GEOMETRIC INVARIANTS FOR SEIFERT FIBRED 3-MANIFOLDS

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Abstract. In this paper, we obtain a formula for the $\eta$-invariant of the signature operator for some circle bundles over Riemannian 2-orbifolds. We then apply it to Seifert fibred 3-manifolds endowed with one of the six Seifert geometries. By using a relation between the Chern-Simons invariant and the $\eta$-invariant, we also derive some elementary formulae for the Chern-Simons invariant of these manifolds. As applications, we show that some families of these manifolds cannot be conformally immersed into the Euclidean space $\mathbb{E}^4$.

1. Introduction

In dimension 3, the work of Thurston [15], [16] indicates that there are essentially eight relevant homogeneous geometries needed for geometric structures on 3-manifolds. Of these, six are the so-called Seifert geometries.

The purpose of this paper is to study two kinds of geometric invariants, the Chern-Simons invariant and the $\eta$-invariant, for a 3-manifold endowed with one of these six Seifert geometries.

The Chern-Simons invariant was first defined in [3] for a closed Riemannian 3-manifold by integrating a certain 3-form over the manifold. It is only defined mod $\mathbb{Z}$.

The $\eta$-invariant of a selfadjoint elliptic operator was originally introduced by Atiyah, Patodi, and Singer in [1] for odd-dimensional Riemannian manifolds in terms of the spectrum of the operator. The $\eta$-invariant we are interested in here is that of the signature operator. Such an invariant has a topological interpretation which allows us to compute it without using analytic tools. As shown in [1], it measures the extent to which the Hirzebruch signature formula fails for a nonclosed $4k$-dimensional Riemannian manifold whose metric is a product near its boundary. In dimension 3, it can be thought of as a real-valued generalization of the Chern-Simons invariant.

The remainder of this paper consists of three sections. In Section 2, we obtain a formula for the $\eta$-invariant of $S^1$-bundles over Riemannian 2-orbifolds. We then, in Section 3, apply our formula to the geometric Seifert fibred 3-manifolds to get explicit expressions of the $\eta$-invariant under these geometries. Finally, in Section 4, we derive some elementary formulae for the Chern-Simons invariant of geometric Seifert fibred 3-manifolds. As applications, we show that some
families of these manifolds cannot be conformally immersed into the Euclidean space $E^4$.

2. **$\eta$-invariant for $S^1$-bundles over Riemannian 2-orbifolds**

In [10], Komuro obtained a formula for the $\eta$-invariant of some circle bundles over Riemann surfaces. In this section, we extend his formula to the case of $S^1$-bundles over Riemannian 2-orbifolds by using the generalized Hirzebruch signature formula as developed by Kawasaki in [8] and [9]. We start with the following topological interpretation for the $\eta$-invariant due to Atiyah, Patodi, and Singer:

**Theorem 2.0** [1]. Let $N$ be a 4k-dimensional Riemannian manifold with $\partial N = M$. Assume that $N$ has a product metric near the boundary. Then

$$\eta(M) = \int_N L_k(P) - \text{Sign}(N)$$

where $L_k$ is the $k$th Hirzebruch $L$-polynomial, $P$ is the Pontrjagin form of $N$ and $\text{Sign}(N)$ is the signature of $N$.

Kawasaki in [8] generalized the above theorem to the case where $M^{4k-1}$ bounds a 4k-dimensional Riemannian orbifold.

To state his result, we need some preliminaries from [8] and [9].

Let $(X, \mathcal{B})$ be a compact orbifold. Denote by $\Sigma X$ the singular set of $X$. For every $x \in \Sigma X$, choose precisely one linear chart $(U_x, G_x, \pi_x)$ such that $\pi_x(0) = x$. For each $g \in G_x$, a local group of $x$, the centralizer $C_{G_x}(g)$ of $g$ in $G_x$ acts on $U^h_x = \{y \in U_x | g \cdot y = y\}$. Let $(1), (h^1_x), \ldots, (h^p_x)$ be all the conjugacy classes in $G_x$. Suppose $y \in U_x \cap \Sigma X$. Then an overlap map $\psi_{yx}$ between $U_y$ and $U_x$ induces a homomorphism $\psi_{yx} : G_y \rightarrow G_x$. The following is a bijection

$$\{(y, (h^j_x)) | y \in U_x \cap \Sigma X, \psi_{yx}^*(h^j_x) = h^j_x \} \rightarrow \tilde{U}^{h^j_x}_x / C_{G_x}(h^j_x),$$

$$(y, (h^j_x)) \mapsto [\psi_{yx}(0)]$$

which defines an orbifold structure on

$$\tilde{\Sigma}X = \{(x, (h^j_x)) | x \in \Sigma X, j = 1, 2, \ldots, p_x\}.$$

The multiplicity of $\tilde{\Sigma}X$ in $X$ at $(x, (h^j_x))$ is the order of the trivially acting subgroup of $C_{G_x}(h^j_x)$ on $\tilde{U}^{h^j_x}_x$.

Let $\tilde{\Sigma}_1X, \ldots, \tilde{\Sigma}_cX$ be the connected components of $\tilde{\Sigma}X$. The multiplicity is locally constant on $\tilde{\Sigma}X$. Thus we can assign a number $m_s = m(\tilde{\Sigma}_sX)$ to each $\tilde{\Sigma}_sX$, called the multiplicity of $\tilde{\Sigma}_sX$ in $X$.

On each orbifold chart $\tilde{U}^{h^s}_x$ of $\tilde{\Sigma}X$, we have the normal bundle $\nu(\tilde{U}^{h^s}_x)$ in $\tilde{U}_x$ and the tangent bundle $\tau(\tilde{U}^{h^s}_x)$. $h$ acts on $\nu(\tilde{U}^{h^s}_x)$. We have the decomposition

$$\nu(\tilde{U}^{h^s}_x) = \bigoplus_{0 < \theta < \pi} \nu^h_\theta$$

where $\nu^h_\theta$ is the bundle of eigenspace for $h$ with eigenvalues $e^{\pm i\theta}$. Introduce complex structures on the bundles $\nu^h_\theta$ such that $h \cdot v = e^{i\theta}v$ if $v \in \nu^h_\theta$. (For
simplicity, we assume here that $\nu^h_n$ can be given a compatible complex structure so that $\theta = \pi$ need not play a special role. The collection of these $C_{G_x}(h)$ bundles form complex vector bundles over $\Sigma X$. Choose a $C_{G_x}(h)$-invariant connection for each bundle. Write the total Chern class

$$c(\nu^h_\theta) = \prod_j (1 + x_j),$$

i.e., $c_n(\nu^h_\theta) = n$th symmetric polynomial in $x_j$'s. Define

$$L_\theta(\nu^h_\theta) = \prod_j \coth(x_j + i\theta \frac{2}{e^{i\theta e^{2x_j}} + 1}).$$

The local characteristic form

$$L^\Sigma(\nu^h_\theta) = L(\nu^h_\theta), \quad 0 < \theta \leq \pi$$

defines an $L$-class in $\Sigma X$.

