CIRCLE ACTIONS ON RATIONAL HOMOLOGY MANIFOLDS
AND DEFORMATIONS OF RATIONAL HOMOTOPY TYPES

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Abstract. The aim of this paper is to follow up the program set in [LR85,
Rau92], i.e., to show the existence of nontrivial group actions ("symmetries")
on certain classes of manifolds. More specifically, given a manifold \( X \) with
submanifold \( F \), I would like to construct nontrivial actions of cyclic groups
on \( X \) with \( F \) as fixed point set. Of course, this is not always possible, and
a list of necessary conditions for the existence of an action of the circle group
\( T = S^1 \) on \( X \) with fixed point set \( F \) was established in [Rau92]. In this
paper, I assume that the rational homotopy types of \( F \) and \( X \) are related by
a deformation in the sense of [All78] between their (Sullivan) models as graded
differential algebras (cf. [Sul77, Hal83]). Under certain additional assumptions,
it is then possible to construct a rational homotopy description of a \( T \)-action
on the complement \( X \setminus F \) that fits together with a given \( T \)-bundle action on
the normal bundle of \( F \) in \( X \). In a subsequent paper [Rau94], I plan to show
how to realize this \( T \)-action on an actual manifold \( Y \) rationally homotopy
equivalent to \( X \) with fixed point set \( F \) and how to "propagate" all but finitely
many of the restricted cyclic group actions to \( X \) itself.

1. Rational cohomology

Given a (smooth) manifold \( X \) and a submanifold \( F \subset X \) whose rational
homotopy types are related in a sense to be made more precise in several
assumptions throughout this paper. In this section, we would like to construct the
rational cohomology of a candidate for the orbit space of a \( T = S^1 \)-action on
the complement \( X \setminus F \) such that its (algebraic) boundary fits to the orbit space
of a fibrewise \( T \)-action on the sphere bundle \( S\nu \) of the normal bundle \( \nu \) of
\( F \) in \( X \).

In order to formulate some assumptions relating \( X \) and \( F \), we need to
describe the rational homotopy types involved as differential graded algebras
(dgas) over \( \mathbb{Q} \) via their minimal models (see, e.g., [BG76, Sul77, GM81, Hal83,
AP93]). The following definition is a modification of that in [Ger64] to the
category of dgas:

Definition 1.1. Let \((\mathcal{A}, d)\) be a differential graded algebra over \( \mathbb{Q} \), and let \( e \)
denote a (formal) variable in dimension two.

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(1) The graded algebra \( A[e] = A \otimes Q[e] \) together with a differential \( d[e] \) is a 1-parameter deformation of \((A, d)\) if \( d[e] \) projects to \( d \) under the augmentation map \( e : A[e] \to A \). More precisely [LR85, Satz 2.8], the differentials \( d[e] \) and \( d \) are related by a derivation \( \tau : A \to A[e] \) such that

- \( d[e](e) = 0 \);
- \( d[e](x) = d(x) + e \cdot \tau(x), \ x \in A \);
- \( d[e] \circ \tau + \tau \circ d = 0 \).

(2) The trivial deformation of \( A \) is characterized by \( \tau = 0 \).

(3) A 1-parameter deformation of a dga morphism \( j : (A, d) \to (B, d') \) is a dga morphism \( j[e] : (A[e], d[e]) \to (B[e], d'[e]) \) which makes the following diagram commute:

\[ \begin{array}{ccc}
A[e] & \xrightarrow{j[e]} & B[e] \\
\downarrow{e} & & \downarrow{e} \\
A & \xrightarrow{j} & B.
\end{array} \]

Note that a 1-parameter deformation of \( A \) induces an "algebraic Gysin sequence" [LR85]

\[ \cdots \to H^{*+1}(A) \xrightarrow{\tau \cdot e} H^*(A[e]) \xrightarrow{\tau} H^{*+2}(A[e]) \xrightarrow{\tau \cdot e} H^{*+3}(A) \to \cdots \]

as the cohomology long exact sequence of the short exact sequence

\[ 0 \to A[e] \xrightarrow{\tau \cdot e} A[e] \xrightarrow{\tau} A \to 0. \]

Similarly, a 1-parameter deformation of a dga morphism induces graded algebra morphisms which fit into a ladder between the corresponding Gysin sequences.

C. Allday defined in [Al1178] the category of \( \mathbb{Z}/2 \)-graded augmented (Koszul-Sullivan)-differential algebras (KS2DGA). Let \( R = Q[e] \), and \( K = Q(e) \).

In our context, the most important example of such a \( \mathbb{Z}/2 \)-graded object is \((A(e), d(e)) = (A[e] \otimes_R K, d[e] \otimes_R K)\), which, as a graded algebra, is equal to \( A \otimes K \).

**Definition 1.2.** A dga morphism \( j[e] : (A[e], d[e]) \to (B[e], d'[e]) \) is called a local isomorphism if and only if

\[ j[e] \otimes_R \text{id}_K : (A(e), d(e)) \to (B(e), d'(e)) \]

is a weak homotopy equivalence of KS2DGAs, i.e., if it induces an isomorphism in homology.

In that case, the latter morphism is in fact a homotopy equivalence in the category of KS2DGAs [Al1178, Proposition 2.3].

From now on, we assume that \( X \) is a smooth closed simply-connected \( n \)-dimensional manifold and that \( F = \bigsqcup F_i \subset X \) consists of finitely many smooth closed (simply-connected) submanifolds \( F_i \) of dimension \( n_i < n \). Their rational minimal models in the sense of [Sul77, GM81, Hal83] are denoted by \((\mathcal{M}(X), d)\), resp. \((\mathcal{M}(F), d')\). Inclusion induces a dga map \( j : (\mathcal{M}(X), d) \to (\mathcal{M}(F), d') \). Note that the minimal model of the space \( F_T = F \times BT \) is given by \((\mathcal{M}(F)[e], d')\) with trivially extended differential. Furthermore we impose
Assumption A. The normal bundle $\nu = \nu(F, X) = \bigsqcup \nu(F_i, X)$ of $F$ in $X$ supports a complex structure.

Assumption B. There is a 1-parameter deformation $(\mathcal{M}(X)[\epsilon], d[\epsilon])$ of the minimal model of $X$ and a 1-parameter deformation $j_\epsilon : (\mathcal{M}(X)[\epsilon], d[\epsilon]) \rightarrow (\mathcal{M}(F)[\epsilon], d')$ of the inclusion map $j$ into the trivial deformation of $\mathcal{M}(F)$, which is a local isomorphism.

Remark 1.3. In the assumption above, we talk about a specific manifold $F$. Instead, one might just require a deformation into the minimal model of a space $F$ that has a lower cohomological dimension than $X$ itself. It can then be shown along the lines of [Rau92] and the references there, that every component $F_i$ is a rational Poincaré complex and that the Poincaré forms of $X$ and $F$ are related. Remark that the proofs in [Rau92] only use the Borel localization theorem, i.e., a situation that is guaranteed by Assumption B. In [Rau94], we shall discuss how to realize those rational Poincaré complexes by manifolds.

The map $j_\epsilon$ in Assumption B should be thought of as an algebraic simulation of the inclusion of Borel spaces $F \hookrightarrow X_T$, where $F$ is the fixed point set of a $T$-action on $X$. The dgas and maps in between them may be realized by rational spaces and maps [BG76, Hal83], which we denote by $j_\epsilon : F[\epsilon] = F(0) \times BT(0) \rightarrow X[\epsilon]$. Also the augmentation maps $\varepsilon : (\mathcal{M}(X)[\epsilon], d[\epsilon]) \rightarrow (\mathcal{M}(X), d)$ and $\varepsilon : (\mathcal{M}(F)[\epsilon], d') \rightarrow (\mathcal{M}(F), d')$ may be realized by maps $p_x : X(0) \rightarrow X[\epsilon]$, resp. $p_F : F(0) \rightarrow F(0) \times BT(0)$.

