ON THE DIMENSION AND THE INDEX
OF THE SOLUTION SET OF NONLINEAR EQUATIONS

P. S. MILOJEVIĆ

Abstract. We study the covering dimension and the index of the solution set to multiparameter nonlinear and semilinear operator equations involving Fredholm maps of positive index. The classes of maps under consideration are (pseudo) $A$-proper and either approximation-essential or equivariant approximation-essential. Applications are given to semilinear elliptic BVP's.

Introduction

It is the purpose of this paper to study the covering dimension and the index of the solution set of (equivariant) operator equations of the form

\[ H(\lambda, x) = f \quad (\lambda, x) \in D \subset \mathbb{R}^m \times X, \ f \in Y \]

as well as of semilinear operator equations of the form

\[ Ax + N(\lambda, x) = f \quad (\lambda, x) \in D \subset \mathbb{R}^m \times D(A) \subset \mathbb{R}^m \times X, \ f \in Y \]

by developing various continuation principles involving $A$-proper homotopies. Here, $X$ and $Y$ are Banach spaces with a scheme $\Gamma_i = \{X_n, Y_n, Q_n\}$, $D \subset \mathbb{R}^m \times X$ is an open subset, $A : D(A) \subset X \to Y$ is a linear Fredholm map of index $i(A) \geq 0$, and $N : D \to Y$ is a suitable nonlinear multiparameter map. These continuation principles are extensions to the multiparameter case of the earlier ones in [Mi-2–8].

The structure and the covering dimension of the branches of solutions to these equations have been studied by many authors ([AA-1], [MP], [FMP-1,2]) and, when the parameter space is infinite dimensional, by [AA-2], [AMP] using cohomology theories (in particular, Čech cohomology). Recently, a simpler method, based on the notion of essential maps, has been used to study these problems in [IMPV] and, for equivariant maps, in [IMV-1] by developing a degree theory for such maps. For an excellent survey, see Ize [I].

Our study is based on the approximation-essential mapping approach as developed in [IMPV] and [Mi-7], and the basic dimension result of [IMV-1] for $G$-equivariant maps for some compact Lie group and of [FMP-1]. The index

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The majority of the results in the paper have been announced in [Mi-6].
results are based on a finite dimensional index theory. As applications of the obtained results, we prove a number of dimension and index results for semilinear equations of the form

\[ Ax + Nx = f \quad (x \in D(A), f \in Y) \]

where $A$ is a Fredholm map of positive index and $A + N$ is $A$-proper. We still get the solvability results if the involved maps are only pseudo $A$-proper. Applications are given to semilinear elliptic boundary value problems.

1. Continuation theory for equivariant $A$-essential maps

Many problems in Applied Mathematics have symmetries. For example, the constitutive equations of continuum mechanics must be independent of the reference systems; rotations and translations cannot change the nature of the problem. A change in the origin in time should not change the nature of the equations in evolution problems (see [I] for more details). In this section we shall develop some continuation theory for approximation-essential maps dealing with the structure and covering dimension and with the index of the solution set of nonlinear operator equations. We assume that the maps have some symmetry property; i.e. they are equivariant relative to some compact Lie group $G$. Let $GL(X)$ be the space of all linear continuous isomorphisms of $X$ equipped with the operator norm. Suppose that $X$ is a Banach $G$-space or a representation of $G$; i.e., there is given a continuous homomorphism $T_g : G \to GL(X)$. We can (and will) assume that $T_g : X \to X$ is an isometry for each $g \in G$. If $G$ acts on Banach spaces $X$ and $Y$ via $\{T_g\}$ and $\{\tilde{T}_g\}$, respectively, then a map $T : X \to Y$ is said to be $G$-equivariant if $T(T_g x) = \tilde{T}_g(T(x))$ for each $x \in X$ and $g \in G$.

