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FACTORING L-FUNCTIONS AS PRODUCTS OF L-FUNCTIONS

DOUGLAS GRENIER

ABSTRACT. We will demonstrate two factorizations of L-functions associated with automorphic forms on $GL(n, \mathbf{R})$, where one factor is a Riemann zeta-function and the other is an L-function associated to an automorphic form for $GL(n-1, \mathbf{R})$. These will be obtained by establishing the commutation of the Hecke operators and the Φ-operator, a homomorphism from automorphic forms on $GL(n, \mathbf{R})$ to automorphic forms on $GL(n-1, \mathbf{R})$.

1. INTRODUCTION

L-functions associated to automorphic forms are among the most intensely studied objects in number theory. Of course, it was Hecke who first systematically put L-functions in this context, in his case with modular forms. Since Hecke's time this area of study has been greatly expanded by many people, too numerous to begin to name here, to automorphic forms on Lie groups. One of the most important examples is with $GL(n, \mathbf{R})$ as the Lie group, and it is the L-functions associated to this type of automorphic form, specifically those forms invariant on $\Gamma_n = GL(n, \mathbb{Z})$, with which we will be concerned. While some of the results herein may be known in principle from the representation theory of $GL(n)$, by sticking closely to the classical style of Hecke it is possible to observe some explicit relations between the parameters of the L-functions, one of our goals.

For $n \geq 2$, let $G = GL(n, \mathbf{R})$, $K = O(n)$ and let C be the center of G which consists of matrices of the form αI, $\alpha \in \mathbf{R} - \{0\}$. ($I$ is the identity matrix; when necessary for clarity we will use I_n to denote the $n \times n$ identity matrix.) Set $H_n = G/CK$. When $n = 2$ this space is well-known as the hyperbolic upper half-plane. To exploit this we will use the following parametrization of H_n, which comes from the Iwasawa decomposition of G. For any $Z \in H_n$, write

(1)

$$Z = \begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 \cdots y_{n-1} & & & & \\ & y_2 \cdots y_{n-1} & & & \\ & & \ddots & & \\ & & & y_{n-1} & \\ & & & & 1 \end{pmatrix},$$

$$x_{ij} \in \mathbf{R}, \quad y_j > 0.$$

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In this setting Γ_n will act on H_n by left multiplication. At certain times we will be particularly interested in a representative in G/ZK of determinant one. When this is the case we write

$$Z = y^{-1}\begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1 \cdots y_{n-1} & & & & \\ & y_2 \cdots y_{n-1} & & & \\ & & \ddots & & \\ & & & y_{n-1} & \\ & & & & 1 \end{pmatrix}.$$

Clearly $y^n = y_1 y_2^2 \cdots y_{n-1}^{n-1}$.

We may define a function on H_n, the power function $p_\nu(Z)$, for $\nu \in \mathbb{C}^{n-1}$, $Z \in H_n$. First, define $r_1, \ldots, r_n \in \mathbb{C}$ by

$$r_i - r_{i+1} = \nu_i - 1/2\,, \qquad i = 1, \ldots, n-1;\ r_1 + \cdots + r_n = 0$$

(these new parameters, originally introduced by Selberg, [11], will be convenient in several places in the sequel.) If in addition, we let $R_j = r_1 + \cdots + r_j - j(n-j)/4$, then the power function is given by

$$p_\nu(Z) = \prod_{j=1}^{n-1} y_j^{R_j}. \tag{2}$$

We will understand an automorphic form of type ν to be a complex-valued function, f, on H_n satisfying:

- **(D1):** f is an eigenfunction of all $GL_n(\mathbf{R})$-invariant differential operators, and if D represents any such operator, $Df = \lambda_D f$ where λ_D is given by $Dp_\nu = \lambda_D p_\nu$.
- **(D2):** $f(gZ) = f(Z)$ for all $g \in GL_n(\mathbb{Z})$ and $Z \in H_n$
- **(D3):** f has at most polynomial growth, i.e., there are constants $c > 0$ and $r \in \mathbf{R}^{n-1}$ such that $|f(z)| \leq cp_r(Z)$ as each $Z_j \to \infty$

This definition generalizes the nonholomorphic modular forms of Maass, that is, when $n = 2$ these automorphic forms are exactly the Maass forms. An automorphic form, f, is called a cusp form if it satisfies for each $m = 1, 2, \ldots, n-1$:

$$\int_{[0,1]^{m\times n-m}} f\left(\begin{pmatrix} I_m & Q \\ O & I_{n-m} \end{pmatrix} Z\right) dQ = 0\,, \qquad dQ = \prod_{\substack{1\leq i\leq m \\ 1\leq j\leq m-n}} dq_{ij}.$$

To define an L-function associated to an automorphic form on H_n, we need the Hecke operators. Much of the information we need on Hecke operators for $GL(n, \mathbf{R})$ comes from the work of Shimura [13] and Tamagawa [14]. We will discuss this in more detail in §4. For now, suffice it to say that we need only consider Hecke operators $T_{p,\ell}$ defined as follows. For any prime p and $1 \leq \ell \leq n$, let $D_{p,\ell}$ be the $n \times n$ matrix

$$D_{p,\ell} = \begin{pmatrix} p & & & & & \\ & \ddots & & & & \\ & & p & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where there are ℓ entries p and $n-\ell$ entries 1. The double coset $\Gamma_n D_{p,\ell} \Gamma_n$ can be decomposed as $\Gamma_n D_{p,\ell} \Gamma_n = \bigcup_{j=1}^{k} \Gamma_n A_j$. Then the Hecke operator $T_{p,\ell}$ maps automorphic forms on H_n to automorphic forms on H_n by

$$T_{p,\ell} f(Z) = \sum_{j=1}^{k} f(A_j Z).$$

If f is an eigenfunction of these Hecke operators write $T_{p,\ell} f = \lambda_{p,\ell} f$, and normalize via $\lambda_{p,\ell} = p^{\ell(n-\ell)/2} \mu_{p,\ell}$. (Why will become clear later.) The L-functions we will be examining may now be defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large as

$$(3) \qquad L_f(s) = \prod_p (1 - \mu_{p,1} p^{-s} + \mu_{p,2} p^{-2s} + \cdots + (-1)^n p^{-ns})^{-1}.$$

The main purpose of this paper is to show that for certain automorphic forms the L-function (3) factors in two different ways into a product of a Riemann zeta function and an L-function associated to an automorphic form on H_{n-1}. This provides a new approach to determining the L-functions associated to certain Eisenstein series for example.

First, we briefly consider the example of Siegel modular forms for comparison and to provide a little motivation. Temporarily, let H_n denote the generalized upper half-plane of Siegel, i.e.

$$H_n = \{Z \in M_n(\mathbb{R}) | {}^T Z = Z;\ Z = X + iY,\ Y \text{ is positive}\}.$$

The symplectic group $Sp(n, \mathbb{R})$, defined by

$$Sp(n, \mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| A, B, C, D \in M_n(\mathbb{R}) \right.$$
$$\left. {}^T M J_n M = J_n,\ J_n \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\},$$

acts on H_n in analogous fashion to the action of $SL(2, \mathbb{R})$ on the classical upper half-plane, that is, $MZ = (AZ + B)(CZ + D)^{-1}$. A Siegel modular form of weight k is a holomorphic function on H_n satisfying the transformation formula

$$f(MZ) = |CZ + D|^k f(Z) \quad \forall M \in Sp(n, \mathbb{Z})$$

where the Siegel modular group $Sp(n, \mathbb{Z}) = Sp(n, \mathbb{R}) \cap M_{2n}(\mathbb{Z})$. When $n = 1$, they coincide with the classical modular forms.

