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ALGEBRAS ASSOCIATED TO THE YOUNG-FIBONACCI LATTICE

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ABSTRACT. The algebra $\mathcal{F}_n$ generated by $E_1, \dots, E_{n-1}$ subject to the defining relations $E_i^2 = x_i E_i$ ($i = 1, \dots, n-1$), $E_{i+1}E_iE_{i+1} = y_iE_{i+1}$ ($i = 1, \dots, n-2$), $E_iE_j = E_jE_i$ ($|i-j| \geq 2$) is shown to be a semisimple algebra of dimension $n!$ if the parameters $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-2}$ are generic. We also prove that the Bratteli diagram of the tower $(\mathcal{F}_n)_{n\geq 0}$ of these algebras is the Hasse diagram of the Young-Fibonacci lattice, which is an interesting example, as well as Young's lattice, of a differential poset introduced by R. Stanley. A Young-Fibonacci analogue of the ring of symmetric functions is given and studied.

INTRODUCTION

In [S1], R. Stanley introduced a class of partially ordered sets called differential posets, whose prototypical example is Young's lattice $\mathbb{Y}$. S. Fomin [F1] independently defined essentially the same class of graphs, called Y-graphs. (See [F2], [S2] for generalization.) Many enumerative results, concerning the counting of chains or Hasse walks in differential posets or Y-graphs, can be derived by using an algebraic approach (see [S1]) and also by applying a combinatorial method such as Robinson-Schensted-type correspondences (see [F1], [F3], [R1], [R2]). In the case of Young's lattice, these properties reflect the representation theory of the symmetric groups and the theory of symmetric functions.

Fomin [F1] and Stanley [S1] also gave another example of a differential poset, $\mathbb{YF}$, called the Young-Fibonacci lattice. (In [S1] this lattice is denoted by $Z(1)$.) And Stanley posed a problem [S1, §6, Problem 8] to give a natural and combinatorial definition of the tower $(\mathcal{F}_n)_{n\geq 0}$ of semisimple algebras, which play the same role to the Young-Fibonacci lattice $\mathbb{YF}$ as the group algebras of the symmetric groups play to Young's lattice $\mathbb{Y}$. This work is motivated to this problem and the first aim of this article is to give a presentation of $\mathcal{F}_n$, which corresponds to that of the symmetric group with respect to the adjacent transpositions. The second aim is to define and study a $\mathbb{YF}$-analogue of the ring of symmetric functions.

Let us explain in more detail. Young's lattice $\mathbb{Y}$ is the set of all partitions ordered by inclusion of Young (or Ferrers) diagrams. It is well known that the irreducible representations of the symmetric group $\mathfrak{S}_n$ are parametrized by $\mathbb{Y}_n$, the set of partitions of n. If we denote by $V^\lambda_{\mathfrak{S}_n}$ the irreducible $\mathfrak{S}_n$-module

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corresponding to a partition λ, then the restriction of V^λ to $\mathfrak{S}_{n-1}$ decomposes as follows:

$$V^\lambda_{\mathfrak{S}_n} \downarrow_{\mathfrak{S}_{n-1}} \cong \bigoplus_\mu V^\mu_{\mathfrak{S}_{n-1}},$$

where μ runs over all partitions whose Young diagrams are obtained from that of λ by deleting one box. Moreover, the direct sum $R(\mathfrak{S}) = \bigoplus_{n\geq 0} R(\mathfrak{S}_n)$ of the character ring $R(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ has a structure of graded algebra and there is an algebra isomorphism from $R(\mathfrak{S})$ to the ring Λ of symmetric functions. Under this isomorphism, the irreducible character χ^λ of $V^\lambda_{\mathfrak{S}_n}$ corresponds to the Schur function s_λ.

The Young-Fibonacci lattice $\mathbb{YF}$ is a differential poset consisting of all words with alphabets $\{1, 2\}$. (See Section 1 for the definition of the partial order on $\mathbb{YF}$.) Let $\mathscr{F}_n$ be the associative algebra (over a field K_0 of characteristic 0) defined by the following presentation:

$$\begin{aligned}
\text{generators} &: E_1, \dots, E_{n-1}, \\
\text{relations} &: E_i^2 = x_i E_i \quad (i = 1, \dots, n-1), \\
& \quad E_{i+1}E_iE_{i+1} = y_i E_{i+1} \quad (i = 1, \dots, n-2), \\
& \quad E_iE_j = E_jE_i \quad (\text{if } |i-j| \geq 2).
\end{aligned}$$

Suppose that the parameters $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-2} \in K_0$ are generic. In Section 2, we will construct irreducible representations of this algebra $\mathscr{F}_n$ and prove that $\mathscr{F}_n$ is semisimple of dimension $n!$ and its irreducible representations are indexed by $\mathbb{YF}_n$, the set of elements of $\mathbb{YF}$ with rank n. If we denote by V_v the irreducible $\mathscr{F}_n$-module corresponding to $v \in \mathbb{YF}_n$, then the branching rule for the restriction to the subalgebra $\mathscr{F}_{n-1} = \langle E_1, \dots, E_{n-2}\rangle$ is described in the same way as in the case of $\mathbb{Y}$:

$$V_v \downarrow_{\mathscr{F}_{n-1}} \cong \bigoplus_w V_w,$$

where w runs over all words covered by v in $\mathbb{YF}$. In Section 3, we define a graded algebra $R = \bigoplus_{n\geq 0} R_n$, whose homogeneous components R_n are the free $\mathbb{Z}$-modules with basis corresponding to (the isomorphism classes of) the irreducible representations of $\mathscr{F}_n$. This algebra can be considered as a $\mathbb{YF}$-analogue of the ring Λ of symmetric functions. We introduce various basis of R, which correspond to Schur functions, complete symmetric functions, and power sum symmetric functions, and study the transition matrices between these basis in Sections 4 and 5. A generalization to the r-Young-Fibonacci lattice will be given in Section 6.

1. Young-Fibonacci lattice

In this section, we collect some notations and properties concerning with the Young-Fibonacci lattice, which will be used in the rest of this paper. The reader is referred to [S1] for the general theory of differential posets and [S1, §5], [S3] for further information of the Young-Fibonacci lattice.

Let r be a positive integer. Let $\mathbb{YF}^{(r)}$ be the set of all finite words (including the empty word $\varnothing$) with alphabets $\{1_0, \dots, 1_{r-1}, 2\}$. For such a word $v = a_1 \dots a_k \in \mathbb{YF}^{(r)}$, we define its rank $|v| = |a_1| + \cdots + |a_k|$, where $|1_m| = 1$. And we put $\mathbb{YF}^{(r)}_n = \{v \in \mathbb{YF}^{(r)} : |v| = n\}$.

We define a partial order on $\mathbb{YF}^{(r)}$ by requiring the following conditions:

$$\varnothing \text{ is the minimum element}, \tag{1.1}$$

$$C^-(1_m v) = \{v\}, \tag{1.2}$$

$$C^-(2v) = C^+(v), \tag{1.3}$$

where $C^-(x)$ (resp. $C^+(x)$) denotes the set of all elements covered by (resp. covering) x. The notation $x \rhd y$ will be used to mean that x covers y. From (1.2) and (1.3), we have

$$C^+(v) = \{1_m v : m = 0, \dots, r-1\} \cup \{2w : w \in C^-(v)\}. \tag{1.4}$$

This poset $\mathbb{YF}^{(r)}$ is is shown to be a graded lattice, and its rank generating function is given by

$$\sum_{n\geq 0} \#\mathbb{YF}_n^{(r)} q^n = (1 - rq - q^2)^{-1}.$$

In particular, $\#\mathbb{YF}_n^{(1)}$ is the nth Fibonacci number F_n. We call $\mathbb{YF}^{(r)}$ the *r-Young-Fibonacci lattice.*

Let $R_n^{(r)}$ be the free $\mathbb{Z}$-module with basis $\{s_v : v \in \mathbb{YF}_n^{(r)}\}$. Put

$$R^{(r)} = \bigoplus_{n\geq 0} R_n^{(r)}$$

and define a scalar product on R by $\langle s_v, s_w\rangle = \delta_{vw}$ for all $v, w \in \mathbb{YF}^{(r)}$. We introduce two linear maps $U, D : R^{(r)} \to R^{(r)}$ by putting

$$Us_v = \sum_{w \rhd v} s_w, \quad Ds_v = \sum_{w \lhd v} s_w.$$

In Sections 3 and 6, we will define a structure of graded algebra on $R^{(r)}$.

