A NOTE ON NORM INEQUALITIES FOR INTEGRAL OPERATORS ON CONES

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Abstract. Norm inequalities for the Riemann-Liouville operator \( R_rf(x) = \int_{(0,x)} \Delta^{-1}_V(x-t)f(t)dt \) and Weyl operator \( W_rf(x) = \int_{(x,\infty)} \Delta^{-1}_V(t-x)f(t)dt \) on cones in \( R^d \) have been obtained in the case \( r \geq 1 \) [7]. In this note, these inequalities are further extended to the case \( r < 1 \). The question of whether the Hardy operator \( Hf(x) = \int_{(0,x)} f(t)dt \) on cones is bounded from \( L^p(\Delta^0_V(x)) \) to \( L^q(\Delta^0_V(x)) \) (\( q < p \)) is also solved.

Let \( V \) be a homogeneous cone in \( R^d \). \( V \) defines a partial ordering in \( R^d \) in such a way that \( x <_V y \) if and only if \( y - x \in V \). The cone interval \( (a, b) \) is thus given by \( (a, b) = \{x \in V: a <_V x <_V b\} \). For \( x \in V \) we define \( \Delta_V(x) = \int_{(0,x)} dy \).

Let \( G(V) \) denote the automorphism group of \( V \), and let \( \Sigma = \{x \in V: |x| = 1\} \), \( \sigma_0 = \sigma_0(V) = \inf\{\alpha: \int_{[x]} \Delta^\alpha(t')dt' < \infty\} \) and \( \sigma(V) = \max(-1, \sigma_0) \). It is known (see [4, 7]) that if \( \alpha > \sigma(V) \), then \( \int_{(0,x)} \Delta^\alpha_V(t)dt \) is finite for all \( x \in V \) and homogeneous of order \( \alpha + 1 \) so that

\[
\int_{(0,x)} \Delta^\alpha_V(t)dt = c\Delta^{\alpha+1}_V(x).
\]

The dual \( V^* \) of \( V \) is defined as \( V^* = \{x \in R^d: x \cdot y > 0, \forall y \in V, y \neq 0\} \). Clearly, \( V^* \) is also a cone. It is known that \( V^{**} = V \).

The *-function on \( V \) is the mapping \( x \rightarrow x^* \) such that \( x^* = -\text{grad} \log(\phi(x)) \), where \( \phi(x) = \int_{V^*} e^{x^*y}dy \) is the characteristic function of \( V \). It is known (see [2, 6]) that the *-function is a one-to-one mapping from \( V \) onto \( V^* \). Let \( G(V \rightarrow V^*) \) be the group of linear transformations mapping \( V \) onto \( V^* \). A homogeneous cone \( V \) is said to be a domain of positivity if there is an element \( S \in G(V \rightarrow V^*) \) so that \( S \) is symmetric and positive definite. It can be shown (see [6, 7]) that for a domain of positivity \( V \), \( x <_V y \) if and only if \( y^* <_{V^*} x^* \).

In this note, we shall continue to consider the Riemann-Liouville operator

\[
R_rf(x) = \int_{(0,x)} \Delta^{-1}_V(x-t)f(t)dt
\]

and Weyl operator

\[
W_rf(x) = \int_{(x,\infty)} \Delta^{-1}_V(t-x)f(t)dt
\]
on cones on $R^d$. It is worth noting that the Riemann-Liouville operators whose kernels are complex power functions associated with the cone $V$ were extensively studied in [1], although they are different from the Riemann-Liouville operator we shall study here.

**Theorem 1.** Let $V$ be a domain of positivity in $R^d$. If $1 \leq p \leq q < \infty$, $r - 1 > \sigma(V)$, and $\gamma < -\sigma(V)(1 + \frac{a}{p}) - \sigma(V^*) + q\left(\frac{1}{p} - r + 1\right) - 3$, then for any $f: V \to R^+$,

$$\left(\int_V \Delta_V^{-q}(x) (R_r f(x))^q \, dx\right)^{1/q} \leq c \left(\int_V f^p(x) \Delta_V^{-(r-1)p + (\gamma + 1)p/q - 1}(x) \, dx\right)^{1/p}. \tag{1}$$