Interpreting $(X[\epsilon], F[\epsilon])$ as a pair of rational spaces, Assumption B shows in particular, that $H^*(X[\epsilon], F[\epsilon])$ is a $\mathbb{Q}[\epsilon]$-torsion module. As in [Rau92], one may prove

**Lemma 1.4.** Under Assumption B, the map $j^\epsilon_\ast : H^n(X[\epsilon]) \rightarrow H^n(X)$ is onto in dimension $n$, if $F \neq \emptyset$.

A choice of a complex structure on $\nu$ induces a (semifree) $T \subset \mathbb{C}^*$-action. After choice of a Hermitian metric on $\nu$ and conjugation with the associated exponential map, this action induces a semifree action on a tubular neighborhood $F \subset U \subset X$ with fixed point set $F$. In particular, $\partial U$ becomes a free $T$-manifold $T$-diffeomorphic to the sphere bundle $SU$ with orbit space $\partial U/T$ diffeomorphic to $CP^n$. Our aim is to construct the homology of a (virtual) orbit space for a (free) $T$-action on the manifold $M = X \setminus U \simeq X \setminus F$ with boundary $CP^n$. The first step is:

**Lemma 1.5.**

$$H^*(X[\epsilon], F[\epsilon]) = H^*(j_\epsilon) = \begin{cases} \mathbb{Q}, & \ast = n - 1, \\ 0, & \ast \geq n. \end{cases}$$

**Proof.** We have to chase the diagram of pairs

$$\begin{array}{cccccccc}
\cdots & \rightarrow & H^{*-1}(X[\epsilon]) & \xrightarrow{j_\epsilon} & H^{*-1}(F[\epsilon]) & \xrightarrow{\delta} & H^{*-2}(j_\epsilon) & \rightarrow & H^{*-2}(X[\epsilon]) & \xrightarrow{j_\epsilon} & H^{*-2}(F[\epsilon]) & \rightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \rightarrow & H^{*+1}(X[\epsilon]) & \xrightarrow{j_\epsilon} & H^{*+1}(F[\epsilon]) & \xrightarrow{\delta} & H^{*+2}(j_\epsilon) & \rightarrow & H^{*+2}(X[\epsilon]) & \xrightarrow{j_\epsilon} & H^{*+2}(F[\epsilon]) & \rightarrow & \cdots 
\end{array}$$

The vertical arrows stand for multiplication by $\varepsilon$, which is an isomorphism on $H^{*}(F[\epsilon])$ for all $* \geq \max\{n_i\} - 1$ and thus for $* \geq n - 3$, and on $H^{*}(X[\epsilon])$ for $* \geq n - 1$. The five lemma shows that it is an isomorphism on $H^{*}(j_\epsilon)$ for...
* \geq n \) Furthermore, Assumption B shows that the map \( j_{[e]}^* \) is an isomorphism for large * and hence for * \geq n \), which allows to conclude that \( H^*(j_{[e]}) = 0 \) for * \geq n \).

It follows from Lemma 1.4, that \( j_{[e]}^{n-1} \) is an isomorphism, too, and hence we have \( H^{n-1}(j_{[e]}) \cong \ker(j_{[e]}^{n-2}) \). On the other hand, by Lemma 1.4, multiplication with \( e \) on \( H^{n-2}(X_{[e]}) \) has a 1-dimensional cokernel, generated by an element \( x_{[e]} \) with \( p_T^*(x_{[e]}) = [X] \in H^n(X) \), the fundamental class. Hence, \( H^{n-1}(j_{[e]}) \) is a 1-dimensional \( \mathbb{Q} \)-vector space generated by \( \delta(\frac{1}{e} \cdot j_{[e]}^*(x_{[e]})) \). □

The cup-product defines \( \mathbb{Q} \)-bilinear pairings

\[ I : H^*(X_{[e]}) \otimes H^{n-1-*}(X_{[e]}, F_{[e]}) \to H^{n-1}(X_{[e]}, F_{[e]}) \cong \mathbb{Q}, \quad 0 \leq * \leq n-1, \]

which in turn yields vector space homomorphisms

\[ i_{[e]} : H^*(X_{[e]}) \to H_{n-1-*}(X_{[e]}, F_{[e]}) = \text{Hom}(H^{n-1-*}(X_{[e]}, F_{[e]}); \mathbb{Q}). \]

From now on, we impose:

**Assumption C.** The inclusion \( j \) of \( F \) into \( X \) induces the trivial map in rational cohomology in positive degrees.

**Proposition 1.6.** Under Assumption C, the maps \( i_{[e]} \) above are onto.

**Proof.** We use the following notation introduced in [Rau92] (in the absolute case): Let

\[ R(X, F) = p^*H^*(X_{[e]}, F_{[e]}) \subset H^*(X, F), \]

and

\[ I(X) = p^*(\text{Tor}(H^*(X_{[e]}))) \]

denote the image of the \( \mathbb{Q}[e] \)-torsion submodule of \( H^*(X_{[e]}) \). First, we show that the map \( k^* : H^*(X, F) \to H^*(X) \) induces an isomorphism

\[ K : H^*(X, F)/R(X, F) \to H^*(X)/I(X), \]

using the commutative diagram

\[ \cdots \to H^*-1(F_{[e]}) \to H^*(X_{[e]}, F_{[e]}) \xrightarrow{k_{[e]}^*} H^*(X_{[e]}) \to H^*(F_{[e]}) \to \cdots \]

\[ \downarrow \quad \downarrow p^* \quad \downarrow \quad \downarrow \]

\[ \cdots 0 \to H^{*-1}(F) \xrightarrow{\delta} H^*(X, F) \xrightarrow{k^*} H^*(X) \to H^*(F) \to \cdots \]

The quotient map \( K \) is well-defined, since, by Assumption B, \( H^*(X_{[e]}, F_{[e]}) \) is a \( \mathbb{Q}[e] \)-torsion module. It is one-to-one, since every torsion element in \( H^*(X_{[e]}) \) comes from \( H^*(X_{[e]}, F_{[e]}) \), and since \( \delta(H^{*-1}(F)) = \delta(R(F)) \subset R(X, F) \). Finally, \( K \) is onto because of Assumption C.

It is shown in [Rau92] that \( R(X) = I(X) \perp \) under the Poincaré duality form on \( H^*(X) \). Hence, the latter factors over a nondegenerate pairing \( P : H^*(X)/I(X) \to R(X)^* \), the \( \mathbb{Q} \)-vector space dual to \( R(X) \). Furthermore, the transgression map \( \tau : H^*(X, F) \to H^{*-1}(X_{[e]}, F_{[e]}) \) induces an isomorphism \( \tau : H^*(X, F)/R(X, F) \to \ker(e) \subset H^{*-1}(X_{[e]}, F_{[e]}) \). Moreover, the duality forms \( P \) and \( I \) are linked together by the transgression \( \tau \) as follows:

1. \( \tau : \mathbb{Q} \cong H^n(X, F) \to H^{*-1}(X_{[e]}, F_{[e]}) \cong \mathbb{Q} \) is an isomorphism.
2. \( \tau(y \cup p^*x) = \tau(y) \cup x, \ y \in H^*(X, F), \ x \in H^*(X_{[e]}) \) [LR85].
In the commutative diagram

\[
\begin{array}{ccc}
H^*(X, F)/R(X, F) & \xrightarrow{\tau} & H^*(X)/I(X) \\
\ker(-e) & \subset & H^{-1}(X[e], F[e]) \xrightarrow{i_{[e]}} H_{n-1-\ast}(X[e]),
\end{array}
\]

the composite map \( H^*(X, F)/R(X, F) \to H_\ast(X[e]) \) is monomorphic. Hence the adjoint of \( i_{[e]} \) has to be injective on \( \tau H_\ast(X, F) = \ker(-e) \). We conclude that \( i_{[e]} : H^\ast(X[e]) \to (\ker(-e))^\ast \), the dual of \( \ker(-e) \), is onto.