**Theorem 2.1** (Kawasaki [8]). Let $N$ be a $4k$-dimensional Riemannian orbifold with $\partial N = M$ a $(4k - 1)$-dimensional Riemannian manifold. Assume that $N$ has a product metric near the boundary. Then

$$\eta(M) = \int_N L_k(P) - \text{Sign}(N) + \sum_{s=1}^c \frac{1}{m_s} \langle L^\Sigma_s, [\Sigma_s N] \rangle.$$  

In Theorems 2.0 and 2.1, it is required that the metric on $N$ is a product near the boundary. But this may not be the case in practice. We need a so-called "boundary correction term". The following formulation is due to Gilkey [4].

Let $C = M \times [0, 1]$ and identify $M \times \{0\}$ with $M = \partial N$. Let $g_0$ be the product metric on $C$. Extend the metric $g$ on $N$ to a metric $g_1$ on $C \cup N$ such that $g_1$ is a product near $M \times \{1\}$ and agrees with $g_0$ near $M \times 1$. Let $\nabla_0$ and $\nabla_1$ be the Riemannian connections determined by $g_0$ and $g_1$ respectively. Denote $\omega_0$, $\omega_1$ and $\Omega_0$, $\Omega_1$ the corresponding connection forms and curvature forms of $\nabla_0$ and $\nabla_1$ respectively.

Let $\omega = \omega_0 - \omega_1$, $\omega_t = (1 - t)\omega_0 + t\omega_1$, and $\Omega_t$ the curvature form of $\omega_t$. Define

$$TL_k = 2k \int_0^1 L_k(\omega_t, \Omega_t, \cdots, \Omega_t) dt.$$  

Then

$$dTL_k = L_k(\Omega_1) - L_k(\Omega_0).$$

Since $L_k(\Omega_0) = 0$, we get

$$dTL_k = L_k(\Omega_1) - L_k(\Omega_1, \cdots, \Omega_1).$$

Thus

$$\int_C L_k(\Omega_1) = \int_C dTL_k = \int_{\partial C} TL_k = \int_{M \times \{0\}} TL_k = -\int_M TL_k.$$
It follows from Theorem 2.1 that

\[
\eta(M) = \int_{C \cup N} L_k(P) - \text{Sign}(C \cup N) + \sum_{s=1}^{c} \frac{1}{m_s} \langle L_{x_s}, [\Sigma_s N] \rangle
\]

\[
= \int_C L_k(\Omega_1) + \int_N L_k(P) - \text{Sign}(N) + \sum_{s=1}^{c} \frac{1}{m_s} \langle L_{x_s}, [\Sigma_s N] \rangle
\]

\[
= -\int_M TL_k + \int_N L_k(P) - \text{Sign}(N) + \sum_{s=1}^{c} \frac{1}{m_s} \langle L_{x_s}, [\Sigma_s N] \rangle.
\]

Now, let \( S(E) \to F \) be a principal \( S^1 \)-bundle over an oriented, Riemannian 2-orbifold \((F, g)\). Let \( p : E \to F \) be the associated \( C^1 \)-bundle.

Choose a fiber metric \( \tilde{g} \) and a \( \tilde{g} \)-preserving connection \( \tilde{\nabla} \). Then we obtain a Riemannian orbifold \((E, g)\) by assuming that

\[
g|_{\text{horizontal}} = p^* \tilde{g} \quad \text{and} \quad g|_{\text{vertical}} = \tilde{g}.
\]

We now determine the Riemannian connection of \( g \) on \( E \).

For any \( x \in F \), choose local orthonormal (horizontal) sections \( e_1, e_2 \) near \( x \), Then they determine a coordinate system \( u^1, u^2 \) along the fiber with \( e_\alpha = \partial / \partial u^\alpha, \alpha = 1,2 \).

Let \( x_3, x_4 \) be a local orthonormal frame of \( F \) near \( x \) and \( e_3, e_4 \) their horizontal lifts with respect to \( \tilde{\nabla} \).

Let \( \hat{R} \) be the curvature tensor of the connection \( \tilde{\nabla} \) and \( \hat{\Omega}_\alpha^\theta \) its components with respect to \( e_1, e_2 \). Then \( \hat{\Omega}_1^2 = \hat{R}_{1234} x_3^* \wedge x_4^* \) where

\[
\hat{R}_{1234} = \tilde{g} (\hat{R}(x_3, x_4) e_1, e_2).
\]

We will denote \( \hat{R} = \hat{R}_{1234} \) for brevity. For convenience, we use the polar coordinate system \((r, \theta)\) in the vertical space so that \( u^1 = r \cos \theta \) and \( u^2 = r \sin \theta \).

A straightforward calculation yields

**Lemma 2.2.** With respect to the basis \( \{e_1 = \frac{1}{r} \frac{\partial}{\partial \theta}, e_2 = \frac{\partial}{\partial r}, e_3, e_4\} \), the connection form \( \omega \) of \( \nabla \) is given by

\[
\omega_1 = d\theta, \quad \omega_3 = -\frac{1}{2} r \hat{R} e_4^*, \quad \omega_4 = \frac{1}{2} r \hat{R} e_3^*,
\]

\[
\omega_2 = \omega_4 = 0, \quad \omega_3 = -\frac{1}{2} r^2 \hat{R} d\theta + p^* \hat{\omega}_3^4
\]

where \( \{r d\theta, dr, e_3^*, e_4^*\} \) is the dual basis of \( \{\frac{1}{r} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial r}, e_3, e_4\} \) and \( \hat{\omega}_3^4 \) is the component of the connection form of \( \tilde{g} \) with respect to \( x_3, x_4 \).

From Lemma 2.2, we have the following
Lemma 2.3. If $\bar{R}$ is constant on $F$, then with respect to the same basis as in Lemma 2.2, the components of the curvature form $\Omega$ of $\nabla$ are given by

\begin{align}
\Omega_1^2 &= \bar{R}e_3^* \wedge e_4^*, \\
\Omega_3^3 &= -\frac{1}{2} \bar{R}e_3^* \wedge dr + \frac{1}{4} r^2 \bar{R}^2 e_3^* \wedge r d\theta, \\
\Omega_4^4 &= \frac{1}{2} \bar{R}e_3^* \wedge dr + \frac{1}{4} r^2 \bar{R}^2 e_4^* \wedge r d\theta, \\
(2.1) \\
\Omega_2^2 &= \frac{1}{2} \bar{R}e_4^* \wedge r d\theta, \\
\Omega_3^3 &= -\frac{1}{2} \bar{R}e_3^* \wedge r d\theta, \\
\Omega_4^4 &= p^* \bar{\Omega}_3^3 - \frac{3}{4} r^2 \bar{R}^2 p^* \omega_F + \bar{R} dr \wedge r d\theta,
\end{align}

where $\bar{\Omega}_3^3$ is the curvature form of $\bar{\omega}_3^3$ and $\omega_F = x_3^* \wedge x_4^*$ is the volume form of $(F, \tilde{g})$. Equation (2) is valid in general.