We show (1) in the case $r \geq 1$ (see [7]). Since $-1 \leq \sigma(V) \leq 0$ for any cone $V$, Theorem 1 extends the result to the case $0 < r < 1$. It is also worth noting that $\Delta_V(x) \to 0$ as $x$ approaches the boundary of $V$. Hence, the kernel of $R_r$ approaches infinity as $t$ approaches any point on the boundary of $x - V$ in the case $r - 1 > \sigma(V)$. In the one dimensional case, where $V = (0, \infty)$, $\sigma(V) = -1$, $r > 0$, Theorem 1 gives the boundedness of the Riemann-Liouville operator on the half line.

In the proof of Theorem 1 we shall deal with integrals on the cone of the form

$$\int_{(0,x)} \Delta_V^\alpha(x-t) \Delta_V^\beta(t) \, dt$$

and

$$\int_{(x,\infty)} \Delta_V^\alpha(t-x) \Delta_V^\beta(t) \, dt$$

where $\alpha, \beta < 0$. The following two lemmas prove that, under certain conditions on $\alpha$ and $\beta$, the integrals are finite for each $x \in V$ so that they can be “integrated” out. When restricted to the one dimensional case, these two lemmas give the best results.

**Lemma 1.** Let $V$ be a homogeneous cone and let

$$g(x) = \int_{(0,x)} \Delta_V^\alpha(x-t) \Delta_V^\beta(t) \, dt, \quad x \in V.$$ 

If $\alpha > \sigma(V)$ and $\beta > \sigma(V)$, then $g(x)$ is finite for each $x \in V$ and is homogeneous of order $\alpha + \beta + 1$. Hence, there is a constant $c$ for which

$$g(x) = c \Delta_V^{\alpha + \beta + 1}(x), \quad x \in V.$$ 

**Proof.** Let $y \in V$. By Fubini’s theorem, we have

$$\int_{(0,y)} g(x) \, dx = \int_{(0,y)} \Delta_V^\beta(t) \left(\int_{(t,y)} \Delta_V^\alpha(x-t) \, dx\right) \, dt$$

$$= \int_{(0,y)} \Delta_V^\beta(t) \left(\int_{(0,y-t)} \Delta_V^\alpha(z) \, dz\right) \, dt$$

$$\leq \int_{(0,y)} \Delta_V^\beta(t) \left(\int_{(0,y)} \Delta_V^\alpha(z) \, dz\right) \, dt$$

$$= \left(\int_{(0,y)} \Delta_V^\beta(t) \, dt\right) \cdot \left(\int_{(0,y)} \Delta_V^\alpha(z) \, dz\right).$$
If \( \alpha > \sigma(V) \) and \( \beta > \sigma(V) \), the two integrals above are finite, and so \( g(x) \) is finite for almost every \( x \in V \).

Let \( x_0 \in V \) be such that \( g(x_0) \) is finite. Since \( V \) is homogeneous, for any \( x \in V \) there exists \( A \in G(V) \) so that \( x = Ax_0 \). Then we have

\[
g(x) = g(Ax_0) = \int_{(0, Ax_0)} \Delta^\alpha_V(Ax_0 - t)\Delta^\beta_V(t) \, dt
\]

\[
= \int_{(0, x_0)} \Delta^\alpha_V(A(x_0 - z))\Delta^\beta_V(Az) |A| \, dz
\]

\[
= \int_{(0, x_0)} |A|^\alpha \Delta^\alpha_V(x_0 - z) |A|^\beta \Delta^\beta_V(z) |A| \, dz = |A|^\alpha + \beta + 1 g(x_0).
\]

Hence, \( g(x) \) is finite for each \( x \in V \) and is homogeneous of order \( \alpha + \beta + 1 \). Therefore, there is a constant \( c \) for which \( g(x) = c\Delta^\alpha + \beta + 1_V(x), \ x \in V \).