Proposition 1.1 [S1, §5]. *The poset $\mathbb{YF}^{(r)}$ is an r-differential poset. Hence we have $DU - UD = r\,\mathrm{Id}$, where* Id *denotes the identity map on $R^{(r)}$.*

For $v \in \mathbb{YF}_n^{(r)}$, let Ω^v be the set of all sequences $(v^{(0)}, \dots, v^{(n)})$ such that $v^{(0)} = \varnothing$, $v^{(n)} = v$, and $v^{(i)}$ covers $v^{(i-1)}$ for all i; that is, Ω^v is the set of all saturated chains from $\varnothing$ to v. We denote the cardinality of Ω^v by $e(v)$. From the general theory of differential posets, we have the following proposition:

Proposition 1.2 [S1, Corollary 3.9]. *For the r-Young-Fibonacci lattice $\mathbb{YF}^{(r)}$, one has*

$$\sum_{v\in\mathbb{YF}_n^{(r)}} e(v)^2 = r^n n!.$$

If $r = 1$, then we omit the superscript (r), so that we write $\mathbb{YF} = \mathbb{YF}^{(1)}$, $\mathbb{YF}_n = \mathbb{YF}_n^{(1)}$, $R = R^{(1)}$, and $R_n = R_n^{(1)}$.

It is convenient to write $v \in \mathbb{YF}$ of the form $1^{m_1}21^{m_2}2\dots 1^{m_r}21^{m_{r+1}}$, where r is the number of 2's appearing in v and $m_i \geq 0$. The number m_1 is denoted by $m(v)$ and it will play a role in Section 5.

2. Algebra $\mathscr{F}_n$ and its representations

Let K_0 be a field of characteristic 0. We work over the base field $K = K_0(x_1, \dots, y_1, \dots)$, the rational function field with indeterminates $x_1, \dots, y_1, \dots$.

Definition. Let $\mathscr{F}_n = \mathscr{F}_n(x_1, \dots, x_{n-1}; y_1, \dots, y_{n-2})$ be the associative K-algebra with identity 1 defined by the following presentation:

$$\text{generators}: E_1, \dots, E_{n-1},$$

$$\text{relations}: E_i^2 = x_i E_i \quad (i = 1, \dots, n-1), \tag{2.1}$$

$$E_i E_j = E_j E_i \quad (\text{if } |i-j| \geq 2), \tag{2.2}$$

$$E_{i+1} E_i E_{i+1} = y_i E_{i+1} \quad (i = 1, \dots, n-2). \tag{2.3}$$

In this section, we will construct irreducible representations of $\mathscr{F}_n$ by using paths in $\mathbb{YF}$ (as in [GHJ, Chapter 2], [KM], [W]) and prove that $\mathscr{F}_n$ is a semisimple algebra of dimension $n!$.

First we show that the monomials in $E_1, \dots, E_{n-1}$ span the algebra $\mathscr{F}_n$.

Lemma 2.1. *We define a sequence of subsets $\mathscr{B}_k$ $(k = 0, 1, \dots, n)$ as follows:*

$$\begin{gathered} \mathscr{B}_0 = \mathscr{B}_1 = \{1\}, \\ \mathscr{B}_m = \{bE_{m-1} \dots E_k : b \in \mathscr{B}_{m-1}, k = 1, \dots, m\}. \end{gathered} \tag{2.4}$$

Here we understand that $E_{m-1} \dots E_k = 1$ if $k = m$. Then $\mathscr{B}_n$ spans $\mathscr{F}_n$. In particular, $\dim_K \mathscr{F}_n \leq n!$.

Proof. Let $\mathscr{F}_m'$ be the K-subspace spanned by $\mathscr{B}_m$. We prove by induction on m that $\mathscr{F}_m'$ is stable under the right multiplication by E_l $(l = 1, \dots, m-1)$. We will show that $a = bE_{m-1} \dots E_k E_l \in \mathscr{F}_m'$ for $b \in \mathscr{F}_{m-1}'$, $k = 1, \dots, m$, and $l = 1, \dots, m-1$. If $l \leq k-2$, then we have $a = bE_l E_{m-1} \dots E_k$ by (2.3). Since $bE_l \in \mathscr{F}_{m-1}'$ by the induction hypothesis, we have $a \in \mathscr{F}_m'$. If $l = k-1$, then it is clear that $a \in \mathscr{F}_m'$. If $l = k$, then by (2.1), we have $a = x_k bE_{m-1} \dots E_k \in I_m$. If $l > k$, then by using (2.2) and (2.3), we have

$$\begin{aligned} a &= bE_{m-1} \dots E_l E_{l-1} E_l E_{l-2} \dots E_k \\ &= y_{l-1} bE_{m-1} \dots E_l E_{l-2} \dots E_k \\ &= y_{l-1} bE_{l-2} \dots E_k E_{m-1} \dots E_l. \end{aligned}$$

It follows from the induction hypothesis that $a \in \mathscr{F}_m'$. □

In order to describe matrix representations of $\mathscr{F}_n$, we associate $\alpha(v) \in K$ to each element $v \in \mathbb{YF}$. Let $(P_l)_{l \geq 0}$ be the sequence of polynomials $P_l(x_1, \dots, x_l; y_1, \dots, y_{l-1})$ given by the following recurrence:

$$P_0 = 1, \quad P_1 = x_1, \quad P_l = x_l P_{l-1} - y_{l-1} P_{l-2}. \tag{2.5}$$

Then $\alpha(v)$ is defined as follows:

$$\begin{gathered} \alpha(1^l) = P_l(x_1, \dots, x_l; y_1, \dots, y_{l-1}), \\ \alpha(1^l 2) = P_{l+1}(y_1, x_3, \dots, x_{l+2}; x_1 y_2, y_3, \dots, y_{l+1}). \end{gathered} \tag{2.6}$$

In general, if v is of the form $1^l 2u$ ($|u| = m$), then we put

$$\alpha(1^l 2u) = \alpha(1^l 2)[x_j \to x_{m+j}, y_j \to y_{m+j}] \alpha(u), \tag{2.7}$$

where $P[z \to w]$ indicates that we substitute w for z in P.

Lemma 2.2. *For* $v \in \mathbb{YF}_n$, *we have*

$$\sum_{u \triangleright v} \alpha(u) = x_{n+1}\alpha(v), \tag{2.8}$$

$$\alpha(2v) = y_{n+1}\alpha(v). \tag{2.9}$$

Moreover, we have

$$\sum_{v \in \mathbb{YF}_n} e(v)\alpha(v) = x_1 \dots x_n, \tag{2.10}$$

where $e(v)$ *is the number of saturated chains from* $\varnothing$ *to* v *in* $\mathbb{YF}$.

Proof. The relation (2.9) is clear from the definition (2.7) and $\alpha(2) = y_1$. We prove (2.8) by induction on $|v|$. First we consider the case where $v = 2w$. Since $C^+(2w) = \{12w\} \cup \{2z : z \triangleright w\}$ by (1.4), we have

$$\sum_{u \triangleright 2w} \alpha(u) = \alpha(12)[x_j \to x_{m+j}, y_j \to y_{m+j}]\alpha(w) + \sum_{z \triangleright w} y_{m+2}\alpha(w),$$

where $|w| = m$. By using $\alpha(12) = x_3y_1 - x_1y_2$ and the induction hypothesis, we get

$$\sum_{u \triangleright 2w} \alpha(u) = (x_{m+3}y_{m+1} - x_{m+1}y_{m+2})\alpha(w) + y_{m+2}x_{m+1}\alpha(w) = x_{m+3}\alpha(2w).$$

If $v = 1^k 2w$ for some $k > 0$, then

$$\sum_{u \triangleright 1^k 2w} \alpha(u) = \alpha(1^{k+1}2)[x_j \to x_{m+j}, y_j \to y_{m+j}]\alpha(w)$$
$$+ y_{m+k+2}\alpha(1^{k-1}2)[x_j \to x_{m+j}, y_j \to y_{m+j}]\alpha(w),$$

where $|w| = m$. Hence it is enough to show

$$\alpha(1^{k+1}2) + y_{k+2}\alpha(1^{k-1}2) = x_{k+3}\alpha(1^k 2).$$

But this is clear from the definition (2.5) and (2.6). Similarly we can check the case where $v = 1^n$.

The remaining equation (2.10) follows from (2.8). □

It follows from the definition of differential posets (see Proposition 1.1) and (1.3) that the $e(v)$'s are uniquely determined by the same recurrence relations as (2.8) with $x_i = i$ and $y_i = i$, and the initial condition $e(\varnothing) = 1$. So we have $\alpha(v)[x_i \to i, y_i \to i] = e(v)$. In particular, $\alpha(v)$ is a nonzero polynomial.

For $v \in \mathbb{YF}_n$, let V_v be the K-vector space with basis Ω^v. Then $\dim V_v = e(v)$. Now define an action $\pi_v(E_i)$ of each generator E_i on the vector space V_v as follows:

(2.11)

$$\pi_v(E_i)(v^{(0)}, \dots, v^{(i-1)}, v^{(i)}, v^{(i+1)}, \dots, v^{(n)})$$
$$= \begin{cases} \displaystyle\sum_{z \triangleright v^{(i-1)}} \frac{\alpha(z)}{\alpha(v^{(i-1)})}(v^{(0)}, \dots, v^{(i-1)}, z, v^{(i+1)}, \dots, v^{(n)}) & \text{if } v^{(i+1)} = 2v^{(i-1)} \\ 0 \quad \text{otherwise.} \end{cases}$$

Lemma 2.3. *The endomorphisms* $\pi_v(E_i)$ *satisfy the defining relations* (2.1)–(2.3) *of* $\mathcal{F}_n$. *Hence we obtain a representation* π_v *of* $\mathcal{F}_n$ *on* V_v.

