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UNIVALENT FUNCTIONS AND THE POMPEIU PROBLEM

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ABSTRACT. In this paper we prove a result on the Pompeiu problem. If the Schwarz function Φ of the boundary of a simply-connected domain $\Omega \subset \mathbb{R}^2$ extends meromorphically into a certain portion E of Ω with a pole at some point $z_0 \in E$, then Ω has the Pompeiu property unless Φ is a Möbius transformation, in which case Ω is a disk.

1. INTRODUCTION

In 1929 the Rumanian mathematician D. Pompeiu formulated the following problem: "*To characterize those bounded domains* $\Omega \subset \mathbb{R}^2$ *for which* $f \equiv 0$ *is the only continuous function such that*

$$\int_{\sigma(D)} f\,dx = 0, \tag{1.1}$$

for every rigid motion σ *of* $\mathbb{R}^2$".

One says that Ω has the Pompeiu property if $f \equiv 0$ is the only continuous function for which (1.1) holds. For a historical introduction to the problem we refer the reader to [GS1]. In that paper we conjectured that (modulo sets of measure zero) the disk is the only simply-connected domain that does not have the Pompeiu property. Chakalov [C] was the first one to realize that the disk fails to have the Pompeiu property. In fact, if one considers the function $f(x_1, x_2) = \sin(ax_1)$, then one has

$$\int_{B_r(x_0)} f(x)\,dx = \frac{2\pi r}{a} \sin(ax_{0,1}) J_1(ar),$$

where $x_0 = (x_{0,1}, x_{0,2})$ is fixed, $B_r(x_0) = \{x \mid |x - x_0| < r\}$, and J_1 is the Bessel function of order one. It is therefore enough to choose $a > 0$, such that $J_1(ar) = 0$, for (1.1) to hold.

This paper contains some progress toward the above conjecture. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain whose boundary $\partial\Omega$ is a piecewise C^1 Jordan curve. By the Riemann mapping theorem there exists a univalent function $h: D \to \Omega$, where $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Moreover, h can be extended in a one-to-one fashion to a continuous map of $\overline{D}$ onto $\overline{\Omega}$. In order

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to state the main result in this paper we need to introduce some definitions. We consider the Schwarz function of $\partial\Omega$ given by

$$\Phi(w) = \overline{h\left(\frac{1}{\overline{h^{-1}(w)}}\right)}.$$

A priori, Φ is well defined on $\partial\Omega$. Given a straight line $L \subset \mathbb{C}$, and a point $z_0 \notin L$, we denote by $\Lambda(L; z_0)$ the open half-plane lying on one side of L and containing z_0. We also let

$$E(L; z_0) = \Lambda(L; z_0) \cap \Omega.$$

The main result in this paper is given by the following

Theorem 1. *Suppose that there exist* $z_0 \in \Omega$ *and a straight line* $L \subset \mathbb{C}$ *such that:*

(i) Φ *can be extended to a holomorphic function in* $E(L; z_0)\backslash\{z_0\}$ *having a pole in* z_0;

(ii) Φ *is not a Möbius transformation.*

Then, Ω *has the Pompeiu property.*

Figure 1 below illustrates the situation.

We now state two remarkable consequences of Theorem 1.

Corollary 2. *Suppose that* h *is univalent in* D *and meromorphic in* $\mathbb{C}$, *with at least one pole in* $\overline{\mathbb{C}}\backslash\overline{D}$. *If, moreover,* h *is not a Möbius transformation, then* $\Omega = h(D)$ *has the Pompeiu property.*

If we specialize Theorem 1 to the class of convex domains we obtain the following partial solution of the Pompeiu problem.