To show that \( i_{[e]} \) is onto \( H_{n-1-\ast}(X[e], F[e]) \), pick an element

\[
z \in \ker(-e) \setminus \ker(-e^{-1}) \subset H^\ast(X, F).
\]

Since \( e^{-1} \cdot z \in \ker(-e) \), there is an element \( x \in H^\ast(X_T) \) such that \( z \cup e^{-1} \cdot p^*x = e^{-1} \cdot z \cup p^*x = 0 \in H^{n-1}(X[e], F[e]). \)

For \( M = X \setminus F \), Alexander duality suggest the following

**Definition 1.7.** \( H^\ast_{[e]}(M) = H_{n-1-\ast}(X[e], F[e]). \)

In particular, \( H^\ast_{[e]}(M) = 0 \) for \( \ast < 0 \) and \( \ast \geq n \). The surjections \( i_{[e]} : H^\ast(X[e]) \to H^\ast_{[e]}(M) \) can be used to give the latter graded \( \mathbb{Q} \)-vector space the structure of a graded ring with a map \((-e)^\ast\) of degree 2 as a graded \( \mathbb{Q}[\epsilon] \)-quotient algebra of \( H^\ast(X[e]). \)

Our next goal is to define connecting homomorphisms in a Gysin type long exact sequence

\[
\cdots \to H^{\ast-1}(M) \xrightarrow{t_M} H^\ast_{[e]}(M) \xrightarrow{\delta} H^\ast_{[e]}(M) \xrightarrow{p_M} H^\ast(M) \cdots.
\]

This is quite easy using Alexander duality \( A : H^\ast(X \setminus F) \cong H^\ast(X \setminus Dv) \cong H_{n-\ast}(X, Dv) \cong H_{n-\ast}(X, F) \). In detail, \( t_M \), resp. \( p_M \), are given by the compositions

\[
t_M : H^{\ast-1}(X \setminus F) \xrightarrow{A} H_{n-\ast+1}(X, F) \xrightarrow{p} H_{n-\ast+1}(X[e], F[e]) \xrightarrow{\delta} H^\ast_{[e]}(X \setminus F),
\]

\[
p_M : H^\ast_{[e]}(X \setminus F) \xrightarrow{\delta} H_{n-\ast-1}(X[e], F[e]) \xrightarrow{t} H_{n-\ast}(X, F) \xrightarrow{A^{-1}} H^\ast(X \setminus F).
\]

As in [LR85, Lemma 2.1.c], one may show inductively

**Lemma 1.8.** The Gysin type sequence (1.2) is exact.

Next, we have to simulate the inclusion of the boundary \( CP^\nu \subset (X \setminus F)/T \) in case of a \( T \)-action by an algebraic counterpart. Geometry imposes an additional requirement: Let \( F = \bigsqcup_i F_i, \ CP^\nu = \bigsqcup_i CP^\nu \) denote decomposition into connected components. Then \( H^{n-2}(CP^\nu) \cong \bigoplus_i H^{n-2}(CP^\nu_i) \cong \bigoplus_i Q \) by evaluation at (properly chosen) fundamental classes. Adding over the components yields a map \( E : H^{n-2}(CP^\nu) \to \bigoplus_i Q \to Q \). Let \( p_{[e]} : CP^\nu \to F_T \) denote the Borel construction applied to the inclusion \( S^\nu \hookrightarrow D^\nu \), which is a map over \( F \).
Assumption D. The sequence

\[ 0 \to H^{n-2}(X[e]) \xrightarrow{j_{[e]}^*} H^{n-2}(F[e]) \xrightarrow{\text{deg}[e]} \mathbb{Q} \to 0 \]

is exact.

Remark 1.9. (1) The proof of Lemma 1.5 shows, that the cokernel of \( j_{[e]}^* \) is isomorphic to \( \mathbb{Q} \) under Assumption B. Hence, the assumption only specifies the image of \( j_{[e]}^* \) in dimension \( n-2 \).

(2) If there is a semifree \( T \)-action on \( X \) with fixed point set \( F \) and with \( M \approx X \setminus F \), then the sequence in Assumption D is in fact exact, since

(a) \( H^{n-2}(X[e], F[e]) \cong H^{n-2}(M/T, CP\nu) \cong H^n(M/T, CP\nu) = 0 \)

(use excision and \( H^{n-1}(M, Su) \cong H_1(M) = 0 \)).

(b) The following diagram commutes:

\[
\begin{array}{ccc}
H^{n-2}(X[e]) & \xrightarrow{j_{[e]}^*} & H^{n-2}(F[e]) \\
\downarrow p_{[e]}^* & & \downarrow \delta \\
H^{n-2}(CP\nu) & \xrightarrow{\delta = E} & H^{n-1}(M/T, CP\nu) \cong \mathbb{Q}
\end{array}
\]

(After proper choices of fundamental classes, the bottom map \( \delta \) corresponds to the map \( E \) above.)

Corollary 1.10. Assumption D implies: \( \ker(i_{[e]}^*) \subseteq \ker(p_{[e]}^* \circ j_{[e]}^*), \) i.e., \( p_{[e]}^* j_{[e]}^* z = 0 \) for all \( z \in H^*(X[e]) \) satisfying \( z \cup H^{n-*+1}(X[e], F[e]) = 0 \in H^{n-1}(X[e], F[e]) \). Hence, there is a well-defined ring homomorphism \( k_{[e]} \) completing the square

\[
\begin{array}{ccc}
H^*(X[e]) & \xrightarrow{j_{[e]}^*} & H^*(F[e]) \\
\downarrow i_{[e]} & & \downarrow p_{[e]}^* \\
H^*(X \setminus F) & \xrightarrow{k_{[e]}} & H^*(CP\nu).
\end{array}
\]

Remark 1.11. If the fixed point set \( F \) is connected, the conclusion above has the following geometric interpretation: \( j_{[e]}^*(z) \in H^*(F[e]) \) is a multiple of the total Chern class

\[ C(\nu) = e^{\nu F} + c_1(\nu F) \cdot e^{\nu F - 1} + c_2(\nu F) \cdot e^{\nu F - 2} + \cdots + c_{n-3}(\nu F). \]

Proof. The existence of \( k_{[e]} \) above is equivalent to the first statement in the corollary. The latter is trivially true for \( s \geq n - 1 \), since \( H^*(CP\nu) = 0 \).

Now, suppose \( z \in H^*(X[e]), s \leq n - 2 \), and \( z \cup H^{n-1}(X[e], F[e]) = 0 \). Let \( \delta \) denote the connecting homomorphism \( \delta : H^*(F[e]) \to H^{s+1}(X[e], F[e]) \). For all \( s \in H^{n-2}(F[e]) \), the condition above implies: \( \delta(j_{[e]}^*(z) \cup s) = z \cup \delta s = 0 \). Hence, for every such \( s \), there is an element \( u \in H^{n-2}(X[e]) \) with \( j_{[e]}^*(u) = j_{[e]}^*(z) \cup s \).