Proof. The relation (2.2) is clear from the definition.

Let $T = (v^{(0)}, \dots, v^{(n)}) \in \Omega^v$. We will check that $\pi_v(E_i)^2 T = x_i \pi_v(E_i) T$. If $v^{(i+1)} \neq 2v^{(i-1)}$, then both sides are 0. If $v^{(i+1)} = 2v^{(i-1)}$, then

$$\begin{aligned}
\pi_v(E_i)^2 T &= \sum_{w \rhd v^{(i-1)}} \frac{\alpha(w)}{\alpha(v^{(i-1)})} \pi_v(E_i) T_w \\
&= \sum_{u \rhd v^{(i-1)}} \sum_{w \rhd v^{(i-1)}} \frac{\alpha(w)}{\alpha(v^{(i-1)})} \frac{\alpha(u)}{\alpha(v^{(i-1)})} T_u \\
&= \sum_{w \rhd v^{(i-1)}} \frac{\alpha(w)}{\alpha(v^{(i-1)})} \pi_v(E_i) T \\
&= x_i \pi_v(E_i) T \quad \text{(by (2.8))}
\end{aligned}$$

where $T_w = (v^{(0)}, \dots, v^{(i-1)}, w, v^{(i+1)}, \dots, v^{(n)})$.

Next we check that $\pi_v(E_{i+1})\pi_v(E_i)\pi_v(E_{i+1})T = y_i \pi_v(E_{i+1})T$. If $v^{(i+2)} \neq 2v^{(i)}$, then both sides are 0. If $v^{(i+2)} = 2v^{(i)}$, then

$$\begin{aligned}
\pi_v(E_{i+1})\pi_v(E_i)\pi_v(E_{i+1})T &= \sum_{w \rhd v^{(i)}} \frac{\alpha(w)}{\alpha(v^{(i)})} \pi_v(E_{i+1})\pi_v(E_i) T_{v^{(i)}, w} \\
&= \frac{\alpha(2v^{(i-1)})}{\alpha(v^{(i)})} \pi_v(E_{i+1})\pi_v(E_i) T_{v^{(i)}, 2v^{(i-1)}} \\
&= \frac{\alpha(2v^{(i-1)})}{\alpha(v^{(i)})} \sum_{u \rhd v^{(i-1)}} \frac{\alpha(u)}{\alpha(v^{(i-1)})} \pi_v(E_{i+1}) T_{u, 2v^{(i-1)}} \\
&= \frac{\alpha(2v^{(i-1)})}{\alpha(v^{(i)})} \frac{\alpha(v^{(i)})}{\alpha(v^{(i-1)})} \pi_v(E_{i+1}) T \\
&= y_i \pi_v(E_{i+1}) T \quad \text{(by (2.9))}
\end{aligned}$$

where $T_{u,w} = (v^{(0)}, \dots, v^{(i-1)}, u, w, v^{(i+2)}, \dots, v^{(n)})$. □

If v covers w, then V_w can be considered as a subspace of V_v by identifying $(v^{(0)}, \dots, v^{(n-2)}, w) \in \Omega^w$ with $(v^{(0)}, \dots, v^{(n-2)}, w, v) \in \Omega^v$.

Lemma 2.4. *If we restrict the representation* π_v *to the subalgebra* $\mathcal{F}_{n-1}$ *generated by* $E_1, \dots, E_{n-2}$, *then* V_v *decomposes as follows:*

$$V_v \downarrow_{\mathcal{F}_{n-1}} \cong \bigoplus_{w \lhd v} V_w.$$

Proof. This is clear from the definition (2.11) of the action of $E_1, \dots, E_{n-2}$. □

Lemma 2.5. *The representations* (π_v, V_v) *of* $\mathcal{F}_n$ *are irreducible and pairwise inequivalent.*

Proof. We proceed by induction on n. First we show the irreducibility of (π_v, V_v). Let $W \neq \{0\}$ be an $\mathcal{F}_n$-submodule of V_v.

If $v = 1v'$, then Lemma 2.4 implies that $V_{1v'} = V_{v'}$ as on $\mathcal{F}_{n-1}$-module. By the induction hypothesis, it is irreducible over $\mathcal{F}_{n-1}$. Hence we have $W = V_{v'} = V_{1v'}$.

If $v = 2v''$, then there exists an element x such that x covers v'' and $V_x \subset W$, because the irreducible decomposition of $V_v \downarrow_{\mathscr{F}_{n-1}}$ is multiplicity-free. Now let $y \neq x$ be an element covering v'' and consider two chains $T = (v^{(0)}, \dots, v^{(n-3)}, v'', x, v)$ and $T' = (v^{(0)}, \dots, v^{(n-3)}, v'', y, v) \in \Omega^v$. Let z_y be the minimal central idempotent of $\mathscr{F}_{n-1}$ corresponding to π_y. Then it follows from the definition of $\pi_v(E_{n-1})$ that

$$\pi_v(z_y)\pi_v(E_{n-1})T = \frac{\alpha(y)}{\alpha(v'')}T' \in W.$$

Hence we have $W \cap V_y \neq \{0\}$. Since V_y is an irreducible $\mathscr{F}_{n-1}$-module by the induction hypothesis, we see that $V_y \subset W$. Recalling that y is arbitrary, we have $W = V_x \oplus \bigoplus_{y \triangleright v'', y \neq x} V_y = V_v$.

Next we show that the (π_v, V_v) are inequivalent. Suppose that $V_v \cong V_w$ as as $\mathscr{F}_n$-module. Then, by Lemma 2.4, we have $C^-(v) = C^-(w)$. Except for the case where $v = 11$ and $w = 2$, it follows from definition (1.2) and (1.3) that $v = w$. In the exceptional case, it follows from $\pi_{11}(E_1) = 0$ and $\pi_2(E_1) = x_1 \,\mathrm{Id}$ that $V_{11} \not\cong V_2$. □

Now we are in position to prove the main theorem.

Theorem 2.6. (1) *The algebra $\mathscr{F}_n$ is semisimple.*

(2) *The set $\mathscr{B}_n$ of monomials defined by* (2.4) *is a basis of $\mathscr{F}_n$. In particular,* $\dim \mathscr{F}_n = n!$.

(3) *The V_v's ($v \in \mathbb{YF}_n$) give a complete set of irreducible $\mathscr{F}_n$-modules.*

Proof. Let $\operatorname{rad}\mathscr{F}_n$ be the radical of $\mathscr{F}_n$. Then, by Lemma 2.5, we have

$$\begin{aligned}\dim(\mathscr{F}_n/\operatorname{rad}\mathscr{F}_n) &\geq \dim\left(\bigoplus_{v\in\mathbb{YF}_n} \pi_v(\mathscr{F}_n)\right) \geq \sum_{v\in\mathbb{YF}_n} (\dim V_v)^2 \\ &= \sum_{v\in\mathbb{YF}_n} e(v)^2 = n!.\end{aligned}$$

Here we have used Proposition 1.2. On the other hand, Lemma 2.1 implies that $\dim \mathscr{F}_n \leq n!$. Therefore we obtain the desired results. □

For $a \in \mathscr{F}_n$, we define

$$\operatorname{Tr}^{(n)}(a) = (x_1 \dots x_n)^{-1} \sum_{v\in\mathbb{YF}_n} \alpha(v) \operatorname{tr}_{V_v}(\pi_v(a)), \tag{2.12}$$

where tr_{V_v} denotes the usual trace on the vector space V_v. Then $\operatorname{Tr}^{(n)}$ has the following properties similar to those of the Markov trace on the Iwahori-Hecke algebra of the symmetric group (see [W, §3]).

Proposition 2.7. *The functional* $\operatorname{Tr}^{(n)}$ *defined by* (2.12) *satisfies the following*:

(1) $\operatorname{Tr}^{(n)}(1) = 1$.

(2) $\operatorname{Tr}^{(n)}(ab) = \operatorname{Tr}^{(n)}(ba)$.

(3) *If $a \in \mathscr{F}_{n-1}$, then* $\operatorname{Tr}^{(n)}(aE_{n-1}) = y_{n-1} \operatorname{Tr}^{(n)}(a)$.

(4) *If $a \in \mathscr{F}_{n-1}$, then* $\operatorname{Tr}^{(n)}(a) = \operatorname{Tr}^{(n-1)}(a)$.

Proof. (1) follows from (2.10). (2) is clear from definition (2.12).