For a subgroup $H$ of $G$, we denote by $\text{Fix}_X(H) = \{x \in X | T_h x = x \text{ for each } h \in H\}$. A subgroup $X_g = \{g \in G | T_g x = x\}$ is called the isotropy subgroup of $G$ at $x$. Define $X_H = \{x \in X | G_x = H\}$ and $X(H) = GX_H = \{x \in X | G_x = g^{-1}Hg \text{ for some } g \in G\}$. If $G$ is a finite group, we define for any irreducible representation $V \subset X$ the number $\mu(V)$ as the greatest common divisor of the numbers $|G/H_i|$, where $H_i$ ranges over all subgroups such that $V_H = \{x \in V | G_x = g^{-1}H_i g \text{ for some } g \in G\} \neq \emptyset$. If $V_1, \ldots, V_k$ are all irreducible representations contained in $X$, then $\mu(X)$ is defined as the greatest common divisor of the numbers $\mu(V_i)$. A subset $S \subset X$ is said to be $G$-invariant if $GS \subset S$. Let $X/G$ be the orbit space of $X$ relative to $G$.

A. Dimension results. Let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of Banach $G$-spaces $X$ and $Y$, respectively, with $\text{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$ and let $Q_n : Y \to Y_n$ be a linear projection onto $Y_n$ with $\delta = \sup \{\|Q_n\| < \infty \}$. Suppose that $\dim X_n - \dim Y_n = i \geq 0$ for each $n$ and some fixed $i$. We say that a scheme $\Gamma_i = \{X_n, Y_n, Q_n\}$ for $(X, Y)$ is $G$-invariant if all $X_n$ and $Y_n$ are $G$-invariant. Let $T_n = Q_n T$.

Let $U \subset X$ be an open, not necessarily bounded, subset and $S$ be an arbitrary subset of $X$. To define the class of admissible (pseudo) $A$-proper maps on $S \cap U$, we recall ([IMPV], [IMV-1]).

Definition 1.1. Let $S \subset X$ be a $G$-invariant subset. A map $T : U \to Y$ is said to be admissible w.r.t. a $G$-invariant $\Gamma_i$ on $S \cap U$ if there is an open,
bounded, and $G$-invariant set $V_0$ such that $T^{-1}(0) \cap S \subset V_0 \subset \overline{V}_0 \subset U$ and $(T_n)^{-1}(0) \cap S \subset V_0 \cap X_n$ for all $n$. If $T : \overline{U} \to Y$ and $G = \{e\}$, then it is admissible w.r.t. $\Gamma_i$ on $S \cap \overline{U}$ if $T^{-1}(0) \cap S \subset V_0 \subset U$ for some open and bounded subset $V_0$ and $(T_n)^{-1}(0) \cap S \subset V_0 \cap X_n$ for all $n$.

**Definition 1.2.** An admissible map $T$ w.r.t. $\Gamma_i$ on $S \cap U$ (resp., $S \cap \overline{U}$) is said to be Approximation-essential (A-essential for short) w.r.t. $\Gamma_i$ on $S \cap U$ (resp., $S \cap \overline{U}$) if, for each $n$, $T_n = Q_n T : U \cap X_n \to Y_n$ is continuous, and for any open, bounded, and $G$-invariant set $V$ such that $(T_n)^{-1}(0) \cap S \subset V \cap X_n \subset \overline{V} \cap X_n \subset U \cap X_n$ (resp., $(T_n)^{-1}(0) \cap S \subset V \cap X_n \subset U \cap X_n$), any continuous $G$-equivariant extension $\tilde{T}_n : \overline{V} \cap X_n \to Y_n$ of $T_n | \partial V \cap X_n \to Y_n$ has a zero in $S \cap V \cap X_n$.

**Remark 1.1.** If $G = \{e\}$, $U$ is bounded, and $S \cap \overline{U}$ is closed, then $T : \overline{U} \to Y$ is A-essential w.r.t. $\Gamma_i$ on $S \cap \overline{U}$ if and only if, for each $n$, the restriction $T_n | S \cap \partial U \cap X_n$ is essential with respect to $S \cap \overline{U} \cap X_n$ in the classical sense (cf. Dugundji-Granas [DG]); i.e., any continuous extension $\tilde{T}_n : S \cap \overline{U} \cap X_n \to Y_n$ has a zero in $S \cap \overline{U} \cap X_n$. Hence, we see that in this case the A-essentiality w.r.t. $\Gamma_i$ on $S \cap \overline{U}$ reduces to the one introduced by the author ([Mi-7]).