**Lemma 2.** Let \( V \) be a domain of positivity and let

\[
h(x) = \int_{(x, \infty)} \Delta^\gamma(t - x)\Delta^\delta_V(t) \, dt, \quad x \in V.
\]

If \( \alpha > \sigma(V) \) and \( \alpha + \beta < -3 - \sigma(V^*) - \sigma(V) \), then \( h(x) \) is finite for each \( x \in V \) and homogeneous of order \( \alpha + \beta + 1 \). Hence, there is a constant \( c \) for which

\[
h(x) = c\Delta^\alpha + \beta + 1_V(x), \quad x \in V.
\]

**Proof.** The condition on \( \alpha + \beta \) gives \(-\sigma(V) > 3 + \sigma(V^*) + \alpha + \beta \). Let \( y \in R \) so that \(-\sigma(V) > y > 3 + \sigma(V^*) + \alpha + \beta \). We have, for \( y \in V \),

\[
\int_{(y, \infty)} \Delta^\gamma(x) h(x) \, dx
\]

\[
= \int_{(y, \infty)} \Delta^\gamma(x) \int_{(x, \infty)} \Delta^\alpha_V(t - x)\Delta^\delta_V(t) \, dt \, dx
\]

\[
= \int_{(y, \infty)} \left( \int_{(y, t)} \Delta^\alpha_V(t - x)\Delta^\gamma(x) \, dx \right) \Delta^\delta_V(t) \, dt
\]

\[
\leq \int_{(y, \infty)} \left( \int_{(0, t)} \Delta^\alpha_V(t - x)\Delta^\gamma(x) \, dx \right) \Delta^\delta_V(t) \, dt.
\]

Since \( \alpha > \sigma(V) \) and \(-\gamma > \sigma(V) \), by Lemma 1, we have

\[
\int_{(y, \infty)} \left( \int_{(0, t)} \Delta^\alpha_V(t - x)\Delta^\gamma(x) \, dx \right) \Delta^\delta_V(t) \, dt = c \int_{(y, \infty)} \Delta^{\alpha + \beta + 1 - \gamma + 1}_V(t) \, dt.
\]

Since \( V \) is a domain of positivity, a change of variable \( t \to t^* \) gives

\[
\int_{(y, \infty)} \Delta^{\alpha + \beta + 1 - \gamma + 1}_V(t) \, dt = \int_{(0, y^*)} \Delta^{\alpha - \beta + \gamma - 3}_V(t) \, dt.
\]

Since \(-\alpha - \beta + \gamma - 3 > \sigma(V^*) \), the last integral is finite. Hence, \( h(x) \) is finite for almost every \( x \in V \). Clearly, \( h(Ax) = |A|^\alpha + \beta + 1 h(x) \) for \( A \in G(V) \). Hence, \( h(x) \) is finite for each \( x \in V \) and is homogeneous of order \( \alpha + \beta + 1 \). Therefore, there is a constant \( c \) for which \( h(x) = c\Delta^\alpha + \beta + 1_V(x), \ x \in V \).
Proof of Theorem 1. Noting the condition on \( \gamma \) in the hypothesis, we can choose \( b \) so that \( \sigma(V) < b < (-3 - \sigma(V^*) - \sigma(V) - \gamma + q(\frac{1}{p} - r + 1))\frac{p'}{q} \).

Using Hölder’s inequality, we have

\[
\int_V \Delta^{\gamma-q}(x)(R_rf(x))^q \, dx
= \int_V \Delta^{\gamma-q}(x) \left( \int_{(0,x)} \Delta_V^{(r-1)/p}(x-t)f(t) \right. \\
   \left. \cdot \Delta_V^{-b/p'}(t)\Delta_V^{(r-1)/p'}(x-t)\Delta_V^{b/p'}(t) \right) \, dx
\leq \int_V \Delta^{\gamma-q}(x) \left( \int_{(0,x)} \Delta_V^{r-1}(x-t)f^p(t)\Delta_V^{-b(p-1)}(t) \right) \, dx
   \cdot \left( \int_{(0,x)} \Delta_V^{r-1}(x-t)\Delta_V^{b}(t) \right) \, dx.
\]

Noting that \( r - 1 > \sigma(V) \) and \( b > \sigma(V) \), by Lemma 1, we have

\[
\int_V \Delta^{\gamma-q}(x)(R_rf(x))^q \, dx
\leq c \int_V \Delta_V^{r-1}(x-t)f^p(t)\Delta_V^{-b(p-1)}(t) \, dx.
\]