(3) Given $T \in \Omega^v$, let p_T be the minimal idempotent of $\mathscr{F}_{n-1}$ such that

$$\pi_w(p_T) = \begin{cases} E_{TT} & (w = v), \\ 0 & (w \neq v), \end{cases}$$

where E_{TT} denotes the matrix unit, i.e., the linear map defined by $E_{TT}(S) = \delta_{S,T}T$ for $S \in \Omega^v$. Since $\sum_{v \in \mathbb{YF}_{n-1}} \sum_{T \in \Omega^v} p_T = 1$, it is enough to show

$$\operatorname{Tr}^{(n)}(aE_{n-1}p_T) = y_{n-1}\operatorname{Tr}^{(n)}(ap_T).$$

Since p_T is a minimal idempotent, there exists a scalar $\gamma(a) \in K$ such that $p_T a p_T = \gamma(a) p_T$. Hence we have

$$\operatorname{Tr}^{(n)}(ap_T) = (x_1 \dots x_n)^{-1}\gamma(a)\alpha(v).$$

On the other hand, if z_v is the minimal central idempotent of $\mathscr{F}_n$ corresponding to π_v, then we have

$$z_v E_{n-1} p_T = \frac{\alpha(v)}{\alpha(v^{(n-2)})} p_T p_{\widetilde{T}},$$

where $T = (v^{(0)}, \dots, v^{(n-2)}, v^{(n-1)})$, $\widetilde{T} = (v^{(0)}, \dots, v^{(n-2)}, v^{(n-1)}, 2v^{(n-2)})$. Hence we have

$$\begin{aligned}\operatorname{Tr}^{(n)}(aE_{n-1}p_T) &= \frac{\alpha(v)}{\alpha(v^{(n-2)})}\gamma(a)\operatorname{Tr}^{(n)}(p_{\widetilde{T}}) \\ &= (x_1 \dots x_n)^{-1}\frac{\alpha(v)}{\alpha(v^{(n-2)})}\gamma(a)\alpha(2v^{(n-2)}) \\ &= (x_1 \dots x_n)^{-1} y_{n-1}\alpha(v)\gamma(a).\end{aligned}$$

Hence we obtain (3).

(4) For $a \in \mathscr{F}_{n-1}$, by using (2.8) and Lemma 2.5, we have

$$\begin{aligned}\operatorname{Tr}^{(n)}(a) &= (x_1 \dots x_n)^{-1} \sum_{|w|=n-1} \sum_{v \rhd w} \alpha(v)\operatorname{tr}_{V_w}(\pi_w(a)) \\ &= (x_1 \dots x_n)^{-1} \sum_{|w|=n-1} x_n \alpha(w)\operatorname{tr}_{V_w}(\pi_w(a)) \\ &= \operatorname{Tr}^{(n-1)}(a). \quad \square\end{aligned}$$

Remark. The proof of this section is on the same line as [KM] and [W].

Finally we mention the specialization of the parameters $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-2}$. The above argument guarantees the following theorem.

Theorem 2.8. *Let $\xi_1, \dots, \xi_{n-1}, \eta_1, \dots, \eta_{n-2}$ be elements of the field K_0. Let $\overline{\mathscr{F}_n} = \mathscr{F}_n(\xi_1, \dots, \xi_{n-1}; \eta_1, \dots, \eta_{n-2})$ be the algebra over K_0 generated by $E_1, \dots, E_{n-1}$ with their fundamental relations given by* (2.1)–(2.3), *where the x_i's and y_j's are replaced by ξ_i's and η_j's respectively. If*

$$\alpha(v)[x_i \to \xi_i, y_j \to \eta_j] \neq 0$$

for all words v with $|v| \leq n-1$, then $\overline{\mathscr{F}_n}$ is a semisimple algebra of dimension $n!$.

Remark. The above argument can be easily generalized to the differential poset $T(N)$, which is obtained from the partial differential poset $\mathbb{Y}_{[N]} = \coprod_{k=0}^{N} \mathbb{Y}_k$ by iterating Wagner's construction. (See [S1, pp. 957–958].) Let $\mathcal{T}(N)_n$ be the associative algebra over the field $K(q)$ with generators $T_1, \ldots, T_{N-1}, E_N, \ldots, E_{n-1}$ and the following defining relations:

$$\begin{aligned}
&(T_i - q)(T_i + q^{-1}) = 0 \quad (i = 1, \ldots, N-1), \\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (i = 1, \ldots, N-2), \\
&T_i T_j = T_j T_i \quad (|i-j| \geq 2), \\
&E_N^2 = [l] E_N, \\
&E_N T_{N-1} E_N = q^l E_N, \\
&E_i^2 = x_i E_i \quad (i = N+1, \ldots, n-1), \\
&E_{i+1} E_i E_{i+1} = y_i E_{i+1} \quad (i = N, \ldots, n-2), \\
&E_i T_j = T_j E_i \quad (|i-j| \geq 2), \\
&E_i E_j = E_j E_i \quad (|i-j| \geq 2),
\end{aligned}$$

where $[l] = (q^l - q^{-l})/(q - q^{-1})$. Then one can show that the Bratteli diagram of the tower $(\mathcal{T}(N)_n)_{n \geq 0}$ is the Hasse diagram of $T(N)$. Note that the algebras $\mathcal{T}(N)_n$ for $n \leq N$ are the Iwahori-Hecke algebra of the symmetric group $\mathfrak{S}_n$. And M. Kosuda and J. Murakami [KM] have shown that, if $l \geq N+1$, then the algebra $\mathcal{T}(N)_{N+1}$ is isomorphic to the centralizer algebra of the quantum group $U_q(\mathfrak{gl}(l, \mathbb{C}))$ on the space $V^{\otimes N} \otimes V^*$, where V is the l-dimensional vector representation of $U_q(\mathfrak{gl}(l, \mathbb{C}))$.

3. $\mathbb{YF}$-ANALOGUE OF THE RING OF SYMMETRIC FUNCTIONS

In this section, we give a definition of a graded algebra structure on $R = \bigoplus_{n \geq 0} R_n$, which becomes a $\mathbb{YF}$-analogue of the ring Λ of symmetric functions. Many of the results in the following sections have counterparts in the theory of symmetric functions. (See [M].)

Let $\mathcal{F}_{m,n}$ be the subalgebra of $\mathcal{F}_{m+n}(x_1, \ldots, x_{m+n-1}; y_1, \ldots, y_{m+n-2})$ generated by $E_1, \ldots, E_{m-1}, E_{m+1}, \ldots, E_{m+n-1}$. Then it follows from Theorem 2.7(2) that

$$\begin{aligned}
\mathcal{F}_{m,n} \cong\ & \mathcal{F}_m(x_1, \ldots, x_{m-1}; y_1, \ldots, y_{m-2}) \\
& \otimes \mathcal{F}_n(x_{m+1}, \ldots, x_{m+n-1}; y_{m+1}, \ldots, y_{m+n-2}).
\end{aligned}$$

So $\mathcal{F}_{m,n}$ is a semisimple algebra. If $|w| = m+n$ and $|u| = m$, then let $\Omega^{w/u}$ be the set of all saturated chains from u to w and $V_{w/u}$ the vector space with basis $\Omega^{w/u}$. Note that $V_{w/u} = \{0\}$ unless $w > u$. We define an action $\pi_{w/u}(E_i)$ on $(v^{(0)}, \ldots, v^{(n)}) \in \Omega^{w/u}$ by the same formula as (2.11) with $\alpha(z)$ and $\alpha(v^{(i-1)})$ replaced by $\alpha(z)[x_j \to x_{m+j}, y_j \to y_{m+j}]$ and $\alpha(v^{(i-1)})[x_j \to x_{m+j}, y_j \to y_{m+j}]$ respectively. Then this action of generators affords a representation $\pi_{w/u}$ of $\mathcal{F}_n(x_{m+1}, \ldots, x_{m+n-1}; y_{m+1}, \ldots, y_{m+n-2})$ on $V_{w/u}$. (See the proof of Lemma 2.3.)

Proposition 3.1. *If* $|w| = m+n$,

$$V_w \downarrow_{\mathcal{F}_{m,n}} \cong \bigoplus_{|u|=m} V_u \otimes V_{w/u}$$

as $\mathcal{F}_{m,n}$*-module.*

Now define a product on R by

$$s_u s_v = \sum_{w \in \mathbb{YF}_{m+n}} c_{uv}^w s_w$$

for $u \in \mathbb{YF}_m$ and $v \in \mathbb{YF}_n$, where the structure constant c_{uv}^w is defined as follows. Let V_u (resp. V_v) be the irreducible $\mathscr{F}_m(x_1, \dots, x_{m-1}; y_1, \dots, y_{m-2})$-module (resp. $\mathscr{F}_n(x_{m+1}, \dots, x_{m+n-1}; y_{m+1}, \dots, y_{m+n-2})$-module) corresponding to $u \in \mathbb{YF}_m$ (resp. $v \in \mathbb{YF}_n$). Then c_{uv}^w is defined to be the multiplicity of the irreducible $\mathscr{F}_{m+n}$-module V_w in the induced module $\mathscr{F}_{m+n} \otimes_{\mathscr{F}_{m,n}} (V_u \otimes V_v)$. By Frobenius reciprocity, we see that c_{uv}^w is the multiplicity of the irreducible $\mathscr{F}_{m,n}$-module $V_u \otimes V_v$ in the restriction $V_w \downarrow_{\mathscr{F}_{m,n}}$. This product makes R an associative graded algebra.