Corollary 3. *Suppose that* $\Omega = h(D)$ *be a convex set. Assume that* h *has a pole on the boundary of the circle of convergence relative to its Taylor expansion at* $z = 0$. *If* h *is not a Möbius transformation, then* Ω *has the Pompeiu property.*

Remark. Corollary 2 contains the result in our paper [GS2] (see also [GS3]) concerned with the case

$$h(z) = \sum_{k=0}^{N} a_k z^k.$$

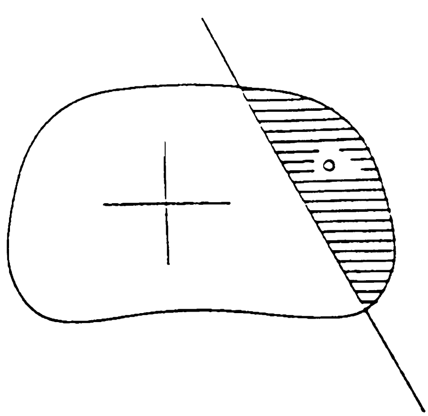

FIGURE 1

Furthermore, it contains a result in a recent paper by Ebenfelt [E]. The latter has proved that if h is a univalent function in D such that $h(z) = \frac{p(z)}{q(z)}$, with p and q polynomials, then $\Omega = h(D)$ has the Pompeiu property, unless h is a Möbius transformation.

Our strategy to prove Theorem 1 is to study, by Riemann's method of the steepest descent, the asymptotic behavior of the (complexified) Fourier transform of the characteristic function of Ω, $\hat{\chi}_\Omega$, along the algebraic variety of $\mathbb{C}^2$, $M_\alpha = \{\zeta_1^2 + \zeta_2^2 = \alpha\}$, $\alpha > 0$. This is due to an important characterization of the Pompeiu property established in 1973 by Brown, Schreiber, and Taylor [BST], see Theorem A in the next section. We mention that Berenstein [B] was the first one to use asymptotic expansions of $\hat{\chi}_\Omega$ in connection with the Pompeiu problem.

2. Preliminary reductions

We begin this section by recalling the above-mentioned characterization of the Pompeiu property due to Brown, Schreiber, and Taylor [BST].

Theorem A. *A bounded domain $\Omega \subset \mathbb{R}^2$ has the Pompeiu property if and only if there exists no $\alpha \in \mathbb{C}\backslash\{0\}$ such that the complexified Fourier transform of the characteristic function of Ω, $\hat{\chi}_\Omega$, vanishes identically on*

$$M_\alpha = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 | \zeta_1^2 + \zeta_2^2 = \alpha\}.$$

It was observed by Berenstein [B] that, when Ω is simply connected, $\alpha \in \mathbb{C}\backslash\{0\}$ in the statement of Theorem A can be replaced by $\alpha > 0$. Furthermore, when $\partial\Omega$ is a rectifiable Jordan curve, then the divergence theorem allows to replace $\hat{\chi}_\Omega$ with $\hat{\chi}_{\partial\Omega}$. Note that for $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$

$$\hat{\chi}_{\partial\Omega} = \int_{\partial\Omega} e^{i\langle\zeta, x\rangle}(dx_1 + i dx_2) \tag{2.1}$$

where we have let $\langle\zeta, x\rangle = \zeta_1 x_1 + \zeta_2 x_2$. Changing ζ in $-i\zeta$ in (2.1) we are thus led to study the following oscillatory integral

$$\int_{\partial\Omega} e^{\langle\zeta, x\rangle}(dx_1 + i dx_2) \tag{2.2}$$

for $\zeta \in M_{-\alpha}$, with $\alpha > 0$. We write ζ in the form

$$\zeta = r(\cos\theta, \sin\theta) + it(-\sin\theta, \cos\theta).$$

The condition $\zeta \in M_{-\alpha}$ becomes

$$t^2 = \alpha + r^2. \tag{2.3}$$

We have

$$\begin{aligned} \langle\zeta, x\rangle &= x_1(r\cos\theta - it\sin\theta) + x_2(r\sin\theta + it\cos\theta) \\ &= r x_1 e^{-i\theta} + i r x_2 e^{-i\theta} - i(t-r)x_1\sin\theta + i(t-r)x_2\cos\theta \\ &= r e^{-i\theta}(x_1 + i x_2) - i(t-r)(x_1\sin\theta - x_2\cos\theta). \end{aligned} \tag{2.4}$$