Since \( q/p \geq 1 \), by the Minkowski integral inequality, we have

\[
\int_V \Delta^{\gamma-q}(x)(R_rf(x))^q \, dx
\leq c \left( \int_V f^p(t) \right. \\
   \left. \cdot \Delta_V^{-b(p-1)}(t) \left( \int_{(t,\infty)} \Delta_V^{(r-1)q/p}(x-t)\Delta_V^{\gamma-q + (r+b)q/p'}(x) \, dx \right)^{p/q} \right) \, dt \right)^{q/p}.
\]

Noting that \( (r - 1)q/p > \sigma(V) \) and \( (r - 1)q/p + \gamma - q + (r + b)q/p' = rq - q/p + \gamma - q + bq/p' < -3 - \sigma(V^*) - \sigma(V) \), by Lemma 2, we have

\[
\int_V \Delta^{\gamma-q}(x)(R_rf(x))^q \, dx
= c \left( \int_V f^p(t)\Delta_V^{(r-1)q/p + \gamma - q + (r+b)q/p'}(t) \, dt \right)^{q/p}
= c \left( \int_V f^p(t)\Delta_V^{(r-1)p + (\gamma + 1)p/q - 1}(t) \, dt \right)^{q/p}.
\]

Using Theorem 1 and the fact that Weyl’s operator is the dual of Riemann-Liouville’s operator, we can prove the following norm inequality for Weyl’s operator on cones.
Theorem 2. Let $V$ be a domain of positivity in $\mathbb{R}^n$. If $1 \leq p \leq q < \infty$, $r - 1 > \sigma(V)$, and $\gamma > \sigma(V)(1 + q/p') + \sigma(V^*)q/p' + q(1 + 2/p')$, then

$$\left( \int_V \Delta_V^{1-q}(x) (W_t f(x))^q dx \right)^{1/q} \leq c \left( \int_V f^p(x) \Delta_V^{(r-1)p+(\gamma+1)p/q-1}(x) dx \right)^{1/p}. $$

Now we consider the Hardy operator

$$Hf(x) = \int_{(0,x)} f(t) dt$$
on cones in $\mathbb{R}^d$. As a corollary of the main theorem in [7], we have shown that if $1 < p < q < \infty$ and $\gamma < -\sigma(V)q/p' - \sigma(V^*) + q/p - 2$, then

$$\left( \int_V \Delta_V^{1-q}(x) (Hf(x))^q dx \right)^{1/q} \leq c \left( \int_V f^p(x) \Delta_V^{(\gamma+1)p/q-1}(x) dx \right)^{1/p}. $$

It is natural to inquire whether there exist appropriate numbers $\alpha$ and $\beta$ so that

$$\left( \int_V \Delta^\alpha(x) (Hf(x))^q dx \right)^{1/q} \leq c \left( \int_V f^p(x) \Delta^\alpha(x) dx \right)^{1/p},$$

holds for all $f \geq 0$ when $1 < q < p < \infty$. In the one dimensional case, the fact that (3) does not hold for any values of $\alpha$ and $\beta$ when $1 < q < p < \infty$ is simply a consequence of a theorem in [3] concerning the Hardy inequality with general weights. This result can be generalized to cones in $\mathbb{R}^d$.

Theorem 3. Let $V$ be a domain of positivity in $\mathbb{R}^d$. If $1 < q < p < \infty$, then for any values $\alpha$ and $\beta$, there is no constant $c > 0$ such that (3) holds for all $f \geq 0$.

Using the Hardy operator with weight, we see immediately that Theorem 3 is equivalent to the following theorem.

Theorem 4. Let

$$H_\alpha f(x) = \int_{(0,x)} f(t) \Delta^\alpha_V(t) dt,$$

and let $V$ be a domain of positivity in $\mathbb{R}^d$. If $1 < q < p < \infty$, then for any values of $\alpha$ and $\beta$, there is no constant $c > 0$ such that

$$\left( \int_V \Delta^\beta(x) (H_\alpha f(x))^q dx \right)^{1/q} \leq c \left( \int_V f^p(x) \Delta^\gamma(x) dx \right)^{1/p},$$

holds for all $f \geq 0$.