Proposition 3.2. *Suppose that $w \in \mathbb{YF}_{m+2}$ and $u \in \mathbb{YF}_m$ satisfy $w > u$. Then, as an $\mathscr{F}_2(x_{m+1})$-module,*

$$V_{w/u} \cong \begin{cases} (V_{11})^{\oplus d-1} \oplus V_2 & \text{if } w = 2u, \\ V_{11} & \text{otherwise}, \end{cases}$$

where $d = \#C^+(u)$.

Proof. This is clear by considering the action of E_1. □

Proposition 3.3.

(1) $s_v s_1 = \sum_{w \rhd v} s_w$.
(2) $s_v s_2 = s_{2v}$.
(3) $s_{1v} = s_v s_1 - \left(\sum_{z \lhd v} s_z\right) s_2$.

Proof. (1) is clear from Lemma 2.4 and Proposition 3.1. (2) follows from Propositions 3.1 and 3.2. (3) is a direct consequence of (1) and (2) because of (1.4). □

The abstract structure of R is given by the following theorem.

Theorem 3.4. *Let $\mathbb{Z}\langle X, Y\rangle$ be the noncommutative polynomial ring with grading given by $\deg X = 1$ and $\deg Y = 2$. Then there exists an algebra isomorphism $\varphi : \mathbb{Z}\langle X, Y\rangle \to R$ such that $\varphi(X) = s_1$ and $\varphi(Y) = s_2$.*

Proof. There exists an algebra homomorphism $\varphi : \mathbb{Z}\langle X, Y\rangle \to R$ such that $\varphi(X) = s_1$ and $\varphi(Y) = s_2$. By Proposition 3.3, this homomorphism φ is surjective. On the other hand, the homogeneous components of degree n in $\mathbb{Z}\langle X, Y\rangle$ and R are both free $\mathbb{Z}$-modules of rank F_n (nth Fibonacci number). Hence φ is an isomorphism. □

Proposition 3.5.

$$s_{u2v} = s_v s_{u2}. \tag{3.1}$$

In particular, for $v = 1^{m_1}21^{m_2}2\dots21^{m_{r+1}}$,

$$s_v = s_{1^{m_{r+1}}} s_{1^{m_r}2} \dots s_{1^{m_1}2}. \tag{3.1'}$$

Proof. We prove (3.1) by induction on $|u|$. If $u = \varnothing$, then (3.1) reduces to Proposition 3.3(2). If $u = 2u''$, then by Proposition 3.3(2) and the induction hypothesis,

$$s_{2u''2v} = s_{u''2v} s_2 = s_v s_{u''2} s_2 = s_v s_{2u''2}.$$

If $u = 1u'$, then it follows from Proposition 3.3(3) and the induction hypothesis that

$$s_{1u'2v} = s_{u'2v}s_1 - \left(\sum_{x \lhd u'2v} s_x\right) s_2, \quad s_v s_{1u'2} = s_v s_{u'2} s_1 - s_v \left(\sum_{y \lhd u'2} s_y\right) s_2.$$

Hence it suffices to show that

$$\sum_{x \lhd w2v} s_x = \sum_{y \lhd w2} s_v s_y. \tag{3.2}$$

We show (3.2) by induction on $|w|$. The case where $w = \varnothing$ follows from Proposition 3.3(1). If $w = 1w'$, then

$$\sum_{x \lhd 1w'2v} s_x = s_{w'2v}, \quad \sum_{y \lhd 1w'2} s_v s_y = s_v s_{w'2}.$$

Here we use the induction hypothesis on $|w|$ to obtain (3.2) for $w = 1w'$. If $w = 2w''$, then

$$\sum_{x \lhd 2w''2v} s_x = s_{1w''2v} + \sum_{t \lhd w''2v} s_t s_2, \quad \sum_{y \lhd 2w''2} s_v s_y = s_v s_{1w''2} + \sum_{z \lhd w''2} s_v s_z s_2.$$

Now from the induction hypothesis on $|u|$ and $|w|$, we have

$$s_{1w''2v} = s_v s_{1w''2}, \quad \sum_{t \lhd w''2v} s_t s_2 = \sum_{z \lhd w''2} s_v s_z s_2.$$

This completes the proof of (3.2), hence (3.1). □

This proposition, together with Proposition 3.3, enables us to express s_v as a "determinant" of the matrix having noncommutative entries s_1, s_2 (and 0, 1).

There is an involutive automorphism ω of the poset $\mathbb{YF}$ such that

$$\omega(v11) = v2, \quad \omega(v2) = v11, \quad \omega(v21) = v21.$$

Then we can define a linear automorphism $\widetilde{\omega}$ of R by $\widetilde{\omega}(s_v) = s_{\omega(v)}$. However $\widetilde{\omega}$ is not an algebra homomorphism: in fact,

$$\widetilde{\omega}(s_v s_1) = \widetilde{\omega}(s_v)s_1, \quad \widetilde{\omega}(s_v s_2) = \widetilde{\omega}(s_v)s_2 \quad (v \neq \varnothing).$$

Hence, for $v \neq \varnothing$, we have $\widetilde{\omega}(s_v s_w) = \widetilde{\omega}(s_v)s_w$.

4. $\mathbb{YF}$-analogue of Kostka numbers and the Littlewood-Richardson rule

Definition. For $w = b_1 \dots b_l \in \mathbb{YF}_n$, we define

$$h_w = s_{b_l} \dots s_{b_1}.$$

Note that the order of product in h_w is reversed to that of w. For $v, w = b_1 \dots b_l \in \mathbb{YF}_n$, let $\mathscr{K}_{vw}$ be the set of sequences $(v^{(0)}, \dots, v^{(l)})$ from $v^{(0)} = \varnothing$ to $v^{(l)} = v$ satisfying

(1) If $b_i = 1$, then $v^{(l-i+1)}$ covers $v^{(l-i)}$.
(2) If $b_i = 2$, then $v^{(l-i+1)} = 2v^{(l-i)}$.

We put $K_{vw} = \#\mathscr{K}_{vw}$ and call this a $\mathbb{YF}$-Kostka number.

By definition, we have $K_{v,1^n} = e(v)$ if $|v| = n$. Then the following proposition is an immediate consequence of Proposition 3.3.

Proposition 4.1. *For $w \in \mathbb{YF}_n$, one has*

$$h_w = \sum_{v \in \mathbb{YF}_n} K_{vw} s_v.$$

This corresponds to the Young's rule for the representation of the symmetric groups (see [JK, 2.8.5]).

Now we introduce a partial order $\succeq$ (called *dominance order*) on each graded component $\mathbb{YF}_n$ of the Young-Fibonacci lattice. For $v = a_1 \dots a_k$, $w = b_1 \dots b_l \in \mathbb{YF}_n$, we define $v \succeq w$ if $a_1 + \dots + a_i \geq b_1 + \dots + b_i$ for all $i = 1, 2, \dots, \min(k, l)$.

Theorem 4.2. *The following are equivalent for $v, w \in \mathbb{YF}_n$:*

(1) $v \succeq w$.
(2) $K_{vw} \neq 0$.
(3) $K_{uv} \leq K_{uw}$ *for all* $u \in \mathbb{YF}_n$.

Proof. (1) $\Rightarrow$ (3) It is enough to consider the case where either

(a) $v = a_1 \dots a_i 21 a_{i+3} \dots a_k$, $w = a_1 \dots a_i 12 a_{i+3} \dots a_k$, or
(b) $v = a_1 \dots a_i 2$, $w = a_1 \dots a_i 11$.

In case (a), by Proposition 3.3(3),

$$\begin{aligned} h_w - h_v &= s_{a_k} \dots s_{a_{i+3}} (s_2 s_1 - s_1 s_2) s_{a_i} \dots s_{a_1} \\ &= s_{a_k} \dots s_{a_{i+3}} s_{12} s_{a_i} \dots s_{a_1}. \end{aligned}$$

Hence $K_{uw} - K_{uv}$ is nonnegative because it is the multiplicity of V^u in the $\mathscr{F}_n$-module induced from $V^{a_k} \otimes \dots \otimes V^{a_{i+3}} \otimes V^{12} \otimes V^{a_i} \otimes \dots \otimes V^{a_1}$. Case (b) is similarly proved by using $s_1^2 - s_2 = s_{11}$.

(3) $\Rightarrow$ (2) If we take $u = v$ in (3), we have $K_{vw} \geq K_{vv} = 1$.

(2) $\Rightarrow$ (1) We proceed by induction on n. Let $v = a_1 \dots a_k$ and $w = b_1 \dots b_l$. And fix a sequence $(v^{(0)}, \dots, v^{(l)}) \in \mathscr{K}_{v,w}$.