**Definition 1.3.** A map $T : U \to Y$ is (pseudo) A-proper w.r.t. $\Gamma_i$ on $S \cap U$ if whenever $V$ is an open and bounded set with $V \subset \overline{V} \subset U$ and $\{x_{nk} \in S \cap V \cap X_{nk}\}$ is such that $Q_n T x_{nk} - Q_n f \to 0$ for some $f \in Y$, then $\{x_{nk}\}$ has a subsequence converging to $x \in S \cap U$ (there is $x \in S \cap U$) with $Tx = f$. If $T : \overline{U} \to Y$, then it is (pseudo) A-proper w.r.t. $\Gamma_i$ on $S \cap \overline{U}$ if it has the above property whenever $V$ is open, bounded, and $V \subset U$.

**Definition 1.4.** A homotopy $H : [0, 1] \times U \to Y$ is admissible w.r.t. a $G$-invariant $\Gamma_i$ on $S \cap U$ if there is an open, bounded and $G$-invariant subset $V_0$ such that for each $t \in [0, 1]$, $H_t = H(t, \cdot)$, $H_t^{-1}(0) \cap S \subset V_0 \subset \overline{V}_0 \subset U$ and $(Q_n H_t)^{-1}(0) \cap S \subset V_0 \cap X_n$ for all $n$. If $H : [0, 1] \times \overline{U} \to Y$, then it is clear how to define its admissibility w.r.t. $\Gamma_i$ on $S \cap \overline{U}$.

Note that the admissibility of $H_t$ means that, for each $t \in [0, 1]$, $H_t$ and $Q_n H_t$ have no zero near the boundaries $\partial U$ and $\partial U \cap X_n$.

**Definition 1.5.** A homotopy $H : [0, 1] \times U \to Y$ is A-proper w.r.t. $\Gamma_i$ on $S \cap U$ if $Q_n H_t : U \cap X_n \to Y_n$ is continuous for each $t$ and $n$, and if $\{x_{nk} \in S \cap V \cap X_{nk}\}$, with $V$ open and bounded such that $V \subset \overline{V} \subset U$, $t_k \in [0, 1]$ with $t_k \to t$ and $Q_n H(t_k, x_{nk}) - Q_n f \to 0$ as $k \to \infty$ for some $f \in Y$, then a subsequence of $\{x_{nk}\}$ converges to $x \in S \cap U$ and $H(t, x) = f$. Similarly, we define the A-properness of $H : [0, 1] \times \overline{U} \to Y$.

The classes of A-proper and pseudo A-proper maps are very general and we refer to [FMP-1, Mi-1-8, Pe-1-2] for many examples of such maps. We note also that (pseudo) A-proper maps w.r.t. $\Gamma_i$ with $i > 0$ have been first studied by the author in [Mi-4, 6] and, independently, by Fitzpatrick-Massabo-Pejsachowicz (cf. [FMP-1]). For a recent survey of the (pseudo) A-proper mapping theory, we refer to [Mi-9].

We have the following transversality result.
Theorem 1.1. Let $D$ be $U$ or $\overline{U}$ and $H : [0, 1] \times D \to Y$ be an admissible and $A$-proper homotopy w.r.t. $\Gamma_i$ on $S \cap D$. Then $H_1$ is $A$-essential w.r.t. $\Gamma_i$ on $S \cap D$ if and only if $H_0$ is such.

Proof. When $U$ is bounded and $S \cap \overline{U}$ is closed, the theorem was proved by the author [Mi-7]. Using similar arguments, it is easy to prove it in this generality (cf. also [IMPV]). □

Definition 1.6. A map $T : U \to Y$ is said to be sectionally proper on a closed (in $X$) subset $C \subset U$ if and only if for any finite dimensional subspace $Y_n$ of $Y$ and any compact set $K \subset Y_n$, the set $T^{-1}(K)$ is compact.