Since from our assumptions in the introduction $\partial\Omega = h(\partial D)$, where h is univalent in $D = \{w \in \mathbb{C} | |w| < 1\}$, we have for $s \in [0, 2\pi]$

$$x_1(s) = \frac{1}{2}h(e^{is}) + \frac{1}{2}k(e^{is}). \tag{2.5}$$

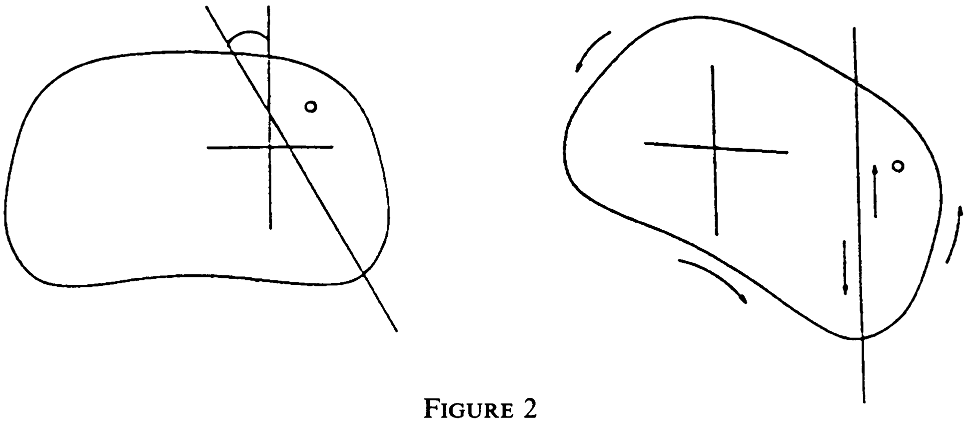

FIGURE 2

Here, we have let

$$k(w) = \overline{h\left(\frac{1}{\overline{w}}\right)}. \tag{2.6}$$

Analogously, we have

$$x_2(s) = \frac{1}{2i} h(e^{is}) - \frac{1}{2i} k(e^{is}). \tag{2.7}$$

Inserting (2.5), (2.7) in (2.4) we obtain

$$\langle \zeta, x \rangle = re^{-i\theta} h - i\frac{(t-r)}{2}[(h+k)\sin\theta + i(h-k)\cos\theta],$$

which, after some easy reductions, gives

$$\langle \zeta, x \rangle = \frac{t+r}{2} e^{-i\theta} h(e^{is}) - \frac{t-r}{2} e^{i\theta} k(e^{is}). \tag{2.8}$$

Taking (2.8) into account, we see that (up to a factor of i) the integral in (2.2) becomes

$$\begin{aligned} I(r) &= \int_{\partial\Omega} \exp\left[\frac{t+r}{2} e^{-i\theta} w - \frac{t-r}{2} e^{i\theta} \Phi(w)\right] dw \\ &= e^{i\theta} \int_{\partial\Sigma} \exp\left[\frac{t+r}{2} w - \frac{t-r}{2} \Psi(w)\right] dw \end{aligned} \tag{2.9}$$

where $\Sigma = e^{-i\theta}\Omega$, $\Psi(w) = e^{i\theta}\Phi(e^{i\theta}w)$. At this point we choose $\theta \in [0, 2\pi]$ in such a way that the straight line $e^{-i\theta}L$, where L is as in the statement of Theorem 1, becomes parallel to the imaginary axis. We let $w_0 = e^{-i\theta}z_0$, $M = e^{-i\theta}L$, where $z_0 \in \Omega$ is as in the assumption of Theorem 1, see Figure 2.

We now have from (2.9)

(2.10)

$$e^{-i\theta} I(r) = \left(\int_{\partial E(M;w_0)} + \int_{\partial[\Sigma \setminus E(M;w_0)]}\right) \exp\left[\left(\frac{t+r}{2}\right) w - \frac{t-r}{2}\Psi(w)\right] dw.$$

3. ASYMPTOTIC EXPANSION OF $\hat{\chi}_{\partial\Omega}$ AND THE POMPEIU PROPERTY

The aim of this section is to establish the asymptotic behavior as $r \to \infty$ of the integral in the right-hand side of (2.10). We begin by analyzing that part of the integral on the set $\partial(\Sigma\backslash E(M; w_0))$. We let $A = \max|\Psi|$ on $\partial(\Sigma\backslash E(M; w_0))$. Then

$$\left|\int_{\partial(\Sigma\backslash E(M; w_0))} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w)\right] dw\right| \leq \int_{\partial(\Sigma\backslash E(M; w_0))} \exp\left[\frac{t+r}{2}\Re e\, w + \frac{t-r}{2}A\right] ds.$$

