to \infty$, $z \in \Pi^-$. Observe that, by (3), functions of this type belong to $L^2(\mu; Y)$. Depending on the particular measure μ, $H^2(\mu; Y)$ may be the same space as $L^2(\mu; Y)$, or it may be different. For example, if $a(t) = \cos t$, then μ consists of two point masses of size $\frac{1}{2}$ at ± 1, and in this case $H^2(\mu; Y) = L^2(\mu; Y)$. On the other hand, if $a(t) = e^{-t}$, then it is easy to show that the functions φ in $L^2(\mu; Y)$ can be identified with the set of functions of the type $(1-z)\psi(z)$, where ψ belongs to the normal unweighted H^2-space over Π^- (see the example at the end of Section 4). This latter case is a more typical one than the former. We shall call $H^2(\mu; Y)$ *the absorbed energy space.*

Many times the equation (1) is written in a slightly different way. If the material is a solid as opposed to a liquid, then, in our formulation, this means that $a(\infty) \neq 0$. Let us separate $a(t)$ into $b(t) + E$, where E, the equilibrium elasticity modulus, is the value of a at infinity. With this change equation (1) becomes

$$(14)\qquad \begin{aligned} &u'(t) = v(t), \quad v'(t) = -D^*\sigma(t) + f(t), \\ &\sigma(t) = \nu Dv(t) + EDu(t) + \int_{-\infty}^{t} b(t-s)Dv(s)ds, \quad t \geq 0. \end{aligned}$$

This is the formulation used in [4] (apart from the fact that there the viscosity ν throughout was taken to be zero, and E was throughout nonzero). If one computes an energy balance equation for (14) in the same way as we did for (1), then one gets an extra term

$$\frac{1}{2}E\|Du(T)\|_Y^2 - \frac{1}{2}E\|Du(0)\|_Y^2$$

which has an obvious interpretation as the change of potential energy, due to the equilibrium elastic response of the material. The same term is in fact present in the previous computation, too, hidden in the double integral term. A nonzero value E of a at infinity corresponds to a point mass in μ of size E at the origin, and thus in our formulation the same term appears as a part of (13) in the form

$$\frac{1}{2}E\|\varphi(T, 0)\|_Y^2 - \frac{1}{2}E\|\varphi(0, 0)\|_Y^2.$$

Thus, the potential energy has been absorbed into (13). Comparing the treatment here to the one in [4] one finds that there is a certain advantage to proceed in the way that we do. The formulation simplifies, since there is no need to keep track of u (which represents position; recall that v stands for velocity), and some of the complications in [4] due to the fact that D may have a nontrivial kernel are avoided. The same formulation (1) that we use here is also used in [1] (except that there $\nu = 0$, and the kernel is allowed to be tensor-valued instead of scalar-valued).

The equation could be even further simplified in the sense that one could absorb the term $\nu Dv(t)$ into the convolution. Formally this corresponds to a point mass of a of size ν at the origin. The measure μ that corresponds to this point mass in the inversion formula (4) is the Lebesgue measure multiplied by $2\pi\nu$ (i.e., a constant function $2\pi\nu$ multiplied by $d\omega$). By adding this constant term to μ we could absorb all stress related terms into φ. This would only require minor modifications below. However, due to the specific nature of this term, we prefer to keep it separate. In particular, in this way we will have $\hat{a}(\lambda) \to 0$ as $\lambda \to \infty$, with λ real (except in Theorem 24).

Thus, the setting that we use is the following. The state space becomes $X \times H^2(\mu; Y)$, and the equation that an element (v, φ) of our state space should satisfy is

$$(15)\qquad \begin{aligned} v'(t) &= -D^*\left(\nu Dv(t) + \int_{-\infty}^{\infty} \varphi(t, i\omega)\,\mu(d\omega)\right) + f(t), \\ \varphi'(t, z) &= z\varphi(t, z) + Dv(t), \quad t \geq 0, \end{aligned}$$

Let us denote the different operators appearing above as follows:

$$(16)\quad B\varphi = \int_{-\infty}^{\infty} \varphi(i\omega)\,\mu(d\omega), \qquad (M\varphi)(z) = z\varphi(z), \qquad (\Lambda v)(z) = v.$$

Then the abstract differential equation (15) can be written in the matrix form

$$\frac{d}{dt}\begin{pmatrix} v \\ \varphi \end{pmatrix} = \begin{pmatrix} -\nu D^*D & -D^*B \\ \Lambda D & M \end{pmatrix}\begin{pmatrix} v \\ \varphi \end{pmatrix} + \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{17}$$

This should really be regarded as a formal equation in the sense that all the different components of $\mathscr{A}(v, \varphi)$ that one gets from the matrix multiplication above, i.e., $-\nu D^*Dv$, $-D^*B\varphi$, ΛDv, and $M\varphi$, need not be defined separately. It is enough if $-D^*(\nu Dv + B\varphi)$ and $\Lambda Dv + M\varphi$ are well defined.

Theorem 1. *Let* $\mathscr{H} = X \times H^2(\mu; Y)$, *let* $\mathscr{A}$ *be the operator*

$$\mathscr{A} = \begin{pmatrix} -\nu D^*D & -D^*B \\ \Lambda D & M \end{pmatrix}$$

mapping $\operatorname{dom}\mathscr{A} \subset \mathscr{H}$ *into* $\mathscr{H}$, *with*

$$\operatorname{dom}\mathscr{A} = \{ (v, \varphi) \in \mathscr{H} \mid v \in \operatorname{dom} D,\ \nu Dv + B\varphi \in \operatorname{dom} D^*, \\ \Lambda Dv + M\varphi \in H^2(\mu; Y) \}.$$

Then $\mathscr{A}$ *is the infinitesimal generator of a* C_0 *contraction semigroup* S *in* $\mathscr{H}$.

We remark that $B\varphi$ is well defined for $(v, \varphi) \in \operatorname{dom}\mathscr{A}$. To see this, observe that φ is locally integrable with respect to μ, and that, if we denote $\Lambda Dv + M\varphi$ by ψ, then at infinity φ can be written in the form $\varphi(z) = (\psi(z) - Dv)/z$. As we observed earlier, functions of this type may be integrated against μ, if one use (7) as a definition of the integral.

3. Proof of well-posedness in the absorbed energy space

According to the Lumer-Phillips theorem (see [13]), $\mathscr{A}$ generates a C_0 contraction semigroup in $\mathscr{H}$ if $\operatorname{dom}\mathscr{A}$ is dense in $\mathscr{H}$, if $\mathscr{A}$ is dissipative, and if there is some $\lambda > 0$ such that the range of $\lambda - \mathscr{A}$ is all of $\mathscr{H}$. Thus, Theorem 1 follows from this theorem, together with Lemmas 2–4 below. These lemmas and their proofs are quite similar to the corresponding lemmas and proofs in [4].

Lemma 2. *The domain of* $\mathscr{A}$ *is dense in* $\mathscr{H}$.

Proof. Since $\operatorname{dom} D^*D$ is dense in X and $H^2(\mu; \operatorname{dom} D^*)$ is dense in $H^2(\mu; Y)$, it suffices to show that $\operatorname{dom}\mathscr{A}$ is dense in

$$\operatorname{dom} D^*D \times H^2(\mu; \operatorname{dom} D^*).$$

Let $v \in \operatorname{dom} D^*D$ and $\varphi \in H^2(\mu; \operatorname{dom} D^*)$, and for each $\lambda > 0$, define

$$\varphi_\lambda(z) = \frac{\lambda\varphi(z)}{\lambda - z} + \frac{Dv}{\lambda - z}.$$

We claim that for each λ, $(v, \varphi_\lambda) \in \operatorname{dom}\mathscr{A}$, and that $\varphi_\lambda \to \varphi$ in $H^2(\mu; \operatorname{dom} D^*)$ as $\lambda \to \infty$. Clearly, this implies our earlier claim that $\operatorname{dom}\mathscr{A}$ is dense in $\operatorname{dom} D^*D \times H^2(\mu; \operatorname{dom} D^*)$. To prove the convergence claim it suffices to observe that

$$\|\varphi_\lambda - \varphi\|_{H^2(\mu;\operatorname{dom} D^*)} \le \|z/(\lambda - z)\varphi\|_{H^2(\mu;\operatorname{dom} D^*)} + \|1/(\lambda - z)Dv\|_{H^2(\mu;\operatorname{dom} D^*)},$$

and to use (3) and Lebesgue's dominated convergence theorem. To show that $(v, \varphi_\lambda) \in \operatorname{dom}\mathscr{A}$ we need to prove that $\nu Dv + B\varphi_\lambda \in \operatorname{dom} D^*$, and that $\Lambda Dv + M\varphi_\lambda \in H^2(\mu; Y)$. For the first of these two assertions, note that by the choice of v, $Dv \in \operatorname{dom} D^*$, and that the function $z \mapsto (\lambda/(\lambda - z))\varphi(z)$ belongs to $L^1(\mu; \operatorname{dom} D^*)$, hence

$$B\varphi_\lambda = \int_{-\infty}^{\infty} \frac{\lambda\varphi(i\omega)\,\mu(d\omega)}{\lambda - i\omega} + \hat{a}(\lambda)Dv \in \operatorname{dom} D^*.$$

For the second assertion, one computes

$$\begin{aligned}\psi_\lambda(z) &= (\Lambda Dv)(z) + (M\varphi_\lambda)(z) = Dv + z\varphi_\lambda(z)\\ &= \frac{\lambda Dv}{\lambda - z} + \frac{\lambda z\varphi(z)}{\lambda - z} \in H^2(\mu; \operatorname{dom} D^*).\end{aligned}$$

Thus, Lemma 2 is true. □

Lemma 3. *The operator* $\mathscr{A}$ *is dissipative. More precisely, for all* $(v, \varphi) \in \operatorname{dom}\mathscr{A}$, $\Re\langle(v, \varphi), \mathscr{A}(v, \varphi)\rangle = -\nu\|Dv\|_Y^2 \le 0$.

The proof of this lemma is closely related to the energy computation that we made in order to motivate the choice of norm.

Proof. Let $(v, \varphi) \in \operatorname{dom}\mathscr{A}$, and denote $\mathscr{A}(v, \varphi)$ by (w, ψ). Then $\psi(z) = Dv + z\varphi(z)$. In the computation that we shall make in a moment, we need to know that

$$\int_{-\infty}^{\infty} \left|\Re\langle\varphi(i\omega), Dv\rangle_Y\right| \mu(d\omega) < \infty.$$

To see that this must be true, observe that

$$\Re\langle\varphi(i\omega), Dv\rangle_Y = \Re\langle\psi(i\omega)/(i\omega), Dv\rangle_Y,$$

where the left-hand side is integrable with respect to μ at zero, and the right-hand side is integrable with respect to μ at infinity. Thus, the whole function in integrable.

Now compute

$$\begin{aligned}&\Re\langle(v, \varphi), \mathscr{A}(v, \varphi)\rangle\\ &\quad= \Re\langle v, -D^*(\nu Dv - B\varphi)\rangle_X + \int_{-\infty}^{\infty} \Re\langle\varphi(i\omega), Dv + i\omega\varphi(i\omega)\rangle_Y\,\mu(d\omega)\\ &\quad= -\nu\|Dv\|_Y^2 - \Re\langle Dv, B\varphi\rangle_Y + \int_{-\infty}^{\infty} \Re\langle\varphi(i\omega), Dv\rangle_Y\,\mu(d\omega)\\ &\quad= -\nu\|Dv\|_Y^2 \le 0. \square\end{aligned}$$

In particular, observe that if $\nu = 0$, then $\Re\langle(v, \varphi), \mathscr{A}(v, \varphi)\rangle = 0$. (If we would have absorbed the term νDv in (1) into the convolution, then the result would have been $\Re\langle(v, \varphi), \mathscr{A}(v, \varphi)\rangle = 0$ also for nonzero ν; cf. the discussion in the preceding section.)

Lemma 4. *Every λ with $\Re\lambda > 0$ belongs to the resolvent set of $\mathscr{A}$, and*

$$(v, \varphi) = [\lambda - \mathscr{A}]^{-1}(w, \psi)$$

may be computed as follows. Define

$$V(\lambda) = [\lambda + (\nu + \hat{a}(\lambda))D^*D]^{-1}, \quad W(\lambda) = [\lambda + (\nu + \hat{a}(\lambda))DD^*]^{-1},$$
$$\widetilde{w}(\lambda) = B(\lambda - z)^{-1}\psi(z) = \int_{-\infty}^{\infty} (\lambda - i\omega)^{-1}\psi(i\omega)\,\mu(d\omega).$$

Then

$$v = V(\lambda)w - D^*W(\lambda)\widetilde{w}(\lambda),$$
$$\varphi(z) = \frac{Dv + \psi(z)}{\lambda - z} = \frac{DV(\lambda)w - DD^*W(\lambda)\widetilde{w}(\lambda) + \psi(z)}{\lambda - z}.$$

Proof. First, let us observe that the operators $V(\lambda)$ and $W(\lambda)$ exist as bounded continuous operators. This follows from the following facts. Since $\nu \geq 0$ and $\Re\lambda > 0$, we have from (6), $\nu + \Re\hat{a}(\lambda) > 0$. The inverses defining $V(\lambda)$ and $W(\lambda)$ then exist if and only if $-\lambda/(\nu + \hat{a}(\lambda))$ does not belong to the spectra of D^*D and DD^*, respectively. Since D^*D and DD^* are positive and selfadjoint, their spectra are contained in $\mathbf{R}^+$. However, due to the fact that $\Re\lambda > 0$ and $\nu + \Re\hat{a}(\lambda) > 0$, the number $-\lambda/(\nu + \hat{a}(\lambda))$ cannot be real and nonnegative. Thus, $V(\lambda)$ and $W(\lambda)$ exist. Moreover, observe that $V(\lambda)$ maps X into $\operatorname{dom} D^*D \subset \operatorname{dom} D \subset X$, and that $W(\lambda)$ maps Y into $\operatorname{dom} DD^* \subset \operatorname{dom} D^* \subset Y$.

To prove that $\lambda - \mathscr{A}$ is one-to-one, hence has an inverse, we argue as follows. Assume that $(\lambda - \mathscr{A})(v, \varphi) = 0$ for some $(v, \varphi) \in \operatorname{dom}\mathscr{A}$. This means that

$$\lambda v + D^*(\nu Dv + B\varphi) = 0, \quad \lambda\varphi(z) - Dv - z\varphi(z) = 0.$$

Thus $\varphi(z) = Dv/(\lambda - z)$, and

$$\lambda v + D^*(\nu Dv + BDv/(\lambda - z)) = \lambda v + D^*(\nu Dv + \hat{a}(\lambda)Dv) = 0.$$

Apply $V(\lambda)$ to this equation to conclude that $v = 0$, from which $\varphi(z) = Dv/(\lambda - z) = 0$ follows. Thus, $(\lambda - \mathscr{A})$ is one-to-one.

Finally, we show that if $(w, \psi) \in \mathscr{H}$, and if (v, φ) is defined as described in Lemma 4, then $(v, \varphi) \in \operatorname{dom}\mathscr{A}$ and $(\lambda - A)(v, \varphi) = (w, \psi)$. First observe that $v \in \operatorname{dom} D$. By (8),

$$B\varphi = \hat{a}(\lambda)Dv + \widetilde{w}(\lambda),$$

hence

$$\begin{aligned}\nu Dv + B\varphi &= (\nu + \hat{a}(\lambda))Dv + \widetilde{w}(\lambda)\\ &= (\nu + \hat{a}(\lambda))DV(\lambda)w - (\nu + \hat{a}(\lambda))DD^*W(\lambda)\widetilde{w}(\lambda) + \widetilde{w}(\lambda)\\ &= (\nu + \hat{a}(\lambda))DV(\lambda)w + \lambda W(\lambda)\widetilde{w}(\lambda).\end{aligned}$$

This shows that $\nu Dv + B\varphi \in \operatorname{dom} D^*$. The first component of $(\lambda - \mathscr{A})(v, \varphi)$ may then be computed, and we get

$$\begin{aligned}\lambda v &+ D^*(\nu Dv + B\varphi) \\ &= \lambda V(\lambda)w - \lambda D^* W(\lambda)\widetilde{w}(\lambda) + (\nu + \hat{a}(\lambda))D^* DV(\lambda)w + \lambda D^* W(\lambda)\widetilde{w}(\lambda) \\ &= w,\end{aligned}$$

as claimed. To check that the second component of $\mathscr{A}(v, \varphi)$ belongs to $H^2(\mu; Y)$, we compute

$$Dv + z\varphi(z) = Dv + z\left(\frac{Dv}{\lambda - z} + \frac{\psi(z)}{\lambda - z}\right) = \frac{\lambda Dv}{\lambda - z} + \frac{z\psi(z)}{\lambda - z} \in H^2(\mu; Y).$$

Furthermore,

$$\lambda\varphi(z) - (Dv + z\varphi(z)) = \lambda\left(\frac{Dv}{\lambda - z} + \frac{\psi(z)}{\lambda - z}\right) - \frac{\lambda Dv}{\lambda - z} - \frac{z\psi(z)}{\lambda - z} = \psi(z). \square$$

The lemmas above are also valid if one replaces $H^2(\mu; Y)$ by $L^2(\mu; Y)$. Thus, $\mathscr{A}$ generates a contraction semigroup in this bigger space, too. The physical interpretation of this semigroup is somewhat different (it seems to be of combined initial and forcing function type).

4. Spaces of observable and unobservable absorbed energy

According to the computation that we made in Section 2 in order to the motivate the norm that we use, the absorbed energy is not dissipated. Instead the μ-weighted L^2-norm of $\varphi(t)$ describes the total amount of energy that has been absorbed by the material up to time t, not counting the energy loss due to pure viscosity. Of course, in a real life situation some of the internal energy must have dissipated, and can no longer be retrieved.

One major problem is that the state space $\mathscr{H}$ may still be too large, in spite of the fact that we already cut it down from the original $X \times L^2(\mu; Y)$ to $X \times H^2(\mu; Y)$. It is not always true that the whole space $\mathscr{H}$ is "observable" and "controllable".

The concepts of observability and controllability rely on the choice of an output operator that determines the observation, and an input operator that determines what type of feedbacks are allowed. As output operator we shall throughout use the operator $\mathscr{C}$ defined by $\mathscr{C}(v, \varphi) = v$. In other words, we observe only the velocity component of the solution. As input operator $\mathscr{B}$ we use the operator indicated in (1), namely

$$\mathscr{B}f = (f, 0).$$

Thus, our feedback is a force resulting in acceleration. A subspace of $\mathscr{H}$ is unobservable if every initial vector h belonging to that subspace results in a zero output $\mathscr{C}S(t)h$. It is uncontrollable if the evolution of the solution of (1) in that subspace is independent of the choice of the control $\mathscr{B}f$.