Before presenting our main theorem, we need some facts about the local structure of a 2-orbifold and a complex line bundle over it.

Let $F$ be an oriented, closed 2-orbifold. Then, by the classification theorem of 2-orbifolds (see [15], for example), the only possible orbifold charts have the form $\mathbb{R}^2/(\mathbb{Z}/\alpha)$ with $\mathbb{Z}/\alpha$ acting on $\mathbb{R}^2$ by multiplication by $e^{2\pi i \beta/\alpha}$ for some $\beta$ prime to $\alpha$. $\Sigma F$ consists of a set of isolated points $\{x_1, \ldots, x_n\}$ in $F$. Thus, $F$ is determined by the data $(g; \alpha_1, \ldots, \alpha_n)$ where $g$ is the genus of $F$ and the $\alpha_i$'s are such that $G_{x_i} = \mathbb{Z}/\alpha_i$. We assume thereafter, without loss of generality, that $\alpha_i > 0$.

Let $E \to F$ be a complex line bundle over $F$ such that $E$ is a 4-orbifold with orbifold chart $(\mathbb{R}^2 \times \mathbb{R}^2)/\mathbb{Z}/\alpha$ where $\mathbb{Z}/\alpha$ acts on the first coordinate by $e^{2\pi i \beta/\alpha}$ and on the second by $e^{2\pi i \gamma/\alpha}$ for some $\beta$ and $\gamma$ prime to $\alpha$. Thus $\Sigma E = \{(x_1, 0), \ldots, (x_n, 0)\}$. We call

$$((\alpha_1; \beta_1, \gamma_1), \ldots, (\alpha_n; \beta_n, \gamma_n))$$

the orbifold data of $E$.

Theorem 2.4. Let $p : E \to F$ be a complex line bundle over an oriented, closed Riemannian 2-orbifold $(F, \tilde{g})$. Suppose that the total space $E$ has orbifold data $((\alpha_1; \beta_1, \gamma_1), \ldots, (\alpha_n; \beta_n, \gamma_n))$. Choose a fiber metric $\tilde{g}$ in $E$ and let $\bar{\nabla}$ be a $\tilde{g}$-preserving connection in $E$. Then $(E, \tilde{g})$ becomes a Riemannian orbifold. Assume that $\bar{R}$ is constant on $F$. Then the $\eta$-invariant of the circle bundle of radius $r$ is given by

$$\eta(S_r E) = \frac{\pi r^2}{\text{Vol}(F)} \chi - \left( \frac{\pi r^2}{\text{Vol}(F)} \right)^2 c_1^2 + \frac{1}{3} c_1 \varepsilon + \sum_{j=1}^n 4 s(\beta_j, \gamma_j; \alpha_j)$$

where $c_1$ is the (rational) Euler number of the bundle $E \to F$. $\chi$ is the (rational) Euler characteristic of the base orbifold $F$. $s(\beta_j, \gamma_j; \alpha_j)$ is the following
generalized Dedekind sum as in [7]:

\[ s(\beta, \gamma; \alpha) = \frac{1}{4\alpha} \sum_{k=1}^{\alpha-1} \cot \left( \frac{k\beta\pi}{\alpha} \right) \cot \left( \frac{k\gamma\pi}{\alpha} \right). \]

\( \varepsilon \) is defined by

\[ \varepsilon = \begin{cases} 
1 & \text{if } c_1 > 0, \\
0 & \text{if } c_1 = 0, \\
-1 & \text{if } c_1 < 0.
\end{cases} \]

Proof. Denote by \( D_r(E) \) the disk-bundle of radius \( r \) of \( E \). Let \( g_0 = g |_{S_r(E)} \times dt^2 \) be the product metric on \( D_{r+1}(E) - D_r(E) = S_r(E) \times [0, 1] \). Let \( \nabla_0 \) be the Riemannian connection determined by \( g_0 \) and \( \omega_0, \Omega_0 \) the connection form and curvature form of \( \nabla_0 \) respectively. Choose a metric \( g_1 \) on \( D_{r+1}(E) - D_r(E) \) such that \( g_1 = g \) on \( D_{r+1}(E) - D_r(E) \) and \( g_1 = g_0 \) on \( D_{r+1}(E) - D_{r+\frac{1}{2}}(E) \). Let \( \nabla_1 \) be the Riemannian connection determined by the metric \( g_1 \) and \( \omega_1, \Omega_1 \) the connection form and curvature form of \( \nabla_1 \) respectively.

Write \( \omega = \omega_1 - \omega_0 \) and \( \omega_t = (1 - t)\omega_0 + t\omega_1 \). Then from Lemma 2.2, with respect to the basis \( \{ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial r}, e_3, e_4 \} \), the components of \( \omega_t \) on \( D_{r+\frac{1}{2}}(E) - D_r(E) \) are given by

\[ (\omega_t)_{\frac{1}{2}} = t\tilde{\theta}, \quad (\omega_t)_{\frac{3}{2}} = -\frac{1}{2}r\tilde{\theta}e_4^*, \quad (\omega_t)_{\frac{4}{2}} = \frac{1}{2}r\tilde{\theta}e_4^*, \]

\[ (\omega_t)^2 = (\omega_t)^3 = 0, \quad (\omega_t)^4 = -\frac{1}{2}r^2\tilde{\theta}d\theta + p^*\hat{\omega}_4^3. \]

It follows that \( (\Omega_t)^2 = tp^*\hat{\Omega}_1^2 \). Also, \( \omega_1^2 = -\tilde{d}\theta \) and \( \omega_2^2 = 0 \) others. From Lemma 2.3 we get

\[ (\Omega_1^2)^2 = (\Omega_2^2)^2 = (\Omega_3^2)^2 = 0, \]

\[ (\Omega_1^3)^2 + (\Omega_4^3)^2 = -\frac{1}{2}r^2\tilde{\theta}e_4^* \wedge e_4^* \wedge rdr \wedge d\theta, \]

\[ (\Omega_3^4)^2 = -\frac{3}{2}r^2\tilde{\theta}e_4^* \wedge e_4^* \wedge rdr \wedge d\theta + 2p^*\hat{\Omega}_3^4 \wedge \tilde{\theta}dr \wedge r\tilde{\theta}. \]

Thus,

\[ L_1(g) = \frac{1}{3}P_1(g) = \frac{1}{12\pi^2} \sum_{s \leq q} (\Omega_s^q)^2 \]

\[ = \frac{1}{6\pi^2} (\tilde{\theta}r^2p^*\hat{\Omega}_3^4 \wedge rdr \wedge d\theta - r^2\tilde{\theta}p^*\omega_F \wedge rdr \wedge d\theta), \]

\[ TL_1 = 2 \int_0^1 L_1(\omega, \Omega_t) dt \]

\[ = 2 \cdot \frac{1}{12\pi^2} \int_0^1 tp^*\hat{\Omega}_1^2 \wedge (-\tilde{\theta}) dt \]

\[ = -\frac{1}{12\pi^2} p^*\hat{\Omega}_1^2 \wedge \tilde{\theta}. \]
Also, note that \( \hat{\Omega}_1^2 \) and \( \hat{\Omega}_1^2 \) represent \( 2\pi \chi(F) \) and \( 2\pi c_1(E) \) respectively. Thus, since \( \tilde{R} \) is constant on \( F \), we have

\[
\tilde{R} = \frac{2\pi c_1}{\text{Vol}(F)}.
\]