Thus, for every \( \sigma_i \in H^{n-*+2}(CP\nu_i) \), Assumption D implies:

\[ E(p_{[e]}^*(j_{[e]}^*(z)) \cup \sigma_i) = 0. \]

Since \( p_{[e]}^*(j_{[e]}^*(z)) \cup \sigma_i \) lives in \( H^{n-2}(CP\nu_i) \), it has to be trivial itself. Using Poincaré duality for \( CP\nu_i \), we conclude that \( p_{[e]}^*(j_{[e]}^*(z)) = 0. \) \( \square \)
We finish this section with two lemmas that show that our algebra behaves as the cohomology of an orbit space with respect to quotient maps and transgressions:

**Lemma 1.12.** Let $k : S\nu \hookrightarrow X \setminus F$ denote the inclusion map. The diagram

$$
\begin{align*}
H_{[e]}^*(X_{[e]}) & \xrightarrow{k_*} H^*(CP\nu) \\
\downarrow p_M^* & \quad \quad \quad \quad \quad \downarrow p_F^* \\
H^*(X \setminus F) & \xrightarrow{k^*} H^*(S\nu)
\end{align*}
$$

commutes. If $F$ is rationally contractible in $X$, i.e., $[\ast] = 0$ for $* > 0$, both compositions are trivial.

**Proof.** The diagram of the lemma embeds into

$$
\begin{align*}
H^*(X_{[e]}) & \xrightarrow{i_{[e]}} H^*(F_{[e]}) \\
\downarrow & \quad \quad \quad \quad \quad \downarrow \\
H_{[e]}^*(X \setminus F) & \xrightarrow{k_{[e]}} H^*(CP\nu) \\
\downarrow p_M^* & \quad \quad \quad \quad \quad \downarrow p_F^* \\
H^*(X \setminus F) & \xrightarrow{k^*} H^*(S\nu) \\
\downarrow & \quad \quad \quad \quad \quad \downarrow \\
H^*(X) & \xrightarrow{j^*} H^*(F).
\end{align*}
$$

Since $i_{[e]}$ is surjective, it is enough to see that all the outer diagrams commute. This is true by definition apart from the left parallelogram, which rewrites as

$$
\begin{align*}
H_{[e]}^*(X_{[e]} \setminus F_{[e]}) & \xrightarrow{k_{[e]}} H_{n-1}^*(X_{[e]} \setminus F_{[e]}) \\
\downarrow & \quad \quad \quad \quad \quad \downarrow t^* \\
H^*(X) & \xrightarrow{j^*} H_{n-1}^*(X, F),
\end{align*}
$$

where the horizontal arrows denote evaluation at the fundamental classes. From the Gysin sequence (1.2) it is easy to see (as in [LR85]) that $t : H^n(X, F) \rightarrow H^{n-1}(X_{[e]}, F_{[e]})$ is an isomorphism, and commutativity of the last diagram thus follows from:

$$
t(p_X^*(y) \cup z) = y \cup t(z), \quad y \in H^*(X_{[e]}), \quad z \in H^{n-*}(X, F),
$$

see [LR85, p. 552]. □

**Lemma 1.13.** The following diagram commutes:

$$
\begin{align*}
H^*(M) & \xrightarrow{k^*} H^*(S\nu) \\
\downarrow t & \quad \quad \quad \quad \quad \downarrow t \\
H_{[e]}^{n-1}(M) & \xrightarrow{k_{[e]}} H^{n-1}(CP\nu).
\end{align*}
$$
Proof. Embed the diagram of the lemma as the center of the following diagram:

\[ \begin{array}{cccccc}
H_n-*(X,F) & \cong & H_n-*(M,S\nu) & \xrightarrow{\delta} & H_{n-1}-(S\nu) \\
\downarrow p_* & & \downarrow & & \downarrow p_* \\
H^*(M) & \xrightarrow{k_*} & H^*(S\nu) & & \\
\downarrow t & & \downarrow t & & \\
H^{*,-1}_(e)(M) & \xrightarrow{k_{\nu}} & H^{*,-1}_(CP\nu) & & \\
\xrightarrow{\cong} & & \xrightarrow{\cong} & & \\
H_n-*(X[e],F[e]) & \xrightarrow{j_{\nu}} & H_n-*(F[e]) & = & H_n-*(CP\nu) \\
\downarrow \partial & & \downarrow \partial & & \\
H_{n-1}-(F[e]) & = & H_{n-1}-(F[e]) \\
\end{array} \]

We have to check commutativity of the outer and of the lower "rectangles"; commutativity of the smaller interior diagrams is by definition or routine.

The outer diagram commutes, since it is dual to part of the following commutative diagram:

\[ \begin{array}{cccc}
H^{*,-1}_(CP\nu) & \xrightarrow{p_{\downarrow \nu}} & H^{*,-1}_(S\nu) \\
\downarrow p_{\downarrow \nu} & & \downarrow \delta \\
H^{*,-1}_(F[e]) & \xrightarrow{p_{\downarrow \nu}} & H^{*,-1}_(F) & \ni \gamma \\
\downarrow \gamma & & \downarrow \delta & & \ni \gamma \\
H^{*,-1}_(X[e],F[e]) & \xrightarrow{p_{\downarrow \nu}} & H^{*,-1}_(X,F) \\
\end{array} \]

Now to the lower part of the diagram: Let \( y \in H^{*,-1}_(X[e]) \). Moving to the left, it corresponds to the linear form \( (z \mapsto y \cup \delta z \in H^{n-1}_(X[e],F[e]) \cong \mathbb{Q}) \) on \( H^{n-2}_(F[e]) \). Under the path to the right it corresponds to the map \( (z \mapsto E(p_{\downarrow \nu} j_{\nu} y \cup p^* z) \in \mathbb{Q}) \). Both come from the map \( (z \mapsto j_{\nu} y \cup z \in H^{n-2}_(F[e])) \) by composition with \( \delta \), resp. \( p_{\nu} \). According to Assumption D, the maps \( \delta \) and \( E \circ p_{\nu} : H^{n-2}_(CP\nu) \to \mathbb{Q} \) agree up to a nontrivial rational factor, which can be eliminated by a change of the fundamental class \( H^{n-1}_(X[e],F[e]) \).

2. Rational homotopy: Perturbing spaces and maps

It was the plan of the preceding section to describe the cohomology of a potential orbit space of a \( T \)-action on \( M = X \setminus F \) as a perturbation (quotient) of the cohomology of the potential Borel space \( X[e] \). Similarly, the cohomology of the inclusion map from \( CP\nu \) into it was obtained by a perturbation (quotient) of the deformation map \( j_{\nu} : H^*(X[e]) \to H^*(F[e]) \). In this section, we are going to refine this method to rational homotopy [Sul77, BG76, GM81, Hal83], which makes additional hypotheses necessary.

First, we show that certain conditions on the (co)-connectivity of \( X \), resp. \( F \), imply that the rational homotopy of \( X \) and \( M = X \setminus F \) are closely related. As a consequence, it turns out, that a minimal model of a space \( M[e] \) may be constructed as a perturbation of \( M(X[e]) \), and similarly a rational map \( CP\nu \to M[e] \).