We now choose $\beta > 0$ such that on the set $\partial(\Sigma\backslash E(M; w_0))$ we have (see Figure 2)

$$\Re e\, w \leq \Re e\, w_0 - \beta.$$

Noting that (2.3) gives

$$\text{(3.1)} \qquad \frac{t+r}{2} = r\left(1 + O\left(\frac{1}{r^2}\right)\right), \quad t-r = \frac{\alpha}{2r}\left(1 + O\left(\frac{1}{r}\right)\right)$$

as $r \to \infty$, it follows that on the set $\partial(\Sigma\backslash E(M; w_0))$ we have uniformly as $r \to \infty$

$$\frac{t+r}{2}\Re e\, w + \frac{t-r}{2}A \leq \frac{t+r}{2}(\Re e\, w_0 - \beta) + \frac{t-r}{2}A = r(\Re e\, w_0 - \beta)(1 + o(1)).$$

From this we derive the estimate for $r \to \infty$

$$\text{(3.2)} \quad \left|\int_{\partial(\Sigma\backslash E(M; w_0))} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}(w)\right] dw\right| \leq C \exp\left[r\left(\Re e\, w_0 - \frac{\beta}{2}\right)\right].$$

We will now analyze the first integral in the right-hand side of (2.10). We have for $\delta > 0$ small by Cauchy's theorem

(3.3)

$$\begin{aligned}\int_{\partial E(M; w_0)} &\exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w)\right] dw \\ &= \exp\left(\frac{t+r}{2}w_0\right)\int_{|w-w_0|=\delta} \exp\left[\frac{t+r}{2}(w-w_0) - \frac{t-r}{2}\Psi(w)\right] dw \\ &= \exp\left(\frac{t+r}{2}w_0\right)\int_{|w|=\delta} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w)\right] dw.\end{aligned}$$

By the assumptions on Φ in Theorem 1, there exists $m \in \mathbb{N}$ such that

$$\text{(3.4)} \qquad \Psi(w_0 + w) = \sum_{k=-m}^{\infty} a_k w^k$$

for $|w| \leq \delta$, with $a_{-m} \neq 0$. We now distinguish two cases.

First case. $m \geq 2$.

Using (2.3) we can write

$$\text{(3.5)} \qquad \frac{t-r}{2} = \frac{\alpha}{2(t+r)}.$$

By (3.4), (3.5) we have on the circle $\{w \in \delta e^{i\tau} | -\pi \leq \tau \leq \pi\}$

$$\text{(3.6)} \quad \begin{aligned}&\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w) \\ &= \frac{t+r}{2}\delta e^{i\tau} - \frac{\alpha a_{-m}}{2(t+r)}\delta^{-m}e^{-im\tau} - \frac{\alpha}{2(t+r)}\sum_{k=-m+1} a_k\delta^k e^{ik\tau}.\end{aligned}$$

We now choose

$$\delta = \left(\frac{t+r}{2}\right)^{-2/(m+1)} .$$

Then, (3.6) becomes as $r \to \infty$

$$\frac{t+r}{2} w - \frac{t-r}{2} \Psi(w_0 + w) = \left(\frac{t+r}{2}\right)^{(m-1)/(m+1)} \left[e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau} + o(1)\right] .$$

From the first equality in (3.1) we conclude that

$$(3.7) \quad \frac{t+r}{2} w - \frac{t-r}{2} \Psi(w_0 + w) = r^{(m-1)/(m+1)} q(r) \left\{e^{i\tau} - \frac{\alpha a_{-m} e^{-im\tau}}{4} + o(1)\right\}$$

with $q(r) \to 1$ as $r \to \infty$, uniformly on the circle $\{w = \delta e^{i\tau} | -\pi \le \tau \le \pi\}$. Taking (3.7) into account, we obtain for (3.3) with some $p(r) \to 1$ as $r \to \infty$ (3.8)

$$\int_{\partial E(M;w_0)} \exp\left[\frac{t+r}{2} w - \frac{t-r}{2} \Psi(w)\right] dw = i r^{-2/(m+1)} p(r) \exp\left(\frac{t+r}{2} w_0\right)$$
$$\cdot \int_{-\pi}^{\pi} \exp\left\{r^{(m-1)/(m+1)} q(r) \left[e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau} + o(1)\right]\right\} e^{i\tau} \, d\tau .$$