Siegel's Φ-operator is a homomorphism from the space of modular forms on H_n onto the space of modular forms on H_{n-1}, defined by

$$\Phi f(Z_1) = \lim_{\alpha\to\infty} f\begin{pmatrix} Z_1 & 0 \\ 0 & i\alpha \end{pmatrix}, \qquad Z_1 \in H_{n-1}. \tag{4}$$

A Siegel modular form is called a cusp form if $\Phi f \equiv 0$.

We use the Hecke operators for the Siegel modular forms to define L-functions associated to these forms. These Hecke operators are somewhat similar to those for $GL(n, \mathbb{Z})$ discussed above. After the L-function of a Siegel modular form f, $L_f(s)$, is suitably defined we come to the work of Andrianov [1] and Zharkovskaya [17]. They demonstrated that the Φ-operator commutes with the Hecke operators (Maass [9] originally proved the commutation formulas in the case $n = 2$) and obtained the following result (Andrianov for $n = 2$ and Zharkovskaya in general): if f is a Siegel modular form which is not a cusp form, then

$$L_f(s) = L_{\Phi f}(s) L_{\Phi f}(s - k + n).$$

Even when $n = 1$ we see this relationship if we let f be the Eisenstein series

$$E_k(z) = \sum_{(m,n)=1} (mz+n)^{-k}, \qquad z \in H.$$

Then $L_f(s) = \zeta(s)\zeta(s-k+n)$ where ζ is Riemann's zeta function.

For the Maass forms setting, if we let f be the nonholomorphic Eisenstein series on H,

$$E_\nu(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^\nu = \sum_{(m,n)=1} \frac{y^\nu}{|mz+n|^{2\nu}},$$

we have $L_f(s) = \zeta(s-\nu+1/2)\zeta(s+\nu-1/2)$. Seeing this similarity, one might expect that we should be able to obtain a result on factoring L-functions for $GL(n, \mathbb{Z})$ somewhat analogous to the Siegel modular forms case.

For the rest of the paper we return to H_n being defined as the symmetric space G/CK.

2. Fourier expansions of automorphic forms

Fourier expansions of automorphic forms were first obtained independently by Shalika [12] and Piatetski-Shapiro [10] in the adelic setting, although the latter was lacking some of the archimedean theory. In describing the Fourier expansions for $GL(n, \mathbb{R})$ we will adopt a style somewhat akin to Bump's in [2, 3]. These series will be in terms of the Whittaker functions of Jacquet [6], which may be defined as follows:

Let

$$\omega = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

Also, recall that we have

$$r_i - r_{i+1} = \nu_i - 1/2, \qquad i = 1, \ldots, n-1; \; r_1 + \cdots + r_n = 0,$$

and let

$$\Lambda_n(\nu) = \prod_{1\le i<j\le n} \Lambda(r_i - r_j + 1/2) \;; \quad \Lambda(r) = \pi^{-r}\Gamma(r)\zeta(2r).$$

Then for $N \in \mathbb{R}^{n-1}$ with no $N_i = 0$ we have

$$W_n(\nu, N, Y) = \Lambda_n(\nu) \int_{\mathbb{R}^{n(n-1)/2}} p_\nu(\omega Z) e\left(-\sum_{i=1}^{n-1} N_i x_{i\,i+1}\right) dX \tag{5}$$

where $e(x) = e^{2\pi ix}$ and $dX = \prod_{1\le i<j\le n} dx_{ij}$. $W_n(\nu, N, Y)$ is easily seen to be an eigenfunction of the G-invariant differential operators as in (D1), satisfies the transformation formula

$$W_n(\nu, N, XY) = e\left(\sum_{i=1}^{n-1} N_i x_{i\,i+1}\right) W_n(\nu, N, Y),$$

and is of at most polynomial growth in the y_i as all $y_i \to \infty$. It was shown by Kostant, [7], (and is also an unpublished result of Casselman and Zuckerman) that there are $n!$ independent solutions to the differential equations of (D1), but the multiplicity one theorem of Shalika, [12], says that (5) is the unique solution, up to constant multiples, satisfying the growth condition, at least in certain circumstances. The referee has pointed out that here one actually needs the more general result of Wallach in [16]. Theorem 8.8 of that paper generalizes Shalika's multiplicity one theorem to the case we need. Certainly the Whittaker function easily meets the polynomial growth condition; it is actually of rapid decay as the $y_i \to \infty$.

To obtain the Fourier expansions we need here, we will have to translate from adelic language to $GL(n, \mathbb{R})$, but at the same time expand these results to include automorphic forms which are not necessarily cusp forms. For a partition $n = n_1 + \cdots + n_k$, let $P(n_1, \dots, n_k)$ be the subgroup of Γ_n of matrices of the form

$$\begin{pmatrix} g_1 & & * \\ & \ddots & \\ 0 & & g_k \end{pmatrix}$$

with $g_j \in \Gamma_{n_j}$, in other words, the intersection of a standard parabolic subgroup of $GL(n, \mathbb{R})$ with Γ_n. Of course, for the trivial partition we have $P(n) = \Gamma_n$. With this notation one may write the Fourier expansion of an automorphic form f.

Theorem 1. *If f is an automorphic form then f has a Fourier series expansion*

$$f(Z) = \sum_{\substack{g\in P(1,\dots,1,i_2-i_1,\dots,n-i_k)\backslash P(1,\dots,1,n-i_1) \\ N_{i_1}\neq 0,\dots,N_{i_k}\neq 0}} a_N(gZ) \tag{6}$$

where the sum is over g as described and $N \in \mathbb{Z}^{n-1}$ with $N_{i_1}, \dots, N_{i_k} \neq 0$ and all other $N_i = 0$ for all $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n-1\}$ with $1 \le i_1 < i_2 <$

$\cdots < i_k \le n-1$. *The coefficients* a_N *satisfy*

$$a_N(Z) = \int_{[0,1]^{n(n-1)/2}} f\left(\begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} Z\right) e\left(-\sum_{i=1}^{n-1} N_i u_{ii+1}\right) dU$$

We sketch a proof of this. We will need

Lemma 1. *Since* $f(Z)$ *is invariant under the subgroup of* Γ_n *consisting of matrices of the form*

$$\begin{pmatrix} 1 & {}^T b \\ 0 & I \end{pmatrix}$$

with $b \in \mathbb{Z}^{n-1}$, *there is a Fourier expression*

$$f(Z) = \sum_{N \in \mathbb{Z}^{n-1}} c_N(Z) \tag{7}$$

where $c_N\left(\left(\begin{smallmatrix} 1 & {}^T u \\ 0 & I \end{smallmatrix}\right) Z\right) = c_N(Z) e({}^T u N)$ *for* $u \in \mathbb{R}^{n-1}$. *For any* $g \in \Gamma_{n-1}$ *we have*

$$c_{g^{-1}N}(Z) = c_N\left(\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} Z\right). \tag{8}$$

There is a proof of this in [4].