We begin by presenting some necessary easy (and probably well-known) lemmas from rational homotopy theory. The first question is: Given a map
f : A → X. To which extent does the rational homotopy of X together with the rational cohomology of the map f determine the rational homotopy of A, or, in other words, is the rational homotopy of A "formal, given that of X"?
The reader should have in mind the case A = X \ F.

**Proposition 2.1.** Let f : A → X be a map between 1-connected CW-complexes with

\[
\begin{align*}
H^*(X; \mathbb{Q}) &= 0, \quad * \leq i; \\
H^*(f; \mathbb{Q}) &= 0, \quad * \leq k; \\
H^*(A; \mathbb{Q}) &= 0, \quad * \geq j.
\end{align*}
\]

For j ≤ \min\{i + k, 2k - 1\}, the rational homotopy of A is determined by that of X and by the (canonically induced) i* : H*(f) → H*(X).

**Proof.** Let \(\mathcal{A}\) denote the functor which to a simplicial complex associates its rational PL de Rham complex [Sul77, Hal83]. Regard the composition \(\mathcal{M}(X) \to \mathcal{A}(X) \to \mathcal{A}(A)\) of a model map for X and the map induced by f. A model for A extending \(\mathcal{M}(X)\) can be obtained from the Postnikov tower of the map f turned into a fibration \(F_f \to A \to X\). This has been made explicit in the thesis of Grivel, see [Hal83]. In our case, \(F_f\) is rationally \((k - 1)\)-connected, and hence, up to dimension 2k - 2, the following diagram consists of isomorphisms:

\[
\begin{align*}
\mathcal{M}(F_f)(2k - 2) &\cong \pi^*(F_f) \cong s^{-1}\pi^*(f) \\
&\uparrow h^* \cong \uparrow h^* \cong \* \leq 2k - 2. \\
H^*(F_f) &\cong s^{-1}H^*(f)
\end{align*}
\]

where \(\pi^* = \text{Hom}(\pi_*; \mathbb{Q})\), \(s\) denotes a degree 1 suspension, and \(h^*\) denotes the dual of the Hurewicz homomorphism. Let \(l : \mathcal{M}(F_f)(2k - 2) \to s^{-1}H^*(f)\) denote the obvious composition. Lift the cohomology of the map

\[
\mathcal{M}(F_f)(2k - 2) \xrightarrow{l} s^{-1}H^*(f) \xrightarrow{d_f} s^{-1}H^*(X)
\]

to get a map \(d_f : \mathcal{M}(F_f)(2k - 2) \to \mathcal{M}(X)\) of degree 1 and define

\[
\mathcal{M}'(A) = (\mathcal{M}(X) \otimes \mathcal{M}(F_f)(2k - 2), d_X \otimes d_f).
\]

It is then easy to write down a dga map \(\mathcal{M}'(A) \to \mathcal{A}(A)\) which is a weak equivalence up to dimension \(\min\{i + k, 2k - 1\}\); the first bound is needed to exclude mixed products in cohomology. Since \(H^*(A; \mathbb{Q}) = 0, * \geq j\), the rational homotopy of A can then be determined from \(\mathcal{M}'(A)\) in a purely formal way [Sul77]. □

**Corollary 2.2.** If \(f_* : \pi_*(A) \otimes \mathbb{Q} \to \pi_*(X) \otimes \mathbb{Q}\) is onto for \(k < * \leq k'\), then \(\mathcal{M}'(A)\) above is a minimal model through dimension \(k' - 1\). If \(f_* : H_*(A; \mathbb{Q}) \to H_*(X; \mathbb{Q})\) is onto for \(k < * \leq k'\), then the differential \(d_f\) may be chosen to be trivial through dimension \(k' - 1\).
Proof. Under the assumptions of the corollary, the lower horizontal maps in the following diagram are trivial:

\[
\begin{array}{ccc}
\mathcal{M}(F_f)(2k-2) & \overset{d_f}{\longrightarrow} & \mathcal{M}(X) \\
\downarrow l & & \downarrow f^* \\
H^{*+1}(f) & \longrightarrow & H^{*+1}(X) \\
\downarrow h^* & & \downarrow h^* \\
\pi^{*+1}(f) & \longrightarrow & \pi^{*+1}(X).
\end{array}
\]

Corollary 2.3. Let \( f : A \to X \) be as in (2.1). If \( X \) is formal, so is \( A \).

Proof. The diagram

\[
\begin{array}{ccc}
\mathcal{M}(X) & \longrightarrow & H^*(X) \\
\downarrow & & \downarrow f^* \\
\mathcal{M}(X) \otimes \mathcal{M}(F_f)) & \longrightarrow & H^*(A)
\end{array}
\]

with a rational homotopy equivalence on top can easily be extended to a weak equivalence up to dimension \( j - 1 \) in the bottom line as in (2.1). An extension to dimensions \( \geq j \) is formal as well. □

Definition 2.4. Let \( X \) be an \( n \)-dimensional connected 1-connected manifold, \( F = \bigsqcup F_j \) a collection of disjoint 1-connected submanifolds, \( \dim F_j = m_j \), \( m = \max \{m_j\} \). Define \( cX = \max \{k|H^k(X; \mathbb{Q}) = 0\} \) and

\[
cF = \begin{cases} 
\max \{k|H^k(F; \mathbb{Q}) = 0\}, & F \text{ connected,} \\
-1, & \text{else.}
\end{cases}
\]

Assumption E. All of the following inequalities are satisfied:

\[
m \leq cX + cF, \quad m < 2 \cdot cX - 1, \quad 2 \cdot m \leq n + 2 \cdot cF - 1, \quad 2 \cdot m \leq n + cX.
\]

Corollary 2.5. Under Assumption E, the rational homotopy type of \( M = X \setminus F \) has cohomological dimension \( d = n - \min\{cX, cF + 1\} \) and is determined by that of \( X \) and by the map \( i^* : H^*(X, M) \to H^*(X) \).

Proof. Use the Thom isomorphism (with \( t(\nu_i) \) denoting the Thom class of \( \nu_i \)), Alexander duality, and excision to obtain:

\[
H^*(X, M) \cong \bigoplus_i H^*(D\nu_i, S\nu_i) \cong \bigoplus_i H^{*-n+m_i}(F_i) \cdot t(\nu_i) = 0, \quad * \leq n - m - 1;
\]

\[
H^*(M) \cong H_{n-*}(M, S\nu) \cong H_{n-*}(X, F) = 0, \quad * \geq n - cF - 1 \text{ and } * \geq n - cX.
\]

Use (2.1) with \( k = n - m - 1 \), \( j = \min\{n - cF - 1, \ n - cX\} \). □

Corollary 2.6. If, in addition to Assumption E, the inclusion of \( F \) into \( X \) is trivial, i.e., it factors over a point in rational homotopy, then a minimal model of \( M = X \setminus F \) is obtained from \( \mathcal{M}(X) \otimes \Lambda s^{-1}H^*(X, M) \) with trivial differential on the second factor by killing cohomology in dimensions greater than or equal to \( n - \min\{cX, cF + 1\} \). □
Corollary 2.7. If, in addition to Assumption E, \( \dim X \leq 4 \cdot cX + 2 \), then both \( X \) and \( M = X \setminus F \) are formal.