There is one particular case where a portion of $\mathscr{H}$ is both unobservable and uncontrollable, namely the case where D^* is not one-to-one. This situation is described in the following theorem:

Theorem 5. *The space $\mathscr{H}$ in Theorem 1 can be split into two orthogonal subspaces $X \times H^2(\mu; (\ker D^*)^\perp)$ and $\{0\} \times H^2(\mu; \ker D^*)$, both of which are invariant subspaces of $\mathscr{A}$. The latter subspace is both unobservable and uncontrollable. For each $(0, \varphi)$ in this subspace we have $S(t)(0, \varphi(z)) = (0, e^{zt}\varphi(z))$.*

Proof. Split $\mathscr{H}$ into $X \times H^2(\mu; (\ker D^*)^\perp) \times H^2(\mu; \ker D^*)$. Then, since the range of D is equal to $(\ker D^*)^\perp$, since $H^2(\mu; (\ker D^*)^\perp)$ and $H^2(\mu; \ker D^*)$ are invariant subspaces for the multiplication operator M, and since B maps $H^2(\mu; (\ker D^*)^\perp)$ into $(\ker D^*)^\perp$ and $H^2(\mu; \ker D^*)$ into $\ker D^*$, the operator $\mathscr{A}$ splits into

$$\mathscr{A} = \begin{pmatrix} -\nu D^*D & -D^*B & 0 \\ \Lambda D & M & 0 \\ 0 & 0 & M \end{pmatrix}.$$

The block-diagonal structure of $\mathscr{A}$ means that $X \times H^2(\mu; (\ker D^*)^\perp)$ and $\{0\} \times H^2(\mu; \ker D^*)$ are invariant subspaces for $\mathscr{A}$, and that the original equation decouples into two independent equations. The output operator vanishes on the latter space, and the range of the input operator is contained in the former space. Thus, the latter space is both unobservable and uncontrollable, as claimed. A direct substitution shows that $S(t)(0, \varphi(z)) = (0, e^{zt}\varphi(z))$ for each (v, φ) in this subspace. □

Note that in the original time domain formulation of the problem the function $e^{zt}\varphi(z)$ that appears in the description of T in the second invariant subspace in Theorem 5 corresponds to a plain left-translation by the amount t of the inverse Laplace transform of φ. Thus, in the time domain interpretation, the unobservable and uncontrollable part is simply left-translated, but otherwise completely ignored.

Remark 6. Because of Theorem 5 we shall throughout in the sequel assume that D^* is one-to-one. In other words, if D^* is not one-to-one, then we replace Y by $(\ker D^*)^\perp$. Thus, we assume that zero belongs to the resolvent set of DD^*.

Theorem 5 does not claim that the subspace $\{0\} \times H^2(Y; \ker D^*)$ is the *maximal unobservable* and *maximal uncontrollable* subspaces of our equation. This subspace may be a part of a larger unobservable subspace, as well as a part of a larger uncontrollable subspace. Indeed, it is often possible to prove that the unobservable subspace is strictly larger than the subspace mentioned in Theorem 5.

Definition 7. The space $\mathscr{U}$ is the maximal subspace of $H^2(\mu; Y)$ with the following property: If $\varphi \in \mathscr{U}$ and $\lambda > 0$, then

$$B(\lambda - z)^{-1}\varphi(z) = \int_{-\infty}^{\infty} \frac{\varphi(i\omega)}{\lambda - i\omega}\, \mu(d\omega) = 0.$$

The space $\mathscr{U}$ is called the *space of unobservable absorbed energy*, and its orthogonal complement $\mathscr{O} = \mathscr{U}^\perp$ in $H^2(\mu; Y)$ is called the *space of observable absorbed energy*. We denote the orthogonal projections of $H^2(\mu; Y)$ onto $\mathscr{U}$ and $\mathscr{O}$ by $P_{\mathscr{U}}$ and $P_{\mathscr{O}}$, respectively.

Equivalently, $\mathscr{U}$ is the intersection over all real $\lambda > 0$ of the null-space of the operator that maps φ into $B(\lambda - z)^{-1}\varphi(z)$. It is also the intersection over all complex λ with $\Re\lambda > 0$ of the null-space of the operator that maps φ into $B(\lambda - z)^{-1}\varphi(z)$ (an analytic function that vanishes on the real axis vanishes everywhere in the right half-plane). Moreover, $\mathscr{U}$ is invariant with respect to multiplication by functions that are analytic in Π^- and tend to a limit at infinity; to see this one argues as follows. First, let us show that $\mathscr{U}$ is invariant with respect to multiplication by the function $(\lambda_0 - z)^{-1}$, where λ_0 is an arbitrary constant with positive real part. To do this we have to show that $B[(\lambda - z)^{-1}(\lambda_0 - z)^{-1}\varphi(z)] = 0$ for every $\lambda > 0$ and for every $\varphi \in \mathscr{U}$. But if $\lambda \neq \lambda_0$, then this follows from the fact that

$$\frac{1}{(\lambda - z)(\lambda_0 - z)} = \frac{1}{\lambda_0 - \lambda}\left[\frac{1}{\lambda - z} - \frac{1}{\lambda_0 - z}\right],$$

and in the case where $\lambda = \lambda_0$ we may let $\lambda \to \lambda_0$ in the formula above to get $B(\lambda_0 - z)^{-2}\varphi(z) = 0$. This proves our claim that $\mathscr{U}$ is invariant with respect to multiplication by the function $(\lambda_0 - z)^{-1}$, where λ_0 is an arbitrary constant with positive real part. By iterating this argument we find that, if $\varphi \in \mathscr{U}$, then $\varphi(z)$ multiplied by arbitrary powers of $(\lambda_0 - z)^{-1}$ must belong to $\mathscr{U}$, hence φ multiplied by any function that can be uniformly approximated by polynomials in $(\lambda_0 - z)^{-1}$ must belong to $\mathscr{U}$. According to Mergelyan's theorem, the set of functions that that are analytic in Π^- and tend to a limit at infinity can be uniformly approximated by polynomials in $(\lambda_0 - z)^{-1}$ (in order to apply the version of Mergelyan's theorem that is given in [18] one should first map the point λ_0 into the point at infinity, and the left half-plane into a compact subset of **C**).

Theorem 8. *The subspace $\{0\} \times \mathscr{U}$ of $\mathscr{H}$ is an invariant subspace of $\mathscr{A}$. This subspace is unobservable, and for each $(0, \varphi)$ in this subspace we have $S(t)(0, \varphi(z)) = (0, e^{zt}\varphi(z))$. More precisely, the following claims are true. Split the space $\mathscr{H}$ into $\mathscr{H} = \mathscr{X} \times \mathscr{U}$, where $\mathscr{X} = X \times \mathscr{O}$, and accordingly, split the semigroup S into four components*

$$S(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix},$$

where (with some abuse of notation) $S_{11}(t) = P_{\mathscr{X}}S(t)P_{\mathscr{X}}$, $S_{12}(t) = P_{\mathscr{X}}S(t)P_{\mathscr{U}}$, $S_{21}(t) = P_{\mathscr{U}}S(t)P_{\mathscr{X}}$, and $S_{22}(t) = P_{\mathscr{U}}S(t)P_{\mathscr{U}}$. Then $S_{12}(t) \equiv 0$, $S_{11} = P_{\mathscr{X}}S(t)$, $S_{22} = S(t)P_{\mathscr{U}}$, and $S_{11}(t)$ and $S_{22}(t)$ are C_0 contraction semigroups in $\mathscr{X}$ and $\mathscr{U}$, respectively. The generators $\mathscr{A}_{11}$ and $\mathscr{A}_{22}$ of these semigroups can be written in the form (with some additional abuse of notation)

$$\mathscr{A}_{11} = P_{\mathscr{X}}\mathscr{A} \quad \text{and} \quad \mathscr{A}_{22} = \mathscr{A}P_{\mathscr{U}} = M.$$

Moreover,

$$\Re\langle(v, \varphi), \mathscr{A}_{11}(v, \varphi)\rangle \leq -\nu\|Dv\|_Y^2 \quad \text{and} \quad \Re\langle(v, \varphi), \mathscr{A}_{22}(v, \varphi)\rangle = 0.$$

Proof. In Lemma 4, split ψ into $\psi_1 + \psi_2$, with $\psi_1 = P_{\mathscr{O}}\psi$ and $\psi_2 = P_{\mathscr{U}}\psi$, and split φ into $\varphi_1 + \varphi_2$, with $\varphi_1 = P_{\mathscr{O}}\varphi$ and $\varphi_2 = P_{\mathscr{U}}\varphi$. Let $\tilde{\psi}_1(\lambda, z) = (\lambda - z)^{-1}\psi_1(z)$, $\tilde{\psi}_2(\lambda, z) = (\lambda - z)^{-1}\psi_2(z)$, $\tilde{w}_1(\lambda) = B\tilde{\psi}_1(\lambda, \cdot)$, and $\tilde{w}_2(\lambda) =$

$B\tilde{\psi}_2(\lambda, \cdot)$. Then $\tilde{\psi}_2(\lambda, \cdot) \in \mathscr{U}$ and $\tilde{w}_2(\lambda) = B\tilde{\psi}_2(\lambda, \cdot) = 0$; hence

$$\begin{aligned} v &= V(\lambda)w - D^* W(\lambda)\tilde{w}_1(\lambda), \\ \varphi_1(z) &= P_{\mathscr{O}}\left[(\lambda - z)^{-1} Dv + \tilde{\psi}_1(\lambda, z)\right], \\ \varphi_2(z) &= P_{\mathscr{U}}\left[(\lambda - z)^{-1} Dv + \tilde{\psi}_1(\lambda, z)\right] + \tilde{\psi}_2(\lambda, z). \end{aligned} \tag{18}$$

Here the crucial fact is that v and φ_1 are independent of ψ_2. Since we can generate the semigroup $S(t)$ from the resolvent operators $[\lambda - \mathscr{A}]^{-1}$ through the formula

$$S(t) = \lim_{n\to\infty} [I - (t/n)\mathscr{A}]^{-n}; \tag{19}$$

see [13], the semigroup has the block-triangular structure that we claimed it to have; i.e., $P_{\mathscr{X}} S(t) P_{\mathscr{U}} \equiv 0$. Obviously S_{11} and S_{22} are strongly continuous, and it is easy to check that they satisfy the semigroup identity. Thus, they are C_0 semigroups in the state spaces $\mathscr{X}$ and $\mathscr{U}$, respectively.

Next we prove that S_{11} and S_{22} are contraction semigroups. To see this we return to the energy identity that motivated the norm on φ in the first place. This energy identity can be deduced directly from the abstract equation as follows: Denote $P_{\mathscr{O}}\varphi$ by φ_1 and $P_{\mathscr{U}}\varphi$ by φ_2. Take the real part of the inner product of (v, φ) and the equation $(v, \varphi)' = \mathscr{A}(v, \varphi)$, and then use Lemma 3 to get

(20)
$$\frac{d}{dt}\frac{1}{2}\|v\|_X^2 + \frac{d}{dt}\frac{1}{2}\|\varphi_1\|_{\mathscr{O}}^2 + \frac{d}{dt}\frac{1}{2}\|\varphi_2\|_{\mathscr{U}}^2 = \Re\langle (v, \varphi), \mathscr{A}(v, \varphi)\rangle = -\nu\|Dv\|_Y^2 \le 0.$$

Integration over $[0, T]$ gives

$$\begin{aligned} &\frac{1}{2}\|v(T)\|_X^2 + \frac{1}{2}\|\varphi_1(T)\|_{\mathscr{O}}^2 + \frac{1}{2}\|\varphi_2(T)\|_{\mathscr{U}}^2 \\ &= \frac{1}{2}\|v(0)\|_X^2 + \frac{1}{2}\|\varphi_1(0)\|_{\mathscr{O}}^2 + \frac{1}{2}\|\varphi_2(0)\|_{\mathscr{U}}^2 - \int_0^T \nu\|Dv(t)\|_Y^2 dt. \end{aligned} \tag{21}$$

This is true for all choices of $(v, \varphi_1, \varphi_2) \in \operatorname{dom}\mathscr{A}$. In particular, if we take $\varphi_2(0) = 0$, then we get

$$\frac{1}{2}\|v(T)\|_X^2 + \frac{1}{2}\|\varphi_1(T)\|_{\mathscr{O}}^2 \le \frac{1}{2}\|v(0)\|_X^2 + \frac{1}{2}\|\varphi_1(0)\|_{\mathscr{O}}^2 - \int_0^T \nu\|Dv(t)\|_Y^2 dt.$$

Divide this by T and let $T \downarrow 0$, to conclude that the generator $\mathscr{A}_{11}$ of S_{11} satisfies

$$\Re\langle (v, \varphi_1), \mathscr{A}_{11}(v, \varphi_1)\rangle \le -\nu\|Dv\|_Y^2.$$

If we instead take $v(0) = 0$ and $\varphi_1(0) = 0$ in (21), then by the block-triangular nature of the semigroup, $v(T) = 0$ and $\varphi_1(T) = 0$. Thus, in this case,

$$\frac{1}{2}\|\varphi_2(T)\|_{\mathscr{U}}^2 = \frac{1}{2}\|\varphi_2(0)\|_{\mathscr{U}}^2.$$

This implies that $\|S_{22}(t)x\|_{\mathscr{U}} = \|x\|_{\mathscr{U}}$ for all $x \in \mathscr{U}$ and all $t \ge 0$, and that the generator $\mathscr{A}_{22}$ of S_{22} satisfies $\Re\langle \varphi_2, \mathscr{A}_{22}\varphi_2\rangle = 0$.

That $\mathscr{A}_{11} = P_{\mathscr{X}}\mathscr{A}$ and $\mathscr{A}_{22} = \mathscr{A} P_{\mathscr{U}}$ follows directly from the corresponding formulas for the semigroups. To see that we may identify $\mathscr{A}_{22}$ with M we may take $v = 0$ and $\varphi_1 = 0$ in (18) to get $\varphi_2(z) = (\lambda - z)^{-1}\psi_2(z)$. From the generation formula (19) one gets $S_{22}(t)\varphi_2(z) = e^{zt}\varphi_2(z)$. The generator of this semigroup is the multiplication operator M. □

It is possible to give a quite natural and very intuitive time domain interpretation of Theorem 8. Let us begin by giving a time domain interpretation of $\mathscr{U}$. Below we argue formally, and freely change the order of integrals, without worrying about their convergence. Define $\check{\varphi}(s) = Dv(s)$ for $s \le 0$, and let

$$b(t) = \int_{-\infty}^{0} a(t-s)\check{\varphi}(s)\,ds, \qquad t \ge 0.$$

Then, at least formally,

$$b(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{0} e^{i\omega(t-s)}\check{\varphi}(s)\,ds\,\mu(d\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\varphi(i\omega)\,\mu(d\omega),$$

where

$$\varphi(z) = \int_{-\infty}^{0} e^{-zs}\check{\varphi}(s)\,ds$$

is the Laplace transform of $\check{\varphi}$. Let us compute the Laplace transform $\hat{b}(\lambda)$ of b at some point λ with $\Re\lambda > 0$. It is (at least formally) given by

$$\begin{aligned}\hat{b}(\lambda) &= \int_{0}^{\infty} e^{-\lambda t}b(t)\,dt = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-(\lambda-i\omega)t}dt\varphi(i\omega)\,\mu(d\omega)\\ &= \int_{-\infty}^{\infty}\frac{1}{\lambda - i\omega}\,\varphi(i\omega)\,\mu(d\omega).\end{aligned}$$

This transform vanishes for all λ with $\Re\lambda > 0$ iff $\varphi \in \mathscr{U}$. On the other hand, the transform vanishes iff b itself vanishes identically. Thus, formally we may interpret $\mathscr{U}$ as the space of Laplace transforms of functions in the kernel of the Hankel operator that maps a function $\check{\varphi}$ defined on $\mathbf{R}^-$ into the function $t \mapsto \int_{-\infty}^{0} a(t-s)\check{\varphi}(s)\,ds$ defined on $\mathbf{R}^+$. In many cases this formal interpretation can be made precise. With this interpretation, the splitting of $H^2(\mu; Y)$ into $\mathscr{O} \oplus \mathscr{U}$ corresponds to the following construction: The initial function $Dv(s)$ is split into two orthogonal parts $Dv = \check{\varphi}_1 + \check{\varphi}_2$, where $\check{\varphi}_2$ satisfies $\int_{-\infty}^{0} a(t-s)\check{\varphi}_2(s) \equiv 0$ for $t \ge 0$, hence it contributes nothing to the stress for $t \ge 0$. In the time-domain the semigroup S_{22} corresponds to plain left-translation of $\check{\varphi}$, with no further action. Thus, the part of $Dv(s)$ that belongs to the null-space of the Hankel operator is simply left-translated, but otherwise ignored.

By the standard properties of Hankel operators, the argument above shows that if Y would be finite-dimensional, and if a is a finite sum of exponentials, then one would expect $\mathscr{O}$ to be finite-dimensional and $\mathscr{U}$ infinite-dimensional. More generally, for infinite-dimensional Y, if a is a finite sum of exponentials, then $\mathscr{O}$ should be a product of finitely many copies of Y, and $\mathscr{U}$ should be the main part of $H^2(\omega; Y)$. It also shows that $\mathscr{U}$ is very large when the equation has a finite total delay. Clearly, if a vanishes on $(-\infty, T)$ and $\check{\varphi}$ is supported on $(-\infty, T]$ for some $T < 0$, then the convolution $\int_{-\infty}^{0} a(t-s)\check{\varphi}(s)$ vanishes on $\mathbf{R}^+$; hence the Laplace transform of $\check{\varphi}$ belongs to the unobservable subspace $\mathscr{U}$. This indicates that the inverse Laplace transforms of functions in $\mathscr{O}$ should be supported on $[T, 0]$.

There is one fundamental consequence of Theorem 8.

Theorem 9. *The total amount of unobservable absorbed energy in* (17), *i.e., the norm of the projection of the solution of* (17) *onto* $\mathscr{U}$, *is nondecreasing, independently of how the forcing function* f *in* (17) *is chosen.*

Proof. Let us redo the computation (20), with the equation $(v, \varphi)' = \mathscr{A}(v, \varphi)$ replaced by the forced equation

$$\begin{pmatrix} v \\ \varphi \end{pmatrix}' = \mathscr{A}\begin{pmatrix} v \\ \varphi \end{pmatrix} + \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

The result is the same, except for an additional term $\Re\langle v, f\rangle$. The same computation with $\mathscr{A}$ replaced by $\mathscr{A}_{11}$ gives

$$\frac{d}{dt}\frac{1}{2}\|v\|_X^2 + \frac{d}{dt}\frac{1}{2}\|\varphi_1\|_{\mathscr{O}}^2 \le -\nu\|Dv\|_Y^2 + \Re\langle v, f\rangle.$$

Subtracting this from the forced version of (20) we find that

$$\frac{d}{dt}\frac{1}{2}\|\varphi_2\|_{\mathscr{U}}^2 \ge 0. \quad \square$$

Theorem 9 has one profound consequence. Our ultimate goal is to stabilize the equation exponentially by replacing f by a feedback term, depending on v. According to Theorem 9, independently of the choice of feedback, the norm of the projection of φ onto $\mathscr{U}$ can never decrease. Thus, if $\mathscr{U} \neq \{0\}$, then exponential stabilization is impossible for the full semigroup. This shows that the smaller state space $X \times \mathscr{O}$ is a more natural one for this problem, and that the only part of the semigroup T that is "interesting" is the part S_{11}.

To make the results above more concrete for the reader, let us end this section with three examples, where $a(t) = \alpha\cos\beta t$, $a(t) = \alpha e^{-\beta t}$, and $a(t) = E + \alpha e^{-\beta t}$, respectively. In all these examples the final results will be quite similar, and very natural, although the intermediate computations will differ substantially from each other.