Proof of Theorem 4. First we show that in order that a $c > 0$ exist for which (4) holds for all $f \geq 0$, $\alpha$ and $\beta$ must satisfy $(\alpha + 1)/p' + (\beta + 1)/q = 0$ and $\beta < -1$.

Assume that (4) holds for some values of $\alpha$ and $\beta$. Then (4) implies that for all $z \in V$,

$$\left( \int_{(z, \infty)} \Delta^\alpha_V(x) \left( \int_{(0,x)} f(t) \Delta^\alpha_V(t) dt \right)^q dx \right)^{1/q} \leq c \left( \int_V f^p(x) \Delta^\alpha_V(x) dx \right)^{1/p}. $$
Further, it implies

$$\left( \int_{(0,z)} f(t)\Delta^\alpha_V(t)dt \right) \cdot \left( \int_{(z,\infty)} \Delta^\beta_V(x)dx \right)^{1/q} \leq c \left( \int_V f^p(x)\Delta^\alpha_V(x)dx \right)^{1/p}. $$

Choose a sequence \( \{ V_n \} \) of nested cone intervals so that \( V_n \subset (0,z) \) and \( V_n \nsubseteq (0,z) \). Note that \( \int_{V_n} \Delta^\alpha_V(t)dt < \infty \). Let \( f_n(x) = \chi_{V_n}(x) \). Substituting \( f_n(x) \) in (5) we have

$$\left( \int_{V_n} \Delta^\alpha_V(t)dt \right)^{1/p'} \cdot \left( \int_{(z,\infty)} \Delta^\beta_V(x)dx \right)^{1/q} \leq c. $$

It follows that \( \int_{V_n} \Delta^\alpha_V(t)dt \) is bounded and so \( \int_{(0,z)} \Delta^\alpha_V(x)dx \) is finite. It also follows from (6) that \( \int_{(z,\infty)} \Delta^\beta_V(x)dx \) is finite for each \( z \). It is known [6] that

$$\Delta^\beta_V(x) \leq \rho|x|^d \quad \text{for some } \rho > 0. $$

Therefore, in order that \( \int_{(z,\infty)} \Delta^\beta_V(x)dx \) be finite for each \( z \) it is necessary that \( \beta < -1 \).

Let \( f(x) = \chi_{(0,z)}(x) \). Substituting \( f(x) \) in (5) we have

$$\int_{(0,z)} \Delta^\alpha_V(t)dt \cdot \left( \int_{(z,\infty)} \Delta^\beta_V(x)dx \right)^{1/q} \leq c, \quad z \in V. $$

Integrating these integrals in (8) and taking the supremum over \( z \in V \), we have

$$\sup_{z \in V} \Delta^{(\alpha+1)/p'}(z)\Delta^{(\beta+1)/q}(z) \leq c. $$

Therefore, it is necessary that \( (\alpha + 1)/p' + (\beta + 1)/q = 0 \).

Next, we show that (4) cannot hold even if \( \alpha \) and \( \beta \) satisfy the aforementioned conditions. First assume that \( 1 < q \). Let \( a = (q - 1)/(p - q) \) and \( b = q/(p - q) \). Note that \( a > 0, b > 0, \) and \( (\alpha + 1)(a + 1/p) + (\beta + 1)b = 0 \). Take \( z_0 \in V \) with \( \Delta_V(z_0) = 1 \). Define

$$f_n(x) = \Delta^{a(\alpha+1)}(x) \min(n, \Delta^{b(\beta+1)}(x))\chi_{(0,nz_0)}(x), \quad x \in V, n = 1, 2, \ldots. $$

Clearly, for each \( n \), \( f^n(x) \) is integrable on \( V \). We show that \( \int_V f^n(x)dx \to \infty \) as \( n \to \infty \).