Proof. Combine [Mil79] with (2.3). □

The perturbation of \( X \) into \( M \simeq X \setminus F \) that we have obtained above can be modified on the “Borel space level” to obtain a rational space \( M[e] \). We use the cohomological constructions from section 1 as a guide just as in [LR85, cf. in particular Satz 2.9]. We construct a minimal dga \( \mathcal{M}(M[e]) \), \( \delta \) such that \( \mathcal{M}(M[e]) \cong \mathcal{M}(M) \otimes \mathcal{M}(BT_{(0)}) \cong \mathcal{M}(M) \otimes \mathbb{Q}[e] \) as a graded algebra, and such that the differential \( \delta \) on \( \mathcal{M}(M[e]) \) satisfies the requirements of Assumption B for a deformation. Furthermore, its cohomology is the same as that of the candidate \( H^*_e(M) \) constructed in section 1.

Proposition 2.8. Under Assumptions C and E, there is a rational space \( M[e] \) over \( BT_{(0)} \) and a rational homotopy equivalence from \( M \) into the pullback of the diagram

\[
* = ET_{(0)} \quad \downarrow \\
M[e] \rightarrow BT_{(0)}.
\]

Furthermore, there is an isomorphism \( \psi : H^*(M[e]) \rightarrow H^*_e(M) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
H^*(M[e]) & \xrightarrow{\psi} & H^*(M) \\
\downarrow & & \downarrow \\
H^{*+1}(M) & \xrightarrow{\iota_M} & H^*_e(M)
\end{array}
\]

Proof. We want to obtain a diagram

\[
\begin{array}{ccc}
\mathcal{M}(X) & \leftarrow & \mathcal{M}(X[e]) \\
\downarrow & & \downarrow \\
\mathcal{M}(M) & \leftarrow & \mathcal{M}(M[e])
\end{array}
\]

where \( \mathcal{M}(M[e]) \cong \mathcal{M}(M) \otimes \mathbb{Q}[e] \) as a graded algebra, and \( \mathcal{M}(M) \cong (\mathcal{M}(X) \otimes \Lambda \delta^{-1} H^*(X, M), \delta_X \otimes 0) \) as a dga, up to the cohomological dimension \( d \) from Corollary 2.5. (From Corollary 2.6 we know that the differential on the second factor is trivial.)

We have to find a perturbed differential on \( \mathcal{M}(M[e]) \): First, we define a new differential \( \delta_2 = e \cdot \tau : s^{-1} H^*(X, M) \rightarrow \mathcal{M}(X[e]) \) such that the map \( \tau^* \) in cohomology makes the following diagram commute:

\[
\begin{array}{ccc}
s^{-1} H^*(X, M) & \xrightarrow{\tau^*} & H^*(M) \\
\downarrow & & \downarrow \\
s^{-1} H^{*+1}(X[e]) & \xrightarrow{i[e]} & H^*_e(M)
\end{array}
\]

Extend \( \tau \) to \( \Lambda s^{-1} H^*(X, M) \) as a derivation, and define \( \delta = \delta_X \otimes \delta_2 \) to be the new differential on \( \mathcal{M}(M[e]) \) up to dimension \( d \). Above the cohomological dimension \( d \), the derivation can be extended formally—killing
cohomology—in the same way as in the proof of [LR85, Satz 2.9]. The resulting cohomology \( H^*(M[e]) = H^*(\mathcal{M}(M[e])) \) is a quotient of \( H^*(X[e]) \) which is isomorphic to \( H^*_e(M) \), since the new differential annihilates precisely the kernel of \( i_e : H^*(X[e]) \to H^*_e(M) \). The map \( i_e \) factors over \( H^*(M[e]) \) to yield the map \( \psi \) above. □

**Remark 2.9.** (1) The proof is by induction on degrees and works only for minimal algebras with *decomposable* differentials. I do not know whether (2.8) is true without Assumption C.

(2) One can show as in [LR85], that the construction above ends up with a formal space \( X[e] \) when feeded with a formal space \( X \).

Note for the sake of completeness:

**Lemma 2.10.** Let \( \nu \downarrow F \) denote a vector bundle over the space \( F \) with minimal model \( \mathcal{M}(F) \).

1. If \( \nu^l \) is a real vector bundle with Euler class \( e(\nu) \in H^k(F) \), a minimal model for the sphere bundle \( S(\nu) \) is given by

\[
\mathcal{M}(S\nu) = (\mathcal{M}(F) \otimes \wedge(s), d \otimes d_1),
\]

where \( |s| = l - 1 \) and \( d_1(s) \) is a cocycle representing \( e(\nu) \) in cohomology.

2. If \( \nu^k \) is a complex vector bundle with total Chern class \( C(\nu) \in H^{**}(F)[e] \) (cf. 1.11), a minimal model of the projective bundle \( CP\nu \) is given by

\[
\mathcal{M}(CP\nu) = (\mathcal{M}(F) \otimes \wedge(e, s), d \otimes d_2),
\]

where \( |e| = 2, |s| = 2k - 1 \), \( d_2(e) = 0 \), and \( d_2(s) \) is a cocycle representing \( C(\nu) \).

We finish this section by lifting the map \( k[e] : H^*_e(M) \to H^*(CP\nu) \) from Corollary 1.10 to rational homotopy.

**Proposition 2.11.** Under Assumptions C–E, there is a dga map \( K_e : \mathcal{M}(M[e]) \to \mathcal{M}(CP\nu) \) inducing \( k[e] \) in cohomology and fitting into the diagram

\[
\begin{array}{ccc}
\mathcal{M}(X[e]) & \xrightarrow{j[e]} & \mathcal{M}(F[e]) \\
\downarrow i[e] & & \downarrow p[e] \\
\mathcal{M}(M[e]) & \xrightarrow{K[e]} & \mathcal{M}(CP\nu) \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
\mathcal{M}(M) & \xrightarrow{k} & \mathcal{M}(S\nu).
\end{array}
\]

**Proof.** Since \( i[e] \) is supposed to be onto, we only have to extend the map \( p[e] \circ j[e] : \mathcal{M}(X[e]) \to \mathcal{M}(CP\nu) \) to \( s^{-1}H^*(f) \) to get a dga map defined on \( \mathcal{M}(M[e]) \), i.e., it has to be defined on the generators of \( s^{-1}H^*(f)(d) \) in a way that commutes with differentials. Above the cohomological dimension, it can always be extended to a dga map.

Let \( y \) denote a generator from \( s^{-1}H^*(X, M) \), representing a cocycle in \( \mathcal{M}(M) \). We want to define \( K[e](y) = k(y) + e \cdot k'(y) \), where \( k' \) has to be defined as a map of degree 2 in such a way that \( K[e] \) commutes with differentials.

Note, that the lower diagram of the proposition commutes by definition. It turns out that \( K[e] \) is a dga map if and only if \( d[e]k'(y) = K[e](\tau y) - \tau k(y) \), where \( d[e] \)
denotes the differential on $\mathcal{M}(CP\nu)$ and $\tau$ denotes the transgressions in that algebra, resp. in $\mathcal{M}(M[e])$. The element on the right side is in fact a coboundary by (1.13).