First, let us take $a(t) = \alpha\cos\beta t$ with $\alpha > 0$ and $\beta > 0$. Then μ consists of two point masses of size $\alpha/2$, located at $\pm\beta$. (In the complex plane, the values $\pm\beta$ corresponds to the points $z = \pm i\beta$.) Thus, $L^2(\mu; Y)$ can be identified with $Y \times Y$. We shall denote an element of $L^2(\mu; Y)$ by $(y_{i\beta}, y_{-i\beta})$, where the former value represents the value at $i\beta$, and the latter the value at $-i\beta$. The square of the norm of this element in $L^2(\mu; Y)$ is

$$(\alpha/2)(\|y_{i\beta}\|_Y^2 + \|y_{-i\beta}\|_Y^2).$$

Clearly, in this case $H^2(\mu; Y) = L^2(\mu; Y)$, and

$$B(y_{i\beta}, y_{-i\beta}) = \alpha/2(y_{i\beta} + y_{-i\beta}).$$

A point $(y_{i\beta}, y_{-i\beta})$ belongs to $\mathscr{U}$ iff

$$(\alpha/2)((\lambda - i\beta)^{-1} y_{i\beta} + (\lambda + i\beta) y_{-i\beta}) = 0$$

for all $\lambda > 0$. By using two different values for λ and solving this equation with respect to $y_{i\beta}$ and $y_{-i\beta}$ one gets $y_{i\beta} = y_{-i\beta} = 0$. Thus, $\mathscr{U} = \{0\}$. The equation (17) becomes

$$\begin{aligned} v' &= -D^* [\nu D v + (\alpha/2)(y_{i\beta} + y_{-i\beta})] + f, \\ y'_{i\beta} &= Dv + i\beta y_{i\beta}, \quad y'_{-i\beta} = Dv - i\beta y_{i\beta}. \end{aligned}$$

Observe that this is the same equation that one would have obtained from (1) by introducing the new variables

$$y_{i\beta}(t) = \int_{-\infty}^{t} e^{i\beta(t-s)} Dv(s)\, ds, \qquad y_{-i\beta}(t) = \int_{-\infty}^{t} e^{-i\beta(t-s)} Dv(s)\, ds.$$

As our second example we take $a(t) = \alpha e^{-\beta t}$ with $\alpha > 0$ and $\beta > 0$. Then

$$\mu(d\omega) = \frac{\alpha\beta}{\pi(\omega^2 + \beta^2)}\, d\omega,$$

where the final $d\omega$ represents the standard Lebesgue measure. Since $|\beta - i\omega|^2 = \omega^2 + \beta^2$, we can make the identifications

$$\begin{aligned} L^2(\mu; Y) &= \{ (z - \beta)\varphi(z) \mid \varphi \in L^2(\mathbf{1}; Y) \}, \\ H^2(\mu; Y) &= \{ (z - \beta)\varphi(z) \mid \varphi \in H^2(\mathbf{1}; Y) \}, \end{aligned}$$

where $L^2(\mathbf{1}; Y)$ and $H^2(\mathbf{1}; Y)$ are the space that one gets by replacing μ by the Lebesgue measure. The inner product is given by

$$\langle \varphi, \psi \rangle_{L^2(\mu; Y)} = \frac{\alpha\beta}{\pi} \langle (z - \beta)^{-1}\varphi(z), (z - \beta)^{-1}\psi(z) \rangle_{L^2(\mathbf{1}; Y)}.$$

Since μ is a scalar multiple of the Poisson kernel for the half-plane, we have

$$B(\lambda - z)^{-1}\varphi(z) = \int_{-\infty}^{\infty} \frac{\varphi(i\omega)}{\lambda - i\omega} \frac{\alpha\beta}{\pi(\omega^2 + \beta^2)}\, d\omega = \frac{\alpha\varphi(-\beta)}{\lambda + \beta},$$

where we in the final expression have evaluated φ at a point located at the negative real axis, instead of (as we have done up to now) evaluating φ only at points located at the imaginary axis; since we assume that $\varphi \in H^2(\mu; Y)$, this is legitimate. It follows from this formula that

$$\mathscr{U} = \{ \varphi \in H^2(\mu; Y) \mid \varphi(-\beta) = 0 \}.$$

If $\varphi \in \mathscr{U}$, and if $\psi(z) = \psi$ is a constant function, then

$$\begin{aligned} \langle \psi, \varphi \rangle_{H^2(\mu; Y)} &= \int_{-\infty}^{\infty} \langle \psi, \varphi(i\omega) \rangle \frac{\alpha\beta}{\pi(\omega^2 + \beta^2)}\, d\omega \\ &= \left\langle \psi, \int_{-\infty}^{\infty} \varphi(i\omega) \frac{\alpha\beta}{\pi(\omega^2 + \beta^2)}\, d\omega \right\rangle \\ &= \alpha \langle \psi, \varphi(-\beta) \rangle = 0. \end{aligned}$$

Since the set of constant functions together with $\mathcal{U}$ spans $H^2(\mu; Y)$, this shows that $\mathcal{O} = \mathcal{U}^\perp$ is the set of all constant functions. The projection operators $P_\mathcal{O}$ and $P_\mathcal{U}$ are given by

$$(P_\mathcal{O}\varphi)(z) = \varphi(-\beta), \qquad (P_\mathcal{U}\varphi)(z) = \varphi(z) - \varphi(-\beta).$$

If we use the notation $y_{-\beta}$ for an element of $\mathcal{O}$, and let φ represent a function in $\mathcal{U}$, then (17) becomes

$$\begin{aligned} v' &= -D^*\left[\nu Dv + \alpha y_{-\beta}\right] + f, \\ y'_{-\beta} &= Dv - \beta y_{-\beta}, \quad \varphi'(z) = \beta y_{-\beta} + z\varphi(z). \end{aligned}$$

If we ignore the last equation, then this is the same equation that one would get from (1) by introducing the new variable

$$y_{-\beta}(t) = \int_{-\infty}^{t} e^{-\beta(t-s)} Dv(s)\, ds.$$

In our third example we take $a(t) = \alpha e^{-\beta t} + E$, where $E > 0$ is a constant. This example is important for the reason that whenever the material that we are describing is a solid as opposed to a liquid, we must have $a(\infty) > 0$. (The value $a(\infty)$ is called the equilibrium elasticity modulus.) The measure μ is the same as in the previous example, plus an additional point mass of size E at the origin. We can this time make the identifications

$$\begin{aligned} L^2(\mu; Y) &= Y \times \{ (z-\beta)\varphi(z) \mid \varphi \in L^2(\mathbf{1}; Y) \}, \\ H^2(\mu; Y) &= Y \times \{ (z-\beta)\varphi(z) \mid \varphi \in H^2(\mathbf{1}; Y) \}, \end{aligned}$$

where the first component represents the value of φ at the origin. The inner product is given by

$$\begin{aligned} &\langle (x, \varphi), (y, \psi) \rangle_{L^2(\mu; Y)} \\ &\quad = E\langle x, y \rangle_Y + \frac{\alpha\beta}{\pi} \langle (z-\beta)^{-1}\varphi(z), (z-\beta)^{-1}\psi(z) \rangle_{L^2(\mathbf{1}; Y)}, \end{aligned}$$

and

$$B(\lambda - z)^{-1}(y, \varphi(z)) = \frac{Ey}{\lambda} + \frac{\alpha\varphi(-\beta)}{\lambda + \beta}.$$

Thus

$$\mathcal{U} = \{ (y, \varphi) \in H^2(\mu; Y) \mid y = 0 \text{ and } \varphi(-\beta) = 0 \}.$$

The orthogonal complement $\mathcal{O}$ of this set can be identified with $Y \times Y$, where the first copy of Y, that we denote by Y_0, corresponds to functions of the type $(y_0, 0)$, and the second copy of Y, that we denote by $Y_{-\beta}$, corresponds to functions of the type $(0, y_{-\beta})$, where $y_{-\beta}$ represents a constant function. We have

$$\begin{aligned} P_\mathcal{O} = P_{Y_0} + P_{Y_{-\beta}}, \quad P_{Y_0}(y, \varphi) = y, \quad P_{Y_{-\beta}}(y, \varphi) = \varphi(-\beta), \\ P_\mathcal{U}(y, \varphi) = (0, \varphi(z) - \varphi(-\beta)), \end{aligned}$$

and (17) becomes

$$\begin{aligned} v' &= -D^*\left[\nu Dv + Ey_0 + \alpha y_{-\beta}\right] + f, \quad y'_0 = Dv, \\ y'_{-\beta} &= Dv - \beta y_{-\beta}, \quad \varphi'(z) = \beta y_{-\beta} + z\varphi(z). \end{aligned}$$

If we ignore the last equation, then this is once more the same equation that one would get from (1) by introducing the two new variables

$$y_0(t) = \int_{-\infty}^{t} Dv(s)\,ds, \qquad y_{-\beta}(t) = \int_{-\infty}^{t} e^{-\beta(t-s)} Dv(s)\,ds.$$

5. The semigroup generated by a completely monotone kernel in internal energy space

The treatments in [1], [2], [4], [10], [11], and [12] are based on the fact that in these papers, the relaxation modulus a was taken to be completely monotone. In the case of a scalar kernel, the one that we treat here, this means that by Bernstein's theorem [6, Theorem 2.5, p. 143], a has a representation

$$a(t) = \int_{\mathbf{R}^-} e^{zt}\,\mu(dz), \quad t > 0, \tag{22}$$

and $\hat{a}(\lambda)$ is given by

$$\hat{a}(\lambda) = \int_{\mathbf{R}^-} \frac{\mu(dz)}{\lambda - z}, \quad \Re\lambda > 0. \tag{23}$$

This time $\int_{\mathbf{R}^-}(1+|z|)^{-1}\,\mu(dz) = \hat{a}(1) < 0$, so the convergence problems that forced us to symmetrize some integrals (see (5) and (7)) do not appear.

As we mentioned earlier, the original semigroup setting for our equation given in [4] is very similar to the one above. The only substantial difference is the use of a different state space, which in [4] (after a simple modification; cf. the discussion in Section 2) is $X \times L^2(\mu; Y)$; here μ refers to the measure in (22) supported on $\mathbf{R}^-$, as opposed to the measure that we used earlier, and that is supported on the imaginary axis.

In [4], Desch and Miller prove a slightly modified version of the following result.

Theorem 10 [4, Theorem 2.3]. *Let a be completely monotone with the representation* (22). *Define the operators B, M, and Λ by*

$$B\varphi = \int_{\mathbf{R}^-} \varphi(z)\,\mu(dz), \qquad (M\varphi)(z) = z\varphi(z), \qquad (\Lambda v)(z) = v.$$

Let $\mathscr{H} = X \times L^2(\mu; Y)$, and let $\mathscr{A}$ be the operator

$$\mathscr{A} = \begin{pmatrix} -\nu D^*D & -D^*B \\ \Lambda D & M \end{pmatrix}$$

mapping $\operatorname{dom}\mathscr{A} \subset \mathscr{H}$ *into* $\mathscr{H}$, *with*

$$\operatorname{dom}\mathscr{A} = \{(v, \varphi) \in \mathscr{H} \mid v \in \operatorname{dom} D,\ \nu Dv + B\varphi \in \operatorname{dom} D^*,\ \Lambda Dv + M\varphi \in L^2(\mu; Y)\}.$$

Then $\mathscr{A}$ is the infinitesimal generator of a C_0 contraction semigroup S in $\mathscr{H}$. Moreover, for all $(v, \varphi) \in \operatorname{dom}\mathscr{A}$,

$$\Re\langle (v, \varphi), \mathscr{A}(v, \varphi)\rangle = -\nu\|Dv\|_Y^2 + \int_{\mathbf{R}^-} z\|\varphi(z)\|_Y^2\,\mu(dz) \le 0.$$

The proof of this theorem is essentially the same as the proof of Theorem 1. The only difference is that formula (13) that we used to motivate our choice of state space changes, as does the coercivity formula in Lemma 3. The new appropriate version of (13) is (this is obvious from the previous computation leading to (13))

$$\begin{aligned}\int_0^T \Re\left\langle v(t), D^* \int_{-\infty}^t a(t-s)Dv(s)\,ds\right\rangle_X dt \\ = \frac{1}{2}\int_{\mathbf{R}^-} \|\varphi(T, z)\|_Y^2\,\mu(dz) - \frac{1}{2}\int_{\mathbf{R}^-} \|\varphi(0, z)\|_Y^2\,\mu(dz) \\ - \int_0^T \int_{\mathbf{R}^-} z\|\varphi(t, z)\|_Y^2\,\mu(dz).\end{aligned} \tag{24}$$

It makes sense to interpret the norm of $\varphi(t, z)$ in $L^2(\mu; Y)$ as internal energy, and we shall call $L^2(\mu; Y)$ the *internal energy space*. Clearly, the internal energy is not conserved; it dissipates at a rate of $\int_{\mathbf{R}^-} z\|\varphi(t, z)\|_Y^2\,\mu(dz)$.

In the case of a completely monotone relaxation modulus we thus have two different semigroups, the one in Theorem 1 in the absorbed energy space, and the one in Theorem 10 in the internal energy space. How do these relate to each other? The answer to this question is simple enough.

Theorem 11. *The space $\mathcal{O}$ of observable absorbed energy is densly imbedded in the space $L^2(\mu; Y)$ of internal energy, and the semigroup S_{11} described in Theorem 8, defined on $X \times \mathcal{O}$, coincides with the semigroup in Theorem 10 when restricted to $X \times \mathcal{O}$.*

In the examples that we looked at in the preceding section, the norms in $\mathcal{O}$ and $L^2(\mu; Y)$ are the same. However, this will not be true in general.

Proof. In this proof, to avoid the easy case where equation (1) becomes identical to the standard wave equation and the two semigroups coincide, let us assume that a is not a constant. We also assume that $E = a(\infty) \neq 0$; this is not essential, but we leave the case $E = 0$ to the reader (it is actually slightly simpler since some of the terms drop out).

By, e.g., [6, Proposition 4.3], a is of strong positive type, i.e., its Laplace transform $\hat{a}$ satisfies

$$\Re\hat{a}(i\omega) \geq \frac{\epsilon}{1+\omega^2}, \tag{25}$$

for some $\epsilon > 0$. The original representation formula (2) for general kernels of positive type is still valid in the same sense as before, for some appropriate measure $\tilde{\mu}$ (different from the measure μ in (22)). This measure consists of a point mass of size E at the origin, plus an absolutely continuous part $(1/\pi)\Re\hat{a}(i\omega)\,d\omega$. Thus, (2) becomes

$$a(t) = E + \int_{-\infty}^{\infty} e^{i\omega t} b(\omega)\,d\omega,$$

where

$$b(\omega) = \frac{1}{\pi}\Re\hat{a}(i\omega) = \frac{1}{\pi}\int_{\mathbf{R}^-} \frac{-x}{x^2+\omega^2}\ \mu(dx).$$

Because of the lower bound (25) that we have on b, and because of the point mass at zero, we can make the identifications

$$L^2(\tilde{\mu}; Y) = Y \times L^2(b; Y), \qquad H^2(\tilde{\mu}; Y) = Y \times H^2(b; Y),$$

where the first components of the spaces represent the value of a function at zero, and

$$L^2(b; Y) \subset \{(z-\beta)\varphi(z) \mid \varphi \in L^2(\mathbf{1}; Y)\},$$
$$H^2(b; Y) \subset \{(z-\beta)\varphi(z) \mid \varphi \in H^2(\mathbf{1}; Y)\}.$$

In particular, functions in $H^2(\tilde{\mu}; Y)$ have analytic extensions to the interior of Π^-, and they are well defined on the negative real axis. Thus, we may compute their norm in $L^2(\mu; Y)$. We have

$$\begin{aligned}
\|\varphi\|^2_{H^2(\tilde{\mu};Y)} &= E\|\varphi(0)\|^2_Y + \int_{-\infty}^{\infty} \|\varphi(i\omega)\|^2_Y b(\omega)\,d\omega \\
&= E\|\varphi(0)\|^2_Y + \int_{-\infty}^{\infty} \|\varphi(i\omega)\|^2_Y \frac{1}{\pi}\int_{(-\infty,0)} \frac{-x}{x^2+\omega^2}\ \mu(dx)d\omega \\
&= E\|\varphi(0)\|^2_Y + \int_{(-\infty,0)}\int_{-\infty}^{\infty} \|\varphi(i\omega)\|^2_Y \frac{1}{\pi}\frac{-x}{x^2+\omega^2}\ d\omega\,\mu(dx) \\
&\quad \text{(a subharmonic function is integrated against the Poisson kernel)} \\
&\geq E\|\varphi(0)\|^2_Y + \int_{(-\infty,0)} \|\varphi(x)\|^2_Y\,\mu(dx) \\
&= \int_{\mathbf{R}^-} \|\varphi(x)\|^2_Y\,\mu(dx) = \|\varphi\|^2_{L^2(\mu;Y)}.
\end{aligned}$$

Thus, the norm in $L^2(\mu; Y)$ is dominated by the norm in $H^2(\tilde{\mu}; Y)$, and we may "imbed" $H^2(\tilde{\mu}; Y)$ continuously into $L^2(\mu; Y)$. Here the citation marks mean that this imbedding need not be injective: those functions in $H^2(\tilde{\mu}; Y)$ that vanish a.e. with respect to μ on the negative real axis are mapped into zero. Are there such nonzero functions? This depends on the support of μ; more precisely, it depends on whether the condition that $\varphi = 0$ a.e. with respect to μ implies that $\varphi = 0$. If the support of μ has a finite cluster point, then this is true. It is not true if the support of μ is finite. In the remaining case the support of μ is a sequence $\{z_n\}$ with $z_n \to -\infty$ as $n \to \infty$, and there do exist nonzero analytic functions vanishing at each point z_n if the sum $\sum 1/(1-z_n)$ converges, cf. [18, pp. 333–334].

We claim that the null-space of the "imbedding mapping" is equal to the space $\mathscr{U}$ of unobservable absorbed energy. To prove this we have to compute

$B(\lambda - z)^{-1}\varphi(z)$, which is done as follows (recall that $(\lambda - z)^{-1}\varphi(z) \in H^1(\mu; Y)$):

$$\begin{aligned} B(\lambda - z)^{-1}\varphi(z) &= E\varphi(0)/\lambda + \int_{-\infty}^{\infty} \frac{\varphi(i\omega)}{\lambda - i\omega} \frac{1}{\pi} \int_{(-\infty,0)} \frac{-x}{x^2+\omega^2}\, \mu(dx) d\omega \\ &= E\varphi(0)/\lambda + \int_{(-\infty,0)} \int_{-\infty}^{\infty} \frac{\varphi(i\omega)}{\lambda - i\omega} \frac{1}{\pi} \frac{-x}{x^2+\omega^2}\, d\omega\, \mu(dx) \\ &\qquad \text{(use the Poisson formula for the half-plane)} \\ &= E\varphi(0)/\lambda + \int_{(-\infty,0)} \frac{\varphi(x)}{\lambda - x}\, \mu(dx) = \int_{\mathbf{R}^-} \frac{\varphi(x)}{\lambda - x}\, \mu(dx). \end{aligned}$$

This shows that the two different definitions of B, the original one in (16), and the second one in Theorem 10 agree. Moreover, it is clear that the space of functions that vanish a.e. with respect to μ must be contained in $\mathscr{U}$. To see that the converse is true as well, note that if $\varphi \in \mathscr{U}$, then φ is orthogonal in $L^2(\mu; Y)$ to all functions of the type $y/(\lambda - z)$, where $y \in Y$ is a constant and $\lambda > 0$, and this set of functions generate a dense subset of $L^2(\mu; Y)$.

Thus, the kernel of the operator "imbedding" $H^2(\tilde{\mu}; Y)$ into $L^2(\mu; Y)$ is $\mathscr{U}$, and we conclude that $\mathscr{O}$ is continuously imbedded (with no citation marks) in $L^2(\mu; Y)$. This imbedding is dense, since functions of the type $y/(\lambda - z)$, where $y \in Y$ is a constant and $\lambda > 0$, belong to $H^2(\tilde{\mu}; Y)$, and they span a dense subset of $L^2(\mu; Y)$.