Recall that, if $K$ is a topological space and $m$ is a positive integer, then $K$ is said to have covering dimension equal to $m$ provided that $m$ is the smallest integer such that whenever $\mathcal{F}$ is a family of open subsets of $K$, whose union covers $K$, then there is a refinement, $\mathcal{F}_1$, of $\mathcal{F}$ whose union also covers $K$, and no subfamily of $\mathcal{F}_1$ consisting of more than $m + 1$ members has nonempty intersection. If $K$ fails to have this refinement property for each positive integer, then $K$ is said to have infinite dimension. When $x \in K$, then $K$ is said to have dimension (covering) at least $m$ at $x$ if each neighborhood of $x$ in $A$, has dimension at least $m$. In the absence of a manifold structure on $K$, the concept of dimension is the natural way to describe its size.

We need the following covering dimension result for $G$-equivariant $A$-proper homotopies of Ize-Massabo-Vignoli [IMV-1].

Theorem 1.2. Let $S$ be closed in $U$ and $H : [0, 1] \times U \to Y$ be a sectionally proper on bounded and closed subsets of $S \cap U$ admissible $G$-equivariant $A$-proper homotopy w.r.t. $G$-invariant scheme $\Gamma_i = \{\Lambda \times R^m \times X_n, Y_n \times R^k, Q_n\}$ on $S \cap U$. Let $H_0$ be $A$-essential w.r.t. $\Gamma_i$ on $S \cap U$. Then there exists an invariant set $\Sigma \subset S \cap U$ which is minimal, closed (in $U$), and

(a) $H_1$ is $A$-essential w.r.t. $\Gamma_i$ on $\Sigma \cap U = \Sigma$ and so $H_1^{-1}(0) \cap \Sigma \neq \emptyset$.

(b) If $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1$ and $\Sigma_2$ are proper, closed, and invariant subsets with $\Sigma_1 \cap \Sigma_2 = \emptyset$, then either $\Sigma_1 = \emptyset$ or $\Sigma_2 = \emptyset$. This is equivalent to saying that $\Sigma/G$ is connected. If $G$ is connected, then so is $\Sigma$.

(c) If $\text{Fix}_\Gamma(G) \neq \{0\}$, then $\Sigma$ is either unbounded or $\Sigma \cap \partial U \neq \emptyset$. Assume that $Y = \text{Fix}_\Gamma(G) \oplus Y_2$, where $Y_2$ is such that $\text{Fix}_\Gamma(G) = \{0\}$, and decompose $H_1$ as $H_1 = (h_1, h_2)$, where $h_1 : U \to Y_1 = \text{Fix}_\Gamma(G)$ and $h_2 : U \to Y_2$. Then there exists an invariant minimal subset $\Sigma$ contained in $h_2^{-1}(0) \cap \Sigma$ such that

(i) $h_1$ is $A$-essential and invariant on $\Sigma \cap U$ w.r.t. $\Gamma_m$ and, in particular, $\Sigma$ is either unbounded, or $\text{cl}(\Sigma) \cap \partial U \neq \emptyset$ (provided $\text{Fix}_\Gamma(G) \neq \{0\}$).

(ii) If $\Sigma/G = \Sigma_1 \cup \Sigma_2$ with $\Sigma_1$, $\Sigma_2$ closed and proper subsets of $\Sigma/G$, then $\dim(\Sigma_1 \cap \Sigma_2 \cap X_n) \geq \dim \text{Fix}_\Gamma(G) - 1$ for infinitely many $n$. In particular, $\Sigma/G$ is connected and has infinite dimension at each point.

When $G = \{e\}$, then every map is $G$-equivariant. Hence, as a particular case of Theorem 1.2 we have the covering dimension result for $A$-proper homotopies due to Ize-Massabo-Pejsachowicz-Vignoli [IMPV] and to Fitzpatrick-Massabo-Pejsachowicz [FMP-1] in a less general form.