Starting with the Fourier expansion (7), for each term $c_N(Z)$ with $N \neq 0$, there is a $g \in \Gamma_{n-1}$ such that N can be written as $N = gN'$ where $N' = {}^T(N_1, 0, \ldots, 0)$, $N_1 > 0$. N_1 is the gcd of N. If $g \in P(1, n-2)$, $gN' = \pm N'$. Thus, combining (7) and (8), and observing that the set of matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right)$ with $g \in P(1, n-2)$ is isomorphic to $P(1, 1, n-2)\backslash P(1, n-1)$, we can now write

$$f(Z) = a_0(Z) + \sum_{N_1 \neq 0} \sum_{g \in P(1,1,n-2)\backslash P(1,n-1)} a_{N_1}(gZ) \tag{9}$$

and $a_{N_1}(Z) = c_{N'}(Z)$ with N' as above. Now for any N_1, including $N_1 = 0$, $a_{N_1}(Z)$ is invariant under the subgroup of Γ_n containing those matrices of the form

$$\begin{pmatrix} 1 & 0 & \\ & 1 & {}^T b \\ & & I \end{pmatrix}, \qquad b \in \mathbb{Z}^{n-2}.$$

Thus we can continue to repeat the above procedure using the corresponding versions of Lemma 1, eventually obtaining Theorem 1.

If f is a cusp form the Fourier expansion is simpler to write. The terms $a_N(gZ)$ in (6) with any of the $N_i = 0$ vanish, reducing (6) to

$$f(Z) = \sum_{\substack{N \in \mathbb{Z}^{n-1} \\ N_i \neq 0}} \sum_{g \in P(1,\ldots,1)\backslash P(1,n-1)} a_N(gZ).$$

In this case by multiplicity one it is seen that $a_N(Z)$ is a constant multiple of $W_n(\nu, N, Z)$ if f is an automorphic form of type ν. For convenience when the Hecke operators enter the picture, it is desirable to write

$$a_N(Z) = a_N \prod_{j=1}^{n-1} |N_j|^{-j(n-j)/2} W_n(\nu, N, Z).$$

When f is not necessarily a cusp form, the $a_N(gZ)$ can still be expressed in terms of Whittaker functions, but the situation is more complicated. If $N \in \mathbb{Z}^{n-1}$ has $N_{j_1} = N_{j_2} = \cdots = N_{j_{n-k-1}} = 0$, $a_N(Z)$ is formed from a product of powers of the y_i from (2) and Whittaker functions W_{j_1}, $W_{j_2-j_1}$, etc. The terms of this form to which we will need to pay particularly close attention will be those with $N_1 = 0$. Of those, the $a_N(gZ)$ with $N_1 = 0$ and all other $N_i \neq 0$ can serve as prototype; the necessary results involving the others would follow by analogy. In this situation it is convenient to rewrite the Iwasawa decomposition of Z, (1), as

$$Z = \begin{pmatrix} 1 & {}^T x \\ 0 & I \end{pmatrix} \begin{pmatrix} z & O \\ O & z^{-1/(n-1)} Z' \end{pmatrix}$$

where $x \in \mathbb{R}^{n-1}$, $z > 0$, and $Z' \in H_{n-1}$ is a representative of determinant one. In terms of these "partial Iwasawa coordinates,"

$$p_\nu(Z) = z^{r(\nu)} p_{\nu'}(Z'),$$

where $r(\nu) = \frac{2}{n-1} \sum_{k=1}^{n-1} (n-k)\nu_k$. It may also be observed that

$$r(\nu) = \frac{n}{2} - \frac{2n}{n-1} r_n,$$

which ties in with how the r_i were defined. Throughout the paper, given a vector $v \in \mathbb{C}^k$, let v' be the vector in $\mathbb{C}^{k-1}$ obtained by dropping the first component of v. We have specifically then, for $N = {}^T(0, N_2, N_3, \ldots, N_{n-1})$ with $N_i \neq 0$, $i = 2, \ldots, n-1$, that

$$\text{(10)} \qquad a_N(Z) = \sum_{\sigma \in S_n} a_N^\sigma \prod_{j=2}^{n-1} |N_j|^{-(j-1)(n-j)/2} z^{\sigma(r(\nu))} W_{n-1}(\sigma(\nu)', N', Z'),$$

where the action of $\sigma \in S_n$, the group of permutations on n letters, on ν is best described as permuting the r_i associated to ν. This was Selberg's motivation for introducing these auxiliary parameters.

Closely related to these terms are the $a_N(Z)$ where only the last component of N is 0. If for any $v \in \mathbb{C}^k$, we denote by v^* the vector obtained by dropping the last component of v, and define Z^* accordingly, we may write for $a_N(Z)$ with only $N_{n-1} = 0$:

$$a_N(Z) = \sum_{\sigma \in s_n} a_N^\sigma \sum_{j=2}^{n-2} |N_j|^{-j(n-j-1)/2} Y^{\sigma(r^*(\nu))} W_{n-1}(\sigma(\nu)^*, N^*, Z^*)$$

where $r^*(\nu) = \frac{2}{n-1} \sum_{k=1}^{n-1} k\nu_k$. (In terms of the auxiliary parameters, $r^*(\nu) = \frac{2n}{n-1} r_1 - \frac{n}{2}$.) These latter will play the most important role in determining the second factorization of the L-functions.

3. Analogues of Siegel's Φ-operator

We briefly summarize the results of [4] introducing an analogue of the Φ-operator and indicate a second such operator closely tied to the first.

The $r(\nu)$ and $r^*(\nu)$ above come from the power function $p_\nu(Z)$. In terms of the notation at the end of the previous section we have already seen that for $Z \in H_n$

$$p_\nu(Z) = z^{r(\nu)} p_{\nu'}(Z')$$

and similarly we have

$$p_\nu(Z) = y^{r^*(\nu)} p_{\nu^*}(Z^*) .$$

Put in other words, $z^{-r(\nu)} p_\nu(Z) = p_{\nu'}(Z')$ a power function on H_{n-1}, a function in this case of $y_2, \ldots, y_{n-1}$. Similarly, $y^{-r^*(\nu)} p_\nu(Z)$ is also a power function on H_{n-1}, but a function of $y_1, \ldots, y_{n-2}$. Since automorphic forms are asymptotically bounded by the power functions we are led to

Theorem 2. *Define Φf on H_{n-1} by*

$$\Phi f(Z') = \lim_{y_1 \to \infty} z^{-r(\nu)} f\left(\begin{pmatrix} z & \\ & z^{-\frac{1}{n-1}} Z' \end{pmatrix}\right) .$$

Φ is a homomorphism of the space of automorphic forms on H_n onto the space of automorphic forms on H_{n-1}. If f is a cusp form, $\Phi f \equiv 0$. However, it is possible to have $\Phi f \equiv 0$ even if f is not a cusp form (see [4]).

We can also define the related operator $\tilde{\Phi}$ by

$$\tilde{\Phi} f(Z') = \lim_{y_{n-1} \to \infty} y^{-r^*(\nu)} f\left(\begin{pmatrix} y^{\frac{1}{n-1}} Z' & \\ & y^{-1} \end{pmatrix}\right) .$$

$\tilde{\Phi}$ is also a homomorphism from the space of automorphic forms on H_n onto the space of automorphic forms on H_{n-1}.