Choose a partition of unity \( \{ f_\alpha \}_{\alpha \in \Lambda} \) subordinate to the open cover \( \mathcal{U} \) of \( F \). Then we get

\[
\int_{D_r(E)} L_1(g) = \sum_{\alpha \in \Lambda} \frac{1}{|G_\alpha|} \int_{E_\alpha} (f_\alpha \circ p) \frac{1}{6\pi^2} (\tilde{R} p^* \hat{\Omega}_1^2 + \tilde{R} d\theta)
\]

\[
= \sum_{\alpha \in \Lambda} \frac{1}{|G_\alpha|} \int_{U_\alpha} f_\alpha \frac{1}{6\pi^2} \left( \pi r^2 \tilde{R} \hat{\Omega}_1^2 - \frac{1}{2} \pi r^4 \tilde{R}^2 \hat{\Omega}_1^2 \right)
\]

\[
= \int_F \frac{1}{6\pi^2} \left( \pi r^2 \tilde{R} \hat{\Omega}_1^2 - \frac{1}{2} \pi r^4 \tilde{R}^2 \hat{\Omega}_1^2 \right) [F]
\]

\[
= \frac{2}{3} c_1 \left( \frac{\pi r^2}{\text{Vol}(F)} \chi - \left( \frac{\pi r^2}{\text{Vol}(F)} \right)^2 c_1 \right).
\]

\[
\int_{S_r(E)} TL_1(g) = -\sum_{\alpha \in \Lambda} \frac{1}{|G_\alpha|} \int_{U_\alpha \times S^1} (f_\alpha \circ p) \frac{1}{12\pi^2} (p^* \hat{\Omega}_1^2 + \tilde{d}\theta)
\]

\[
= -\sum_{\alpha \in \Lambda} \frac{2\pi}{|G_\alpha|} \int_{U_\alpha} f_\alpha \frac{1}{12\pi^2} \hat{\Omega}_1^2 = -\frac{1}{6\pi} \int_F \hat{\Omega}_1^2
\]

\[
= -\frac{1}{6\pi} 2\pi c_1(E) [F] = -\frac{1}{3} c_1.
\]

To determine \( \text{Sign}(D_r(E)) \), we examine the following diagram:

\[
H^2(D_r(E), S_r(E)) \otimes H^2(D_r(E), S_r(E)) \rightarrow H^4(D_r(E), S_r(E))
\]

\[
\rightarrow \cong \rightarrow \cong \rightarrow \cong
\]

\[
H_2(D_r(E)) \otimes H_2(D_r(E)) \rightarrow H_0(D_r(E))
\]

\[
\rightarrow \cong \rightarrow \cong \rightarrow \cong
\]

\[
Z(a) \otimes Z(a) \rightarrow Z(1)
\]

\[
a \rightarrow a \rightarrow c_1
\]

where the vertical arrows are Poincaré-Lefschetz duality with \( Q \) coefficients and the last row is an orbifold version of the Euler characteristic of \( E \) (see
Thus, we get
\[
\text{Sign}(D_r(E)) = \varepsilon = \begin{cases} 
1 & \text{if } c_1 > 0, \\
0 & \text{if } c_1 = 0, \\
-1 & \text{if } c_1 < 0.
\end{cases}
\]

Next, we determine the term
\[
\sum \frac{1}{m_s} \langle L, [\Sigma D_r(E)] \rangle.
\]

By the orbifold data, we have
\[
\Sigma D_r(E) = \{(x_j,(g^j_k))| j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, \alpha_j - 1\}
\]
and
\[
m^k_j = \alpha_j \quad \text{for } k = 1, 2, \ldots, \alpha_j - 2.
\]

Then,
\[
\nu_{(x_j,(g^j_k))} = T_{(x_j,(g^j_k))}E = C_1 \oplus C_2
\]
with \(g^j_k\) acting on \(C_1\) and \(C_2\) by multiplication by \(e^{2\pi \beta_j k i/\alpha_j}\) and \(e^{2\pi \gamma_j k i/\alpha_j}\) respectively. It follows that
\[
\langle L, [\Sigma D_r(E)] \rangle = \prod_{0 < \theta \leq \pi} L_\theta(\nu_{(x_j,(g^j_k))}[(x_j,(g^j_k))])
\]
\[
= \frac{e^{2\pi \beta_j k i/\alpha_j} + 1}{e^{2\pi \beta_j k i/\alpha_j} - 1} \cdot \frac{e^{2\pi \gamma_j k i/\alpha_j} + 1}{e^{2\pi \gamma_j k i/\alpha_j} - 1}
\]
\[
= - \cot \frac{k \beta_j \pi}{\alpha_j} \cot \frac{k \gamma_j \pi}{\alpha_j}.
\]

Therefore,
\[
\sum \frac{1}{m_s} \langle L, [\Sigma D_r(E)] \rangle = - \sum_{j=1}^{n} \frac{1}{\alpha_j} \sum_{k=1}^{\alpha_j-1} \cot \frac{k \beta_j \pi}{\alpha_j} \cot \frac{k \gamma_j \pi}{\alpha_j}
\]
\[
= \sum_{j=1}^{n} \frac{1}{\alpha_j} (4\alpha_j s(\beta_j, \gamma_j; \alpha_j))
\]
\[
= 4 \sum_{j=1}^{n} s(\beta_j, \gamma_j; \alpha_j).
\]

Finally, from formula (2.1) we get
\[
\eta(S_rE) = \int_{D_r(E)} L_1 - \int_{S_rE} T L_1 - \text{Sign}(D_r(E)) + \sum \frac{1}{m_s} \langle L, [\Sigma D_r(E)] \rangle
\]
\[
= 2 \, c_1 \left\{ \frac{\pi r^2}{\text{Vol}(F)} \chi - \left( \frac{\pi r^2}{\text{Vol}(F)} \right)^2 c_1 \right\} + \frac{1}{3} c_1 - \varepsilon + 4 \sum_{j=1}^{n} s(\beta_j, \gamma_j; \alpha_j). \quad \Box
\]
3. Application to geometric Seifert fibred 3-manifolds

This section is devoted to providing some explicit formulae for the $\eta$-invariant of geometric Seifert fibred 3-manifolds. We refer to [11] and [13] for basic material on Seifert fibred 3-manifolds and their relevant geometries.

A Seifert fibred 3-manifold can be viewed as an $S^1$-fibration $M \to F$ over a closed 2-orbifold $F$.