At this point we observe that the integral on the right-hand side of (3.8) is of the type studied in the paper [GS2]. By virtue of the work done in [GS2] we can conclude that the asymptotic behavior, as $r \to \infty$, of the above-mentioned integral is as follows

$$(3.9) \quad \begin{aligned} &\int_{-\pi}^{\pi} \exp\left\{r^{(m-1)/(m+1)} q(r) \left[e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau} + o(1)\right]\right\} e^{i\tau} \, d\tau \\ &\quad = r^{-(m-1)/2(m+1)} A(r) \exp[r^{(m-1)/(m+1)} B(r)], \end{aligned}$$

where, having let $\varphi(\tau) = e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau}$ for $\tau \in \mathbb{C}$, one has for $r \to \infty$

$$A(r) \to \frac{e^{i\tau_0}}{\sqrt{2\varphi''(\tau_0)}} = A_0 \neq 0, \quad B(r) \to \varphi(\tau_0) .$$

Here, τ_0 is a suitable simple critical point of the function φ. Inserting (3.9) in (3.8) and recalling (3.1), we obtain

$$(3.10) \quad \begin{aligned} &\int_{\partial E(M;w_0)} \exp\left[\frac{t+r}{2} w - \frac{t-r}{2} \Psi(w)\right] dw \\ &\quad = r^{-(m+3)/2(m+1)} A_1(r) \exp[r w_0 + r^{(m-1)/(m+1)} B(r)], \end{aligned}$$

where $A_1(r) \to i A_0$, as $r \to \infty$.

Using (3.2), (3.10) in (2.10) we finally conclude for $r \to \infty$

$$(3.11) \quad \begin{aligned} e^{-i\theta} I(r) &= r^{-(m+3)/2(m+1)} A_1(r) \exp[r w_0 + r^{(m-1)/(m+1)} B(r)] \\ &\cdot \left\{1 + O\left(r^{(m+3)/2(m+1)} \exp\left[-\frac{\beta}{2} r + C r^{(m-1)/(m+1)}\right]\right)\right\}, \end{aligned}$$

for some number $C > 0$. Observing now that $0 < \frac{m-1}{m+1} < 1$, we infer that

$$O\left(r^{(m+3)/2(m+1)} \exp\left[-\frac{\beta}{2} r + C r^{(m-1)/(m+1)}\right]\right) = o(1)$$

as $r \to \infty$. In conclusion, we obtain from (3.11)

$$e^{-i\theta} I(r) = r^{-(m+3)/2(m+1)} A_2(r) \exp[r w_0 + r^{(m-1)/(m+1)} B(r)], \tag{3.12}$$

where $A_2(r) \to i A_0$ as $r \to \infty$.

To conclude the study of the asymptotic behavior of the integral $e^{-i\theta} I(r)$ in (2.10) we need to analyze the case in which Ψ has as simple pole in w_0, i.e., the case in which $m = 1$ in (3.4).

Second case. $m = 1$.

We consider again the integral on the circle $\{w \in \delta e^{i\tau} | -\pi \le \tau \le \pi\}$ in the right-hand side of (3.3). By the assumptions in Theorem 1, the function Ψ is not a Möbius transformation. If $w = \sigma e^{i\tau}$, with $|\sigma| = \delta$, then we have from (3.5)

$$\begin{aligned} \frac{t+r}{2} w - \frac{t-r}{2} \Psi(w_0 + w) &= \frac{1}{2}\left[(t+r)w - \frac{\alpha}{t+r}\Psi(w_0+w)\right] \\ &= \frac{1}{2}\left\{(t+r)\sigma e^{i\tau} - \frac{\alpha}{t+r}\left[\frac{a_{-1}}{\sigma} e^{-i\tau} + a_0 + \sum_{k=q}^{\infty} a_k \sigma^k e^{ik\tau}\right]\right\} \end{aligned} \tag{3.13}$$