Our final claim was that the two semigroups, the one in Theorem 1, and the one in Theorem 10, agree on $X \times \mathscr{O}$. But this is clear, since the two different operators B that we defined agree; hence the generators of the two semigroups agree. □

6. A NEW CLASS OF RELAXATION MODULI GENERATING WELL-POSED PROBLEMS

It is a quite common assumption that the relaxation modulus of a material is completely monotone. A common textbook approach is to assume that the material consists of several different types of molecules, whose resistance to deformations contribute to the stress. Each part of the system is modelled with a combination of one ideal spring and one ideal dashpot, and the stresses due to the different parts are supposed to add up to the total stress. A model of this type lead to a relaxation modulus a of the form

$$a(t) = \sum_{j=1}^{N} \alpha_j e^{-\beta_j t},$$

where α_j and β_j are positive constants (one of the constants β_j is allowed to be zero; this gives rise to the equilibrium elastic response). If one allows infinitey many such subsystems, then the sum should be replaced by an integral of the type (22), i.e., by a general completely monotone kernel.

In the preceding argument we have ignored the fact that all molecules have a nonzero mass. If one takes inertia effects into account, then one has to add a nonzero mass to each subsystem. Then the response of the subsystem will look like the response of a damped second order oscillator, and it will not necessarily be of the type $\alpha_j e^{-\beta_j t}$, unless the system is overdamped. An underdamped system will give solutions of the form $\Re\alpha_j e^{\beta_j t}$, where α_j and β_j are complex, with $\Re\beta_j < 0$. Combining several such subsystems we get the total response

$$a(t) = \sum_{j=1}^{N} \Re\alpha_j e^{\beta_j t} = \sum_{j=1}^{N} \frac{1}{2}\left(\alpha_j e^{\beta_j t} + \overline{\alpha}_j e^{\overline{\beta}_j t}\right),$$

with complex α_j and β_j.

There are some restrictions on α_j and β_j related to the fact that the sum must be of positive type. The simplest way to guarantee this is to require that each α_j is real and nonnegative, and that each β_j has a nonpositive real part. We shall throughout make this assumption in the sequel, since it leads to a meaningful mathematical theory; kernels where α_j are complex seem to be more difficult to deal with. If we again allow the system to have infinitely many subsystems, then we get a relaxation modulus of the type

$$a(t) = \int_\Gamma e^{zt}\,\mu(dz), \tag{26}$$

where $\Gamma \subset \Pi^-$ is the support of μ. Note that this time the positive measure μ is supported on a two-dimensional set Γ instead of on the one-dimensional sets in (2) and (22). Like the measure in (2) it must be symmetric with respect to the real axis, i.e., $\mu(\overline{E}) = \mu(E)$ for each Borel set E. This is the class of kernels that we shall discuss in the sequel. Usually we shall add some restrictions on the support Γ. For example, for exponential stabilizability we will have to require that Γ is ultimately bounded away from the imaginary axis as the imaginary part tends to infinity.

Formally, $\hat{a}$ should be given by

$$\hat{a}(\lambda) = \int_\Gamma \frac{\mu(dz)}{\lambda - z}, \quad \Re\lambda > 0. \tag{27}$$

Unfortunately, this integral need not converge absolutely. We are saved by the fact that there is a natural growth restriction on the measure μ in (26), just like in (2) and (22). To show this we symmetrize the formula above to get

(28)

$$\hat{a}(\lambda) = \frac{1}{2}\int_\Gamma \left(\frac{1}{\lambda - z} + \frac{1}{\lambda - \overline{z}}\right)\mu(dz) = \int_\Gamma \frac{\lambda - \Re z}{(\lambda - z)(\lambda - \overline{z})}\,\mu(dz), \quad \Re\lambda > 0.$$

Note that for λ real and positive the integrand is positive; hence there can be no cancellation of positive and negative values, and for these values of λ the integral must converge absolutely. Thus (take $\lambda = 1$),

$$\int_\Gamma \frac{1 + |\Re z|}{|1 + z|^2}\,\mu(dz) < \infty. \tag{29}$$

But this means that the integral in (28) in fact converges absolutely for all λ with $\Re\lambda > 0$.

We shall later need to split (28) into its real and imaginary parts, given by

$$\text{(30)}\quad \Re\hat{a}(\lambda) = \int_\Gamma \frac{\Re(\lambda - z)\big((\Re(\lambda - z))^2 + (\Im\lambda)^2 + (\Im z)^2\big)}{|\lambda - z|^2|\lambda - \overline{z}|^2}\,\mu(dz), \quad \Re\lambda > 0,$$

and

$$\Im\hat{a}(\lambda) = \int_\Gamma \frac{-\Im\lambda((\Re(\lambda - z))^2 + (\Im\lambda)^2 - (\Im z)^2)}{|\lambda - z|^2|\lambda - \overline{z}|^2}\, \mu(dz), \quad \Re\lambda > 0. \tag{31}$$

The former of these two formulas shows that a is of positive type. One may also split (27) into its real and imaginary parts, instead of splitting (28). This leads to the two alternative formulas

$$\Re\hat{a}(\lambda) = \int_\Gamma \frac{\Re(\lambda - z)}{|\lambda - z|^2}\, \mu(dz) \geq 0, \quad \Re\lambda > 0, \tag{32}$$

and

$$\Im\hat{a}(\lambda) = \int_\Gamma \frac{-\Im(\lambda - z)}{|\lambda - z|^2}\, \mu(dz) \geq 0, \quad \Re\lambda > 0. \tag{33}$$

Note that the former of these two formulas converges absolutely, so it is always valid. The latter formula, like (27), is valid only when

$$\int_\Gamma \frac{\mu(dz)}{1 + |z|} < \infty. \tag{34}$$

Indeed, note that this stronger boundedness requirement follows from (29) whenever Γ is contained in a sector $|\arg(-z)| \leq \theta$, where $\theta < \pi/2$.

In the sequel we shall throughout assume that $\hat{a}$ has a representation of the type described above, i.e., we make the following hypothesis.

Hypothesis 12. The Laplace transform $\hat{a}$ of a is given by

$$\hat{a}(\lambda) = \frac{1}{2}\int_\Gamma \left(\frac{1}{\lambda - z} + \frac{1}{\lambda - \overline{z}}\right) \mu(dz), \quad \Re\lambda > 0,$$

where μ is a positive measure supported on a subset Γ of the closed left half-plane Π^-, satisfying the moment condition

$$\int_\Gamma \frac{1 + |\Re z|}{|1 + z|^2}\, \mu(dz) < \infty.$$

Whenever necessary due to lack of absolute convergence, we interpret the integral $\int_\Gamma \varphi(z)\, \mu(dz)$ as

$$\int_\Gamma \varphi(z)\, \mu(dz) = \frac{1}{2}\int_\Gamma (\varphi(z) + \varphi(\overline{z}))\, \mu(dz).$$

We have already constructed two different semigroups corresponding to the two different representations that we had for the kernel: a general one for kernels of positive type in Section 2, and a more particular one for completely monotone kernels in Section 5. It should come as no surprise to the reader that the same construction can be carried out for arbitrary kernels of the general type (26), i.e., for arbitrary measures μ.

Theorem 13. *Let Hypothesis* 12 *hold. Define the operators* B, M, *and* Λ *by*

$$B\varphi = \int_\Gamma \varphi(z)\,\mu(dz), \qquad (M\varphi)(z) = z\varphi(z), \qquad (\Lambda v)(z) = v.$$

Let $\mathscr{H} = X \times H^2(\mu; Y)$, *and let* $\mathscr{A}$ *be the operator*

$$\mathscr{A} = \begin{pmatrix} -\nu D^*D & -D^*B \\ \Lambda D & M \end{pmatrix}$$

mapping $\operatorname{dom}\mathscr{A} \subset \mathscr{H}$ *into* $\mathscr{H}$, *with*

$$\operatorname{dom}\mathscr{A} = \{ (v, \varphi) \in \mathscr{H} \mid v \in \operatorname{dom} D,\ \nu Dv + B\varphi \in \operatorname{dom} D^*, \\ \Lambda Dv + M\varphi \in H^2(\mu; Y) \}.$$

Then $\mathscr{A}$ *is the infinitesimal generator of a* C_0 *contraction semigroup* S *in* $\mathscr{H}$. *Moreover, for all* $(v, \varphi) \in \operatorname{dom}\mathscr{A}$,

$$\Re\langle (v, \varphi), \mathscr{A}(v, \varphi)\rangle = -\nu\|Dv\|_Y^2 + \int_\Gamma \Re z \|\varphi(z)\|_Y^2\,\mu(dz) \le 0.$$

As before, we define $H^2(\mu; Y)$ to be the closure in $L^2(\mu; Y)$ of the set of functions φ that are analytic in Π^- with $z\varphi(z)$ tending to a finite limit as $|z| \to \infty$, $z \in \Pi^-$. Again the proof is essentially the same as the proof of Theorem 1. This time formula (13) becomes

$$\begin{aligned} &\int_0^T \Re\left\langle v(t), D^*\int_{-\infty}^t a(t-s)Dv(s)\,ds\right\rangle_X dt \\ &\quad = \frac{1}{2}\int_\Gamma \|\varphi(T, z)\|_Y^2\,\mu(dz) - \frac{1}{2}\int_\Gamma \|\varphi(0, z)\|_Y^2\,\mu(dz) \\ &\qquad - \int_0^T \int_\Gamma \Re z\,\|\varphi(t, z)\|_Y^2\,\mu(dz). \end{aligned} \tag{35}$$

The analogy of Theorem 11 is true as well, in the sense that the observable subspace $\mathscr{O}$ for the semigroup in Theorem 1 is continuously imbedded in the observable subspace $\mathscr{O}$ for the semigroup in Theorem 13.

There is one drawback with the representation (26): it is not unique in the sense that there are several measures that produce the same function a. We have seen this already in Section 5, where we had two different representations for a, i.e., (2) and (22). As we shall see in Section 7, one would like the (connected) component of the complement of the support of μ containing the right half-plane to be as large as possible. In particular, if the kernel is completely monotone, then (22) is the preferred representation. If the kernel is a finite sum of the type $a(t) = \sum_{j=1}^N \Re\alpha_j e^{\beta_j t}$, where each $\alpha_j > 0$ and $\Re\beta_j \le 0$, then the preferred measure μ consists of point-masses at β_j and $\overline{\beta}_j$. It is a very interesting open problem how to find a measure μ with minimal support. This problem is a typical ill-posed inverse problem: given a vector field in Π^+ (the function $\hat{a}$), how does one determine its source (the measure μ), located somewhere in Π^-?

7. Eigenvalues and essential spectrum of the generator

An isolated point λ_0 in the spectrum $\sigma(\mathscr{A})$ of $\mathscr{A}$ is called an *eigenvalue of finite multiplicity* if λ_0 is an eigenvalue of $\mathscr{A}$, and if the generalized eigenspace $\bigcup_{n\in\mathbf{N}}\ker(\lambda_0-\mathscr{A})^n$ is finite-dimensional. In this case we also call λ_0 a *finite-dimensional pole* of the resolvent $(\lambda-\mathscr{A})^{-1}$, due to the fact that $(\lambda-\mathscr{A})^{-1}$ can be expanded into a finite sum

$$(\lambda-\mathscr{A})^{-1}=\mathscr{A}_0(\lambda)+\sum_{j=1}^{k}\mathscr{A}_{-j}(\lambda-\lambda_0)^{-j},$$

where $\mathscr{A}_0$ is analytic at λ_0, and each $\mathscr{A}_{-j}$ is a degenerate (finite rank) operator, i.e., an operator with finite-dimensional range. The *essential spectrum* $\sigma_{\text{ess}}(\mathscr{A})$ of $\mathscr{A}$ is the spectrum of $\mathscr{A}$ after all eigenvalues of finite multiplicity have been removed.

The essential spectrum of $\mathscr{A}$ is related to the *essential growth rate* of the semigroup $S(t)$ generated by $\mathscr{A}$. This growth rate can be defined in several different equivalent ways. Here we shall use the same definition as in [4]. First of all, the *growth rate* $\omega(S)$ of the semigroup S is the infimum over all ω for which S satisfies a growth estimate of the type $\|S(t)\|\leq Me^{\omega t}$, for some constant M. Equivalently, since the function $t\mapsto|S(t)|$ is submultiplicative (cf. [6, Lemma 4.1, p. 120])

$$\omega(S)=\inf_{t>0}\ln|S(t)|=\lim_{t\to\infty}\ln|S(t)|.$$

To get the essential growth rate $\omega_{\text{ess}}(S)$ one decomposes the state space $\mathscr{H}$ into two parts, $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$, where each part is an invariant subspace of $S(t)$, and the space $\mathscr{H}_2$ is finite-dimensional. By definition, the essential growth rate $\omega_{\text{ess}}(S)$ of S is the infimum over all decompositions of the preceding type of the growth rate of the restriction of $S(t)$ to $\mathscr{H}_1$. Clearly, $\omega_{\text{ess}}(S)\leq\omega(S)$.

It is easy to show that if the growth rate $\omega(S)$ of S differs from the essential growth rate $\omega_{\text{ess}}(S)$ of S, then $\mathscr{A}$ has at least one eigenvalue λ_0 of finite multiplicity with $\Re\lambda_0=\omega(S)$. Moreover, each point λ with $\Re\lambda>\omega_{\text{ess}}(S)$ that belongs to the spectrum of $\mathscr{A}$ is an eigenvalue of finite multiplicity, and each half-plane $\Re\lambda\geq\delta>\omega_{\text{ess}}(S)$ contains only finitely many such eigenvalues. (The converse statement is not true; we shall return to the converse statement in Section 8.)

Thus, if one wants to determine or to get estimates on the essential growth rate of T, then the first step is to investigate the behavior of the resolvent operator $(\lambda-\mathscr{A})^{-1}$. This is what we do next. In order to be able to say something about the spectrum of $\mathscr{A}$ in the left half-plane (the open right half-plane always belongs to the resolvent set of $\mathscr{A}$), we shall work in a state space induced by the representation (26), where we assume that, apart from a finite number of isolated points (possibly none) on the imaginary axis, the set Γ should be contained in a half-plane $\Re z\leq\delta<0$.

The way in which we have set up the problem leads to the possibility that the spectrum of $\mathscr{A}$ may be quite large. For example, in the absorbed energy setting the whole left half-plane will usually belong to the spectrum of $\mathscr{A}$, due to the fact that the functions $\varphi\in H^2(\mu;Y)$ usually will have to be analytic in the whole left half-plane. The same phenomenon shows up in other cases, too, depending on the measure μ.

The following construction is nonstandard, but it is very natural for our particular problem.

Definition 14. Let $\Gamma \subset \Pi^-$ be the support of μ, and let Ω be the component of $\mathbf{C} \setminus \Gamma$ that contains the open right half-plane. Then $\overline{\Gamma}$ is defined to be the set $\mathbf{C} \setminus \Omega$. In other words, $\overline{\Gamma}$ is the union of Γ and all but one of the components of $\mathbf{C} \setminus \Gamma$ (the one containing the open the right half-plane). The essential support Γ_{ess} of μ consists of all the cluster points of $\overline{\Gamma}$.

In the next theorem we are able to give a complete description of the spectrum of $\mathscr{A}$ in $\mathbf{C} \setminus \Gamma_{\text{ess}}$.

Theorem 15. *Let Hypothesis* 12 *hold. Suppose that the support Γ of μ is a subset of Π^-, but not a subset of the imaginary axis (i.e, there is some point $z \in \Gamma$ with $\Re z < 0$; thus we rule out the absorbed energy semigroup of Section* 2). *Define $\overline{\Gamma}$ and Γ_{ess} as in Definition* 14, *and define $\mathscr{A}$ and $\mathscr{H}$ as in Theorem* 13.

(1) *The essential spectrum of $\mathscr{A}$ in $\mathbf{C} \setminus \Gamma_{\text{ess}}$ is contained in $\mathbf{C} \setminus \overline{\Gamma}$, and it consists of those points $\lambda \notin \overline{\Gamma}$ where $\nu + \hat{a}(\lambda) = 0$. Moreover, the following two claims are true. If Γ is contained in a half-plane $\Re z \le \delta \le 0$, then $\mathscr{A}$ has no essential spectrum in $\Re\lambda > \delta$. If Γ has no cluster point on the imaginary axis, then $\mathscr{A}$ has no essential spectrum in the closed right half-plane $\Re z \ge 0$.*

(2) *An isolated point λ of $\overline{\Gamma}$ belongs either to the resolvent set of $\mathscr{A}$ or is an eigenvalue of finite multiplicity of $\mathscr{A}$, depending on whether* 0 *belongs to the spectrum of D^*D or not.*

(3) *A point $\lambda \notin \overline{\Gamma}$ that does not belong to the essential spectrum of $\mathscr{A}$ (i.e., $\nu + \hat{a}(\lambda) \neq 0$) belongs either to the the resolvent set of $\mathscr{A}$ or is an eigenvalue of finite multiplicity of $\mathscr{A}$, depending on whether the point $-\lambda/(\nu + \hat{a}(\lambda))$ belongs to the spectrum of D^*D or not.*

This theorem says nothing about the behavior of $(\lambda - \mathscr{A})^{-1}$ on Γ_{ess}. However, in every example that we know of it is true that $\sigma_{\text{ess}}(\mathscr{A}) \supset \Gamma_{\text{ess}}$. This is, in particular, true for the examples discussed in Section 9.

The assumption that Γ contains some point with $\Re z < 0$ is not really important, and most of Theorem 15 is true without it. It is basically used to insure that $\nu + \hat{a}(z) \neq 0$ for all $z \notin \Gamma$ with $\Re z = 0$. If Γ is a subset of the imaginary axis, then there are two possibilities: Either Γ occupies all of the imaginary axis, or not. If it does, then the conclusion of Theorem 15 becomes trivial. If not, then the complement of Γ is connected, and $\Gamma = \overline{\Gamma}$. In this case everything remains true, except for the last claim in Part 1, and except for the fact that 0 need not belong to the essential spectrum of $\mathscr{A}$ when $0 \notin \Gamma$ and $\nu + \hat{a}(0) = 0$. It does belong to the spectrum in this case, but it is an eigenvalue of finite multiplicity of $\mathscr{A}$ if $\hat{a}'(0) \neq 0$. We leave the proof of this to the reader.

Proof. First, suppose that $\lambda_0 \notin \overline{\Gamma}$, that $\nu + \hat{a}(\lambda_0) \neq 0$, and that $-\lambda_0/(\nu + \hat{a}(\lambda_0))$ belongs to the resolvent set of DD^* and D^*D (recall that DD^* and D^*D have the same spectra, except that zero may belong to the spectrum of D^*D but not to the spectrum of DD^*). Then our theorem claims that λ_0 belongs to the resolvent set of $\mathscr{A}$. The proof of this is the same as the proof of Lemma 4, except for one complication. Here do not require that $\Re\lambda_0 > 0$. Although it is obvious that the function $(\lambda_0 - z)^{-1}$ belongs to $L^2(\mu; \mathbf{C})$, it is not obvious that it belongs to $H^2(\mu; \mathbf{C})$. (This is the reason why we have to replace Γ by $\overline{\Gamma}$.) Recall that only those functions in $L^2(\mu; \mathbf{C})$ that can be approximated by functions φ that are analytic in Π^- with $z\varphi(z)$ tending to a finite limit as $|z| \to \infty$, $z \in \Pi^-$, belong to $H^2(\mu; \mathbf{C})$. Fortunately, the function $(\lambda_0 - z)^{-1}$ is of this type; to see this one argues as follows. According to Runge's theorem (see [18]), the function $(1-z)/(\lambda_0 - z)$ may be uniformly approximated on $\overline{\Gamma}$ by polynomials φ_n in, e.g., $(1-z)^{-1}$. This implies that $(1-z)^{-1}\varphi_n(z)$ tends to $(\lambda_0 - z)^{-1}$ in $L^2(\mu; \mathbf{C})$ as $n \to \infty$, hence $(\lambda_0 - z)^{-1}$ indeed belongs to $H^2(\mu; \mathbf{C})$. Thus, we conclude that those points $\lambda_0 \notin \overline{\Gamma}$ for which $\nu + \hat{a}(\lambda_0) \neq 0$ and for which $-\lambda_0/(\nu + \hat{a}(\lambda_0))$ belongs to the resolvent set of DD^* and D^*D belong to the resolvent set of $\mathscr{A}$.