If $a_1 = b_1 = 1$, then $(v^{(0)}, \dots, v^{(l-1)}) \in \mathscr{K}_{v',w'}$, where $v' = a_2 \dots a_l$ and $w' = b_2 \dots b_k$. By the induction hypothesis, we have $a_2 + \dots + a_i \geq b_2 + \dots + b_i$ for all i, Hence we have $v \succeq w$. If $b_1 = 2$, then $v = v^{(l)} = 2v^{(l-1)}$, so that $a_1 = 2$. Then we can conclude $v \succeq w$ in a similar way.

Suppose that $a_1 = 2$ and $b_1 = 1$. Since $v^{(l-1)}$ is covered by v, we have either

(a) $v^{(l)} = 2^p a_{p+1} \dots a_l$, $v^{(l-1)} = 2^{p-1} 1 a_{p+1} \dots a_l$, or
(b) $v^{(l)} = 2^{p-1} 1 a_{p+1} \dots a_l$, $v^{(l-1)} = 2^{p-1} a_{p+1} \dots a_l$.

Let $v^{(l-1)} = c_1 \dots c_m$. In case (a), by the induction hypothesis, we have $c_1 + \dots + c_i \geq b_2 + \dots + b_{i+1}$. Since $a_j \geq c_j$ for all j, we have

$$\begin{aligned} a_1 + \dots + a_i &\geq c_1 + \dots + c_i \geq b_2 + \dots + b_i + b_{i+1} \\ &\geq b_2 + \dots + b_i + 1 = b_1 + \dots + b_i. \end{aligned}$$

In case (b), by the induction hypothesis, we have $c_1 + \dots + c_i \geq b_2 + \dots + b_{i+1}$. If $i \leq p - 1$, then the proof is similar to that of case (a). If $i \geq p$, then we see that

$$\begin{aligned} a_1 + \dots + a_i &= c_1 + \dots + c_{p-1} + 1 + c_p + \dots + c_{i-1} \\ &\geq b_2 + \dots + b_i + 1 = b_1 + \dots + b_i. \quad \square \end{aligned}$$

There are recurrence formulas for the $\mathbb{YF}$-Kostka numbers K_{vw}.

Proposition 4.3.

(1) $K_{1v,1w} = K_{v,w}$.
(2) $K_{1v,2w} = 0$.
(3) $K_{2v,1w} = \sum_{u \rhd v} K_{u,w}$.
(4) $K_{2v,2w} = K_{v,w}$.

Proof. Easily follows from the definition. □

All matrices considered in the following have rows and columns indexed by $\mathbb{YF}_n$ in dominance order. We put $K_n = (K_{v,w})_{v,w \in \mathbb{YF}_n}$. For example,

$$K_5 = \begin{array}{c} \\ 221 \\ 212 \\ 2111 \\ 122 \\ 1211 \\ 1121 \\ 1112 \\ 11111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 221 & 212 & 2111 & 122 & 1211 & 1121 & 1112 & 11111 \end{array} \\ \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 3 & 4 & 8 \\ & 1 & 1 & 1 & 1 & 1 & 3 & 4 \\ & & 1 & 0 & 1 & 1 & 1 & 4 \\ & & & 1 & 1 & 1 & 2 & 3 \\ & & & & 1 & 1 & 1 & 3 \\ & & & & & 1 & 1 & 2 \\ & & & & & & 1 & 1 \\ & & & & & & & 1 \end{pmatrix} \end{array}.$$

Let $D_n = (D_{uv})_{u \in \mathbb{YF}_{n-1}, v \in \mathbb{YF}_n}$ be the matrix describing the covering relation between $\mathbb{YF}_n$ and $\mathbb{YF}_{n-1}$, so that

$$D_{uv} = \begin{cases} 1 & \text{if } u \lhd v, \\ 0 & \text{otherwise.} \end{cases}$$

By definition (1.2) and (1.3), D_{n+1} is of the form

$$D_{n+1} = ({}^tD_n \quad I_{F_n}), \tag{4.1}$$

where I_k is the $k \times k$ identity matrix. Then we can rewrite Proposition 4.3 in matrix form:

$$K_{n+1} = \begin{pmatrix} K_{n-1} & D_n K_n \\ 0 & K_n \end{pmatrix}. \tag{4.2}$$

Remark. Recently T. Halverson and A. Ram [HR] show that the matrix K_n appears as the character table of $\mathscr{F}_n$. Namely, if we define an element $e_w \in \mathscr{F}_n$ by $e_\varnothing = e_1 = 1$ and

$$e_w = \begin{cases} e_{w'} & \text{if } w = 1w', \\ \frac{1}{x_{n-1}} E_{n-1} e_{w''} & \text{if } w = 2w'', \end{cases}$$

then we have $\mathrm{tr}_{V_v}(\pi_v(e_w)) = K_{vw}$.

Definition. Let u, v, w be three elements of $\mathbb{YF}$ satisfying $|u| + |v| = |w|$, and write $v = a_1 \dots a_k = 1^{m_1} 2 \dots 2 1^{m_{r+1}}$. Then we define $\mathscr{L}_{w/u,v}$ to be the set of all sequences $(w^{(0)}, \dots, w^{(k)})$ from $u = w^{(0)}$ to $w = w^{(k)}$ satisfying

(1) If $a_i = 1$, then $w^{(k-i+1)}$ covers $w^{(k-i)}$.
(2) If $a_i = 2$, then $w^{(k-i+1)} = 2w^{(k-i)}$.
(3) The triple $(w^{(j-1)}, w^{(j)}, w^{(j+1)})$ is not of the form $(w^{(j-1)}, 1w^{(j-1)}, 2w^{(j-1)})$ for any $j = 1, \dots, m_{r+1} - 1$.
(4) If $a_i = 1$ and $i \le k - m_{r+1} - 1$, then $w^{(k-i+1)} = 1w^{(k-i)}$.

Theorem 4.4.
$$c^w_{uv} = \#\mathscr{L}_{w/u,v}.$$

Proof. It follows from (3.1') that
$$s_u s_v = \sum_x c^x_{u,1^{m_{r+1}}} s_{1^{m_1}2\dots 1^{m_r}2x}.$$

And, by definition, we have
$$\#\mathscr{L}_{w/u,v} = \begin{cases} \#\mathscr{L}_{x/u,1^{m_{r+1}}} & \text{if } w = 1^{m_1}2\dots 1^{m_r}2x, \\ 0 & \text{otherwise.}\end{cases}$$

Hence it suffices to show the claim in the case where $v = 1^m$. Now we proceed by induction on m. If $m = 0$ or 1, then it is easy to see that
$$c^w_{u,\varnothing} = \#\mathscr{L}_{w/u,\varnothing} = \delta_{u,w},$$
$$c^w_{u,1} = \#\mathscr{L}_{w/u,1} = \begin{cases} 1 & \text{if } w \rhd u, \\ 0 & \text{otherwise.}\end{cases}$$

If $m \geq 1$, then we have, from Proposition 3.3,
$$\begin{aligned} s_u s_{1^{m+1}} &= s_u(s_{1^m}s_1 - s_{1^{m-1}2}) \\ &= \sum_y c^y_{u,1^m} s_y s_1 - \sum_z c^z_{u,1^{m-1}} s_z s_2 \\ &= \sum_w \left(\sum_{y \lhd w} c^y_{u,1^m}\right) s_w - \sum_z c^z_{u,1^{m-1}} s_{2z}. \end{aligned}$$

Hence we have
$$c^w_{u,1^{m+1}} = \begin{cases} c^{w'}_{u,1^m} & \text{if } w = 1w', \\ \sum_{y \rhd w''} c^y_{u,1^m} - c^{w''}_{u,1^{m-1}} & \text{if } w = 2w''. \end{cases}$$

On the other hand, $\mathscr{L}_{1w'/u,1^{m+1}}$ consists of the sequences $(w^{(0)}, \dots, w^{(m)}, 1w')$ such that $(w^{(0)}, \dots, w^{(m)}) \in \mathscr{L}_{w'/u,1^m}$ and $\mathscr{L}_{2w''/u,1^{m+1}}$ consists of the sequences $(w^{(0)}, \dots, w^{(m)}, 2w'')$ such that $(w^{(0)}, \dots, w^{(m)}) \in \mathscr{L}_{y/u,1^m}$ for some $y \rhd w''$ and that $(w^{(m-1)}, w^{(m)}, 2w'')$ is not of the form $(w'', 1w'', 2w'')$. Therefore we obtain the same recurrence:
$$\#\mathscr{L}_{w/u,1^{m+1}} = \begin{cases} \#\mathscr{L}_{w'/u,1^m} & \text{if } w = 1w', \\ \sum_{y \rhd w''} \#\mathscr{L}_{y/u,1^m} - \mathscr{L}_{w''/u,1^{m-1}} & \text{if } w = 2w''. \end{cases}$$

So we have $c^w_{uv} = \#\mathscr{L}_{w/u,v}$. □

5. $\mathbb{YF}$-ANALOGUE OF POWER SUM SYMMETRIC FUNCTIONS

Definition. For $v = 1^{m_1}21^{m_2}\dots 1^{m_r}21^{m_{r+1}}$, we define
$$p_v = p_{21^{m_{r+1}}} p_{21^{m_r}} \cdots p_{21^{m_2}} p_{1^{m_1}},$$
where
$$p_{1^k} = s_1^k, \quad p_{21^k} = s_1^k(s_1^2 - (k+2)s_2).$$

We remark that

(5.1) $$p_{1v} = p_v p_1, \quad p_{2v} = p_v(s_1^2 - (m(v)+2)s_2),$$

where $m(v)$ is the number of 1's at the head of v. Let $T = (T_{vw})$ be the transition matrix from p to h:

$$p_v = \sum_w T_{vw} h_w.$$

Then T is the diagonal sum of matrices $T_n = (T_{vw})_{v, w \in \mathbb{YF}_n}$. We use (5.1) to obtain the following recurrences for T_{vw}.