for some $q \in \mathbb{N}$, with $a_q \neq 0$. At this point we choose

$$\sigma = \frac{i\sqrt{\alpha a_{-1}}}{t+r}. \tag{3.14}$$

It follows from (3.13), (3.14) that for $w = \sigma e^{i\tau}$ one has

$$\begin{aligned} \frac{t+r}{2} w - \frac{t-r}{2} \Psi(w_0 + w) &= -\frac{\alpha a_0}{t+r} + i\sqrt{\alpha a_{-1}} \cos\tau \\ &+ \frac{C}{(t+r)^{q+1}} e^{iq\tau} + O\left(\frac{1}{(t+r)^{q+2}}\right), \end{aligned} \tag{3.15}$$

for some $C \neq 0$. We conclude

(3.16)

$$\begin{aligned} &\int_{|w|=\delta} \exp\left[\frac{t+r}{2} w - \frac{t-r}{2}\Psi(w_0+w)\right] dw \\ &= i\sigma\varepsilon^{-\alpha a_0/(t+r)} \int_{-\pi}^{\pi} e^{i\sqrt{\alpha a_{-1}}\cos\tau} \left\{1 + \frac{C}{(t+r)^{q+1}} e^{iq\tau} + O\left(\frac{1}{(t+r)^{q+2}}\right)\right\} e^{i\tau}\, d\tau. \end{aligned}$$

We now recall the integral representation of the Bessel function J_n (see [L])

$$J_n(z) = \frac{i^{-n}}{2\pi} \int_{-\pi}^{\pi} e^{iz\cos\tau} e^{in\tau}\, d\tau, \qquad n \in \mathbb{Z}.$$

Using this we can rewrite (3.16) as follows

(3.17)

$$\begin{aligned} &\int_{|w|=\delta} \exp\left[\frac{t+r}{2} w - \frac{t-r}{2}\Psi(w_0+w)\right] dw \\ &= -\frac{\sqrt{\alpha a_{-1}}}{t+r} e^{-\alpha a_0/(t+r)} \left\{ 2\pi i J_1(\sqrt{\alpha a_{-1}}) \right. \\ &\qquad \left. + \frac{2\pi C i^q J_q(\sqrt{\alpha a_{-1}})}{(t+r)^{q+1}} + O\left(\frac{1}{(t+r)^{q+2}}\right)\right\}. \end{aligned}$$

Since, from (3.1), $t + r(1 + o(1))$ as $r \to \infty$, and by a theorem of Siegel, J_1 and J_q have no common zeros (see [W, p. 485]). (3.17) implies

$$\begin{aligned}\int_{|w|=\delta} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w)\right] dw \\ = \frac{E_1(r)}{r}\left[J_1(\sqrt{\alpha a_{-1}}) + \frac{E_2}{r^{q+1}} J_q(\sqrt{\alpha a_{-1}})\right],\end{aligned} \tag{3.18}$$

where $E_1(r) \to E_0 \neq 0$ as $r \to \infty$, and $E_2 \neq 0$. From (2.10), (3.2) and (3.18) we finally obtain

$$\varepsilon^{-i\theta} I(r) = \varepsilon^{rw_0} \frac{E_3(r)}{r}\left[J_1(\sqrt{\alpha q_{-1}}) + \frac{E_2}{r^{q+1}} J_q(\sqrt{\alpha a_{-1}})\right] \tag{3.19}$$

with $E_3(r) \to E_0$ as $r \to \infty$ (of course, in this estimate we have used again (3.1)).

We are now ready to conclude the proof of Theorem 1. We recall that from the reductions in §2, the oscillatory integral $\hat{\chi}_{\partial\Omega}(\zeta)$, with ζ moving out to infinity along a special path of $M_{-\alpha}$, was shown to equal $e^{-i\theta} I(r)$ in (2.10) (up to a factor of i).