It is possible to describe explicitly the relation between Φ and $\tilde{\Phi}$. The map $Z \to \omega\, {}^T Z^{-1} \omega$ is an involution of H_n. For an automorphic form f on H_n, there is the so-called contragredient form $\tilde{f}$ associated to f via $\tilde{f}(Z) = f(\omega\, {}^T Z^{-1} \omega)$. Bringing in the Φ-operators gives $\tilde{\Phi} \tilde{f} = \widetilde{(\Phi f)}$ or equivalently $\tilde{\Phi} f = \widetilde{(\Phi \tilde{f})}$. This can be seen as follows:

$$\begin{aligned}
\tilde{\Phi} \tilde{f}(Z_1) &= \lim_{y_{n-1} \to \infty} \tilde{f}\left(\begin{pmatrix} y^{\frac{1}{n-1}} Z_1 & \\ & y^{-1} \end{pmatrix}\right) \\
&= \lim_{y_{n-1} \to \infty} f\left(\omega_n \begin{pmatrix} y^{\frac{1}{n-1}}\, {}^T Z_1 & \\ & y^{-1} \end{pmatrix}^{-1} \omega_n\right) \\
&= \lim_{y_1 \to \infty} f\left(\begin{pmatrix} y & \\ & y^{-\frac{1}{n-1}} \omega_{n-1}\, {}^T Z_1^{-1} \omega_{n-1} \end{pmatrix}\right) \\
&= \Phi f(\omega_{n-1}\, {}^T Z_1^{-1} \omega_{n-1}) = \widetilde{(\Phi f)}(Z_1) .
\end{aligned}$$

It is also possible to define other Φ-operators, as touched on in [4], one for each y_i so that there are $n-1$ of them. These other Φ-operators will not be discussed further here, but they and the corresponding L-function results are the subject of the forthcoming [5].

4. Hecke operators

In this section we collect the necessary information on the Hecke operators that act on automorphic forms for Γ_n. As indicated earlier, most of this is widely known from the work of Tamagawa [14] and Shimura (see [13]). The most comprehensive survey of the theory of Hecke algebras seems to be [8], so we will primarily follow the notation therein, and any details omitted here may be found in that reference.

First we need to briefly describe the Hecke algebra. Let $\mathrm{Inv}(n, \mathbb{Z})$ denote the semigroup of $n \times n$ nonsingular integral matrices. The Hecke algebra associated with the pair $(\Gamma_n, \mathrm{Inv}(n, \mathbb{Z}))$ which will be denoted $\mathscr{H}(\Gamma_n, \mathrm{Inv}(n, \mathbb{Z}))$ is the set of all finite linear combinations of double cosets $\Gamma_n \backslash M / \Gamma_n$, $M \in \mathrm{Inv}(n, \mathbb{Z})$, i.e. an element T of the Hecke algebra looks like

$$T = \sum_{A\Gamma_n \backslash M/\Gamma_n} t(\Gamma_n A \Gamma_n)\Gamma_n A \Gamma_n$$

where A runs through a set of representatives in M of the double cosets, $t(\Gamma_n A \Gamma_n) \in \mathbb{Z}$ and $t(\Gamma_n A \Gamma_n) = 0$ for all but finitely many A. This $\mathbb{Z}$-algebra has the obvious addition and a somewhat more complicated multiplication. However, for $A, B \in \mathrm{Inv}(n, \mathbb{Z})$ with $|A|$ and $|B|$ relatively prime, the multiplication is simple:

$$(\Gamma_n A \Gamma_n) \cdot (\Gamma_n B \Gamma_n) = \Gamma_n (AB) \Gamma_n .$$

The Hecke operators come from this Hecke algebra by considering the decomposition of a double coset into right cosets. For example, if

$$\Gamma_n A \Gamma_n = \bigcup \Gamma_n A_i$$

then the Hecke operator T_A acting on an automorphic form f is defined by

$$T_A f(Z) = \sum f(A_i Z) . \tag{11}$$

Two particular types of elements of the Hecke algebra will be of concern here. First, for a positive integer m, let $D_n(m)$ be the set given by

$$D_n(m) = \{A \in \mathbb{Z}^{n\times n} : \|A\| = m\}$$

where $\|A\|$ means the absolute value of the determinant of A, $|A|$. Now set

$$T(m) = \sum_{A : \Gamma_n \backslash D_n(m)/\Gamma_n} \Gamma_n A \Gamma_n .$$

There is also multiplicativity here: if $(\ell, m) = 1$, $T(\ell) \cdot T(m) = T(\ell m)$. The second type of element of $\mathscr{H}(\Gamma_n, \mathrm{Inv}(n, \mathbb{Z}))$ needed was defined in the introduction, namely

$$T(p, \ell) = \Gamma_n D_n(p, \ell)\Gamma_n$$

where

$$D_n(p, \ell) = \begin{pmatrix} p & & & & & \\ & \ddots & & & & \\ & & p & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} \ell \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} n-\ell \end{array}$$

The latter are important since $\mathscr{H}(\Gamma_n, \mathrm{Inv}(n, \mathbf{Z}))$ decomposes into a tensor product

$$\mathscr{H}(\Gamma_n, \mathrm{Inv}(n, \mathbf{Z})) = \bigotimes_p \mathscr{H}(n, p)$$

where $\mathscr{H}(n, p)$ is the Hecke algebra $\mathscr{H}(\Gamma_n, \mathrm{Inv}(n, \mathbf{Z})_p)$ with

$$\mathrm{Inv}(n, \mathbf{Z})_p = \{A \in \mathbf{Z}^{n\times n} : \|A\| = p^\ell ; \ell = 0, 1, 2, \ldots\}.$$

It turns out that $T(p, 0), \ldots, T(p, n)$ generate the $\mathbf{Z}$-algebra $\mathscr{H}(n, p)$. The $T(m)$ play a role because they are easier to work with when determining the action of the associated Hecke operators on automorphic forms. The two types are related by the Rationality Theorem of Tamagawa, [14]:

Theorem 3. *If X is an indeterminate, the following formal identity holds:*

$$\sum_{k=0}^{\infty} T(p^k)X^k = \left(\sum_{j=0}^{n}(-1)^j p^{j(j-1)/2} T(p, j)X^j\right)^{-1}.$$

Corollary. *The identity*

$$\sum_{m=1}^{\infty} T(m)m^{-s} = \prod_p \left(\sum_{j=0}^{n}(-1)^j T(p, j)p^{(j(j+1)/2)-js}\right)^{-1}$$

holds formally in s.

This is the formal Euler product which leads to the definition of the L-function associated to an automorphic form. Later we will apply the Rationality Theorem to obtain another useful relation between the two types of elements above.