The following continuation theorem gives some conditions on $H$ when Theorem 1.2 can be applied.
Theorem 1.3. Let $U$ be an open subset of $X$, $S$ be closed in $U$, and $H : [0, 1] \times U \to Y$ be a sectionally proper on bounded and closed subsets of $S \cap U$ $A$-proper homotopy w.r.t. $\Gamma_t$ on $S \cap U$. Suppose that there is an open and bounded subset $D$ of $X$ such that

(i) $H(t, x) \neq f$ for $x \in \partial D \cap S$, $t \in [0, 1]$.  
(ii) $H(0, x) \neq tf$ for $x \in \partial D \cap S$, $t \in [0, 1]$.  
(iii) $H_0$ is $A$-essential w.r.t. $\Gamma_t$ on $S \cap U$.

Then the conclusions of Theorem 1.2 hold for $H_1 - f$ with $G = \{e\}$.

Proof. Since $H_t$ is sectionally proper on closed and bounded subsets of $S \cap U$ and $H_t^{-1}(f) \cap S \cap D$ is such a set, then $H_t^{-1}(f) \cap S \cap D$ is a compact set contained in $D$ by (ii) for each $t \in [0, 1]$. Hence, for each $t \in [0, 1]$, there is an open and bounded subset $V_t$ of $X$ such that $H_t^{-1}(f) \cap S \subset V_t \subset \overline{V_t} \subset D$. Define $V = \bigcup_{t \in [0, 1]} V_t$. Then $V$ is open and bounded and $H_t^{-1}(f) \cap S \subset V \subset D$ for each $t \in [0, 1]$; i.e., $H_t$ is admissible on $S \cap \overline{D}$.

Next, set $F_t = H_t - f$. Then $F_t^{-1}(0) = H_t^{-1}(f)$ and $F_t$ is admissible on $S \cap \overline{D}$. Since $H_t$ is an $A$-proper homotopy, then arguing by contradiction and using the admissibility of $H_t$ we get that $F_t$ is admissible w.r.t. $\Gamma_t$ on $S \cap \overline{D}$. Next, set $G_t = H_t - tf$. Then, (ii) implies that $G_t^{-1}(0) \cap S = H_0^{-1}(tf) \cap S$ is compact and contained in $D$ for each $t \in [0, 1]$. Hence, there is an open and bounded set $W_t$ such that $G_t^{-1}(0) \cap S \subset W_t \subset \overline{W_t} \subset D$. Then $W = \bigcup_{t \in [0, 1]} W_t$ is open and bounded and, as above, we get that $G_t$ is admissible w.r.t. $\Gamma_t$ on $S \cap D$.

Finally, set $W_0 = V \cup W$. Then $W_0$ is open and bounded with $W_0 \subset D$ and $G_t$ is an admissible and $A$-proper homotopy w.r.t. $\Gamma_t$ on $S \cap \overline{D}$. Since $G_0 = H_0$ is $A$-essential w.r.t. $\Gamma_t$ on $S \cap \overline{D}$, so is $G_t = F_0$. Hence, since $F_t$ is also an admissible and $A$-proper homotopy w.r.t. $\Gamma_t$ on $S \cap \overline{D}$, it follows that $F_t = H_t - f$ is $A$-essential w.r.t. $\Gamma_t$ on $S \cap \overline{D}$. Therefore, the conclusions of the theorem follow from Theorem 1.2 with $G = \{e\}$.  

When $H_t$ is just pseudo $A$-proper, we can still get the solvability of $H(1, x) = f$. Let $V \subset X$ be a dense subspace, $D \subset R^m \times X$ be an open subset. We say that $H : [0, 1] \times \overline{D} \cap (R^m \times V) \to Y$ is an $m$-parameter $A$-proper homotopy w.r.t. $\Gamma_m = \{R^m \times X_n, Y_n, Q_n\}$ for $(R^m \times X, Y)$ if whenever $\{(\lambda_{nk}, x_{nk}) | (\lambda_{nk}, x_{nk}) \in \overline{D} \cap R^m \times X_n\}$ is bounded and $t_k \in [0, 1]$ such that $\lambda_{nk} \to \lambda$ and $t_k \to t$ and $Q_n H(t, \lambda_{nk}, x_{nk}) \to f$, then a subsequence $x_{nk(i)} \to x$ and $H(t, \lambda, x) = f$.