Next, suppose that we still have $\lambda_0 \notin \overline{\Gamma}$ and $\nu + \hat{a}(\lambda_0) \neq 0$, but that $\alpha_0 = -\lambda_0/(\nu + \hat{a}(\lambda_0))$ belongs to the spectrum of D^*D. Then the resolvent operator $(\alpha - D^*D)^{-1}$ of D^*D is of the form (D^*D is selfadjoint; hence every generalized eigenvector is in fact an eigenvector)

$$(\alpha - D^*D)^{-1} = C_0(\alpha) + C_{-1}(\alpha - \alpha_0)^{-1},$$

where C_0 is analytic at λ_0, and C_{-1} is a degenerate operator, that does not vanish identically. If we here replace α by $-\lambda/(\nu + \hat{a}(\lambda))$ and expand $(-\lambda/(\nu + \hat{a}(\lambda)) - \alpha_0)^{-1}$ into a Laurent series at the point λ_0, then we get an expansion for the operator $V(\lambda)$ in the resolvent formula in Lemma 4 of the type

$$V(\lambda) = -(\nu + \hat{a}(\lambda))^{-1}\left[-\lambda/(\nu + \hat{a}(\lambda)) - D^*D\right]^{-1} = R_0(\lambda) + \sum_{j=1}^{m} R_{-j}(\lambda - \alpha_0)^{-j}.$$

Here R_0 is analytic at λ_0, and the coefficients R_{-j} are scalar multiples of C_{-1}, hence they are degenerate. Moreover, the highest order coefficient R_{-m} does not vanish (where m is the highest exponent in the Laurent expansion of $(-\lambda/(\nu + \hat{a}(\lambda)) - \alpha_0)^{-1}$). Thus, $V(\lambda)$ has a (nontrivial) finite-dimensional pole at λ_0. The same argument can be repeated with D^*D replaced by DD^* to show that $W(\lambda)$ has a finite-dimensional pole at λ_0, too (except when $\lambda_0 = 0$, in which case w is analytic at λ_0). to λ_0. All the other functions that appear in the resolvent formulas given in Lemma 4 are analytic at λ_0. Thus, either $(\lambda - \mathscr{A})^{-1}$ has a finite-dimensional pole at λ_0, or $(\lambda - \mathscr{A})^{-1}$ is analytic at λ_0. To see that the latter alternative is impossible, observe that if we take $\psi = 0$, then $v = V(\lambda)w$, and $V(\lambda)$ is not analytic at λ_0. Thus, λ_0 is an eigenvalue of $\mathscr{A}$ of finite multiplicity, as claimed.

Next, suppose that λ_0 is an isolated point of $\overline{\Gamma}$. Then we may interpret the value of $\varphi \in H^2(\mu; Y)$ at the point λ_0 as a separate variable, that we shall call u. The state space becomes $X \times Y \times H^2(\mu'; Y)$, where the middle component represents the value of φ at λ_0, and μ' is the measure that one gets from μ by removing the point mass at λ_0. Denote this point mass by E, and note that $E > 0$. With the new notation, the abstract evolution equation becomes

$(v, u, \varphi)' = \mathscr{A}(v, u, \varphi)$, where the generator $\mathscr{A}$ is given by

$$\mathscr{A} = \begin{pmatrix} -\nu D^*D & -D^*E & -D^*B \\ D & \lambda_0 & 0 \\ \Lambda D & 0 & M \end{pmatrix}, \tag{36}$$

and B, Λ, and M are defined in the usual way, but with μ replaced by μ'. The formulas for the computation of

$$(v, u, \varphi) = [\lambda - \mathscr{A}]^{-1}(w, y, \psi)$$

change as follows. Define $V(\lambda)$, $W(\lambda)$, and $\widetilde{w}(\lambda)$ as in Lemma 4. Then, for $\lambda \notin \overline{\Gamma}$,

$$\begin{aligned} v &= V(\lambda)w - E(\lambda-\lambda_0)^{-1}D^*W(\lambda)y - D^*W(\lambda)\widetilde{w}(\lambda), \\ u &= \frac{Dv+y}{\lambda-\lambda_0} = \frac{DV(\lambda)w - E(\lambda-\lambda_0)^{-1}DD^*W(\lambda)y + y - DD^*W(\lambda)\widetilde{w}(\lambda)}{\lambda-\lambda_0} \\ &= \frac{DV(\lambda)w + \lambda W(\lambda)y + (\nu+\hat{b}(\lambda))DD^*W(\lambda)y - DD^*W(\lambda)\widetilde{w}(\lambda)}{\lambda-\lambda_0}, \\ \varphi(z) &= \frac{Dv+\psi(z)}{\lambda-z} \\ &= \frac{DV(\lambda)w - E/(\lambda-\lambda_0)^{-1}DD^*W(\lambda)y - DD^*W(\lambda)\widetilde{w}(\lambda) + \psi(z)}{\lambda-z}, \end{aligned}$$

where $\hat{b}(\lambda) = \hat{a}(\lambda) - E/(\lambda-\lambda_0) = \int_{\Gamma\setminus\{0\}} e^{zt}\mu'(dz)$. We leave it to the reader to check that, for all $\lambda \notin \overline{\Gamma}$, these formulas are identical to those in Lemma 4, if one takes into account the fact that u represents the value of φ at λ_0, and y the value of ψ at λ_0.

Now, suppose that 0 does not belong to the spectrum of D^*D. Then we claim that λ_0 belongs to the resolvent set of $\mathscr{A}$. Let us prove this. Substitute $\hat{a}(\lambda) = \hat{b}(\lambda) + E/(\lambda-\lambda_0)$ in the formulas for V and W in Lemma 4 to get

$$\begin{aligned} V(\lambda) &= (\lambda-\lambda_0)\big[\lambda(\lambda-\lambda_0) + ((\lambda-\lambda_0)(\nu+\hat{b}(\lambda)) + E)D^*D\big]^{-1}, \\ W(\lambda) &= (\lambda-\lambda_0)\big[\lambda(\lambda-\lambda_0) + ((\lambda-\lambda_0)(\nu+\hat{b}(\lambda)) + E)DD^*\big]^{-1}, \end{aligned}$$

Thus, $V(\lambda)/(\lambda-\lambda_0)$ and $W(\lambda)/(\lambda-\lambda_0)$ are analytic at λ_0 (since 0 belongs to the resolvent set of DD^* and D^*D), and at the point $\lambda = \lambda_0$ we may replace $V(\lambda)/(\lambda-\lambda_0)$ and $W(\lambda)/(\lambda-\lambda_0)$ in the formulas for v, u, and φ above by $[ED^*D]^{-1}$ and $[EDD^*]^{-1}$, respectively. Substituting these values for $V(\lambda)/(\lambda-\lambda_0)$ and $W(\lambda)/(\lambda-\lambda_0)$ we get at the point $\lambda = \lambda_0$ (note that $[D^*]^{-1}$ and D^{-1} exist and are continuous)

$$\begin{aligned} v &= -D^{-1}y, \\ u &= \Big[[D^*]^{-1}w + (\nu + \hat{b}(\lambda_0) + \lambda_0[DD^*]^{-1})y - \widetilde{w}(\lambda_0)\Big]/E, \\ \varphi(z) &= [Dv+\psi(z)]/(\lambda_0 - z) = [-y+\psi(z)]/(\lambda_0-z). \end{aligned}$$

We leave it to the reader to check that these formulas indeed give the correct result.

The proof of the fact that an isolated point λ_0 of $\overline{\Gamma}$ is an eigenvalue of finite multiplicity of $\mathscr{A}$ if 0 belongs to the spectrum of D^*D is left to the reader. It is the same proof that we gave above for the case where a point $\lambda_0 \notin \overline{\Gamma}$ is an eigenvalue of finite multiplicity of $\mathscr{A}$, except for the fact that one replaces the formulas of Lemma 4 by those given above for an isolated point of $\overline{\Gamma}$.

To complete the proof of the main part of Theorem 15 we still have to show that $\lambda_0 \notin \overline{\Gamma}$ belongs to the essential spectrum of $\mathscr{A}$ whenever $\nu + \hat{a}(\lambda_0) = 0$. Assume that $\lambda_0 \notin \overline{\Gamma}$ and $\nu + \hat{a}(\lambda_0) = 0$. Then $\lambda_0 \neq 0$ (see the discussion immediately following Theorem 15), and the function $\alpha(\lambda) = -\lambda/(\nu + \hat{a}(\lambda))$ maps every punctured complex neighborhood of λ_0 onto a neighborhood of complex ∞. Since the spectra of DD^* and D^*D have a cluster point at $+\infty$, by Part 3 this means that every punctured neighborhood of λ_0 contains an infinity number of (finite-dimensional) poles of $(\lambda - \mathscr{A})^{-1}$. Thus, being a cluster point of eigenvalues of $\mathscr{A}$, λ_0 belongs to the essential spectrum of $\mathscr{A}$.

Finally, let us check the two additional claims in Part 1. If Γ is contained in a half-plane $\Re z \leq \delta < 0$, then (32) shows that $\Re\hat{a}(\lambda) > 0$ for $\Re\lambda > \delta$, hence $\nu + \hat{a}(\lambda) \neq 0$ for these λ. The second claim is proved in the same way, since in that case $\Re\hat{a}(\lambda) > 0$ for all $\lambda \notin \overline{\Gamma}$ with $\Re\lambda \geq 0$, and since isolated points of $\overline{\Gamma}$ do not belong to the essential spectrum of $\mathscr{A}$. □

The importance of the role played by isolated points of $\overline{\Gamma}$ should not be underestimated. I virtually all cases of interest, $\overline{\Gamma}$ will have an isolated point at zero. This is true throughout in [2], [4], and [12] (actually, in [12] the set Γ_{ess} contains at most one point). It is also true whenever $E = a(\infty) > 0$, and the function $e^{\delta t}(a(t) - E)$ is of positive type for some $\delta > 0$. Then $\hat{a}(\lambda)$ is defined for $\Re\lambda > -\delta$, and we may use Bochner's theorem to represent a as an integral

$$(37) \qquad a(t) = E + \int_{-\infty}^{\infty} e^{-\delta t} \cos \omega t \, \mu(d\omega), \quad t > 0.$$

This is an integral of the type (26) with a measure supported on the line $\Re z = -\delta$, apart from a point mass at the origin in case $E \neq 0$. We shall call the corresponding H^2-space the *shifted absorbed energy space.* In this case it makes sense to interpret the value of φ at zero as a separate variable, as we did in the proof of Theorem 15.

Let us end this section by looking at the modal observability and stabilizability of $\mathscr{A}$, i.e., at the observability and stabilizability of each generalized eigenspace corresponding to an eigenvalue of finite multiplicity of $\mathscr{A}$. It is a standard fact in control theory that any finite number of eigenvalues of finite multiplicity may be relocated by means of a finite-dimensional dynamic feedback iff these eigenvalues are observable and controllable.

Theorem 16. *Make the same assumption as in Theorem* 15. *Define the input operator* $\mathscr{B}$ *and the output operator* $\mathscr{C}$ *as in Section* 4. *Then all the eigenvalues of* $\mathscr{A}$ *of finite multiplicity in* $\mathbf{C} \setminus \Gamma_{ess}$ *are observable and controllable.*

Proof. Below we treat only the case where the eigenvalue λ is an isolated point of $\overline{\Gamma}$; the easier case where $\lambda \notin \overline{\Gamma}$ is left to the reader. We shall use the same notation as in the proof of Part 2 of Theorem 15, and work in the state space $\mathscr{H} = X \times Y \times H^2(\mu'; Y)$.

As is well known, an eigenvalue is observable iff the intersection of the corresponding eigenspace with the null-space of $\mathscr{C}$ is $\{0\}$. Thus, we have to show that the equations

$$-D^*(\nu Dv + Eu + B\varphi) = \lambda v, \quad Dv = 0,$$
$$Dv + z\varphi(z) = \lambda\varphi(z), \quad v = 0,$$

imply that $u = 0$ and $\varphi = 0$. Clearly, the last two equations imply $\varphi = 0$. This, plus the first and last equations and our assumption that D^* is one-to-one (see Remark 6) imply that $u = 0$.

It is also well known that an eigenvalue λ is controllable iff the range of $\mathscr{A} - \lambda$ together with the range of $\mathscr{B}$ span the whole space $\mathscr{H}$. Since the range of $\mathscr{B}$ is $X \times \{0\} \times \{0\}$, this means that one must show that for every $(y, \psi) \in Y \times H^2(\mu; Y)$ it is possible to find $(v, u, \varphi) \in \mathscr{H}$ such that

$$Dv = y, \quad Dv + z\varphi(z) - \lambda\varphi(z) = \psi(z).$$

Recall that we assume that the range of D is Y (see Remark 6). Thus, one possible choice of (v, u, φ) is to choose $v \in X$ such that $Dv = y$, to take $u = 0$, and $\varphi(z) = (\psi(z) - y)/(z - \lambda) \in H^2(\mu; Y)$. □

8. Exponential stabilizability and spectrum determined growth

As we saw in Section 7, the growth rate $\omega(S)$ and the essential growth rate $\omega_{\text{ess}}(S)$ of S satisfy

$$\omega \geq \sup \sigma(\mathscr{A}), \qquad \omega_{\text{ess}} \geq \sup \sigma_{\text{ess}}(\mathscr{A}).$$

If the converse inequalities are true as well, i.e., if

$$\omega = \sup \sigma(\mathscr{A}), \qquad \omega_{\text{ess}} = \sup \sigma_{\text{ess}}(\mathscr{A}),$$

then we say that the growth rate of S is determined by the spectrum of its generator $\mathscr{A}$.

Theorem 17. *Make the same assumption as in Theorem* 15. *Then*

$$\omega_{\text{ess}}(S) \leq \max\left\{ \sup \sigma_{\text{ess}}(\mathscr{A}),\ -\frac{1}{2}\liminf_{|\omega|\to\infty} d(i\omega, \Gamma),\ \frac{1}{2}\frac{\int_\Gamma \Re z\,\mu(dz)}{\int_\Gamma \mu(dz)} \right\}$$

where $d(i\omega, \Gamma)$ is the distance from $i\omega$ to Γ, and $\int_\Gamma \Re z\,\mu(dz)/\int_\Gamma \mu(dz)$ should be replaced as $-\infty$ if $\nu > 0$ or $\int_\Gamma \Re z\,\mu(dz) = -\infty$. In particular, if Γ has no cluster point on the imaginary axis, and if $\liminf_{|\omega|\to\infty} d(i\omega, \Gamma) > 0$, then S is exponentially stabilizable. Furthermore, if

$$d(i\omega, \Gamma) \to \infty \quad \textit{as } |\omega| \to \infty$$

and

$$\nu > 0, \quad \textit{or} \quad \int_\Gamma \Re z\,\mu(dz) = -\infty,$$

then the growth rate of S is determined by the spectrum of $\mathscr{A}$.

It is a remarkable fact that this exactly reproduces the main part of [4, Theorem 3.1]. That theorem says that

$$\omega_{\text{ess}}(S) = \max\left\{ \sup \sigma_{\text{ess}}(\mathscr{A}),\ \frac{1}{2}\frac{a'(0)}{a(0)} \right\},$$

where $a'(0)/a(0)$ is interpreted as $-\infty$ if $a'(0) = -\infty$ (in addition, it identifies $\sup \sigma_{\text{ess}}(\mathscr{A})$). Of course, for a completely monotone kernel, $d(i\omega, \Gamma) \geq \omega \to \infty$ as $\omega \to \infty$, so that term drops out. Furthermore, for a completely monotone kernel it is easy to show that

$$a(0) = \int_\Gamma \mu(dz) \quad \text{and} \quad a'(0) = \int_\Gamma \Re z\, \mu(dz),$$

independently of whether these integrals are finite or not. Thus, the estimate in Theorem 17 becomes the same as the one in [2, Theorem 3.1], apart from the fact that we give an inequality instead of an equality.

As we show in Section 10, in our more general case, too, it is true that $a(0) = \int_\Gamma \mu(dz)$. However it is not true in general that $a'(0) = \int_\Gamma \Re z\, \mu(dz)$; to see this observe that it would imply $a'(0) = 0$ whenever we use the absorbed energy setting of Section 2. It is true, however, that $a'(0) \leq \int_\Gamma \Re z\, \mu(dz)$, whenever $a'(0)$ exists. See Theorem 26 for more details.

As the result cited shows, the constant $\frac{1}{2}$ in front of $\int_\Gamma \Re z\, \mu(dz) / \int_\Gamma \mu(dz)$ is optimal. However, as we show with an example in Section 9, in our case the inequality cannot always be replaced by an equality, as was the case in [4, Theorem 3.1]. We do not know if the constant $\frac{1}{2}$ in front of $\liminf_{|\omega|\to\infty} d(i\omega, \Gamma)$ is optimal or not. Note, however, that in the shifted absorbed energy setting with $a(\infty) = 0$, where a has a representation of the form (37) with $E = 0$, we have

$$\liminf_{|\omega|\to\infty} d(i\omega, \Gamma) = \delta = -\frac{\int_\Gamma \Re z\, \mu(dz)}{\int_\Gamma \mu(dz)},$$

whenever the integral converges. For several different examples illustrating and testing the conclusion of Theorem 17, see Section 9.

We have split the proof of Theorem 17 into a sequence of lemmas, i.e., Lemmas 18–23 below. Some of these lemmas are of independent interest in the sense that they will be used in Section 9 in our more detailed analysis of the conclusion of Theorem 17 by means of some examples. Indeed, as the reader may easily check, Theorem 17 follows directly from these lemmas. The general outline of the proof is the same as in [4], but the proofs of some of the lemmas differ significantly from the proofs in [4], due to the fact that we allow a more general class of kernels.

Lemma 18. *Denote* $(\lambda - \mathscr{A})^{-1}$ *by* $R(\lambda)$ *for all those* λ *where the inverse exists. Then*

(1) *If for some* $\delta_0 \in \mathbf{R}$,

$$\lim_{|\omega|\to\infty} \|R(\delta_0 + i\omega)\| = 0,$$

then $\lim_{|\omega|\to\infty} \|R(\delta + i\omega)\| = 0$ *uniformly for* δ *in compact subsets of* $\mathbf{R}$, *and the growth rate of* S *is determined by the spectrum of* $\mathscr{A}$.

(2) *If for all* δ *in an interval* $[\delta_0, 0]$, *it is true that*

$$\limsup_{|\omega|\to\infty} \|R(\delta + i\omega)\| = M < \infty,$$

then $\omega_{\text{ess}}(S) \leq \max\{\sup \sigma_{\text{ess}}(\mathscr{A}), \delta_0 - 1/M\}$.