Proposition 5.1.

(1) $T_{1v, 1w} = T_{vw}$.
(2) $T_{1v, 2w} = 0$.
(3) $T_{2v, 12w} = 0$.
(4) $T_{2v, 11w} = T_{vw}$.
(5) $T_{2v, 2w} = -(m(w) + 2) T_{v, w}$.

Hence, if $T_{vw} \neq 0$, then w is a refinement of v, i.e., w is obtained by replacing some 2's in v by 11. In particular, T_n is a triangular matrix with respect to the dominance order.

Let $V_n = (V_{uv})_{u \in \mathbb{YF}_{n-1}, v \in \mathbb{YF}_n}$ be the $F_{n-1} \times F_n$ matrix defined by

$$V_{uv} = \begin{cases} 1 & \text{if } v = 1u, \\ 0 & \text{otherwise.} \end{cases}$$

That is, V_n is of the form

$$V_n = (0 \quad I_{F_{n-1}}). \tag{5.2}$$

And let M_n be the diagonal matrix whose (v, v)-entry is $m(v)$. Then we have

$$M_{n+1} = \begin{pmatrix} 0 & 0 \\ 0 & M_n + I \end{pmatrix}. \tag{5.3}$$

Also we can rewrite Proposition 5.1 in matrix form:

$$T_{n+1} = \begin{pmatrix} -(M_{n-1} + 2I) T_{n-1} & T_{n-1} V_{n-1} \\ 0 & T_n \end{pmatrix}. \tag{5.4}$$

Let $X = (\chi_w^v)_{w, v \in \mathbb{YF}}$ be the transition matrix from p to s:

$$p_w = \sum_v \chi_w^v s_v.$$

Then X is the diagonal sum of matrices $X_n = (\chi_w^v)_{w, v \in \mathbb{YF}_n}$ and X_n is given by

$$X_n = T_n {}^t K_n. \tag{5.5}$$

Proposition 5.2.

$$X_{n+1} = \begin{pmatrix} -X_{n-1} & X_{n-1} D_n \\ X_n {}^t D_n & X_n \end{pmatrix}, \tag{5.6}$$

$$X_{n-1} D_n = V_{n-1} X_n, \tag{5.7}$$

$$X_n {}^t D_n = {}^t V_{n-1} (M_{n-1} + I) X_{n-1}. \tag{5.8}$$

Proof. First we note that

$$V_{n-1}{}^tK_n = {}^tK_{n-1}D_n, \tag{5.9}$$

$$ {}^tV_{n-1}(M_{n-1}+I)V_{n-1} = M_n. \tag{5.10}$$

These are clear from (4.2) and (5.2)–(5.4).

We will prove by induction on n. From (4.2) and (5.4), we have

$$X_{n+1} = \begin{pmatrix} -(M_{n-1}+2I)T_{n-1}{}^tK_{n-1} + T_{n-1}V_{n-1}{}^tK_n{}^tD_n & T_{n-1}V_{n-1}{}^tK_n \\ T_n{}^tK_n{}^tD_n & T_n{}^tK_n \end{pmatrix}.$$

Using (5.9) and the induction hypothesis ((5.7) and (5.8)), we see

$$T_{n-1}V_{n-1}{}^tK_n{}^tD_n = X_{n-1}D_n{}^tD_n = V_{n-1}{}^tV_{n-1}(M_{n-1}+I)X_{n-1},$$

$$T_{n-1}V_{n-1}{}^tK_n = X_{n-1}D_n.$$

Hence we obtain (5.6). The relations (5.7) and (5.8) can be shown by matrix computation. □

For example,

$$X_5 = \begin{array}{c} \\ 221 \\ 212 \\ 2111 \\ 122 \\ 1211 \\ 1121 \\ 1112 \\ 11111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 221 & 212 & 2111 & 122 & 1211 & 1121 & 1112 & 11111 \end{array} \\ \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 & -1 & 1 \\ -2 & -1 & -1 & 3 & 3 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 2 & 1 & 1 \\ -1 & 1 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -2 & 2 & -1 & 1 & 0 & -1 & 1 \\ 8 & 4 & 4 & 3 & 3 & 2 & 1 & 1 \end{pmatrix} \end{array}.$$

We can rewrite (5.6) into the recurrence relations:

$$\chi^{2v}_{2w} = -\chi^v_w, \quad \chi^{1v}_{2w} = \sum_{u \lhd v} \chi^u_w, \quad \chi^{2v}_{1w} = \sum_{z \rhd v} \chi^z_w, \quad \chi^{1v}_{1w} = \chi^v_w.$$

By using the induction and these recurrence relations, we see that, for $v, w \in \mathbb{YF}_n$,

$$\chi^{1^n}_v = 1, \quad \chi^v_{1^n} = e(v),$$

$$\chi^{1^{n-2}2}_v = \begin{cases} 1 & \text{if } v \text{ ends with } 1, \\ -1 & \text{if } v \text{ ends with } 2, \end{cases}$$

$$\chi^{\omega(w)}_v = \varepsilon(v)\chi^w_v,$$

where $\varepsilon(v) = \chi^{1^{n-2}2}_v$. Here ω is a poset automorphism of $\mathbb{YF}$ defined at the end of Section 3. From the last equation we have $\widetilde{\omega}(p_v) = \varepsilon(v)p_v$.

For $v = 1^{m_1}21^{m_2}2\dots 21^{m_{r+1}} \in \mathbb{YF}$, we define

$$z(v) = m_1!(m_2+2)m_2!\dots(m_{r+1}+2)m_{r+1}!.$$

Then $|v|!/z(v) \in \mathbb{Z}$ and $\sum_{v \in \mathbb{YF}_n} n!/z(v) = n!$. Let Z_n be the diagonal matrix whose (v, v)-entry is $z(v)$. Then we have

$$Z_{n+1} = \begin{pmatrix} (M_{n-1}+2I)Z_{n-1} & 0 \\ 0 & (M_n+I)Z_n \end{pmatrix}.$$

Proposition 5.3.

$$X_n{}^tX_n = Z_n.$$

Therefore we have

$$\langle p_v , p_w \rangle = \delta_{vw} z(v).$$

Proof. Induction on n. By (5.6), we have

$$X_{n+1}{}^tX_{n+1} = \begin{pmatrix} X_{n-1}{}^tX_{n-1} + X_{n-1}D_n{}^tD_n{}^tX_{n-1} & 0 \\ 0 & X_n{}^tX_n + X_n{}^tD_nD_n{}^tX_n \end{pmatrix}.$$

Here we use (5.7), (5.8), and (5.10) to obtain

$$X_{n+1}{}^tX_{n+1} = \begin{pmatrix} (M_{n-1}+2I)Z_{n-1} & 0 \\ 0 & (M_n+I)Z_n \end{pmatrix} = Z_{n+1}. \quad \square$$

Rewriting (5.7) and (5.8) in terms of p_v, we obtain the following proposition.

Proposition 5.4.

$$Up_v = p_{1v}, \quad Dp_{1v} = m(1v)p_v, \quad Dp_{2v} = 0.$$

In particular, for any $v \in YF_n$, p_v is an eigenvector for $UD|_{R_n} : R_n \to R_n$ belonging to the eigenvalue $m(v)$. The p_v's give a complete set of orthogonal eigenvectors for $UD|_{R_n}$.

In the case of Young's lattice or the ring of symmetric functions, the transition matrix $M(p, h)$ (resp. $M(h, s)$) from the power sum symmetric functions to the complete symmetric functions (resp. from the complete symmetric functions to the Schur functions) is a triangular matrix under a suitable ordering (dominance order) of rows and columns. And the character table of the symmetric groups is given by $M(p, s)$, the transition matrix from power sum symmetric functions to the Schur functions. Then Proposition 5.3 corresponds to the orthogonality relations for characters. Proposition 5.4 is a $\mathbb{YF}$-analogue of [S1, Proposition 4.7].