Theorem 1.4. Let $D \subset R^m \times X$ be open and bounded, $f \in Y$, and $H : [0, 1] \times \overline{D} \cap (R^m \times V) \to Y$ be an $A$-proper homotopy on $[0, e] \times \partial D \cap (R^m \times V)$ w.r.t. $\Gamma_m = \{R^m \times X_n, Y_n, Q_n\}$ for each $\epsilon \in (0, 1)$. Let $H(1, \cdot, \cdot, \cdot)$ be pseudo $A$-proper w.r.t. $\Gamma_m$, $H(t, \lambda, x)$ be continuous at $1$ uniformly at $(\lambda, x) \in \overline{D} \cap (R^m \times V)$, and

(i) $H(t, \lambda, x) \neq f$ for $(\lambda, x) \in \partial D \cap (R^m \times V)$, $t \in [0, 1]$.
(ii) $H(0, \lambda, x) \neq tf$ for $(\lambda, x) \in \partial D \cap (R^m \times V)$, $t \in [0, 1]$.
(iii) $H_0$ is $A$-essential w.r.t. $\Gamma_m$ on $\overline{D}$.

Then the equation $H(1, \lambda, x) = f$ is solvable.

Proof. Set $H_t = H(t, \cdot, \cdot, \cdot)$. Arguing by contradiction, it is easy to see that the $A$-properness of $H_0$ and (ii) imply that there is an $n_1 \geq 1$ such that

(4) $Q_{n}H(0, \lambda, x) \neq t Q_n f$ for $x \in \partial D \cap (R^m \times V)$, $t \in [0, 1]$, $n \geq n_1$.

Hence, $H_0 - f$ is $A$-essential on $\overline{D}$.  

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Now, let $\epsilon \in (0, 1)$ be fixed. Then, arguing again by contradiction, we see that the $A$-properness of $H$ on $[0, \epsilon] \times \partial D \cap (R^m \times X)$ and (i) imply that there is an $n_2 = n_2(\epsilon) \geq n_1$ such that for each $n \geq n_2$

$$Q_n H(t, \lambda, x) \neq Q_n f \quad \text{for} \quad (\lambda, x) \in \partial D \cap (R^m \times X_n), \quad t \in [0, \epsilon],$$

with $n_1(\epsilon_1) \geq n_2(\epsilon_2)$ whenever $\epsilon_1 > \epsilon_2$. Using this and the homotopy $F_n : [0, 1] \times D \cap (R^m \times X_n) \to Y_n$, given by $F_n(t, \lambda, x) = Q_n H(\epsilon t, \lambda, x) - Q_n f$, we get that $H - f$ is $A$-essential on $D$. Therefore, for each such $n \geq n_2$ there is an $(\lambda_n, x_n) \in D \cap (R^m \times X_n)$ such that $Q_n H(\epsilon, \lambda_n, x_n) = Q_n f$.

Next, let $\epsilon_k \in (0, 1)$ be increasing and $\epsilon_k \to 1$ and $(\lambda_k, x_k) \in D \cap (R^m \times X_k)$ be such that $Q_k H(\epsilon_k, \lambda_k, x_k) = Q_k f$ for $k \geq 1$. By the continuity of $H_t$ at 1 uniformly for $(\lambda, x)$, we get that $Q_n H(1, \lambda_n, x_n) \to f$, and $H(1, \lambda, x) = f$ for some $(\lambda, x)$ by the pseudo $A$-properness of $H_1$. □

Next, we prove an extension of the first Fredholm theorem to nonlinear pseudo $A$-proper maps. We say that a map $A : V \subset X \to Y$ is $\alpha$-positively homogeneous if $A(tx) = t^\alpha A(x)$ for all $x \in V$, $t > 0$, and some $\alpha > 0$.