Proof. Recall that R satisfies the resolvent equation

$$R(\lambda) = R(\lambda_0) - (\lambda - \lambda_0)R(\lambda)R(\lambda_0).$$

Together with the contraction mapping principle this implies that $R(\lambda)$ exists whenever $R(\lambda_0)$ exists and $|\lambda - \lambda_0| < 1/\|R(\lambda_0)\|$, and that

$$\|R(\lambda)\| \leq \frac{\|R(\lambda_0)\|}{1 - |\lambda - \lambda_0|\|R(\lambda_0)\|}.$$

Thus, if for some real constant δ_0, $\limsup_{|\omega|\to\infty}\|R(\delta_0 + i\omega)\| = M < \infty$, then

$$\limsup_{|\omega|\to\infty} \sup_{\delta\in[\delta_0-\alpha/M, \delta_0+\alpha/M]} \|R(\delta + i\omega)\| \leq M/(1-\alpha), \qquad 0 < \alpha < 1.$$

In the former case considered in the lemma, we have for each finite K,

$$\limsup_{|\omega|\to\infty} \sup_{\delta\in[-K, K]} \|R(\delta + i\omega)\| = 0,$$

and in the latter we get

$$\limsup_{|\omega|\to\infty} \sup_{\delta\in[\delta_0-\alpha/M, \alpha/M]} \|R(\delta + i\omega)\| \leq M/(1-\alpha),$$

since the interval $[\delta_0, 0]$ can be covered by a finite number of intervals of length $2\alpha/M$.

The rest of the proof is now the same as the proof given on [4, p. 422]. It is a fairly directly consequence of Gerhart's theorem (see [5] or [15]). Basically, one splits the state space into two invariant parts; a finite-dimensional one, and an infinite-dimensional one on which Gerhart's theorem applies. □

Frequently it suffices to apply this lemma with $\delta_0 = 0$, and by doing so we could prove the exponential stabilizability, as well as the final claim about spectrum determined growth. However, to get the more precise statement of Theorem 17 in the case where $\liminf_{|\omega|\to\infty} d(i\omega, \Gamma)$ or $\nu = 0$ and $\int_\Gamma \Re z\, \mu(dz)$ is finite, we need to apply it with δ_0 arbitrarily close to the desired bound.

Lemma 19. *Make the same assumption as in Theorem* 15, *and let* $\lambda \in \rho(\mathscr{A}) \setminus \overline{\Gamma}$. *Let* V *and* W *be defined as in Lemma* 4. *Then*

$$V(\lambda)D^* = D^*W(\lambda) \text{ and } DV(\lambda) = W(\lambda)D.$$

Moreover, if β *and* γ *are defined by*

$$\beta(\lambda) = \sup_{\zeta\in\sigma(DD^*)} \left|\frac{1}{\lambda + \zeta(\nu + \hat{a}(\lambda))}\right|,$$

$$\gamma(\lambda) = \sup_{\zeta\in\sigma(DD^*)} \left|\frac{\zeta}{\lambda + \zeta(\nu + \hat{a}(\lambda))}\right|,$$

then

$$\|V(\lambda)\| = \|W(\lambda)\| = \beta(\lambda),$$
$$\|D^*DV(\lambda)\| = \|DD^*W(\lambda)\| = \gamma(\lambda),$$
$$\|DV(\lambda)\| \le \sqrt{\beta(\lambda)\gamma(\lambda)},$$
$$\|D^*W(\lambda)\| \le \sqrt{\beta(\lambda)\gamma(\lambda)}.$$

The proof of this lemma is left to the reader. It is very similar to but slightly simpler than the proof of the corresponding Lemma 3.7 in [4]. The simplification is due to the fact that we assume D^* to be one-to-one.

Lemma 20. *Define* α *through the identity*

$$\alpha^2(\lambda) = \int_\Gamma \frac{\mu(dz)}{|\lambda - z|^2} = \left\| \frac{1}{\lambda - z} \right\|^2_{H^2(\mu;\mathbf{C})},$$

and let β *and* γ *be the functions defined in Lemma* 19. *Then there is a constant* M *such that, for all nonzero* $\lambda \in \rho(\mathscr{A}) \setminus \overline{\Gamma}$, *the function* $R(\lambda) = (\lambda - \mathscr{A})^{-1}$ *satisfies*

$$R(\lambda) \le M\left[\beta(\lambda) + \alpha^2(\lambda)\gamma(\lambda) + 1/d(\lambda, \Gamma)\right],$$

where $d(\lambda, \Gamma)$ *represents the distance from* λ *to* Γ.

Proof. To prove this, let us return to the proof of Lemma 4. Clearly, it suffices to show that the same claim is true when $R(\lambda)$ is replaced by the two linear operators given in that lemma that map the pair (w, ψ) into v and φ, respectively.

First we observe that, by Lemma 19, and the definition of v in Lemma 4,

$$\begin{aligned}\|v\| &\le \beta(\lambda)\|w\| + \sqrt{\beta(\lambda)\gamma(\lambda)}\|\tilde{w}\| \le \beta(\lambda)\|w\| + \sqrt{\beta(\lambda)\gamma(\lambda)}\alpha(\lambda)\|\psi\| \\ &\le \beta(\lambda)\|w\| + \tfrac{1}{2}\left[\beta(\lambda) + \alpha^2(\lambda)\gamma(\lambda)\right]\|\psi\|.\end{aligned}$$

This takes care of one of the two operators. For the other one we need to estimate Dv, which may be done in a similar way:

$$\|Dv\| \le \sqrt{\beta(\lambda)\gamma(\lambda)}\|w\| + \gamma(\lambda)\|\tilde{w}\| \le \sqrt{\beta(\lambda)\gamma(\lambda)}\|w\| + \gamma(\lambda)\alpha(\lambda)\|\psi\|.$$

Thus, since $\varphi(z) = (Dv + \psi)/(\lambda - z)$, we get

$$\begin{aligned}\|\varphi\| &\le \alpha(\lambda)\|Dv\| + 1/d(\lambda, \Gamma)\|\psi\| \\ &\le \tfrac{1}{2}\left[\beta + \alpha^2(\lambda)\gamma(\lambda)\right]\|w\| + \left[\alpha^2(\lambda)\gamma(\lambda) + 1/d(\lambda, \Gamma)\right]\|\psi\|.\end{aligned}$$

□

The next step is to obtain estimates on the functions β and γ in terms of the Laplace transform $\hat{a}$ of a.

Lemma 21. *Let* $\lambda \notin \Gamma$, *and suppose that* $0 \ne \Im\lambda(\nu + \Re\hat{a}(\lambda)) \ne \Re\lambda\Im\hat{a}(\lambda)$. *Then*

$$\beta(\lambda) \le \frac{\sqrt{1/|\Im\lambda|^2 + \kappa^2(\lambda)}}{|1 - \Re\lambda\kappa(\lambda)|} \quad \text{and} \quad \gamma(\lambda) \le \frac{|\lambda|/|\Im\lambda(\nu + \Re\hat{a}(\lambda))|}{|1 - \Re\lambda\kappa(\lambda)|},$$

where

$$\kappa(\lambda) = \frac{\Im \hat{a}(\lambda)}{\Im \lambda(\nu + \Re \hat{a}(\lambda))}.$$

Proof. We have

$$\begin{aligned}\beta(\lambda) &\leq \frac{1}{\inf_{\zeta \in \mathbf{R}} (|\lambda + \zeta(\nu + \hat{a}(\lambda))|)} = \frac{1}{\inf_{\zeta \in \mathbf{R}} |\nu + \hat{a}(\lambda)| \, (|\lambda/(\nu + \hat{a}(\lambda)) + \zeta|)} \\ &= \frac{1}{|\nu + \hat{a}(\lambda)| |\Im[\lambda/(\nu + \hat{a}(\lambda))]|} = \frac{|\nu + \hat{a}(\lambda)|}{\Im \lambda(\nu + \Re \hat{a}(\lambda)) - \Re \lambda \Im \hat{a}(\lambda)} \\ &= \frac{\sqrt{1/|\Im \lambda|^2 + \kappa^2(\lambda)}}{|1 - \Re \lambda \kappa(\lambda)|}.\end{aligned}$$

The estimate for γ is proved in a similar way (begin by dividing the numerator and denominator in the expression defining γ by ζ). □

Thus, to get an upper bound on β we need an upper bound on κ, and to get an upper bound on $\alpha^2\gamma$ we need, in addition, an upper bound on $\alpha^2/(\nu + \Re \hat{a}(\lambda))$. This can be achieved as follows.

Lemma 22. *Let* $\delta_1 \in (-\liminf_{|\omega| \to \infty} d(i\omega, \Gamma), 0]$ *satisfy*

$$\inf_{\delta \in [\delta_1, 0]} \liminf_{|\omega| \to \infty} \omega^2(\nu + \Re \hat{a}(\delta + i\omega)) > 0, \tag{38}$$

and let $\delta_2 \in (-\liminf_{|\omega| \to \infty} d(i\omega, \Gamma), 0]$ *satisfy*

$$\sup_{\delta \in [\delta_2, 0]} \limsup_{|\omega| \to \infty} |\kappa(\delta + i\omega)| < 1/|\delta_2| \tag{39}$$

(*where we interpret* $1/|\delta_2|$ *as* ∞ *if* $\delta_2 = 0$). *Then*

$$\sup_{\delta \in [\delta_1, 0]} \limsup_{|\omega| \to \infty} \frac{\alpha^2(\delta + i\omega)}{\nu + \Re \hat{a}(\delta + i\omega)} < \infty, \tag{40}$$

and $R(\lambda)$ (*defined in Lemma* 18) *satisfies*

$$\sup_{\max\{\delta_1, \delta_2\} \leq \delta \leq 0} \limsup_{|\omega| \to \infty} \|R(\delta + i\omega)\| < \infty.$$

The nice thing about (38) and (39) is that they depend only on ν and $\hat{a}$ i.e., they are independent of the particular measure μ that we use in the representation formula (28) (not counting the fact that δ_1 and δ_2 are bounded from below by $-\liminf_{|\omega| \to \infty} d(i\omega, \Gamma)$). Thus, they may be checked without any reference to a representation formula of the type (28) (as long as such a representation exists, with $\liminf_{|\omega| \to \infty} d(i\omega, \Gamma) > 0$). This will be important when we discuss specific examples. For some time domain monotonicity type conditions that can be used to verify (39), see [7].

Proof. Most of this follows from Lemmas 20 and 21; the only new thing is (40). Clearly, because of (38), in order to prove (40), it suffices to show that, for some constant K, for all $\delta \in [\delta_1, 0]$, and for large $|\omega|$,

$$\alpha^2(\delta + i\omega) \leq K(\nu + \Re \hat{a}(\delta + i\omega) + \omega^{-2}). \tag{41}$$

Define $\delta' = \frac{1}{2}(\delta_1 - \liminf_{|\omega| \to \infty} d(i\omega, \Gamma))$, and split the set Γ into $\Gamma_1 = \Gamma \cap \{\Re z \leq \delta'\}$ and $\Gamma_2 = \Gamma \cap \{\Re z > \delta'\}$. Then Γ_2 is bounded, and $\delta - \Re z \geq$

$\delta_1 - \delta' > 0$ for all $\delta \geq \delta_1$ and all $z \in \Gamma_1$. For $j = 1$ and $j = 2$, define

$$\Re \hat{a}_j(\lambda) = \int_{\Gamma_j} \frac{\Re(\lambda - z)}{|\lambda - z|^2} \mu(dz), \qquad \alpha_j^2(\lambda) = \int_{\Gamma_j} \frac{1}{|\lambda - z|^2} \mu(dz).$$

Then, by (32), $\Re \hat{a} = \Re \hat{a}_1 + \Re \hat{a}_2$, and by the definition of α, $\alpha^2 = \alpha_1^2 + \alpha_2^2$. Clearly, by the formulas above, for each $\delta \geq \delta_1$ and for all ω,

$$\alpha_1^2(\delta + i\omega) \leq (\delta_1 - \delta')^{-1} \Re \hat{a}_1(\delta + i\omega). \tag{42}$$

Moreover, since Γ_2 is bounded, the limits $\lim_{|\omega| \to \infty} \omega^2 \Re \hat{a}_2(\delta + i\omega) = \int_{\Gamma_2} (\delta - \Re z)\, \mu(dz)$ and $\lim_{|\omega| \to \infty} \omega^2 \alpha_2^2(\delta + i\omega) = \int_{\Gamma_2} \mu(dz)$ exist, and are uniformly bounded for $\delta \in [\delta_2, 0]$. Thus, for some appropriately chosen constant K, for all $\delta \in [\delta_1, 0]$, and for large $|\omega|$,

$$\begin{gathered} |\Re \hat{a}_2(\delta + i\omega)| \leq K\omega^{-2}, \quad \alpha_2^2(\delta + i\omega) \leq K\omega^{-2}, \\ \Re \hat{a}_1(\delta + i\omega) \leq \Re \hat{a}(\delta + i\omega) + K\omega^{-2} \leq \nu + \Re \hat{a}_1(\delta + i\omega) + K\omega^{-2}. \end{gathered} \tag{43}$$

Combining (42) and (43) we get (41). □

The following lemma is our final tool to use in the proof of Theorem 17.

Lemma 23. *If* $\nu > 0$ *or* $\int_\Gamma \Re z\, \mu(dz) = -\infty$, *define*

$$q = -\liminf_{|\omega| \to \infty} d(i\omega, \Gamma),$$

otherwise define

$$q = \max\left\{ -\liminf_{|\omega| \to \infty} d(i\omega, \Gamma), \frac{\int_\Gamma \Re z\, \mu(dz)}{\int_\Gamma \mu(dz)} \right\}.$$

Then the condition (38) *is satisfied for every* $\delta_1 \in (q, 0]$, *and the condition* (39) *is satisfied for every* $\delta_2 \in (q/2, 0]$.

Proof. By symmetrizing the formula defining $\alpha(\lambda)$ we get

$$\begin{aligned} \alpha^2(\lambda) &= \frac{1}{2} \int_\Gamma \left(\frac{1}{|\lambda - z|^2} + \frac{1}{|\lambda - \overline{z}|^2} \right) \mu(dz) \\ &= \int_\Gamma \frac{(\Re(\lambda - z))^2 + (\Im \lambda)^2 + (\Im z)^2}{|\lambda - z|^2 |\lambda - \overline{z}|^2} \mu(dz). \end{aligned}$$

Comparing this to (31), we observe that

$$|\Im \hat{a}(\lambda)| \leq |\Im \lambda| \alpha^2(\lambda),$$

hence

$$|\kappa(\lambda)| \leq \frac{\alpha^2(\lambda)}{\nu + \Re \hat{a}(\lambda)}. \tag{44}$$

Thus, in particular, we may replace $|\kappa|$ by $\alpha^2/(\nu + \hat{\Re a})$ in (39).

We claim that, for each $\delta \in (q, 0]$, we have

$$\liminf_{|\omega|\to\infty} \omega^2\Big[\nu + \Re\hat{a}(\delta + i\omega)\Big] \geq \int_\Gamma (\delta - \Re z)\,\mu(dz) > 0 \tag{45}$$

(where one should interpret $\int_\Gamma (\delta - \Re z)\,\mu(dz)$ as $+\infty$ if $\nu > 0$ or the integral $\int_\Gamma \Re z\,\mu(dz)$ diverges), and that

$$\limsup_{|\omega|\to\infty} \frac{\alpha^2(\delta + i\omega)}{\nu + \Re\hat{a}(\delta + i\omega)} \leq (\delta - q)^{-1} \tag{46}$$

(where one interprets $(\delta - q)^{-1}$ as 0 if $q = -\infty$). Clearly, this together with (44) and the fact that $(\delta - q)^{-1} < \delta^{-1}$ if $\delta \in (q/2, 0]$ shows that (38) is satisfied for every $\delta_1 \in (q, 0]$, and that (39) is satisfied for every $\delta_2 \in (q/2, 0]$.

Let us proceed to prove (45) and (46). Take some $\delta \in (q, 0]$ and $\delta' \in (q, \delta]$. Split Γ into two parts Γ_1 and Γ_2, and split $\hat{a}$ into $\hat{a}_1 + \hat{a}_2$ and α^2 into $\alpha_1^2 + \alpha_1^2$ in the same way as in the proof of Lemma 22. Then

$$\Re\hat{a}_1(\delta + i\omega) + (\delta' - \delta)\alpha_1^2(\delta + i\omega) = \int_{\Gamma_1} \frac{\delta' - \Re z}{|\delta + i\omega - z|^2}\,\mu(dz).$$

This together with Fatou's lemma implies that

$$\liminf_{|\omega|\to\infty} \omega^2\Big[\Re\hat{a}_1(\delta + i\omega) + (\delta' - \delta)\alpha_1^2(\delta + i\omega)\Big] \geq \int_{\Gamma_1} (\delta' - \Re z)\,\mu(dz).$$

By the dominated convergence theorem,

$$\lim_{|\omega|\to\infty} \omega^2\Big[\Re\hat{a}_2(\delta + i\omega) + (\delta' - \delta)\alpha_2^2(\delta + i\omega)\Big] = \int_{\Gamma_2} (\delta' - \Re z)\,\mu(dz).$$

Thus, we have

$$\liminf_{|\omega|\to\infty} \omega^2\Big[\nu + \Re\hat{a}(\delta + i\omega) + (\delta' - \delta)\alpha^2(\delta + i\omega)\Big] \geq \int_\Gamma (\delta' - \Re z)\,\mu(dz),$$

where one should interpret $\int_\Gamma (\delta' - \Re z)\,\mu(dz)$ as $+\infty$ if $\nu > 0$ or the integral $\int_\Gamma \Re z\,\mu(dz)$ diverges. In particular, by taking $\delta' = \delta$ we get the claim (45). Indeed, for the final inequality in (45), observe that if the integral $\int_\Gamma \Re z\,\mu(dz)$ converges, then so does the integral $\int_\Gamma \mu(dz)$, and

$$\delta \int_\Gamma \mu(dz) > \int_\Gamma \Re z\,\mu(dz)$$

because of the way in which we chose δ.

To prove (46) we take $q < \delta' < \delta$. Then, by the same argument,

$$\liminf_{|\omega|\to\infty} \omega^2\Big[\nu + \Re\hat{a}(\delta + i\omega) + (\delta' - \delta)\alpha^2(\delta + i\omega)\Big] \geq \int_\Gamma (\delta' - \Re z)\,\mu(dz) > 0.$$

Thus, for $|\omega|$ large enough, we have

$$(\delta - \delta')\alpha^2(\delta + i\omega) \leq \nu + \Re\hat{a}(\delta + i\omega).$$

Clearly, this means that

$$\limsup_{|\omega|\to\infty} \frac{\alpha^2(\delta+i\omega)}{\nu+\Re\hat{a}(\delta+i\omega)} \le (\delta-\delta')^{-1}.$$

Here we may let $\delta' \to q$ to get (46). □

Up to now we have supposed that the relaxation modulus a in (1) is a locally integrable function. However, there is nothing that prevents us from taking a to be an arbitrary passive distribution:

Theorem 24. *Let a be a distribution supported on $\mathbf{R}^+$ with a distribution Laplace transform $\hat{a}$ that has a positive real part in the open half-plane $\Re z > 0$, satisfying $\hat{a}(\lambda) = o(\lambda)$ as $\lambda \to +\infty$. Moreover, suppose that $\hat{a}$ has a representation of the type* (28), *where μ is a positive measure supported on a subset Γ of the closed left half-plane Π^-, satisfying the moment condition* (29) (*it is always possible to choose a representation of the type* (5), *where the measure μ is supported on the imaginary axis, if one cannot find any other representation*). *Then Theorems* 1, 5, 8, 9, 13, 15, 16, *and* 17 *remain true.*

The purpose of the growth restriction on $\hat{a}$ is to rule out the possibility that a contains a distribution derivative of a positive point mass at zero. It would not actually hurt to have such a term, but the nature of the equation then changes completely, and the formulas change slightly.