Associated to $M$ is the Seifert invariant $(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n))$.

The Euler number of the Seifert fibration is

$$e(M \to F) = -\sum_{j=1}^{n} \frac{\beta_j}{\alpha_j}.$$  

The Euler characteristic of the base orbifold $F$ is

$$\chi = 2 - 2g - \sum_{j=1}^{n} \frac{\alpha_j - 1}{\alpha_j}.$$ 

The relevant geometry of a Seifert fibred 3-manifold is determined by $e$ and $\chi$ as follows.

<table>
<thead>
<tr>
<th>$\chi &gt; 0$</th>
<th>$\chi = 0$</th>
<th>$\chi &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e = 0$</td>
<td>$S^2 \times E^1$</td>
<td>$E^3$</td>
</tr>
<tr>
<td>$e \neq 0$</td>
<td>$S^3$</td>
<td>Nil</td>
</tr>
</tbody>
</table>

In what follows, we assume that the base space $F$ is oriented.

Let $M = M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)) \to F$ be a geometric Seifert fibred 3-manifold and $E \to F$ be the associated $C^1$-bundle. Then the set of singular points of the orbifold $E$ is given by $\Sigma = \{(x_1, 0), \cdots, (x_n, 0)\}$. The local group $G_x = \mathbb{Z}/\alpha_j$ acts on the first and the second coordinate by multiplication by $e^{2\pi i \beta_j/\alpha_j}$ and $e^{2\pi i / \alpha_j}$ respectively.

Thus, in the expression of $\eta(M)$ in Theorem 2.4,

$$s(\beta_j, \gamma_j; \alpha_j) = s(\beta_j, 1; \alpha_j) = s(\beta_j, \alpha_j).$$

Choose a fiber metric $\tilde{g}$ on $E$ such that the induced metric on $M$ is the one from the corresponding Seifert geometry. Let $\tilde{\nabla}$ be a $\tilde{g}$-preserving connection on $E$.

**Lemma 3.1.** If $M$ is locally symmetric, then under the above assumption, $\tilde{R}$ is constant on $F$.

**Proof.** Choose the same local basis $\{e_1 = \frac{\partial}{\partial r}, e_2 = \frac{\partial}{\partial \theta}, e_3, e_4\}$ as in Section 2. For every $x \in M$, denote $I_x$ the local reflection about $x$. Since $M = S_r(E)$ is locally symmetric, $I_x$ is an isometry. Hence $dI_x = -Id$ commutes with $\nabla R$.

Thus

$$-\nabla_{e_\alpha} R(e_3, e_4) e_1 = dI_x(\nabla_{e_\alpha} R(e_3, e_4) e_1)$$

$$= \nabla_{-e_\alpha} R(-e_3, -e_4)(-e_1) = \nabla_{e_\alpha} R(e_3, e_4) e_1 \quad \text{for} \quad \alpha = 3, 4.$$
Therefore,
\[ \nabla_{e_{\alpha}} R(e_3, e_4)e_1 = 0 \quad \text{for} \quad \alpha = 3, 4. \]

From Lemma 2.2, we have \( \omega_3^2 = \omega_4^2 = 0 \). Thus,
\[ \nabla_{e_{\alpha}} e_2 = 0 \quad \text{for} \quad \alpha = 3, 4. \]

It follows that
\[
e_{\alpha}(R_{1234}) = e_{\alpha}(g(R(e_3, e_4)e_1, e_2))
\]
\[= g(\nabla_{e_{\alpha}} R(e_3, e_4)e_1, e_2) + g(R(e_3, e_4)e_1, \nabla_{e_{\alpha}} e_2)
\]
\[= 0 \quad \text{for} \quad \alpha = 3, 4. \]

Also, from formula (2) in Section 2, we have
\[ \hat{R} \circ \pi = R_{1234} \circ \pi = R_{1234}. \]

Thus we get \( x_{\alpha}(\hat{R}) = e_{\alpha}(R) = 0 \) for \( \alpha = 3, 4 \). Hence, \( \hat{R} \) is constant on \( F \). \( \square \)

Now, we are in position to apply Theorem 2.4 to the geometric Seifert fibred 3-manifolds.

(a) \( M \) is modeled on \( S^2 \times E^1- \), \( E^3- \) or \( H^2 \times E^1 \)-geometry.

Any Seifert manifold \( M(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)) \) with one of these geometries is locally symmetric and \( c_1 = e = 0 \). Thus we have by Theorem 2.4,
\[ \eta(M) = 4 \sum_{j=1}^{n} s(\beta_j, \alpha_j). \]

(b) \( M \) is modeled on \( S^3 \)-geometry.

Clearly, every Seifert manifold with this geometry is locally symmetric.

(1) \( n \geq 3 \). \( F \) has an \( S^2 \)-geometry. We have
\[
2\pi \chi \cdot 2\pi r = \text{Vol}(M) = \frac{\text{Vol}(S^3(2))}{|\pi_1(M)|} = \frac{16\pi^2}{4|\chi|} = 4\pi^2 \frac{\chi^2}{|e|}. 
\]

Hence, \( r = |\chi/e| \). Thus Theorem 2.4 yields
\[
\eta(M) = \frac{1}{6} \frac{\chi^2}{e} + \frac{1}{3} e - \text{sgn}(e) + 4 \sum_{j=1}^{n} s(\beta_j, \alpha_j). 
\]

(2) \( n \leq 2 \). We have \( M((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = L(p, q) \) where
\[
\epsilon p = \alpha_1 \beta_2 + \alpha_2 \beta_1, \quad \epsilon q = \alpha_1 \beta_2' + \alpha_2' \beta_1, \quad 1 = \alpha_2 \beta_2' - \alpha_2' \beta_2, 
\]
and \( \epsilon = \pm 1 \).

In order to use Theorem 2.4, we need a "geometric fibration" \( M \rightarrow F \) such that \( F \) possesses a geometry from \( S^2 \) as a 2-orbifold. This is equivalent to requiring that \( \alpha_1 = \alpha_2 \).
Lemma 3.2. Any lens space $L(p, q)$ possesses exactly two geometric Seifert fibrations

$$L(p, q) = M((\alpha, \gamma_1), (\alpha, \gamma_2))$$

where

$$\alpha = \frac{p}{\gcd(p, q - 1)} , \quad \gamma_1 + \gamma_2 = \gcd(p, q - 1), \quad \text{and}$$

$$\gamma_2 \frac{q - 1}{\gcd(p, q - 1)} \equiv -1 \pmod{\alpha} \quad \text{or}$$

$$L(p, q) = M((\alpha', \gamma'_1), (\alpha', \gamma'_2))$$

where

$$\alpha' = \frac{p}{\gcd(p, q + 1)} , \quad \gamma'_1 + \gamma'_2 = -\gcd(p, q + 1), \quad \text{and}$$

$$\gamma'_2 \frac{q + 1}{\gcd(p, q + 1)} \equiv -1 \pmod{\alpha'} .$$

Proof. We will show it in one case. The other case is similar.