Denote by T_m and $T_{p,\ell}$ (or by T_m^n and $T_{p,\ell}^n$ when considering different n) the Hecke operators corresponding to $T(m)$ and $T(p, \ell)$ respectively. Applying (11), T_m can be defined as

$$T_m f(Z) = \sum_k f(A_k Z) \tag{12}$$

where it is well-known from the elementary divisor theory of $n \times n$ integral matrices that the representatives A_k from $\Gamma_n D_n(m)\Gamma_n = \bigcup_k \Gamma_n A_k$ may be taken to be the set of matrices of the form (d_{ij}), $d_{ij} = 0$ if $i > j$, $d_{ii} > 0$,

$\prod_{i=1}^{n} d_{ii} = m$, and $0 \le d_{ij} < d_{jj}$ for $i < j$. From here on, set $d_i = d_{ii}$. It is also easily possible to find a set of representatives for the right coset decomposition of $T(p, j)$ so as to define the Hecke operators $T_{p,j}$, but instead we will write the $T_{p,j}$ in terms of the Hecke operators (12) by use of the following:

Lemma 2.

$$(-1)^{\ell-1}p^{\ell(\ell-1)/2}T(p, \ell) = \sum_{\substack{1\le k_1, \dots, k_j \le \ell \\ k_1+\dots+k_j=\ell}} (-1)^{j-1}T(p^{k_1})\cdots T(p^{k_j}).$$

Proof. When $\ell = 1$ the formula reduces to $T(p, 1) = T(p)$. According to Theorem 3,

$$\left(\sum_{j=0}^{n}(-1)^j p^{j(j-1)/2}T(p, j)X^j\right)\left(\sum_{k=0}^{\infty}T(p^k)X^k\right) = \Gamma_n .$$

Computing the coefficient of X^m for $1 \le m \le n$ as in [8] gives

$$\sum_{j=0}^{\ell}(-1)^j p^{j(j-1)/2}T(p, j)\cdot T(p^{\ell-j}) = 0$$

which is equivalent to

$$(13) \qquad (-1)^{\ell-1}p^{\ell(\ell-1)/2}T(p, \ell) = \sum_{j=0}^{\ell-1}(-1)^j p^{j(j-1)/2}T(p, j)T(p^{\ell-j}) .$$

Induction on ℓ completes the proof.

5. Action of the Hecke operators on automorphic forms

The Hecke operator T_m defined in (12) sends automorphic forms to automorphic forms and cusp forms to cusp forms. If f is an automorphic form of type ν with the Fourier expansion (6), write

$$T_n f(Z) = \sum_{\substack{g\in P(1, \dots, 1, i_2-i_1, \dots, n-i_k)\backslash P(1, \dots, 1, n-i_1) \\ N_1\ne 0, \dots, N_{i_k}\ne 0}} b_N(gZ) .$$

We wish to determine the coefficients b_N in terms of the a_N from (6). As mentioned earlier, those with $N_1 = 0$ and all other $N_i \ne 0$ are the ones we need examine for the present purposes. However, the relation between b_N and a_N for those N with all $N_i \ne 0$ will be helpful. While the computations are straightforward (albeit somewhat tedious), we were unable to find them written down anywhere, at least for $n > 3$. They are included here for the sake of completeness.

Lemma 3. With b_N as defined above, and N with all $N_i \neq 0$,

$$b_N = m^{\frac{n-1}{2}} \sum_{\substack{d_1 \cdots d_n = m \\ d_i | N_i}} a_{\frac{d_2 N_1}{d_1}, \dots, \frac{d_n N_{n-1}}{d_{n-1}}}$$

Bump, [2] obtained this formula for $n = 3$. We extend those arguments to all n. Recalling the Iwasawa decomposition (1), $Z = XY$, and using the right coset decomposition described in (12).

$$\prod_{j=1}^{n-1} |N_j|^{-j(n-j)/2} b_N W\left(\begin{pmatrix} N_1 & \cdots & N_{n-1} & & \\ & & \ddots & & \\ & & & N_{n-1} & \\ & & & & 1 \end{pmatrix} Y\right)$$

$$= \sum_{\substack{d_1 \cdots d_n = m \\ d_{ij} \bmod d_j}} \int_0^1 \cdots$$

$$\int_0^1 f\left(\begin{pmatrix} d_1 & & d_{ij} \\ & \ddots & \\ 0 & & d_n \end{pmatrix}\begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} Y\right) e\left(\sum_{i=1}^{n-1} N_i x_{ii+1}\right) dX$$

$$= m^{-\frac{n(n-1)}{2}} \sum \int_0^m \cdots$$

$$\int_0^m f\left(\begin{pmatrix} d_1 & & d_{ij} \\ & \ddots & \\ 0 & & d_n \end{pmatrix}\begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} Y\right) e\left(\sum_{i=1}^{n-1} N_i x_{ii+1}\right) dX \; .$$

After determining u_{ij} from

$$\begin{pmatrix} d_1 & & d_{ij} \\ & \ddots & \\ 0 & & d_n \end{pmatrix}\begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

we can change variables in the integral. We find (remembering that $d_i = d_{ii}$)

$$u_{ij} = d_j^{-1}\left(\sum_{k=i}^{j-1} d_{ik} x_{kj} + d_{ij}\right) \; .$$

Making the change of variables and using the periodicity of the integral, this last integral becomes

$$m^{-n(n-1)/2} \sum_{d_1 \cdots d_n = m} \prod_{j=1}^{n} d_j^{-(n-2j+1)} \sum_{d_{ij} \bmod d_j} e\left(\sum_{i=1}^{n-1} \frac{N_i d_{ii+1}}{d_i}\right) \int_0^{m\frac{d_1}{d_2}} \cdots$$

$$\int_0^{m\frac{d_{n-1}}{d_n}} f\left(\begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} Y\right) e\left(-\sum_{i=1}^{n-1} \frac{N_i d_{i+1}}{d_i} u_{ii+1}\right) dU$$

where the integral corresponding to du_{ij} has the limits 0, $m\frac{d_i}{d_j}$. Because of the term

$$e\left(-\sum_{i=1}^{n-1}\frac{N_i d_{i+1}}{d_i}u_{ii+1}\right)$$

the integral vanishes unless $d_i | N_i d_{i+1}$ for $i = 1, \ldots, n-1$. When this is the case the exponential sum from the integral can be evaluated as

$$\sum_{d_{ij} \bmod d_j} e\left(\sum_{i=1}^{n-1}\frac{N_i d_{ii+1}}{d_i}\right) = \begin{cases} \prod_{j=2}^{n} d_j^{j-1} & \text{if } d_i|N_i \text{ for } i = 1, \ldots, n-1, \\ 0 & \text{otherwise.}\end{cases}$$

Collecting all of the above gives

$$\begin{aligned}
\mathrm{LHS} = {} & m^{-n(n-1)/2} \sum_{\substack{d_1\cdots d_n = m \\ d_i|N_i}} \prod_{j=1}^{n} d_j^{-(n-2j+1)} \prod_{j=2}^{n} d_j^{j-1} \int_0^{m\frac{d_1}{d_2}} \cdots \\
& \int^{m\frac{d_{n-1}}{d_n}} f\left(\begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1\end{pmatrix}\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n\end{pmatrix} Y\right) e\left(-\sum_{i=1}^{n-1}\frac{N_i d_{i+1}}{d_i}u_{ii+1}\right) dU \\
= {} & m^{-n(n-1)/2} \sum \prod_{j=1}^{n} d_j^{-(n-2j+1)} \prod_{j=2}^{n} d_j^{j-1} m^{n(n-1)/2} \\
& \times \prod_{j=1}^{n} d_j^{n-2j+1} \prod_{j=1}^{n-1}\left|\frac{d_{j+1}N_j}{d_j}\right|^{-j(n-j)/2} \\
& \times a_{\frac{d_2 N_1}{d_1}, \ldots, \frac{d_n N_{n-1}}{d_{n-1}}} W\left(\begin{pmatrix} N_1 & \cdots & N_{n-1} & \\ & & \ddots & \\ & & & 1\end{pmatrix} Y\right) \\
= {} & m^{\frac{n-1}{2}} \prod_{j=1}^{n-1} |N_j|^{-j(n-j)/2} \sum_{\substack{d_1\cdots d_n = m \\ d_i|N_i}} a_{\frac{d_2 N_1}{d_1}, \ldots, \frac{d_n N_{n-1}}{d_{n-1}}} \\
& \times W\left(\begin{pmatrix} N_1 & \cdots & N_{n-1} & \\ & & \ddots & \\ & & & 1\end{pmatrix} Y\right)
\end{aligned}$$

finally completing the proof.