Proof. All the proofs remain the same. The only claim that needs to be checked is that it is always possible to choose a representation of the type (5). This follows from the following argument. Since $\Re\hat{a}$ is a positive harmonic function in the right half-plane, it has a Poisson representation (see [9, p. 143])

$$\Re\hat{a}(\lambda) = \alpha\lambda + \int_{-\infty}^{\infty} \frac{\Re\lambda}{(\Re\lambda)^2 + (\Im\lambda-\omega)^2}\,\mu(d\omega), \quad \Re\lambda > 0,$$

where $\alpha \ge 0$, and μ satisfies the growth condition (3). The extra growth condition on $\Re\hat{a}$ in the theorem implies that $\alpha = 0$ (divide by λ, let $\lambda \to \infty$, and use the dominated convergence theorem). Thus, $\Re\hat{a}$ has a representation of the type (6). The right-hand side of (5) defines an analytic function for $\Re\lambda > 0$, that coincides with $\hat{a}(\lambda)$ on the real axis. Thus, (5) is valid. □

9. Some examples

In this section we give a number of examples illustrating the conclusion of Theorem 17.

As our first example, let us assume that $\hat{a}$ has a representation (28) with a measure μ supported in a sector $\{\,|\arg -z| \le \theta < \pi/2\,\}$. Then, according to Part 1 of Theorem 27, a must be analytic in a sector $\{\,|\arg t| < \pi/2-\theta\,\}$. By Parts 1 and 4 of Theorem 26, $a(0) = \int_\Gamma \mu(dz)$ and $a'(0) = \int_\Gamma \Re z\,\mu(dz)$. Thus, the estimate in Theorem 17 becomes

$$\omega_{ess}(S) \le \max\{\,\sup\sigma_{ess}(\mathscr{A}),\ a'(0)/(2a(0))\,\}.$$

This is the same estimate as Desch and Miller get for a completely monotone kernel in [4, Theorem 3.1], except for the fact that we have an inequality instead of an equality. It seems plausible that it should be possible here to strengthen the inequality to an equality. Of course, if $a'(0) = -\infty$ then it does not matter, because for that case both theorems say that the growth rate of S is determined by the spectrum of $\mathscr{A}$, i.e., $\omega_{ess}(S) = \sup \sigma_{ess}(\mathscr{A})$.

One nice property of the distribution class of kernels in Theorem 24 is that it is easy to find examples and counterexamples of this type that illustrate various parts of the theory. One such example is the following. We let a consist of two point masses, one at zero and another at, e.g., the point $t = T$. Then the equation turns into a delay equation

$$v'(t) = -D^*\sigma(t) + f(t), \quad \sigma(t) = \nu Dv(t) + \nu_1 Dv(t-T) \quad (t \geq 0). \tag{47}$$

The Laplace transform of this kernel is

$$\hat{a}(\lambda) = \nu + \nu_1 e^{-\lambda T}.$$

If $|\nu_1| < \nu$, then $\Re\hat{a}(\lambda) \geq 0$ for $\Re\lambda \geq \delta^*$, with

$$\delta^* = \ln(|\nu|_1/\nu) < 0,$$

and, $\Re\hat{a}(\delta^* + i\omega) = \nu(1 \pm \cos\omega T)$, where the sign depends on the sign of ν_1. Thus, we may use a shifted absorbed energy setting, with a measure $\mu(dz) = (\nu/\pi)(1 \pm \cos\omega T)\,d\omega$ supported on the line $\Re z = \delta^*$. According to (28),

$$\hat{a}(\lambda) = (\nu/\pi)\int_{-\infty}^{\infty} \frac{\lambda - \delta^*}{(\lambda - \delta^*)^2 + \omega^2}(1 \pm \cos\omega T)\,d\omega, \quad \Re\lambda > \delta^*.$$

In a certain sense this is the optimal representation formula: it is unique within the class of formulas where the support Γ of μ is contained in the half-plane $\{\Re z \leq \delta^*\}$. This follows from the fact that, if Γ is contained in $\{\Re z \leq \delta^*\}$, and if Γ contains some point z with $\Re z < \delta^*$, then it follows from (30) that for all ω, we would have $\liminf \Re\hat{a}(\lambda + i\omega) > 0$ as λ tends to δ^* from the right, which is not true for $\omega = 2\pi k/T$.

In this example, $\mathscr{A}$ has no essential spectrum in $\Re z > \delta^*$; this follows from Part 1 of Theorem 15. On the other hand, every point on the line $\Re z = \delta^*$ will belong to the essential spectrum of $\mathscr{A}$; this follows from the fact the image of dom$(\mathscr{A})$ under $\mathscr{A}$ projected onto the second component $H^2(\mu; Y)$ of the state space is not all of $H^2(\mu; Y)$. (In addition, at each point on this line where $1 \pm \cos\omega T$ vanishes, there is a singularity in the X-component of the resolvent operator, and it seems like this singularity could not be avoided by any choice of state space, even if it would be of a completely different type than our state space $X \times H^2(\mu; Y)$; any reasonable state space must contain a copy of X.) Thus, $\sup \sigma_{ess}(\mathscr{A}) = \delta^*$. Clearly,

$$\int_\Gamma \Re z\mu(dz) = (\delta^*\nu/\pi)\int_{-\infty}^{\infty}(1 \pm \cos\omega T)\,d\omega = -\infty.$$

According to Theorem 17, $\omega_{ess}(S) \leq \delta^*/2$.

We claim that, for this example, the estimate in Theorem 17 is not sharp. The true essential growth rate is $\omega_{ess}(S) = \delta^*$. This follows easily from Lemma 22. Clearly, we may choose δ_1 in (38) arbitrarily close to δ^*, since $\Re\hat{a}(\delta + i\omega) = \nu + \nu_1 e^{\delta}\cos\omega T \geq \nu - |\nu_1|e^{\delta}$. For the same reason, and since $\Im\hat{a}(\delta + i\omega) =$

$\nu_1 e^\delta \sin\omega$ is bounded, we may choose δ_2 in (39) arbitrarily close to δ^*, too. Thus, from Lemmas 18 and 22, and from the fact that $\sup\sigma_{ess}(\mathscr{A}) = \delta^*$, we get $\omega_{ess}(S) = \delta^*$.

In the remainder of this section we shall look at examples of the type

$$a(t) = b(t) + E,$$

where $E \geq 0$ and $e^t b(t)$ is of positive type. This makes it possible to use the shifted absorbed energy setting, with a measure μ supported on the union of the line $\Re z = -1$ and the point 0, i.e., we have

$$\hat{a}(\lambda) = E/\lambda + \int_{-\infty}^{\infty} \frac{\lambda+1}{(\lambda+1)^2+\omega^2}\,\mu(d\omega), \quad \Re\lambda > -1, \quad \lambda \neq 0.$$

First suppose that b is unbounded at zero. Then, by Part 1 of Theorem 26, $\int_\Gamma \mu(dz) = \infty$, hence $\int_\Gamma \Re z\,\mu(dz) = -\infty$. Thus, Theorem 17 tells us that

$$\omega_{ess}(S) \leq \max\{\sup\sigma_{ess}(\mathscr{A}), -\tfrac{1}{2}\}.$$

At least in some cases this estimate may be improved to

$$\omega_{ess}(S) = \sup\sigma_{ess}(\mathscr{A}).$$

Suppose, for example, that both the function $e^t b(t)$ and its derivative are convex, and that the derivative is unbounded at zero. Then it follows from [7, Section 5] that we may choose both δ_1 in (38) and δ_2 in (39) arbitrarily close to -1. In this case, too, the whole line $\Re z = -1$ belongs to the essential spectrum of $\mathscr{A}$ (cf. the delay example). Thus, we get indeed, $\omega_{ess}(S) = \sup\sigma_{ess}(\mathscr{A})$.

Another possibility is that a is so smooth that both the first and the second derivative of $e^t b(t)$ are integrable. Then

$$(48)\quad \hat{a}(\delta+i\omega) = a(0)/(\delta+i\omega) + a'(0)/(\delta+i\omega)^2 + o(\omega^{-2}), \qquad \delta > -1,$$

as one can show with two integrations by parts (the term $o(\omega^{-2})$ is uniform in δ for $\delta \geq \delta_1 > -1$). In particular,

$$\Im\hat{a}(\delta+i\omega) = -a(0)/\omega + O(\omega^{-2}),$$
$$\Re\hat{a}(\delta+i\omega) = (\delta a(0) - a'(0))/\omega^{-2} + o(\omega^{-2}).$$

Thus, we may choose δ_1 in (38) arbitrarily close to $\max\{-1, a'(0)/a(0)\}$ and δ_2 in (39) arbitrarily close to $\max\{-1, \tfrac{1}{2}a'(0)/a(0)\}$, and find that

$$\omega_{ess}(S) \leq \max\{\sup\sigma_{ess}(\mathscr{A}), -\tfrac{1}{2}a'(0)/a(0)\},$$

as expected (the term -1 has been dropped because of the fact that the whole line $\Re z = -1$ belongs to $\sigma_{ess}(\mathscr{A})$). Moreover, for this particular case, one may repeat the proof used by Desch and Miller for the converse inequality in [2, Theorem 3.1] (that proof depends only on the fact that (48) holds) to get

$$\omega_{ess}(S) = \max\{\sup\sigma_{ess}(\mathscr{A}), -\tfrac{1}{2}a'(0)/a(0)\}.$$

Let us finally consider an example with a somewhat different behavior. For this example, the estimate that one gets for $\omega_{ess}(S)$ is no longer independent of the operator D, as it has been up to now. Instead it sensitive to the singular value distribution of D. Of course, there is still a worst possible estimate that applies to any singular value distribution, but even this estimate is better than the one given by Theorem 17. In order to keep this example reasonably short we leave the verification of many of the details to the reader.

Let

$$a(t) = \begin{cases} E + \alpha(1 - t/T)e^{-t}, & 0 \le t \le 1, \\ E, & \text{otherwise.} \end{cases}$$

Here $\alpha > 0$, $T > 0$, and $E \ge 0$. The Laplace transform of this kernel is the entire function

$$\hat{a}(\lambda) = \frac{E}{\lambda} + \frac{\alpha(e^{-(\lambda+1)T} - 1 + (\lambda + 1)T)}{(\lambda + 1)^2 T}.$$

As always, the term E/λ corresponds to a point mass of size E at zero. The real part of the remainder has a sequence of zeros on the line $\Re\lambda = -1$, but it is strictly positive for $\Re\lambda > -1$. Thus, we may use a shifted absorbed energy setting, with a measure μ supported on the union of the line $\Re z = -1$ and the point 0, and this choice of the support of μ is optimal in the same sense as it was optimal in the delay equation example that we discussed above. We get the representation formulas

$$a(t) = E + (\alpha/\pi) \int_{-\infty}^{\infty} e^{-t} \cos \omega t \, \frac{1 - \cos \omega T}{\omega^2 T} \, d\omega, \quad t \ge 0,$$

and

$$\hat{a}(\lambda) = E/\lambda + (\alpha/\pi) \int_{-\infty}^{\infty} \frac{\lambda + 1}{(\lambda + 1)^2 + \omega^2} \, \frac{1 - \cos \omega T}{\omega^2 T} \, d\omega, \quad \Re\lambda > -1, \quad \lambda \ne 0.$$

In this setting, the whole line $\Re z = -1$ will belong to the essential spectrum of $\mathscr{A}$ (this is due to the $H^2(\mu; Y)$-component of the space; one might be able to get rid of this part of the spectrum by factoring out the unobservable subspace).

For this example the estimate in Theorem 17 becomes

$$\begin{aligned} \omega_{\text{ess}}(S) &\le \max\left\{ \sup \sigma_{\text{ess}}(\mathscr{A}), -\frac{1}{2}, \frac{1}{2} \frac{\int_\Gamma \Re z \, \mu(dz)}{\int_\Gamma \mu(dz)} \right\} \\ &= \max\left\{ \sup \sigma_{\text{ess}}(\mathscr{A}), -\frac{\alpha}{2(\alpha + E)} \right\}. \end{aligned}$$

If we here would try to replace $\int_\Gamma \Re z \, \mu(dz) / \int_\Gamma \mu(dz)$ by $a'(0)/a(0)$ (as we have seen, this is allowed if a is smooth enough or monotone enough; for the moment we ignore the term $-\frac{1}{2}$ in the estimate), then we would get

$$\omega_{\text{ess}}(S) \le \max\left\{ \sup \sigma_{\text{ess}}(\mathscr{A}), -\frac{\alpha(1 + 1/T)}{2(\alpha + E)} \right\}.$$

As we shall see in a moment, the true (guaranteed) estimate lies between these two extremes.

As usual, to get an estimate that does not depend on a particular representation formula we turn to Lemma 22. For this lemma we need to know the asymptotic behavior of $\hat{a}$, but this is no problem since we have an analytic expression for $\hat{a}$. An asymptotic expansion shows that

$$\Im\hat{a}(\delta+i\omega) = -\frac{\alpha+E}{\omega} + O(|\omega|^{-2}),$$
$$\Re\hat{a}(\delta+i\omega) = \frac{1}{\omega^2}\left[(\alpha+E)\delta+\alpha+\frac{\alpha}{T}\left(1-e^{-(\delta+1)T}\cos\omega T\right)\right] + O(|\omega|^{-3}).$$

These expansions show that we may choose δ_1 in (38) arbitrarily close to the maximum of -1 and the unique real solution $\underline{\delta}$ of the equation

$$(\alpha+E)\underline{\delta}+\alpha+\frac{\alpha}{T}\left(1-e^{-(\underline{\delta}+1)T}\right) = 0.$$

and that we may choose δ_2 in (39) arbitrarily close to the maximum of -1 and the unique real solution $\overline{\delta}$ of the equation

$$2(\alpha+E)\overline{\delta}+\alpha+\frac{\alpha}{T}\left(1-e^{-(\overline{\delta}+1)T}\right) = 0.$$

Clearly $\underline{\delta} < \overline{\delta}$, and

$$-\frac{\alpha(1+1/T)}{2(\alpha+E)} < \overline{\delta} < -\frac{\alpha}{2(\alpha+E)}.$$

Thus, the correct estimate for $\omega_{ess}(S)$ is

$$\omega_{ess}(S) \le \max\left\{\sup\sigma_{ess}(\mathscr{A}),\, \overline{\delta}\right\},$$

which lies between the two bounds that we originally suggested.

This time too, it is possible to prove a converse inequality in order to get

$$\omega_{ess}(S) = \max\left\{\sup\sigma_{ess}(\mathscr{A}),\, \overline{\delta}\right\}$$

provided the singular value distribution of D is of the "worst possible type". In a proof of a converse inequality one needs the upper bound on β in Lemma 21 to be more or less exact for large values of $\Im\lambda$. Note that, for large values of ω, we have

$$\delta+i\omega+\zeta\hat{a}(\delta+i\omega) = \delta+i\omega+\zeta a(0)/(\delta+i\omega)+O(\omega^{-2}),$$

so the point ω that minimizes the absolute value of this expression is approximately equal to $\omega = \sqrt{\zeta a(0)}$ for large values of ζ. If one wants to show that the estimate above on $\omega_{ess}(S)$ is sharp, then one should choose the sequence ζ_k of eigenvalues of DD^* in such a way that the the function $\kappa(\delta+i\omega)$ is as large as possible at the points $\omega_k = \sqrt{\zeta_k a(0)}$. The bad points of κ are the points where $\omega_k T = 2\pi k$. Thus, if, for example, the eigenvalues of DD^* is some infinite subset of the set

$$\zeta_k = \frac{(2\pi k)^2}{T^2 a(0)},$$

then it looks like we would get the converse inequality, too. We leave it to the reader to verify that this is, indeed, the case.

However, for a different set of eigenvalues of DD^* one gets a different essential growth rate. The best possible case is where the eigenvalues of DD^* are some infinite subset of the set

$$\zeta_k = \frac{\pi^2(2k+1)^2}{T^2 a(0)}.$$

This will revert the sign of the exponential in the definitions of $\underline{\delta}$ and $\overline{\delta}$, and one gets a much better estimate on the essential growth rate of the semigroup. We leave it to the reader to check that it will, in fact (at least for some values of α, E, and T), be even better than the earlier "optimistic" estimate

$$\omega_{\mathrm{ess}}(S) \leq \max\left\{ \sup \sigma_{\mathrm{ess}}(\mathscr{A}), -\frac{\alpha(1+1/T)}{2(\alpha+E)} \right\}.$$

10. Time domain properties of the relaxation modulus

In this section we investigate how certain properties of the measure μ in the representation (26) is reflected in properties of a. Recall that we throughout assume that Γ is contained in the closed left half-plane Π^-, and that the measure μ satisfies (29). At several occasions we shall constrain the support Γ of μ to lie in a certain sector, and restrict the conclusion to some other sector, and in this connection we use the following notation.

Definition 25. For each $\alpha \in \mathbf{R}$ and $0 < \theta \leq \pi$, let $S(\alpha, \theta)$ be the open sector

$$S(\alpha, \theta) = \{ \lambda \in \mathbf{C} \mid \lambda \neq \alpha \text{ and } |\arg(\lambda - \alpha)| < \theta \}.$$

We begin by looking at conditions relating the behavior of a at zero to certain moments of μ.

Theorem 26. *Let a have the representation* (26), *with a positive measure μ satisfying* (29). *Then*

(1) *a is bounded on $\mathbf{R}^+$ if and only if $\int_\Gamma \mu(dz) < \infty$, in which case $a(0) = \int_\Gamma \mu(dz)$,*
(2) *if $\int_\Gamma |z|^n \mu(dz) < \infty$ for some integer $n \geq 0$, then a is n times continuously differentiable on $\mathbf{R}^+$ and*

$$a^{(n)}(t) = \int_\Gamma z^n e^{zt} \mu(dz) = \int_\Gamma \Re[z^n e^{zt}] \mu(dz);$$

in particular, $a'(0) = \int_\Gamma \Re z \, \mu(dz)$ and $a''(0) = \int_\Gamma [(\Re z)^2 - (\Im z)^2] \mu(dz)$ for $n = 1$ and $n = 2$, respectively,
(3) *if a is bounded and a' is essentially bounded from below on $\mathbf{R}^+$, then $\int_\Gamma \Re z \, \mu(dz) \geq -\operatorname{ess\,sup}_{t \in \mathbf{R}^+} -a'(t) > -\infty$; in particular,*

$$a'(0) \leq \int_\Gamma \Re z \, \mu(dz)$$

whenever $a'(0)$ exists (or more precisely, whenever 0 is a Lebesgue point of a'),
(4) *if $\Gamma \cap S(\alpha, \theta) = \emptyset$ for some $\alpha \in \mathbf{R}$ and some $\theta > \pi/2$ (where $S(\alpha, \theta)$ is defined in Definition 25), then a and a' are bounded on $\mathbf{R}^+$ if and only if $\int_\Gamma |z| \mu(dz) < \infty$, in which case $a'(0) = \int_\Gamma z \, \mu(dz) = \int_\Gamma \Re z \, \mu(dz)$,*

(5) *if* $\Gamma \cap S(0, 3\pi/4) = \emptyset$, *if* a *and* a' *are bounded, and if* a'' *is essentially bounded from above on* $\mathbf{R}^+$, *then* $\int_\Gamma [(\Re z)^2 - (\Im z)^2]\, \mu(dz) \le \operatorname{ess\,sup}_{t\in\mathbf{R}^+} a''(t) < \infty$; *in particular,*

$$a''(0) \ge \int_\Gamma [(\Re z)^2 - (\Im z)^2]\, \mu(dz)$$

whenever $a''(0)$ *exists (or more precisely, whenever* 0 *is a Lebesgue point of* a''),

(6) *if* $\Gamma \cap S(\alpha, \theta) = \emptyset$ *for some* $\alpha \in \mathbf{R}$ *and some* $\theta > 3\pi/4$, *then* a, a' *and* a'' *are bounded on* $\mathbf{R}^+$ *if and only if* $\int_\Gamma |z|^2 \mu(dz) < \infty$, *in which case* $a''(0) = \int_\Gamma z^2\, \mu(dz) = \int_\Gamma [(\Re z)^2 - (\Im z)^2]\, \mu(dz)$.