Suppose that $\alpha, \gamma_1, \gamma_2$ satisfies

$$p = \alpha \gamma_2 + \alpha \gamma_1 , \quad q = \alpha \beta'_2 + \alpha \beta'_1 \gamma_1 , \quad 1 = \alpha \beta'_2 - \alpha \beta'_1 \gamma_2 .$$

Then we have

$$1 = \alpha \beta'_2 - \frac{q - 1}{\gamma_1 + \gamma_2} \gamma_2 = \frac{p}{\gamma_1 + \gamma_2} \beta'_2 - \frac{q - 1}{\gamma_1 + \gamma_2} \gamma_2 .$$

It follows that

$$\gamma_1 + \gamma_2 = \gcd(p, q - 1), \quad \gamma_2 \frac{q - 1}{\gcd(p, q - 1)} \equiv -1 \pmod{\alpha} , \quad \text{and}$$

$$\alpha = \frac{p}{\gcd(p, q - 1)} .$$

The converse is straightforward. \( \square \)

For the geometric fibration $L(p, q) = M((\alpha', \gamma'_1), (\alpha', \gamma'_2))$, we have

$$e = -\frac{\gamma'_1 + \gamma'_2}{\alpha} = -\left(\frac{\gcd(p, q - 1)}{p}\right)^2 \quad \text{and} \quad \chi = 2 - 2\frac{\alpha - 1}{\alpha} = 2\frac{\gcd(p, q - 1)}{p} .$$

Thus, from Theorem 2.4, we get

$$\eta(L(p, q)) = \eta(M((\alpha, \gamma_1), (\alpha, \gamma_2)))$$

$$= \frac{1}{6} \chi^2 + \frac{1}{3} e + \text{sgn}(p) + 4 \sum_{j=1}^{2} s(\gamma_j, \alpha)$$

$$= -\frac{2}{3p} - \frac{1}{3} \left(\frac{\gcd(p, q - 1)}{p}\right)^2 + \text{sgn}(p) + 4 \sum_{j=1}^{2} s(\gamma_j, \alpha) .$$

where

$$\alpha = \frac{p}{\gcd(p, q - 1)} , \quad \gamma_1 + \gamma_2 = \gcd(p, q - 1), \quad \text{and}$$
\[ \gamma_2 \frac{q - 1}{\gcd(p, q - 1)} \equiv -1 \pmod{\alpha}. \]

Similar computation yields

\[ \eta(L(p, q)) = \eta(M((\alpha', \gamma'_1), (\alpha', \gamma'_2))) = \frac{2}{3p} + \frac{1}{3} \left( \frac{\gcd(p, q + 1)^2}{p} \right) - \text{sgn}(p) + 4 \sum_{j=1}^{2} s'(\gamma'_j, \alpha'). \]

where

\[ \alpha' = \frac{p}{\gcd(p, q + 1)}, \quad \gamma'_1 + \gamma'_2 = -\gcd(p, q + 1), \quad \text{and} \]

\[ \gamma'_2 \frac{q + 1}{\gcd(p, q + 1)} \equiv -1 \pmod{\alpha'}. \]

On the other hand, as computed by Atiyah-Patodi-Singer in [2]

\[ \eta(L(p, q)) = -4s(q, p). \]

Thus, by equating the above three formulae, we get the following interesting identities about the Dedekind sums:

\[ s(q, p) = -\sum_{j=1}^{2} s(\gamma_j, \alpha) + \frac{1}{6p} + \frac{1}{12} \left( \frac{\gcd(p, q - 1)^2}{p} \right) - \frac{1}{4} \text{sgn}(p) \]

\[ = -\sum_{j=1}^{2} s(\gamma'_j, \alpha') - \frac{1}{6p} - \frac{1}{12} \left( \frac{\gcd(p, q + 1)^2}{p} \right) + \frac{1}{4} \text{sgn}(p) \]

where

\[ \alpha = \frac{p}{\gcd(p, q - 1)}, \quad \gamma_1 + \gamma_2 = \gcd(p, q - 1), \quad \text{and} \]

\[ \gamma_2 \frac{q - 1}{\gcd(p, q - 1)} \equiv -1 \pmod{\alpha} \]

and

\[ \alpha' = \frac{p}{\gcd(p, q + 1)}, \quad \gamma'_1 + \gamma'_2 = -\gcd(p, q + 1), \quad \text{and} \]

\[ \gamma'_2 \frac{q + 1}{\gcd(p, q + 1)} \equiv -1 \pmod{\alpha'}. \]

In particular, let \( q = 1 \) and \( p > 0 \) in the first equality or \( q = -1 \) and \( p < 0 \) in the second equation, we get

\[ (3.1) \quad s(1, p) = \frac{1}{12p}(p - \text{sgn}(p))(p - 2\text{sgn}(p)). \]

Thus, we have

\[ \sum_{k=1}^{\lfloor p \rfloor - 1} \left( \cot \left( \frac{\pi k}{p} \right) \right)^2 = \frac{1}{3}(p - \text{sgn}(p))(p - 2\text{sgn}(p)). \]

(c) \( M \) is modeled on \( \widetilde{PSL} \)-geometry. Equip \( H^2 \) with the standard hyperbolic metric. Then we have a natural metric on \( T(H^2) \). The identification
between $PSL(2, \mathbb{R})$ and the unit tangent bundle $T^1(\mathbb{H}^2)$ gives rise to a (left-invariant) metric on $PSL(2, \mathbb{R})$ which induces a metric on $\overline{PSL}$.

For a Seifert manifold $M \to F$ with this geometry and the given metric on $\overline{PSL}$, we have $\hat{R} = -1$ on $F$. Thus, Theorem 2.4 implies.

From a homomorphism $\pi_1(M) \to \text{Isom}(\overline{PSL})$ giving a geometric structure on $M$, we get $r = |\chi/e|$ (see [11]). It follows from Theorem 2.4 that

$$\eta(M) = -\frac{1}{2} \frac{\chi^2}{e} + \frac{1}{3}e - \text{sgn}(e) + 4 \sum_{j=1}^{n} s(\beta_j, \alpha_j).$$

From the above discussion, we have the following

**Corollary 3.3.** Under the above five geometries, if we fix the metric in each universal cover as above, then the $\eta$-invariant depends only on the topology.

Finally, under the Nil-geometry, the volume of the base orbifold is indeterminate, so the Seifert invariant alone is not sufficient to express $\eta(M)$.