Assuming f is an eigenfunction of all the Hecke operators T_m, normalized so that $a_{1,\ldots,1} = 1$, we have $b_{1,\ldots,1} = \lambda_m$. In general, $\lambda_m a_N = b_N$. Meanwhile, Lemma 3 shows that $b_{1,\ldots,1} = m^{\frac{n-1}{2}} a_{1,\ldots,1,m}$, giving $\lambda_m = m^{\frac{n-1}{2}} a_{1,\ldots,1,m}$.

Recall that for $N = {}^{T}(0, N_2, \ldots, N_{n-1})$ with $N_i \neq 0$ for $i = 2, \ldots, n-1$, the $a_N(Z)$ have the form (10):

$$a_N(Z) = \sum_{\sigma \in S_n} a_N^\sigma \prod_{j=2}^{n-1} |N_j|^{-(j-1)(n-j)/2} z^{\sigma(r(\nu))} W_{n-1}(\sigma(\nu)', N', Z').$$

Beginning with the same approach as in Lemma 3, we obtain the next lemma.

Lemma 4. *If* $N = {}^T(0, N_2, \dots, N_{n-1})$ *with* $N_i \neq 0$ *for* $i = 2, \dots, n-1$,

$$b_N^\sigma = m^{\frac{n-1}{n}\sigma(r(\nu))} \sum_{k|m} k^{\sigma(\frac{n}{2} - r(\nu))} \sum_{\substack{d_2 \cdots d_n = k \\ d_i | N_i}} a^\sigma_{0, \frac{d_3 N_2}{d_2}, \dots, \frac{d_n N_{n-1}}{d_{n-1}}}.$$

Using the same notation, Lemma 3 gets us as far as

$$b_N(Y) = \prod_{j=2}^{n-1} |N_j|^{-(j-1)(n-j)/2} \sum_{\substack{d_1 \cdots d_n = m \\ d_i | N_i}} \left(\frac{m}{d_1}\right)^{\frac{n}{2}} \sum_{\sigma \in S_n} a^\sigma_{0, \frac{d_3 N_2}{d_2}, \dots, \frac{d_n N_{n-1}}{d_{n-1}}} z(DY)^{\sigma(r(\nu))}$$

$$\times W_{n-1}\left(\sigma(\nu)', \begin{pmatrix} N_2 & \cdots & N_{n-1} & & \\ & N_3 & \cdots & N_{n-1} & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} Y'\right)$$

where $D = \operatorname{diag}(d_1, \dots, d_n)$ and $z(Y) = z$. It is easy to see that $z(DY) = m^{\frac{-1}{n}} d_1 z$. Now let $k = \frac{m}{d_1}$ so that $z(DY) = k^{-1} m^{\frac{n-1}{n}} z$ and substitute in the above. This gives

$$b_N(Y) = \prod_{j=2}^{n-1} |N_j|^{-(j-1)(n-j)/2} \sum_{\sigma \in S_n} m^{\frac{n-1}{n}\sigma(r(\nu))} \sum_{k|m} k^{\sigma(\frac{n}{2} - r(\nu))} \sum_{\substack{d_2 \cdots d_n = m \\ d_i | N_i}}$$

$$\times a^\sigma_{0, \frac{d_3 N_2}{d_2}, \dots, \frac{d_n N_{n-1}}{d_{n-1}}} z^{\sigma(r(\nu))}$$

$$\times W_{n-1}\left(\sigma(\nu)', \begin{pmatrix} N_2 & \cdots & N_{n-1} & & \\ & N_3 & \cdots & N_{n-1} & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} Y'\right).$$

Comparison with $b_N(Y)$ in the form (10) yields the desired result. Lemma 3 together with Lemma 2 explains the normalization used in defining the $\mu_{p,\ell}$ of the L-function (3) in the introduction. This should become even clearer in the next section.

6. Factoring the L-functions

As usual, the L-function associated to the automorphic form f may be defined via a Mellin transform. If f has the Fourier expansion (6), let

$$f^0(Z) = \sum_{m \neq 0} a_{m,1,\dots,1}(Z).$$

Then the Mellin transform

$$\int_0^\infty f^0\begin{pmatrix} y & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix} y^{s-1}\frac{dy}{y} = 2\Psi(s)L_f(s)$$

converges for $\operatorname{Re}(s)$ sufficiently large where

$$\Psi(s) = \int_0^\infty W\begin{pmatrix} y & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix} y^{s-2}\frac{dy}{y}.$$

$L_f(s)$ may be analytically continued to all s. From the above we see that

$$L_f(s) = \sum_{m=1}^{\infty} a_{m,1,\dots,1}m^{-s}. \tag{14}$$

By the corollary to Theorem 3 this has the Euler product

$$L_f(s) = \prod_p (1 - \mu_{p,1}p^{-s} + \mu_{p,2}p^{-2s} + \cdots + (-1)^n p^{-ns})^{-1}$$

with $\mu_{p,\ell} = p^{-\ell(n-\ell)/2}\lambda_{p,\ell}$ as in (3). Since $\lambda_{p,1} = \lambda_p$ we have $\mu_{p,1} = a_{1,\dots,1,p}$. It is also useful to factor each term in the product above so that

$$L_f(s) = \prod_p \prod_{\ell=1}^{n} (1 - \alpha_{p,\ell}p^{-s})^{-1}.$$

Of course, the $\mu_{p,\ell}$ are related to the $\alpha_{p,\ell}$ by the elementary symmetric polynomials. Specifically, let $E_\ell(X_1, \dots, X_n)$, $\ell = 1, \dots, n$ be the elementary symmetric polynomial of degree ℓ in the variables $X_1, \dots, X_n$, i.e.

$$E_\ell(X_1, \dots, X_n) = \sum_{1\le i_1<\cdots<i_\ell\le n} X_{i_1}\cdots X_{i_\ell}.$$

Then $\mu_{p,\ell} = E_\ell(\alpha_{p,1}, \dots, \alpha_{p,n})$, with $\mu_{p,n} = 1$ so that $\prod_{\ell=1}^n \alpha_{p,\ell} = 1$.

From here on let $T_m^{(n)}$ be the Hecke operator defined in (12) acting on automorphic forms for Γ_n and similarly for $T_m^{(n-1)}$ and define $\lambda_m^{(n)}$ by $T_m^{(n)}f = \lambda_m^{(n)}f$ and $\lambda_m^{(n-1)}$ by $T_m^{(n-1)}\Phi f = \lambda_m^{(n-1)}\Phi f$. We can now describe the relation between these Hecke eigenvalues for f and Φf.