3. Excision and duality

For the future development in [Rau94], it is important to ensure that the rational homotopy type of the pair given by the map $K[e]$ of Proposition 2.11 from $CP\nu$ into $M$ behaves like a manifold with boundary, i.e., satisfies Alexander duality. Furthermore, it is preferable to have excision properties at hand as in the situation of an actual $T$-action. In this section we assume the situation of Propositions 2.8 and 2.11, in particular, there is a commutative diagram of (rational) spaces:

$$
\begin{array}{ccc}
S\nu & \rightarrow & M \\
\downarrow & & \downarrow \\
CP\nu & \rightarrow & M[e] \\
\downarrow & & \nearrow \\
F[e]
\end{array}
$$

**Lemma 3.1.** The bottom maps give rise to an isomorphism $H^*(X[e], F[e]) \rightarrow H^*(M[e], CP\nu)$. In particular,

$$
H^*(M[e], CP\nu) \cong \begin{cases}
0, & * \geq n, \\
\mathbb{Q}, & * = n - 1.
\end{cases}
$$

**Proof.** By Proposition 2.11, there is an algebraic Gysin sequence

$$
\cdots H^*(M, S\nu) \rightarrow H^{*+1}(M[e], CP\nu) \rightarrow \cdots
$$

which is connected to the Gysin sequence of $(X, F)$ and $(X[e], F[e])$ by a ladder of homomorphisms. Excision yields an isomorphism $H^*(X, F) \cong H^*(M, S\nu)$ at every third term. As in [LR85], p. 564, the lemma follows by induction. □

**Lemma 3.2.** Evaluation at a fundamental class in $H^*(M[e], CP\nu)$ yields an isomorphism

$$
H^*(M[e]) \rightarrow H_{n-1-*}(M[e], CP\nu) \cong H_{n-1-*}(X[e], F[e]).
$$

**Proof.** Algebraic Gysin sequence, Alexander duality on total spaces, and induction as in [LR85], p.568. □

Lemma 3.2 is in fact the justification for choosing $H_{n-1-*}(X[e], F[e])$ as the “cohomology candidate” $H^*[e](M)$.

**Lemma 3.3.** The map $H^*(X[e], M[e]) \rightarrow H^*(F[e], CP\nu)$ is an isomorphism.

**Proof.** Form the obvious ladder between the two Gysin sequences, use excision $H^*(X, M) \cong H^*(F, S\nu)$ on every third term, and induction. □

As in [Rau92], Proposition 3.1.3, one can conclude easily that the long exact sequence of the pair $(X[e], M[e])$ splits into short exact pieces

$$
0 \rightarrow H^*(X[e], M[e]) \rightarrow H^*(X[e]) \rightarrow H^*(M[e]) \rightarrow 0.
$$
4. Example: Actions on highly-connected manifolds with isolated fixed points

In [LR85], we showed the existence of $T_0$-actions on every CW-complex with the rational homotopy type of a sufficiently connected manifold with vanishing Euler-characteristic and index (and satisfying an additional technical assumption concerning Pontryagin classes). In this section, we want to generalize these results to manifolds with nonnegative Euler-characteristic and vanishing index, cf. Proposition 4.5. In [Rau94], they will be applied to the construction of semifree actions of cyclic groups with isolated fixed points on such manifolds. We recall from [LR85]:

Definition 4.1. A simply-connected CW-complex $X$ is called an FC2-space, if $X_0$ is formal and has cup-length at most 2, i.e., $\tilde{H}^*(X; \mathbb{Q}) \cup \tilde{H}^*(X; \mathbb{Q}) \cup \tilde{H}^*(X; \mathbb{Q}) = 0$.

Remark 4.2. (1) [LR85] If $X$ is rationally homotopy equivalent to a sphere or to a connected sum of products of spheres, i.e.,

$$X \simeqq \sum_j \mathbb{Q}^{s_{ij} \times \mathbb{Q}^{n-i_j}}, \quad 2 \leq i_j \leq n - i_j,$$

then $X$ is an FC2-space.

(2) [LR85] If $M^n$ is a 1-connected manifold (Poincaré duality space suffices) with $H_i(M; \mathbb{Q}) = 0$ for $i \leq k$ with $3k + 1 \geq n$, then $M$ is an FC2-space.

(3) If $M^n$ is a 1-connected manifold with $H_i(M; \mathbb{Q}) = 0$ for $i \leq k$ with $3k + 1 \geq n$ and $\text{index}(M) = 0$, then $M$ is rationally homotopy equivalent to a sphere or a connected sum of products of spheres as in (1) above.

In the following, we shall concentrate on 1-connected manifolds $X^n$ with

$$n \text{ even, } \chi(X) > 0, \text{ index}(X) = 0;$$

(4.1)

$$H_i(X; \mathbb{Q}) = 0 \text{ for } i \leq k \text{ with } 3k + 1 \geq n.$$%

The corresponding case with $\chi(X) = 0$, in particular with $n$ odd, is already treated in [LR85].

Some ideas and notations from [Rau92] are relevant in the following: Let $X^n$ denote a manifold as in (4.1), on which $T$ acts with $X^T \neq \emptyset$. The $\mathbb{Q}[e]$-module $H^*(X_T; \mathbb{Q})$ contains a $\mathbb{Q}[e]$-torsion submodule $\text{Tor}(H^*(X_T; \mathbb{Q}))$. In [Rau92] we introduced and investigated the subspaces $I(X) = p^*(\text{Tor}(H^*(X_T; \mathbb{Q}))) \subseteq R(X) = p^*H^*(X_T; \mathbb{Q}) \subseteq H^*(X; \mathbb{Q})$. An analysis of the derivation $\tau : H^*(X) \rightarrow H^*-1(X_T)$ in the Gysin sequence as in [LR85, Rau92] yields a $\mathbb{Q}$-linear map $\tau : H^*(X)/R(X) \rightarrow I(X)[e]$ that, combined with evaluation $ev_1$ at $e = 1$, yields an isomorphism $ev_1 \circ \tau : H^*(X)/R(X) \rightarrow I(X)$ of $\mathbb{Q}$-vector spaces (of odd negative degree). Similar to [LR85, Chapter 2], this gives rise to an isomorphism of $\mathbb{Q}[e]$-vector spaces

$$\text{Tor}(H^*(X_T)) \cong I(X)[e]/e \cdot \tau(H^*(X)/R(X)),$$

and hence,

(4.2) $$H^*(X_T) \cong I(X)[e]/e \cdot \tau(H^*(X)/R(X)) \oplus (R(X)/I(X))[e],$$
where we consider $I(X)$ and $R(X)/I(X)$ as $\mathbb{Q}$-vector spaces generating $H^*(X_T)$ as a $\mathbb{Q}[e]$-module. We can also describe the trivial part of the cup-product structure on $H^*(X_T)$:

**Lemma 4.3.** Tor$(H^*(X_T)) \cup \hat{H}^*(X_T) = \{0\} \subseteq H^*(X_T)$ for a $T$-manifold $X$ satisfying 4.1.

**Proof.** All elements in $I(X) \cdot \hat{H}^*(X_T)$ are $\mathbb{Q}[e]$-torsion in dimensions $\geq \frac{2n+4}{3}$; since $\tau$ is trivial in this range of dimensions, such a torsion element has to be trivial on the nose. □

Now, we proceed to construct the cohomology of a possible Borel space for some $T$-action on $X$, denoted as $H^*_{[e]}(X)$: If $X$ is a rational homology sphere, we define

$$H^*_{[e]}(X) = H^*(S(n/2 \cdot U \oplus R_T)) \to 2 \cdot H^*(BT),$$

where $U$ denotes a 1-dimensional free complex $T$-representation, and the map is induced from the inclusion map of the two fixed points of the action induced on the sphere at the Borel space level.

If $X$ is not a rational homology sphere, (4.2) yields a rational homotopy equivalence

$$\chi(X) = 2l + 2.$$

We introduce an auxiliary space $S = \sum_{1 \leq j \leq l} (S^{i_j} \times S^{n-i_j})$, on which $T$ acts semifreely with $2l + 2$ isolated fixed points as follows: Start with a linear action with two isolated fixed points of the form $S(U \oplus R)$, $U$ a free $T$-representation, on each of the spheres. This produces actions with four isolated fixed points with the same tangential representations on every component in the connected sum. Taking a connected sum by identifying and eliminating fixed points in pairs yields the desired action with $\chi(S) = 2l + 2$ isolated fixed points.