We conclude this section with the following

**Corollary 3.4.** Equip $PSL(2, \mathbb{R})$ with the above metric. Let $\Gamma \subseteq PSL(2, \mathbb{R})$ be a co-compact Fuchsian group of signature $\{ g; \alpha_1, \ldots, \alpha_n \}$. Then

$$\eta(PSL(2, \mathbb{R})/\Gamma) = \frac{1}{6} (2g + 4 + 7n) - \sum_{j=1}^{n} \left( \frac{1}{3} \alpha_j + \frac{5}{6\alpha_j} \right).$$

**Proof.** As shown in [11],

$$PSL(2, \mathbb{R})/\Gamma = M(g; (1, 2g-2), (\alpha_1, \alpha_1-1), \ldots, (\alpha_n, \alpha_n-1)).$$

Thus,

$$\chi = e = 2 - 2g - \sum_{j=1}^{n} \frac{\alpha_j - 1}{\alpha_j}.$$

Also, from (3.1), we get

$$s(\alpha_j - 1; \alpha_j) = -s(1, \alpha_j) = -\frac{1}{12\alpha_j}(\alpha_j - 1)(\alpha_j - 2).$$

It follows from (3.2) that

$$\eta(PSL(2, \mathbb{R})/\Gamma) = -\frac{1}{2} \chi + \frac{1}{3} \chi + 1 + 4 \sum_{j=1}^{n} s(\alpha_j - 1, \alpha_j)$$

$$= -\frac{1}{6} \chi + 1 + 4 \sum_{j=1}^{n} \left( -\frac{1}{12\alpha_j}(\alpha_j - 1)(\alpha_j - 2) \right)$$

$$= \frac{1}{6} (2g + 4 + 7n) - \sum_{j=1}^{n} \left( \frac{1}{3} \alpha_j + \frac{5}{6\alpha_j} \right). \square$$

**Remark 1.** A similar formula for the the $\eta$-invariant of $PSL(2, \mathbb{R})/\Gamma$ associated to the Dirac operator was obtained in [14].
4. CHERN-SIMONS INVARIANT AND CONFORMAL IMMERSIONS

Let $M$ be a closed, oriented Riemannian 3-manifold. Chern and Simons defined a mod 1 invariant of $M$ in [3], now commonly denoted by $CS(M)$, and showed that $CS(M) \equiv 0 \pmod{1}$ if $M$ conformally immerses into $\mathbb{E}^4$.

A surprising relation between $CS(M)$ and $\eta(M)$ is demonstrated by the following

**Theorem 4.1** (Atiyah-Patodi-Singer [2]). Let $M$ be a closed, oriented Riemannian 3-manifold. Then

$$CS(M) \equiv \frac{3}{2} \eta(M) + \frac{1}{2} \sigma(H_1(M; \mathbb{Z})) \pmod{1}$$

where $\sigma(H_1(M; \mathbb{Z})) = \#$ of 2-primary summands in $H_1(M; \mathbb{Z})$.

In this section, we derive some elementary formulae for the Chern-Simons invariant of the geometric Seifert fibred 3-manifolds and show that some families of them cannot be conformally immersed into $\mathbb{E}^4$. We begin with the following

**Lemma 4.2.** Let $p, q$ be a pair of coprime positive integers. Choose $r, s$ such that $ps + qr = 1$, $q + r$ is even and $s$ is odd if $p$ is even. Then

$$6s(q, p) \equiv \frac{q + r}{2p} \pmod{1}.$$ 

**Proof.** Case 1. $p$ is odd. We have

$$6ps(q, p) = \frac{q + r_0}{2} + \frac{1}{2} p I(p, q) \in \mathbb{Z}$$

where $r_0$ is such that $qr_0 \equiv 1 \pmod{p}$, $-1 < r_0/p \leq 0$, and $I(p, q) \in \mathbb{Z}$ (see [7], for instance). Since $p$ is odd, $q + r_0$ and $I(p, q)$ have the same parity. It follows that

$$6s(q, p) \equiv \frac{q + r_0 + \delta p}{2p} \pmod{1}$$

with

$$\delta = \begin{cases} 0 & \text{if } q + r_0 \text{ is even}, \\ 1 & \text{if } q + r_0 \text{ is odd}. \end{cases}$$

Therefore we have

$$6s(q, p) \equiv \frac{q + r}{2p} \pmod{1}$$

with $qr \equiv 1 \pmod{p}$ and $q + r$ even.
Case 2. \( p \) is even. Then \( q \) is odd. 

From the Dedekind reciprocity law and Case 1, we get

\[
6s(q, p) = -6s(p, q) + \frac{p^2 + q^2 + 1 - 3pq}{2pq} \\
\equiv \frac{-p - t}{2q} + \frac{p^2 + q^2 + 1 - 3pq}{2pq} \pmod{1}, \quad pt \equiv 1 \pmod{q}, \ t \text{ even} \\
\equiv \frac{q^2 + 1 - ps}{2pq} \pmod{1}, \quad ps \equiv 1 \pmod{q}, \ s \text{ odd} \\
\equiv \frac{q + r}{2p} \pmod{1}, \quad ps + qr = 1, \ s \text{ odd}. \quad \Box
\]

Lemma 4.3. Let

\[ M = M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n) \]

be a Seifert fibred 3-manifold and \( l \) the number of even \( \alpha_j \)'s. Then

\[
\sigma(H_1(M; \mathbb{Z})) = \begin{cases} 
1 - 1 & \text{if } l \geq 1, \\
1 & \text{if } l = 0 \text{ and } \sum \beta_j \text{ is even}, \\
0 & \text{if } l = 0 \text{ and } \sum \beta_j \text{ is odd}.
\end{cases}
\]

Proof. Arrange the Seifert invariant such that

\[ M = M(g; (1, a), (\alpha_1, \beta_1), \cdots, (\alpha_l, \beta_l), (\alpha_{l+1}, \beta_{l+1}), \cdots, (\alpha_n, \beta_n)) \]

where \( \alpha_j \) is even for \( j \leq l \), \( \alpha_j \) is odd for \( j > l \), \( \beta_j' \) is even for \( j > l \), and \( a \) is the number of odd \( \beta_j \)'s for \( j > l \). Then we have

\[
H_1(M; \mathbb{Z}) = \mathbb{Z}^{2g} \oplus \text{Cok}(\begin{array}{cccccc}
1 & 1 & \cdots & 1 & \cdots & 1 & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & a \\
\alpha_1 & & & & & & \\
\alpha_l & & & & & & \\
\alpha_{l+1} & & & & & & \\
\alpha_n & & & & & & \\
\end{array}
\).
Therefore

\[
\sigma(H_1(M; \mathbb{Z})) = \sigma(\text{Cok}(\begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a \\
\alpha_1 & & & & & & & \\
& \alpha_l & & & & & & \\
& & \alpha_{l+1} & & & & & \\
& & & \beta_l & & & & \\
& & & & \beta_{l+1} & & & \\
& & & & & \beta_n & & \\
0 & & & & & & 1 & \\
& & & & & & 0 & 1 \\
& & & & & & 1 & 0 \\
& & & & & & & 1 \\
\end{pmatrix}))
\]

\[
= \sigma(\text{Cok}(\begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a \\
0 & & & & & & 1 & \\
& & \alpha_{l+1} & & & & 0 & \\
& & & \beta_l & & & 1 & \\
& & & & \beta_{l+1} & & 0 & \\
& & & & & \beta_n & & 1 \\
& & & & & & 1 & 0 \\
& & & & & & & 1 \\
\end{pmatrix}))
\]

It follows that

\[
\sigma(H_1(M; \mathbb{Z})) = \begin{cases}
 l - 1 & \text{if } l \geq 1 , \\
 1 & \text{if } l = 0 \text{ and } a \text{ is even} , \\
 0 & \text{if } l = 0 \text{ and } a \text{ is odd} ,
\end{cases}
\]

\[
= \begin{cases}
 l - 1 & \text{if } l \geq 1 , \\
 1 & \text{if } l = 0 \text{ and } \sum \beta_j \text{ is even} , \\
 0 & \text{if } l = 0 \text{ and } \sum \beta_j \text{ is odd} .
\end{cases}
\]

By virtue of the formulae for \( \eta(M) \) in Section 3, Theorem 4.1, Lemma 4.2 and Lemma 4.3, we derive the following elementary formulae for the Chern-Simons invariant of the geometric Seifert fibred 3-manifolds.