Choose \( \alpha_n \) so that

$$\Delta^{b(\beta+1)}(\alpha_n z_0) = \alpha_n^{b(\beta+1)} = n. $$

Then we have

$$\int_V f^n(x)\Delta^\alpha_V(x)dx \geq \int_{\{\alpha_n z_0, n z_0\}} \Delta^{a(\alpha+1)p}(x) \cdot \Delta^{b(\beta+1)p}(x)\Delta^\alpha_V(x)dx$$

$$= \int_{\{\alpha_n z_0, n z_0\}} \Delta^{-1}(x)dx. $$
Since $\beta + 1 < 0$, it follows that $\alpha_n \to 0$ and $\langle \alpha_n z_0, nz_0 \rangle \not\to V$ as $n \to \infty$. Noting that $\Delta^{-1}_V(x)$ is not integrable on $V$, we have that

$$\int_V f_n^p(x) \Delta^q_V(x) \, dx \to \infty \quad \text{as} \quad n \to \infty.$$ 

Using Fubini's theorem and noting that $\int_{(x, \infty)} \Delta^\beta_V(y) \, dy$ is finite for every $x \in V$, we have

$$\int_V \left( \int_{(0,y)} f_n(z) \Delta^q_V(z) \, dz \right)^q \Delta^\beta_V(y) \, dy \geq \int_V \left( \int_{(0,y)} f_n(x) \Delta^q_V(x) \left( \int_{(0,x)} f_n(z) \Delta^q_V(z) \, dz \right)^{q-1} \Delta^\beta_V(y) \, dy \right) \, dx$$

$$= \int_V f_n(x) \Delta^q_V(x) \left( \int_{(0,x)} f_n(z) \Delta^q_V(z) \, dz \right)^{q-1} \Delta^\beta_V(y) \, dy \geq c \int_V f_n(x) \Delta^q_V(x) \left( \int_{(0,x)} f_n(z) \Delta^q_V(z) \, dz \right)^{q-1} \Delta^{\beta+1}_V(x) \, dx,$$

where $c$ is a constant independent of $f_n$.

Since $\beta + 1 < 0$, $f_n(x) \Delta^\alpha V_a(a+1) V(x)$ is a decreasing function of $x \in V$ in the partial ordering defined by $V$. Further, we have that

$$\int_{(0,x)} f_n(z) \Delta^q_V(z) \, dz \geq f_n(x) \Delta^\alpha V_a(a+1) V(z) \int_{(0,x)} \Delta^\alpha V_a(a+1) V(z) \, dz.$$

By (7), in order that $\int_{(0,x)} \Delta^q_V(z) \, dz$ be finite, it is necessary that $\alpha > -1$. Thus, $a(\alpha + 1) > 0$ and $\int_{(0,x)} \Delta^\alpha V_a(a+1) V(z) \, dz$ is finite. So the last integral in (10) equals $c \Delta^\alpha V_a(a+1) V(x)$ and

$$\int_{(0,x)} f_n(z) \Delta^q_V(z) \, dz \geq c f_n(x) \Delta^{\beta+1}_V(x),$$

where $c$ does not depend on $f_n$. 

Therefore, (9) becomes

\[
\int_V \left( \int_{(0,y)} f_n(z) \Delta^\alpha_V(z) dz \right) \Delta^\beta_V(y) dy
\geq c \int_V f_n(x) \Delta^\alpha_V(x) \left( \int_{(0,x)} f_n(z) \Delta^\alpha_V(z) dz \right)^{q-1} \Delta^\beta_V(x) dx
\geq c \int_V f_n(x) \Delta^\alpha_V(x) f_n^{-1}(x) \Delta^{(\alpha+1)(q-1)}_V(x) \Delta^\beta_V(x) dx.
\]

Noting that
\[
\Delta^{(\alpha+1)(q-1)+(\beta+1)}_V(x) \geq f_n^{p-q}(x),
\]
we finally have

\[
\int_V \left( \int_{(0,y)} f_n(z) \Delta^\alpha_V(z) dz \right)^q \Delta^\beta_V(y) dy \geq c \int_V f_n^p(x) \Delta^\beta_V(x) dx.
\]