Inclusion of the fixed point set induces a map $i_S : (2l + 2) \cdot BT \to ST$ on the Borel space level, and the corresponding cohomology homomorphism $i_S^*: H^*(S_T) \to (2l + 2) \cdot H^*(BT)$ becomes an isomorphism after inverting $e$ by the Borel localization theorem.

The cohomology fundamental classes of the spheres in $N$ (cf. 4.3) will be denoted $x_j, y_j, x'_j, y'_j, x''_j, y''_j$, resp. $y''_j$. Then $x_j \cup x'_j = x''_j \cup y''_j = [N]$, the cohomology fundamental class, whereas all other products vanish. Define

$$I(X) = \langle x_j, x'_j \rangle_{\mathbb{Q}}, \quad R(X) = I(X) \oplus \langle x''_j, y''_j \rangle_{\mathbb{Q}},$$

and

$$\tau : H^*(X)/R(X) \to I(X)[e]$$

by

$$\tau([y_j]) = x'_j \cdot e^{\frac{n-i_j-i'_j-1}{2}}, \quad \tau([y'_j]) = x_j \cdot e^{\frac{n-i'_j-i_j-1}{2}}.$$

We define, in accordance with (4.2) as a $\mathbb{Q}[e]$-module,

$$H^*_{[e]}(X) = I(X)[e]/e \cdot (\tau(H^*(X)/R(X)))[e] \oplus \langle y_j, y'_j \rangle_{\mathbb{Q}[e]}.$$
Remark that $\langle y_j, y'_j \rangle_{\mathbb{Q}[e]} \cong H^*(S_T)$ as a $\mathbb{Q}[e]$-module; we use this isomorphism together with Lemma 4.3 to provide $H^*_e(X)$ with a graded product structure. There is a projection homomorphism $P : H^*_e(X) \to H^*(S_T)$ with kernel $I(X)[e]/(e \cdot \tau(H^*(X)/R(X))[e])$, which gets isomorphic after inverting $e$. Together with the map $i^*_e$ above, we have constructed a graded ring homomorphism

$$i^*_e \circ P : H^*_e(X) \to H^*(S_T) \to (2l + 2) \cdot H^*(BT),$$

which gets an isomorphism after inverting $e$.

Finally, to get back from cohomology algebras to deformations of rational homotopy types, we apply

**Lemma 4.4.** Let $:\mathcal{H}_i^*, 1 \leq i \leq 3$, denote graded 1-connected $\mathbb{Q}$-algebras, $\mathcal{H}_2^*$ moreover a $\mathbb{Q}[e]$-algebra, $|e| = 2$, which fit into an exact sequence

$$\cdots \to \mathcal{H}_2^{*-2} \to \mathcal{H}_2^{*-1} \xrightarrow{e} \mathcal{H}_2^{*-1} \xrightarrow{\tau} \mathcal{H}_2^{*-1} \to \mathcal{H}_2^{*-1} + 1 \to \cdots ,$$

where $P$ is a ring homomorphism and $\tau$ a derivation. Let furthermore

$$\psi : \mathcal{H}_2^* \to \mathcal{H}_1^* \otimes \mathbb{Q}[e]$$

denote a $\mathbb{Q}[e]$-algebra homomorphism preserving $e$, which becomes an isomorphism after inverting $e$.

Then there are (formal) rational spaces $F, X, X[e]$ and maps $X \xrightarrow{j} X[e]$, resp. $F \times BT(0) \xrightarrow{j_{[e]}} X[e]$, that induce $P$, resp. $\psi$, in cohomology. In particular, $j_{[e]} : \mathcal{M}(X[e]) \to \mathcal{M}(F \times BT(0))$ satisfies Assumption B.

**Proof.** Let $\phi_i : (\mathcal{A}_i, d_i) \to \mathcal{H}_i^*, i = 1, 2$, denote (formal 1-connected) dga minimal models. $\mathcal{A}_2$ contains a (unique) cocycle $e$ of degree 2 representing the element of the same name in $\mathcal{H}_2^*$. Hence, the cokernel of the map $-e : \mathcal{A}_2 \to \mathcal{A}_2$ is a (1-connected) dga $(\mathcal{A}_3, d_3)$, and $\phi_2$ induces a dga map $\phi_3 : \mathcal{A}_2 \to \mathcal{H}_3^*$, hence $\phi_2$ is an isomorphism in cohomology.

In fact, the quotient map $\mathcal{A}_2 \to \mathcal{A}_3$ induces $P$. Realizing the latter by rational spaces and maps [GM81, BG76] yields the map $p : X \to X[e]$. According to [GM81], the map $\psi$ can be lifted to give a dga map $\Psi : \mathcal{A}_2 \to \mathcal{A}_3 \otimes \mathbb{Q}[e]$. A realization of this map gives us the map $j_{[e]} : F \times BT(0) \to X[e]$.

The following final result does not yet give us $T$-actions on the manifolds considered here, but should be considered as a first approximation step: Let $X$ be a manifold satisfying (4.1) with $\chi = \chi(X) > 0$. Let $U_i, 1 \leq i \leq \chi$, denote disjoint open disk neighbourhoods of $\chi$ points in $X$, and define $U = \bigcup U_i$, and $M = X \setminus U$. Its boundary $\partial M$ may be (nonequivariantly) identified with the space $S = \bigcup S_i$, where $S_i = S(V)$, $1 \leq i \leq \chi$, and $V$ denotes a complex free $T$-representation of real dimension $n$.

**Proposition 4.5.** There is a rational space $\tilde{M}$ rationally homotopy equivalent to $M$ supporting a free $T_{(0)}$-action. Furthermore, there is a $T_{(0)}$-equivariant map $I : S_{(0)} \to \tilde{M}$ that, up to rational cohomology, corresponds to the inclusion $S = \partial M \subset M$. 

Proof: Lemma 4.4 is applied to the case $\mathcal{H}_1^* = (2l + 2) \cdot H^*(BT)$, $\mathcal{H}_2^* = H^*_e(X)$, $\mathcal{H}_3^* = H^*(X)$, and $\Psi = \iota^*_5 \circ \alpha$ from 4.4. As a result, we obtain a map $j[e] : \chi \cdot BT \rightarrow X[e]$, which satisfies Assumption B. We want to apply Propositions 2.8 and 2.11, and have to check Assumptions A through E. The inclusion of $\chi$ points into $X$ obviously satisfies Assumptions A, C, and E. Assumption D is certainly satisfied (cf. Remark 1.9) when $X[e]$ is replaced by $S_T$, cf. (4.4). Since the images of $H^*(X[e])$ and of $H^*(S_T)$ in $H^*(\chi \cdot BT)$ coincide, Assumption D is satisfied for $X$ as well.

Hence, Propositions 2.8 and 2.11 yield maps

$$\chi \cdot CP^{2^l-1} \rightarrow M[e] \rightarrow BT(0).$$

The pullback along these maps of the classifying fibre bundle $ET(0) \downarrow BT(0)$ yields the required equivariant map of free $T(0)$-spaces $S(0) \rightarrow Q M$.

Remark 4.6. For manifolds satisfying (4.1), but with negative Euler characteristic, one can perform a similar construction with $F = \sum_j (S^3 \times S^3)$. Modulo fundamental group problems, one might also choose $F$ to be an orientable surface.

REFERENCES


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