Lemma 5. *Let f be an automorphic form of type ν which is an eigenfunction of all the Hecke operators $T_m^{(n)}$ so that $T_m^{(n)}f = \lambda_m^{(n)}f$. Then Φf is an eigenfunction of all the Hecke operators $T_m^{(n-1)}$, $T_m^{(n-1)}\Phi f = \lambda_m^{(n-1)}\Phi f$, and if Φf is not identically 0 we have*

$$\lambda_m^{(n)} = m^{\frac{n-1}{n}r(\nu)} \sum_{k|m} k^{1-r(\nu)}\lambda_k^{(n-1)}. \tag{16}$$

Proof. When $N = {}^T(0, N_2, \dots, N_{n-1})$ and $N' = {}^T(N_2, \dots, N_{n-1})$ set $a_{N'} = a_N^{\mathrm{id}}$ where id is the identity element in S_n.

$$\sum_{N' \in \mathbf{Z}^{n-2}} b_{N'}(Z') = \Phi T_m^{(n)} f(Z') = \Phi \lambda_m^{(n)} f(Z') = \lambda_m^{(n)} \Phi f(Z').$$

From Lemma 4 we obtain the relation

$$\Phi T_m^{(n)} f(Z') = \sum_{N' \in \mathbf{Z}^{n-2}} m^{\frac{n-1}{n} r(\nu)} \sum_{k|m} k^{1-r(\nu)} \cdot k^{\frac{n-2}{2}} \sum_{\substack{d_2 \cdots d_n = k \\ d_i | N_i}} a^{id}_{0, \frac{d_3 N_2}{d_2}, \dots, \frac{d_n N_{n-1}}{d_{n-1}}}.$$

and hence

$$\Phi T_m^{(n)} f = m^{\frac{n-1}{n} r(\nu)} \sum_{k|m} k^{1-r(\nu)} T_k^{(n-1)} \Phi f$$

from which we find that Φf is a Hecke eigenform. (16) now follows immediately. Terras, [15], obtained this last formula in the case where f was a certain type of Eisenstein series. From Lemma 5 we have in particular $\lambda_p^{(n)} = p^{\frac{n-1}{n} r(\nu)} (1 + p^{1-r(\nu)} \lambda_p^{(n-1)})$. Now we need to obtain the relation between $\lambda_{p,\ell}^{(n)}$ and $\lambda_{p,\ell}^{(n-1)}$. From Lemma 5 and Lemma 2 it follows that Φf is an eigenfunction of all the Hecke operators $T_{p,\ell}^{(n-1)}$ and we have the following for the eigenvalues:

Lemma 6. *With f as before and $\Phi f \not\equiv 0$ set $T_{p,\ell}^{(n)} f = \lambda_{p,\ell}^{(n)} f$ and $T_{p,\ell}^{(n-1)} \Phi f = \lambda_{p,\ell}^{(n-1)} \Phi f$, for $\ell = 1, \dots, n-1$. (Recall that $\lambda_{p,n}^{(n)} = 1$ and $\lambda_{p,0}^{(n)} = 1$ by definition of the Hecke operators.) We have*

$$\lambda_{p,\ell}^{(n)} = p^{\ell - \frac{\ell}{n} r(\nu)} \lambda_{p,\ell}^{(n-1)} + p^{\frac{n-\ell}{n} r(\nu)} \lambda_{p,\ell-1}^{(n-1)}.$$

Proof is by induction on ℓ. The case $\ell = 1$ is by (16) as previously noted. Now, assume the lemma is true for $1 \le j < \ell$. From (13),

$$(-1)^{\ell-1} p^{\ell(\ell-1)/2} \lambda_{p,\ell}^{(n)} = \sum_{j=0}^{\ell-1} (-1)^j p^{j(j-1)/2} \lambda_{p,j}^{(n)} \lambda_{p^{\ell-j}}^{(n)}.$$

By the induction hypothesis and (16) this becomes

$$\begin{aligned}
\lambda_{p^\ell}^{(n)} \sum_{j=1}^{\ell-1} (-1)^j p^{j(j-1)/2} &\left(p^{j - \frac{j}{n} r(\nu)} \lambda_{p,j}^{(n-1)} + p^{\frac{n-j}{n} r(\nu)} \lambda_{p,j-1}^{(n-1)} \right) p^{\frac{(\ell-j)(n-1)}{n} r(\nu)} \\
&\times \sum_{k=0}^{\ell-j} p^{k(1-r(\nu))} \lambda_{p^k}^{(n-1)} = p^{\frac{\ell(n-1)}{n} r(\nu)} \sum_{k=0}^{\ell} p^{k(1-r(\nu))} \lambda_{p^k}^{(n-1)} \\
&+ \sum_{j=1}^{\ell-1} (-1)^j p^{\frac{j(j+1)}{2} + \frac{(\ell-j)(n-1)-j}{n} r(\nu)} \lambda_{p,j}^{(n-1)} \sum_{k=0}^{\ell-j} p^{k(1-r(\nu))} \lambda_{p^k}^{(n-1)} \\
&+ \sum_{j=1}^{\ell-1} (-1)^j p^{\frac{j(j-1)}{2} + \frac{(\ell-j)(n-1)+n-j}{n} r(\nu)} \lambda_{p,j-1}^{(n-1)} \sum_{k=0}^{\ell-j} p^{k(1-r(\nu))} \lambda_{p^k}^{(n-1)}.
\end{aligned}$$

Putting the first two terms together and re-indexing the sum in the third gives

$$\begin{aligned}\text{LHS} &= \sum_{j=0}^{\ell-1}(-1)^j p^{\frac{j(j+1)}{2}+\frac{(\ell-j)(n-1)-j}{n}r(\nu)}\lambda_{p,j}^{(n-1)}\sum_{k=0}^{\ell-j}p^{k(1-r(\nu))}\lambda_{p^k}^{(n-1)}\\ &\quad+\sum_{j=0}^{\ell-2}(-1)^{j+1}p^{\frac{j(j+1)}{2}+\frac{(\ell-j-1)(n-1)+n-j-1}{n}r(\nu)}\lambda_{p,j}^{(n-1)}\sum_{k=0}^{\ell-j-1}p^{k(1-r(\nu))}\lambda_{p^k}^{(n-1)}\\ &=\sum_{j=0}^{\ell-1}(-1)^j p^{\frac{j(j+1)}{2}+\frac{(\ell-j)(n-1)-j}{n}r(\nu)+(\ell-j)(1-r(\nu))}\lambda_{p,j}^{(n-1)}\lambda_{p^{\ell-j}}^{(n-1)}\\ &\quad+\sum_{j=0}^{\ell-1}(-1)^j p^{\frac{j(j+1)}{2}+\frac{(\ell-j)(n-1)-j}{n}r(\nu)}\lambda_{p,j}^{(n-1)}\sum_{k=0}^{\ell-j-1}p^{k(1-r(\nu))}\lambda_{p^k}^{(n-1)}\\ &\quad+\sum_{j=0}^{\ell-2}(-1)^{j+1}p^{\frac{j(j+1)}{2}+\frac{(\ell-j-1)(n-1)+n-j-1}{n}r(\nu)}\lambda_{p,j}^{(n-1)}\sum_{k=0}^{\ell-j-1}p^{k(1-r(\nu))}\lambda_{p^k}^{(n-1)}\,.\end{aligned}$$

By (13) again the first sum is just $(-1)^{\ell-1}p^{\ell(\ell-1)/2}p^{\ell-\frac{\ell}{n}r(\nu)}\lambda_{p,\ell}^{(n-1)}$, and after observing that $(n-j-1)+(\ell-j-1)(n-1)=(\ell-j)(n-1)-j$ the second and third sums cancel except for $j=\ell-1$ in the second which leaves $(-1)^{\ell-1}p^{\ell(\ell-1)/2}p^{\frac{n-\ell}{n}r(\nu)}\lambda_{p,\ell-1}^{(n-1)}$.