It is possible to proceed even further and prove similar estimates for higher order derivatives of a. We leave this to the reader.

Proof. If $\int_\Gamma \mu(dz) < \infty$, then clearly a is bounded and continuous, and $a(0) = \int_\Gamma \mu(dz)$. If $\int_\Gamma |z|^n \mu(dz) < \infty$, then we may differentiate under the integral sign to get the given formula for $a^{(n)}$. This proves Claim 2 and one half of Claim 1.

To prove the converse part of the first claim, suppose that a is essentially bounded from above. Then, for all $\lambda > 0$,

$$\lambda \hat{a}(\lambda) = \lambda \int_{\mathbf{R}^+} e^{-\lambda t} a(t)\, dt \le M,$$

where $M = \operatorname{ess\,sup}_{t\in\mathbf{R}^+} a(t)$. However, by (32) and Fatou's lemma,

$$M \ge \liminf_{\lambda\to\infty} \lambda \hat{a}(\lambda) = \liminf_{\lambda\to\infty} \int_\Gamma \frac{\lambda(\lambda - \Re z)}{|\lambda - z|^2}\, \mu(dz) \ge \int_\Gamma \mu(dz).$$

This proves that $\int_\Gamma \mu(dz)$ is finite, and completes the proof of Claim 1.

To prove Claim 3 we suppose that a is bounded and that a' is essentially bounded from below. Recall that the Laplace transform of a' is given by $\lambda\hat{a}(\lambda) - a(0)$; hence, for all $\lambda > 0$,

$$\lambda\,(\lambda \hat{a}(\lambda) - a(0)) = \lambda \int_{\mathbf{R}^+} e^{-\lambda t} a'(t)\, dt \ge -M,$$

where this time $M = \operatorname{ess\,sup}_{t\in\mathbf{R}^+} -a'(t)$. By Claim 1, $a(0) = \int_\Gamma \mu(dz)$. Once more we may use (32) and Fatou's lemma to conclude that

$$\begin{aligned} M &\ge \liminf_{\lambda\to\infty} -\lambda\,(\lambda\hat{a}(\lambda) - a(0)) \\ &= \liminf_{\lambda\to\infty} \int_\Gamma -\lambda \left[\frac{\lambda(\lambda - \Re z)}{|\lambda - z|^2} - 1\right] \mu(dz) \\ &= \liminf_{\lambda\to\infty} \int_\Gamma \frac{[-\lambda \Re z(\lambda - \Re z) + \lambda(\Im z)^2]}{|\lambda - z|^2}\, \mu(dz) \\ &\ge \int_\Gamma -\Re z\, \mu(dz). \end{aligned}$$

This proves Claim 3.

To prove Claim 4, note that the assumption gives an upper bound of the type $|\Im z| \le K(|\Re z| + 1)$ for some constant K. Thus Claim 4 follows from Claims 1, 2 and 3.

The proof of Claim 5 is similar to the proof of Claims 1 and 3. First one finds that

$$\lambda\left(\lambda^2\hat{a}(\lambda) - \lambda a(0) - a'(0)\right) = \lambda\int_{\mathbf{R}^+} e^{-\lambda t}a''(t)\,dt \le M,$$

where $M = \operatorname{ess\,sup}_{t\in\mathbf{R}^+} a''(t)$. Thus (the integrand below is positive because of the sector condition)

$$\begin{aligned} M &\ge \liminf_{\lambda\to\infty} \lambda\left(\lambda^2\hat{a}(\lambda) - \lambda a(0) - a'(0)\right) \\ &= \liminf_{\lambda\to\infty} \int_\Gamma \frac{\lambda^2((\Re z)^2 - (\Im z)^2) - \lambda\Re z|z|^2}{|\lambda - z|^2}\,\mu(dz) \\ &\ge \int_\Gamma [(\Re z)^2 - (\Im z)^2]\,\mu(dz). \end{aligned}$$

This proves Claim 5.

The final Claim 6 follows from Claims 1, 2, and 5, since the sector condition implies that $|z|^2 \le K\left[(\Re z)^2 - (\Im z)^2 + 1\right]$ for some constant K. □

Next we observe that a sector condition of the type assumed in Part 4 of Theorem 26 implies that a must be analytic.

Theorem 27. *Let a have the representation* (26), *with a positive measure μ satisfying* (34), *and with support Γ satisfying $\Gamma \cap S(0, \theta) = \emptyset$ for some $\theta > \pi/2$. Then the following claims are true.*

(1) *a is analytic in the sector $S(0, \theta - \pi/2)$, and, for all $t \in S(0, \theta - \pi/2)$, $a'(t) = \int_\Gamma z^n e^{zt}\,\mu(dz)$.*
(2) *In each proper subsector $S(0, \zeta)$, $\zeta < \theta - \pi/2$, and for all $n \ge 0$, $a^{(n)}(t) = O(|t|^{-n-1})$ as $|t| \to 0$ and $a^{(n)}(t) = O(|t|^{-n})$ as $|t| \to \infty$.*
(3) *If, for some $\beta \in [0, 1]$, μ satisifes $\int_\Gamma 1/(1 + |z|^\beta)\,\mu(dz) < \infty$ (note that this is always true for $\beta = 1$), then, in each proper subsector $S(0, \zeta)$, $\zeta < \theta - \pi/2$, and for all $n \ge 0$, $a^{(n)}(t) = O(|t|^{-n-\beta})$ as $|t| \to 0$.*
(4) *If, for some $\beta \in [0, 1]$, μ satisifes $\int_\Gamma 1/(|z|^\beta + |z|)\,\mu(dz) < \infty$ (note that this is always true for $\beta = 0$), then, in each proper subsector $S(0, \zeta)$, $\zeta < \theta - \pi/2$, and for all $n \ge 0$, $a^{(n)}(t) = O(|t|^{-n-\beta})$ as $|t| \to \infty$.*
(5) *Claim* 1 *and all the claims about the asymptotic behavior of a as $|t| \to 0$ remain true if the sector condition on Γ is relaxed to the assumption that $\Gamma \cap S(\alpha, \theta) = \emptyset$ for some $\alpha \in \mathbf{R}^+$ and some $\theta > \pi/2$.*

Proof. Because of the sector condition, for all $z \in \Gamma$ and for all $t \in S(0, \zeta)$ with $\zeta < \theta - \pi/2$, we have

$$\Re zt \le -\epsilon|zt|, \quad \text{where } \epsilon = \sin(\theta - \zeta - \pi/2) > 0.$$

Thus $|e^{zt}| \le e^{-\epsilon|zt|}$, and we may differentiate under the integral sign to show that a is analytic in $S(0, \theta - \pi/2)$, and that

$$a^{(n)}(t) = \int_\Gamma z^n e^{zt}\,\mu(dz).$$

Moreover,

$$|a^{(n)}(t)| \le \int_\Gamma |z|^n e^{-\epsilon|zt|}\,\mu(dz).$$

Since

$$|z|^n(1+|z|)e^{-\epsilon|zt|} = \left(|t|^{-n}|tz|^n + |t|^{-n-1}|tz|^{n+1}\right)e^{-\epsilon|zt|} \le K\left(|t|^{-n} + |t|^{-n-1}\right),$$

where $K = \max_{x\in\mathbf{R}^+} \max\{x^n e^{-\epsilon x}, x^{n+1}e^{-\epsilon x}\}$, we get

$$|a^{(n)}(t)| \le \left(|t|^{-n} + |t|^{-n-1}\right)\int_\Gamma \frac{K\mu(dz)}{1+|z|}.$$

This proves Claim 2.

The proof of Claims 3 and 4 are similar to the proof of Claim 2, since it is true for all nonnegative β and γ that

$$|z|^n(|z|^\beta + |z|^\gamma)e^{-\epsilon|zt|} \le K\left(|t|^{-n-\beta} + |t|^{-n-\gamma}\right),$$

where $K = \max_{x\in\mathbf{R}^+} \max\{x^{n+\beta}e^{-\epsilon x}, x^{n+\gamma}e^{-\epsilon x}\}$.

To prove the final Claim 5 we split Γ into two parts, one bounded part Γ_1, and another part Γ_2 that satisfies $\Gamma_2 \cap S(0, \theta') = \emptyset$; here $\theta' < \theta$ can be chosen arbitrarily close to θ. The function $a_1(t) = \int_{\Gamma_1} e^{zt}\,\mu(dz)$ is entire, and to the function $a_2(t) = \int_{\Gamma_2} e^{zt}\,\mu(dz)$ we can apply the earlier results. Thus, Claim 1 and all the statements about the asymptotic behavior of a at zero remain true. □

11. Estimates on the Laplace transform of the relaxation modulus

As we saw in Section 7, it is crucial to get estimates on $\hat{a}(\lambda)$ that imply that $\nu + \hat{a}(\lambda) \ne 0$, since every point where this function vanishes belongs to the essential spectrum of $\mathscr{A}$. Some estimates of this type are given below.

Theorem 28. *Let $\hat{a} \not\equiv 0$ have the representation* (28), *and suppose that, for some $\alpha \in \mathbf{R}$ and $\pi/2 \le \theta \le \pi$, $\Gamma \cap S(\alpha, \theta) = \emptyset$, where $S(\alpha, \theta)$ is defined in Definition* 25 (*in particular, this is always true for $\theta = \pi/2$*). *Then*

$$\hat{a}(\lambda) \in S(0, \theta) \quad \textit{for all } \lambda \in S(\alpha, \theta),$$

and, unless a is a constant times $e^{\alpha t}$,

$$\Im(\lambda - \alpha)\hat{a}(\lambda) > 0 \quad \textit{for all } \lambda \in S(\alpha, \theta) \textit{ with } \Im\lambda > 0.$$

In particular, the generator $\mathscr{A}$ has no essential spectrum in $S(\alpha, \theta)$ (cf. Theorem 15).

Proof. Without loss of generality, take $\alpha = 0$; i.e., replace $z - \alpha$ by z, $\lambda - \alpha$ by λ, and $e^{\alpha t}a(t)$ by $a(t)$. In particular, the extra condition in the second claim means that a should not be a constant.

Let us begin the proof of the first claim by observing that $\hat{a}(\lambda) \in S(0, \theta)$ if and only if $\Im e^{i\theta}\hat{a}(\lambda) > 0$ (i.e., $-\theta < \arg\hat{a}(\lambda) < \pi - \theta$) or $\Im e^{-i\theta}\hat{a}(\lambda) < 0$ (i.e., $\theta - \pi < \arg\hat{a}(\lambda) < \theta$), and that $\lambda \in S(0, \theta)$ if and only if $\Im e^{i\theta}\lambda > 0$ or $\Im e^{-i\theta}\lambda < 0$ (the trivial case where $\theta = \pi$ and the arguments are zero is left to the reader). Moreover, since $\Gamma \cap S(0, \theta) = \emptyset$, every $z \in \Gamma$ satisfies $\Im e^{i\theta} z \le 0$

and $\Im e^{-i\theta} z \geq 0$, and the same inequalities are true with z replaced by $\overline{z}$. If $\Im e^{-i\theta}\lambda < 0$, then we use (28) to get

$$\begin{aligned}\Im e^{i\theta}\hat{a}(\lambda) &= \frac{1}{2}\Im\int_\Gamma\left(\frac{1}{e^{-i\theta}(\lambda - z)} + \frac{1}{e^{-i\theta}(\lambda-\overline{z})}\right)\mu(dz)\\ &= \frac{1}{2}\int_\Gamma\left(\frac{-\Im(e^{-i\theta}\lambda - e^{-i\theta}z)}{|\lambda - z|^2} + \frac{-\Im(e^{-i\theta}\lambda - e^{-i\theta}\overline{z})}{|\lambda-\overline{z}|^2}\right)\mu(dz)\\ &\geq 0.\end{aligned}$$

If for some λ we do not have strict positivity, then the function above must vanish identically, due to the fact that a nonnegative harmonic function cannot vanish in the interior of its domain without vanishing identically. This means that $\hat{a}$ must be a constant, i.e., $\hat{a}$ must be zero, but this contradicts the assumption that a is nonzero. Thus, in this case $\hat{a}(\lambda) \in S(0, \theta)$. As similar computation shows that $\Im e^{-i\theta}\hat{a}(\lambda) < 0$ whenever $\Im e^{i\theta}\lambda > 0$. Thus, in both cases, $\hat{a}(\lambda) \in S(0, \theta)$.

Next we turn to the second claim. Assume first that $\pi - \theta < \arg\lambda < \theta$, and recall that for all $z \in \Gamma$, $\theta \leq \arg z \leq 2\pi - \theta$. This implies that, for all such λ and z,

$$0 \leq \arg(z/\lambda) \leq \pi,$$

i.e.,

$$\Im(z/\lambda) \geq 0.$$

Now use this and (28) to get

$$\begin{aligned}\Im\lambda\hat{a}(\lambda) &= \frac{1}{2}\Im\int_\Gamma\left(\frac{\lambda}{\lambda - z} + \frac{\lambda}{\lambda - \overline{z}}\right)\mu(dz)\\ &= \frac{1}{2}\Im\int_\Gamma\left(\frac{1}{1 - z/\lambda} + \frac{1}{1-\overline{z}/\lambda}\right)\mu(dz)\\ &\geq 0.\end{aligned}$$

Again, we must in fact have strict inequality unless $\lambda\hat{a}(\lambda)$ is a constant. Thus, we have proved the second claim for those values of λ where $\pi - \theta < \arg\lambda < \theta$. To complete the proof we still have to treat (for example) the case where $0 < \arg\lambda < \pi/2$. This means that, for the rest of this proof, we may assume that $\theta = \pi/2$.

To simplify the proof in the remaining case where $\theta = \pi/2$ we shall shift to a different representation for a, namely (2), and use the symmetrized formula (5) for $\hat{a}$ (up to now we have not really used the symmetry of μ in this proof, except to ensure convergence of the integrals, but it now becomes crucial). Then

$$\lambda\hat{a}(\lambda) = \int_{-\infty}^{\infty}\frac{\lambda^2}{\lambda^2+\omega^2}\,\mu(d\omega), \quad \Re\lambda > 0.$$

Since $0 < \arg\lambda < \pi/2$, we have for all $\omega \neq 0$, $0 < \arg(\lambda^2 + \omega^2) < \arg\lambda^2 < \pi$. Thus, the integrand in the formula above satisfies $0 < \arg[\lambda^2/(\lambda^2+\omega^2)] < \pi$; i.e., $\Im[\lambda^2/(\lambda^2+\omega^2)] > 0$. Since μ is not allowed to vanish completely in $\mathbf{R}\setminus\{0\}$, we get $\Im\lambda\hat{a}(\lambda) > 0$ for $0 < \arg\lambda < \pi/2$. □

It seems to be a fairly well established experimental fact that the loss angle, i.e., $-\arg \hat{a}(\lambda)$, should lie between 0 and $\pi/2$ for $0 < \arg\lambda < \pi/2$. This is true for all completely monotone kernels (see the proof of the first claim in Theorem 28 when $\theta = \pi$). The fact that all our relaxation modules are of positive type guarantees that the loss angle lies between $-\pi/2$ and $\pi/2$, but it does not guarantee that it is positive. For this we need an additional condition.

Theorem 29. *Define* $S(\alpha, \theta)$ *as in Definition* 2.5. *Let* $a \not\equiv 0$ *have the representation* (26). *Let* $\pi/2 \le \theta \le \pi$ *and* $3\pi/4 \le \vartheta \le \pi$, *and suppose that* $|\tan\vartheta| \le \sin\theta$. *If* $\Gamma \cap S(\alpha, \vartheta) = \emptyset$, *then* $\Im\hat{a}(\lambda) < 0$ *for all* $\lambda \in S(\alpha, \theta)$ *with* $\Im\lambda > 0$. *In particular, if* $\Gamma \cap S(\alpha, 3\pi/4) = \emptyset$, *then* $\Im\hat{a}(\lambda) < 0$ *for all* $\lambda \ne \alpha$ *with* $0 < \arg(\lambda - \alpha) < \pi/2$.

Proof. Again we assume without loss of generality that $\alpha = 0$. As we did at the end of the proof of Theorem 28, we now use a symmetrized version of the formula for $\hat{a}$, i.e., we use (28). Clearly, by (31), if we can show that

(49)
$$-\Im\lambda\big((\Re(\lambda - z))^2 + (\Im\lambda)^2 - (\Im z)^2\big) \le 0, \quad z \notin S(0, \vartheta), \quad 0 < \arg(\lambda - \alpha) < \theta,$$

then (31) gives $\Im\hat{a}(\lambda) \le 0$ for all $\lambda \in S(0, \theta)$. As usual, we get strict inequality because $\hat{a}$ is not allowed to be a constant. Thus, it only remains to prove (49).

Split z into $z = x + iy$ and λ into $\lambda = \gamma + i\delta$. Then

$$-\Im\lambda\big((\Re(\lambda - z))^2 + (\Im\lambda)^2 - (\Im z)^2\big) = -\delta((\gamma - x)^2 + \delta^2 - y^2).$$

Since $\delta > 0$, we have proved (49) if we can show that $(\gamma - x)^2 + \delta^2 - y^2 \ge 0$. Let us here substitute $\gamma = s\cos\zeta$ and $\delta = s\sin\zeta$, where $s > 0$ and $0 < \zeta < \theta$. Then the problem is to show that

$$(s\cos\zeta - x)^2 + s^2\sin^2\zeta - y^2 = s^2 + x^2 - y^2 - 2sx\cos\zeta \ge 0.$$

Letting $s \to 0$ we see that a necessary condition is that $|x| \ge |y|$; this condition is satisfied because of our assumptions that $\vartheta \ge 3\pi/4$ and that $\Gamma \cup S(0, \vartheta) = \emptyset$. Since this condition holds, and since $x \le 0$, the inequality above is true if $\cos\zeta \ge 0$, i.e., if $\zeta \le \pi/2$. If not, then we complete the square to get

$$s^2 + x^2 - y^2 - 2sx\cos\zeta = (s - x\cos\zeta)^2 + x^2\sin^2\zeta - y^2 \ge x^2\sin^2\zeta - y^2.$$

Recall that, by our assumption on Γ, $|y| \le |x|\tan\vartheta$, and observe that $\sin\zeta \ge \sin\theta$. Moreover, we assumed that $|\tan\vartheta| \le \sin\theta$. Thus, we can continue the inequality above to get

$$x^2\sin^2\zeta - y^2 \ge x^2(\sin^2\theta - \tan^2\vartheta) \ge 0.$$

The proof is complete. □

The condition on the sectors in Theorem 29 is sharp. This is easily seen from the example $a(t) = e^{\alpha t}\cos\beta t$. Denote $\alpha + i\beta$ by z, and split λ into $\lambda = \delta + i\omega$. Then the imaginary part of $\hat{a}(\lambda)$ is (cf. (31))

$$\Im\hat{a}(\lambda) = \frac{-\omega((\delta - \alpha)^2 + \omega^2 - \beta^2)}{|\lambda - z|^2|\lambda - \overline{z}|^2}.$$

If, for example, $\beta > \alpha$, then $\Im\hat{a}(i\omega)$ will be positive for small positive ω.

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