From (3.12), (3.19) we see that, under the assumptions in Theorem 1, the latter cannot vanish identically on $M_{-\alpha}$. From Theorem A we conclude that Ω has the Pompeiu property. □

4. Proofs of Corollaries 2 and 3

The proof of Corollary 2 follows immediately from Theorem 1 by observing that if Φ is a Möbius transformation, then so is h. Moreover, if h has at least one pole, then Φ has at least one pole and at most one essential singularity ($w = 0$).

As for the proof of Corollary 3 we observe that if $\Omega = h(D)$ is convex, then by Study's theorem [S, Theorem 2.4] so is $h(D_r)$ for $0 < r \leq 1$, where $D_r = \{z \in \mathbb{C} \mid |z| \leq r\}$. Set $S = \{x \in \mathbb{C} \mid |z| < R\}$ with $R > 1$, and denote $S^{-1} = \{\frac{1}{z} \mid z \in S\}$. Assume that h is holomorphic in S with a pole z_0 on ∂S. Then Φ is holomorphic on $h(S^{-1})$ and has a pole in $h(\frac{1}{z_0})$. Since $h(S^{-1})$ is convex we are in a position to apply Theorem 1, see Figure 3.

We close this paper with an example of a one-parameter family of domains which fall within the scope of Theorem 1, but are not included in any previous result on the Pompeiu problem.

Example. Consider for $0 < \lambda < 2$ the map $h_\lambda : D \to \mathbb{C}$ given by $h_\lambda(x) = \frac{e^\lambda}{\lambda_z - 2}$. In Figure 4, we represent $\Omega_\lambda = h_\lambda(D)$ for some values of λ. There exists $\lambda_0 \in (0, 2)$ such that for $0 < \lambda < \lambda_0$ the domain Ω_λ is convex. Furthermore, one verifies that for $\lambda \in (0, \lambda_0)$

$$\min \operatorname{diam} \Omega_\lambda > \frac{1}{2} \max \operatorname{diam} \Omega_\lambda$$

so that the result in [BK] cannot be invoked.

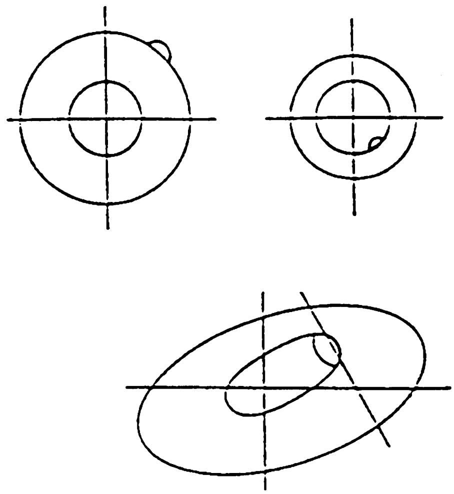

FIGURE 3

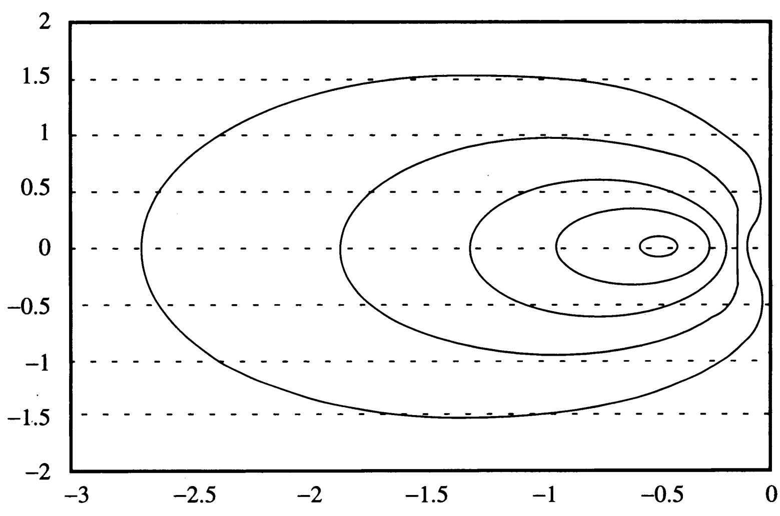

FIGURE 4

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