Now the main result can be stated.

Theorem 4. *If f is an automorphic form of type ν on H_n and Φf is not identically 0, then the following relation between the L-functions holds:*

$$L_f(s)=\zeta\left(s-\frac{n-1}{n}r(\nu)+\frac{n-1}{2}\right)L_{\Phi f}\left(s+\frac{1}{n}r(\nu)-\frac{1}{2}\right).$$

Proof. Recall that when the L-functions are written in the form (15)

$$L_f(s)=\prod_p\prod_{\ell=1}^{n}(1-\alpha_{p,\ell}^{(n)}p^{-s})^{-1}$$

and

$$L_{\Phi f}(s)=\prod_p\prod_{\ell=1}^{n-1}(1-\alpha_{p,\ell}^{(n-1)}p^{-s})^{-1}$$

we have $\mu_{p,\ell}^{(k)}=E_\ell(\alpha_{p,1}^{(k)},\dots,\alpha_{p,k}^{(k)})$ where $E_\ell(X_1,\dots,X_k)$ denotes an elementary symmetric polynomial. By Lemma 6,

$$\mu_{p,\ell}^{(n)}=p^{\frac{\ell}{2}-\frac{\ell}{n}r(\nu)}\mu_{p,\ell}^{(n-1)}+p^{\frac{n-\ell}{n}r(\nu)-\frac{n-\ell}{2}}\mu_{p,\ell-1}^{(n-1)}$$

for $\ell=1,\dots,n-1$. Note that $\mu_{p,0}^{(n-1)}=1$. In terms of the $\alpha_{p,\ell}^{(n-1)}$,

$$\begin{aligned}\mu^{(n)}_{p,\ell} &= p^{\frac{\ell}{2}-\frac{\ell}{n}r(\nu)}\left(\mu^{(n-1)}_{p,\ell}+p^{r(\nu)-\frac{n}{2}}\mu^{(n-1)}_{p,\ell-1}\right)\\ &= p^{\frac{\ell}{2}-\frac{\ell}{n}r(\nu)}\left(E_\ell\left(\alpha^{(n-1)}_{p,1},\dots,\alpha^{(n-1)}_{p,n-1}\right)\right.\\ &\quad\left.+p^{r(\nu)-\frac{n}{2}}E_{\ell-1}\left(\alpha^{(n-1)}_{p,1},\dots,\alpha^{(n-1)}_{p,n-1}\right)\right).\end{aligned}$$

Observe that $E_\ell(X_1,\dots,X_n)=E_\ell(X_1,\dots,X_{n-1})+X_nE_{\ell-1}(X_1,\dots,X_{n-1})$. Thus,

$$\begin{aligned}\mu^{(n)}_{p,\ell} &= p^{\frac{\ell}{2}-\frac{\ell}{n}r(\nu)}E_\ell\left(\alpha^{(n-1)}_{p,1},\dots,\alpha^{(n-1)}_{p,n-1},p^{r(\nu)-\frac{n}{2}}\right)\\ &= E_\ell\left(p^{\frac{1}{2}-\frac{1}{n}r(\nu)}\alpha^{(n-1)}_{p,1},\dots,p^{\frac{1}{2}-\frac{1}{n}r(\nu)}\alpha^{(n-1)}_{p,n-1},p^{\frac{n-1}{n}r(\nu)-\frac{n-1}{2}}\right).\end{aligned}$$

Now we have

$$\begin{aligned}L_f(s) &= \prod_p\prod_{\ell=1}^{n-1}(1-p^{\frac{1}{2}-\frac{1}{n}r(\nu)}\alpha^{(n-1)}_{p,\ell}p^{-s})^{-1}\cdot\prod_p(1-p^{\frac{n-1}{n}r(\nu)-\frac{n-1}{2}-s})^{-1}\\ &= L_{\Phi f}\left(s+\frac{1}{n}r(\nu)-\frac{1}{2}\right)\zeta\left(s-\frac{n-1}{n}r(\nu)+\frac{n-1}{2}\right).\end{aligned}$$

Recalling the definition of $\tilde{\Phi}$ we obtain a second factorization of $L_f(s)$ quite similar to that above. If Φf is not identically 0 then neither is $\tilde{\Phi} f$, so for f as above we have

Corollary.

$$L_f(s)=\zeta\left(s-\frac{n-1}{n}r^*(\nu)+\frac{n-1}{2}\right)L_{\tilde{\Phi} f}\left(s+\frac{1}{n}r^*(\nu)-\frac{1}{2}\right).$$

These formulas can be applied easily to the analogue of the Maass wave form Eisenstein series. Define

$$E_n(\nu,Z)=\sum_{\gamma\in P(1,\dots,1)\backslash\Gamma_n}p_\nu(\gamma Z),\qquad \mathrm{Re}(\nu_i)>1.$$

This was meromorphically continued to all ν by Selberg (see [11]). Clearly $\Phi E_n(\nu,Z)=E_{n-1}(\nu',Z')$. By applying Theorem 4 repeatedly, we obtain,

$$L_{E_n}(s)=\prod_{i=1}^n\zeta(s-2r_i).$$

This is well-known, but we have no specific reference.

Eisenstein series for $GL(n,\mathbb{R})$ may be defined more generally. If P is the parabolic subgroup $P(n_1,\dots,n_k)$ and v_j are automorphic forms on H_{n_j}, and we write $v=(v_1,\dots,v_k)$, we may define an Eisenstein series by:

$$E_P(v;\xi,Z)=\sum_{\gamma\in\Gamma_n/P}\prod_{j=1}^k|a_j(\gamma Z)|^{2\xi_j}v_j(a_j(\gamma Z))$$

where $\xi =^T (\xi_1, \dots, \xi_{k-1}) \in \mathbb{C}^{k-1}$, $\xi_k = 0$, and $a_j(Z) \in M_{n_j}(\mathbb{R})$ is defined by

$$Z = \begin{pmatrix} a_1(Z) & & & \\ & a_2(Z) & * & \\ & * & \ddots & \\ & & & a_k(Z) \end{pmatrix}$$

This series converges if $\mathrm{Re}(\xi_j)$ is sufficiently large. Consider, in particular, Eisenstein series for $P(n-1, 1)$. If ϕ is an automorphic form of type ν^* on H_{n-1}, we have an Eisenstein series $E_{(n-1,1)}(\phi, u, Z)$, defined as above, with the series converging for $\mathrm{Re}(u) > n/2$. To find ν so that $f(Z) = E_{(n-1,1)}(\phi, u, Z)$ is of type ν, consider that this Eisenstein series must behave somewhat like the power function $p_\nu(Z) = y^{r^*(\nu)} p_{\nu^*}(Z^*)$, so it quickly becomes apparent that $u = r^*(\nu)$. It is also easy to see that $\tilde{\Phi} f = \phi$. Applying the Corollary now yields

$$L_f(s) = \zeta\left(s - \frac{n-1}{n}u + \frac{n-1}{2}\right) L_\phi\left(s + \frac{1}{n}u - \frac{1}{2}\right).$$

Other cases where $n_1 = 1$ or $n_k = 1$ also can be handled easily by Theorem 4 or its corollary respectively. For other than these we must defer to [5].

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DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218

Current address: Department of Mathematics, University of Texas of the Permian Basin, Odessa, Texas 79762

E-mail address: grenier_d@gusher.pb.utexas.edu