Without loss of generality, we assume in what follows that our Seifert invariant is in a normal form \(((1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))\). For every pair of \((\alpha_j, \beta_j)\), choose \(r_j, s_j\) such that \(\alpha_j s_j + \beta_j r_j = 1\), \(\beta_j + r_j\) is even and \(s_j\) is odd if \(\alpha_j\) is even.

(a) \(M\) is modeled on \(\mathbb{E}^3, S^2 \times \mathbb{E}^1\), or \(H^2 \times \mathbb{E}^1\)-geometry. We have

\[
CS(M) \equiv 6 \sum_{j=1}^{n} s(\beta_j, \alpha_j) + \frac{1}{2} \sigma(H_1(M; \mathbb{Z})) \pmod{1}
\]

\[
\equiv \begin{cases}
 \sum_{j=1}^{n} r_j/(2\alpha_j) + (l - 1)/2 \pmod{1} & \text{if } l \geq 1 , \\
 \sum_{j=1}^{n} r_j/(2\alpha_j) + 1/2 \pmod{1} & \text{if } l = 0 \text{ and } b + \sum \beta_j \text{ is even} , \\
 \sum_{j=1}^{n} r_j/(2\alpha_j) \pmod{1} & \text{if } l = 0 \text{ and } b + \sum \beta_j \text{ is odd} .
\end{cases}
\]
(b) \( M \) is modeled on \( \mathbb{S}^3 \)-geometry.

(i) \( n \geq 3 \). We have
\[
CS(M) \equiv \frac{1}{4} \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} + \frac{1}{2} l \quad (\text{mod 1}) \quad \text{if } l \leq 1,
\]
\[
\equiv \frac{1}{4} \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} - \frac{1}{2} l \quad (\text{mod 1}) \quad \text{if } l = 0 \text{ and } b + \sum \beta_j \text{ is even},
\]
\[
\equiv \frac{1}{4} \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} + \frac{1}{2} l \quad (\text{mod 1}) \quad \text{if } l = 0 \text{ and } b + \sum \beta_j \text{ is odd}.
\]

(ii) \( n \leq 2 \). We have \( M((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = L(p, q) \) and
\[
CS(L(p, q)) \equiv -6s(q, p) + \frac{1}{2} \delta(L(p, q)) \quad (\text{mod 1})
\]
\[
\equiv -\frac{q + r}{2p} \quad (\text{mod 1})
\]
where \( ps + qr = 1 \) with \( q + r \) even and \( s \) even if \( p \) is even.

(c) \( M \) is modeled on \( \text{PSL} \)-geometry.

\[
CS(M) \equiv -\frac{3}{4} \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} + \frac{1}{2} l \quad (\text{mod 1}) \quad \text{if } l \geq 1,
\]
\[
\equiv -\frac{3}{4} \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} - \frac{1}{2} l \quad (\text{mod 1}) \quad \text{if } l = 0 \text{ and } b + \sum \beta_j \text{ is even},
\]
\[
\equiv -\frac{3}{4} \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} + \frac{1}{2} l \quad (\text{mod 1}) \quad \text{if } l = 0 \text{ and } b + \sum \beta_j \text{ is odd}.
\]

As Hirsch showed in [6], all compact 3-manifolds immerse in \( \mathbb{R}^4 \). We will show that some families of the geometric Seifert fibred 3-manifolds cannot be conformally immersed into \( \mathbb{E}^4 \).

**Corollary 4.4.** Let \( F_g \) be the surface of genus \( g > 1 \) with a hyperbolic geometry. If \( g \) is even, then \( T^1(F_g) = M(g, (1, 2g - 2)) \) with the induced metric from \( \mathbb{H}^2 \) doesn't conformally immerse into \( \mathbb{E}^4 \).
Proof. We have \( e = \chi = 2 - 2g \). It follows from (4.2) that

\[
CS(T^1(F_g)) \equiv \frac{g}{2} - \frac{1}{2} \pmod{1} \neq 0 \quad \text{if} \quad g \text{ is even}. \quad \square
\]

**Corollary 4.5.** Equip \( SO(3) \) with a bi-invariant metric. Let \( \Gamma \) be a finite subgroup of \( SO(3) \). Then \( SO(3)/\Gamma \) doesn’t conformally immerse into \( E^4 \).

**Proof.** By the conformal invariance of the Chern-Simons invariant, we can assume that \( SO(3) \) possesses the standard metric from \( S^3(1) \).

From (5) we have

\[
CS(SO(3)) = CS(L(2, 1)) \equiv -\frac{1}{2} \pmod{1}.
\]

Thus, \( SO(3) \) cannot be conformally immersed into \( E^4 \). Hence \( SO(3)/\Gamma \) cannot be conformally immersed into \( E^4 \). \( \square \)

**Remark.** Heitsch and Lawson in [5] showed that a similar result holds in general for \( SO(2k + 1)/\Gamma \) where \( \Gamma \) is a discrete subgroup of \( SO(2k + 1) \) and \( SO(2k + 1) \) is equipped with a bi-invariant metric.

**Corollary 4.6.** \( L(p, q) \) with the standard metric cannot be conformally immersed into \( E^4 \) except possibly when \( q^2 + 1 \equiv 0 \pmod{p} \) and \( p \) is odd.

**Proof.** If \( p = 2k \) is even, then

\[
L(p, q) = S^3/(Z/2k) = SO(3)/(Z/k).
\]

Thus, Corollary 4.5 implies that \( L(p, q) \) cannot be conformally immersed into \( E^4 \);

If \( p \) is odd, then from (4.1) we get

\[
CS(L(p, q)) \equiv -\frac{q + r}{2p} \pmod{1}
\]

with \( qr \equiv 1 \pmod{p} \) and \( q + r \) even. Thus, \( CS(L(p, q)) \equiv 0 \pmod{1} \) implies \( q + r \equiv 0 \pmod{p} \). Hence \( q^2 + 1 \equiv 0 \pmod{p} \). \( \square \)

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**References**


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