Therefore,

\[
\left( \int_V \left( \int_{(0,y)} f_n(z) \Delta^\alpha_V(z) dz \right)^q \Delta^\beta_V(y) dy \right)^{1/q} \geq c \left( \int_V f_n^p(x) \Delta^\beta_V(x) dx \right)^{1/p},
\]

where \( c \) is independent of \( f_n(x). \) Since \( \int_V f_n^p(x) \Delta^\beta_V(x) dx \to \infty \) as \( n \to \infty \) and \( q < p, \) there is no constant \( c \) such that for all \( f_n, \)

\[
\left( \int_V \left( \int_{(0,y)} f_n(z) \Delta^\alpha_V(z) dz \right)^q \Delta^\beta_V(y) dy \right)^{1/q} \leq c \left( \int_V f_n^p(x) \Delta^\beta_V(x) dx \right)^{1/p}.
\]

So we proved Theorem 4 in the case \( 1 < q. \) If \( q = 1, \) we define

\[
f_n(x) = \min(n, \Delta^{b+1}_V(x)) \chi_{(0,nz_0)}(x), \quad x \in V, \ n = 1, 2, \ldots,
\]

and (9) becomes the following simple inequality:

\[
\int_V \left( \int_{(0,y)} f_n(z) \Delta^\alpha_V(z) dz \right) \Delta^\beta_V(y) dy
= \int_V f_n(x) \Delta^\alpha_V(x) \left( \int_{(x, \infty)} \Delta^\beta_V(y) dy \right) dx
= c \int_V f_n(x) \Delta^{\alpha+\beta+1}_V(x) dx \geq c \int_V f_n^p(x) \Delta^\beta_V(x) dx.
\]

The theorem is proved.

Now we consider the Hardy operator of the form

\[
\tilde{H}_\alpha f(x) = \int_{(x, \infty)} f(t) \Delta^\beta_V(t) dt.
\]

For \( \tilde{H}_\alpha, \) we expect the following similar result:
Theorem 5. Let $V$ be a domain of positivity in $\mathbb{R}^d$. If $1 < q < p < \infty$, then for any values of $\alpha$ and $\beta$, there is no constant $c > 0$ such that

$$\left( \int_V \Delta^\beta_P(x)(\vec{H}_\alpha f(x))^q dx \right)^{1/q} \leq c \left( \int_V f^p(x)\Delta^\alpha_P(x)dx \right)^{1/p}$$

holds for all $f \geq 0$.

Proof. Assume that for some $\alpha$ and $\beta$ there is a constant $c > 0$ such that (11) holds for all $f \geq 0$. Let $g \geq 0$ be a function defined on $V$ with $\int_V g^p(y)\Delta^\alpha_P(y)dy = 1$. Then, for $f \geq 0$,

$$\int_V \left( \int_{(0,y)} f(x)\Delta^\beta_P(x)dx \right) g(y)\Delta^\alpha_P(y)dy$$

$$= \int_V \left( \int_{(x,\infty)} g(y)\Delta^\alpha_P(y)dy \right) f(x)\Delta^\beta_P(x)dx$$

$$\leq \left( \int_V \left( \int_{(x,\infty)} g(y)\Delta^\alpha_P(y)dy \right)^q \Delta^\beta_P(x)dx \right)^{1/q} \left( \int_V f^q(y)\Delta^\alpha_P(y)dy \right)^{1/q'}$$

$$\leq c \left( \int_V g^p(y)\Delta^\alpha_P(y)dy \right)^{1/p} \left( \int_V f^q(y)\Delta^\alpha_P(y)dy \right)^{1/q'}$$

$$= c \left( \int_V f^q(y)\Delta^\alpha_P(y)dy \right)^{1/q'} .$$

Thus, for $f \geq 0$,

$$\left( \int_V \Delta^\alpha_P(y) \left( \int_{(0,y)} f(x)\Delta^\beta_P(x)dx \right)^p dy \right)^{1/p'} \leq c \left( \int_V f^q(y)\Delta^\alpha_P(y)dy \right)^{1/q'} .$$

But this is impossible by Theorem 4. So there are no $\alpha$ and $\beta$ so that (11) holds for all $f \geq 0$.

References

2. M. Koecher, *Positivitatsbereiche im $\mathbb{R}^n$*, Amer. J. Math. 79 (1957), 575–596. (German)