As is shown in [O], each homogeneous component R_n admits a structure of associative commutative algebra satisfying the following properties:

(1) If we denote by $*$ the product in R_n, then $s_u * s_v = \sum_{w\in\mathbb{YF}_n} g_{uv}^w s_w$ with nonnegative integers g_{uv}^w.
(2) s_{1^n} is the identity element of R_n.
(3) $R_n \otimes_{\mathbb{Z}} \mathbb{Q}$ is a semisimple algebra with minimal idempotents $\frac{1}{z(v)}p_v$ ($v \in \mathbb{YF}_n$).

This algebra structure on R_n gives an example of fusion algebra at algebraic level. The notion of fusion algebra is a generalization of the character ring of a finite group. (See [B] for fusion algebras at algebraic level.)

6. Algebras associated to $\mathbb{YF}^{(r)}$

Finally we consider the r-Young-Fibonacci lattice $\mathbb{YF}^{(r)}$. Let K_0 be a field of characteristic 0 such that K_0 contains a primitive rth root ζ of unity. We will work with the base field $K = K_0(x_{i,k}, y_i : i = 1, 2, \ldots, k = 0, 1, \ldots, r-1)$.

Let $\mathscr{F}_n^{(r)}$ be the K-algebra defined by the following presentation:

$$\begin{aligned}
\text{generators}: &\ E_1, \dots, E_{n-1}, t_1, \dots, t_n, \\
\text{relations}: &\ E_i t_i^k E_i = x_{i,k} E_i \quad (i = 1, \dots, n-1, k = 0, \dots, r-1), \\
&\ E_i E_j = E_j E_i \quad (\text{if } |i-j| \geq 2), \\
&\ E_{i+1} E_i E_{i+1} = y_i E_{i+1} \quad (i = 1, \dots, n-2), \\
&\ E_i t_{i+1} = t_{i+1} E_i = E_i \quad (i = 1, \dots, n-2), \\
&\ E_i t_j = t_j E_i \quad (j \neq i, i+1), \\
&\ t_i^r = 1 \quad (i = 1, \dots, n), \\
&\ t_i t_j = t_j t_i \quad (i, j = 1, \dots, n).
\end{aligned}$$

We will construct irreducible representations of $\mathscr{F}_n^{(r)}$ on the K-vector space $V_v^{(r)}$ with basis Ω^v ($v \in \mathbb{YF}_n^{(r)}$). Define endomorphisms $\pi_v^{(r)}(E_i)$ and $\pi_v^{(r)}(t_i)$ on $V_v^{(r)}$ by putting, for a basis element $T = (v^{(0)}, \dots, v^{(n)}) \in \Omega^v$,

$$\pi_v^{(r)}(E_i)(v^{(0)}, \dots, v^{(i-1)}, v^{(i)}, v^{(i+1)}, \dots, v^{(n)})$$
$$= \begin{cases} \displaystyle\sum_{w \rhd v^{(i-1)}} \frac{\alpha^{(r)}(w)}{\alpha^{(r)}(v^{(i-1)})} (v^{(0)}, \dots, v^{(i-1)}, w, v^{(i+1)}, \dots, v^{(n)}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } v^{(i+1)} = 2v^{(i-1)}, \\ 0 \quad \text{otherwise}, \end{cases}$$

$$\pi_v^{(r)}(t_i)(v^{(0)}, \dots, v^{(i-1)}, v^{(i)}, \dots, v^{(n)})$$
$$= \begin{cases} \zeta^k (v^{(0)}, \dots, v^{(i-1)}, v^{(i)}, \dots, v^{(n)}) & \text{if } v^{(i)} = 1_k v^{(i-1)}, \\ (v^{(0)}, \dots, v^{(i-1)}, v^{(i)}, \dots, v^{(n)}) & \text{otherwise.} \end{cases}$$

Here the coefficients $\alpha^{(r)}(v)$ ($v \in \mathbb{YF}^{(r)}$) are defined as follows: First we introduce a family of polynomials $P_l^{k_1, \dots, k_l}$ by the following recurrence:

$$P_0 = 1, \quad P_1^k = \alpha_{1,k}, \quad P_l^{k_1, \dots, k_l} = \alpha_{l,k_1} P_{l-1}^{k_2, \dots, k_l} - \delta_{k_1,0} y_1 P_{l-2}^{k_3, \dots, k_l},$$

where $\alpha_{l,j} = \frac{1}{r} \sum_{k=0}^{r-1} \zeta^{jk} x_{l,k}$. Then $\alpha^{(r)}(v)$ is defined by

$$\alpha^{(r)}(1_{k_1} \dots 1_{k_l}) = P_l^{k_1, \dots, k_l},$$
$$\alpha^{(r)}(1_{k_1} \dots 1_{k_l} 2) = P_{l+1}^{k_1, \dots, k_l, 0} \begin{bmatrix} x_{1,k} \to \delta_{k0} y_1, & x_{i,k} \to x_{i+1,k} \ (i \geq 2) \\ y_1 \to x_{1,0} y_2, & y_i \to y_{i+1} \ (i \geq 2) \end{bmatrix}.$$

In general, for $u \in \mathbb{YF}_m$,

$$\alpha^{(r)}(1_{k_1} \dots 1_{k_l} 2u) = \alpha^{(r)}(1_{k_1} \dots 1_{k_l})[x_{i,k} \to x_{m+i,k}, y_i \to y_{m+i}] \alpha(u).$$

Then we can check that $\pi_v^{(r)}(E_i)$'s and $\pi_v^{(r)}(t_i)$'s satisfy the fundamental relations of $\mathscr{F}_n^{(r)}$. Hence we obtain a representation $\pi_v^{(r)}$ of $\mathscr{F}_n^{(r)}$ on $V_v^{(r)}$.

Theorem 6.1. (1) *The algebra $\mathscr{F}_n^{(r)}$ is semisimple and of dimension $r^n n!$.*

(2) *The $V_v^{(r)}$'s ($v \in \mathbb{YF}_n^{(r)}$) give a complete set of irreducible $\mathscr{F}_n^{(r)}$-modules.*

In the same way as in Section 3, we can define a product on $R^{(r)} = \bigoplus_{n \geq 0} R_n^{(r)}$, where $R_r^{(r)}$ is the free $\mathbb{Z}$-module with basis $\{s_v : v \in \mathbb{YF}_n^{(r)}\}$, and make $R^{(r)}$ an associative graded algebra.

Proposition 6.2.

(1) $s_v s_{1_0} = s_{1_0 v} + \sum_{w \lhd v} s_{2w}$.
(2) $s_v s_{1_k} = s_{1_k v}$ *if* $k \neq 0$.
(3) $s_v s_2 = s_{2v}$.

Theorem 6.3. *Let* $\mathbb{Z}\langle X_0, \dots, X_{r-1}, Y\rangle$ *be the noncommutative polynomial ring with grading given by* $\deg X_k = 1$ *and* $\deg Y = 2$. *Then there exists an algebra isomorphism* $\varphi : \mathbb{Z}\langle X_0, \dots, X_{r-1}, Y\rangle \to R^{(r)}$ *such that* $\varphi(X_k) = s_{1_k}$ $(k = 0, 1, \dots, r-1)$ *and* $\varphi(Y) = s_2$.

Put $R^{(r)}_{\mathbb{C}} = R^{(r)} \otimes_{\mathbb{Z}} \mathbb{C}$ and extend the scalar product $\langle\ ,\ \rangle$ on $R^{(r)}$ to the Hermitian form $\langle\ ,\ \rangle$ on $R^{(r)}_{\mathbb{C}}$. A correspondent to the power sum symmetric functions is defined as follows:

$$p_\varnothing = 1, \quad p_{1_k} = \sum_{j=0}^{r-1} \zeta^{jk} s_{1_j},$$
$$p_{1_k v} = p_v p_{1_k}, \quad p_{2v} = p_v(p_{1_0}^2 - r(m^0(v) + 2)s_2),$$

where $m^0(v)$ is the number of 1_0's at the head of v. And we define $z^{(r)}(v)$ ($v \in \mathbb{YF}^{(r)}$) by the following recurrence:

$$z^{(r)}(\varnothing) = 1,$$
$$z^{(r)}(1_k v) = \begin{cases} r(m^0(v) + 1)z^{(r)}(v) & \text{if } k = 0, \\ r z^{(r)}(v) & \text{if } k \neq 0, \end{cases}$$
$$z^{(r)}(2v) = r^2(m^0(v) + 2)z^{(r)}(v).$$

Then we have

Proposition 6.4. *For* $v, w \in \mathbb{YF}^{(r)}$, *we have*

$$\langle p_v, p_w\rangle = \delta_{vw} z^{(r)}(v).$$

Moreover we have

$$U p_v = p_{1_0 v}, \quad D p_{1_k v} = \delta_{k0} r m^0(1_0 v) p_v, \quad D p_{2v} = 0.$$

In particular, for any $v \in \mathbb{YF}^{(r)}_n$, p_v *is an eigenvector for* $UD|_{R^{(r)}_n} : R^{(r)}_n \to R^{(r)}_n$ *belonging to the eigenvalue* $m^0(v)$. *And the* p_v *'s give a complete set of orthogonal eigenvectors for* $UD|_{R^{(r)}_n}$.

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