**Theorem 1.5.** Let $V$ be a dense subspace of $X$ and $A : V \to Y$ be a positively $\alpha$-homogeneous for some $\alpha > 0$ and $A$-essential $A$-proper map w.r.t. $\Gamma = \{X_n, Y_n, Q_n\}$ on $V$. Suppose that $Ax = 0$ implies $x = 0$ and $N : X \to Y$ is quasibounded; i.e.,

$$|N| = \limsup_{\|x\| \to \infty} \|Nx\|/\|x\|^\alpha < \infty$$

and $|N|$ is sufficiently small. Then, if $A + N$ is pseudo $A$-proper w.r.t. $\Gamma$, the equation $Ax + Nx = f$ is solvable for each $f \in Y$.

**Proof.** By Lemma 2.1 in [Mi-1] there is a $c > 0$ and an $n_0 \geq 1$ such that $\|Q_n Ax\| \geq c\|x\|^{\alpha}$ for each $x \in X_n$ and $n \geq n_0$. Let $H_f(t, x) = Ax + tNx - tf$ for each $f \in X$. Then, arguing by contradiction, it is easy to show that there exists an $r_f > 0$ such that $Q_n H_f(t, x) \neq 0$ for all $x \in \partial B(0, r_f) \cap X_n$, $t \in [0, 1]$, and $n \geq n_0$. Since $H_f(0, \cdot) = A$ is $A$-essential, by Theorem 1.1 $H_f(1, \cdot)$ is $A$-essential w.r.t. $\Gamma$ on $S \cap \overline{U}$. Hence, there is an $x_n \in S \cap \overline{U} \cap X_n$ such that $Q_n H(1, x_n) = 0$ for each $n \geq n_0$. Since $H_f(1, \cdot)$ is pseudo $A$-proper, there is an $x \in S \cap \overline{U}$ such that $Ax + Nx = f$. □

**Special cases.** Now, we shall derive several special cases of the above results using either the $G$-degree, or complementing maps or the homotopy degree. The $G$-degree has been defined, studied, and applied by many authors for various groups $G$ (Dancer, Ize, Massabo, Vignoli, Geba and others, see [IMV-1,2], [IV], [GKW] and [I] for references). The $G$-degree has been developed in [IMV-1,2] for general $G$ and is used below.

We have the following continuation result.

**Theorem 1.6.** (a) Let $D$ be an open bounded $G$-invariant subset of $\Lambda \times R^m \times X$ and $H : D \to Y \times R^k$ be a $G$-equivariant and $A$-proper homotopy w.r.t. a $G$-invariant scheme $\Gamma = \{\Lambda \times R^m \times X_n, Y_n \times R^k, Q_n\}$. Suppose that for some $\lambda_0 \in \text{Fix}_A(G)$, $Q_n H(\lambda_0, x) \neq 0$ on $\partial D_{\lambda_0} \cap (\Lambda \times R^m \times X_n) = \partial \{\lambda_0, x\} \in D \cap (\Lambda \times R^m \times X_n)$ and $\deg_G(Q_n H_{\lambda_0}, D_{\lambda_0} \cap (\Lambda \times R^m \times X_n), 0) \neq 0$
for all large \( n \). Assume that the \( G \)-equivariant Freudenthal suspension theorem applies and that \( \Lambda = \text{Fix}_A(G) \oplus \Lambda_1 \) with \( \dim \text{Fix}_A(G) > 0 \). Then there exists a “continuum” \( C \) of solutions of the equation \( H(\lambda, x) = 0 \) with \( \lambda \in \text{Fix}_A(G) \), such that \( C \cap \partial D \neq \emptyset \) and \( C/G \) is connected and has dimension at each point at least \( \dim \text{Fix}_A(G) \).

(b) \([\text{IMPV}]\) let \( V \) be a dense subspace of \( X \), \( B_1(0, R_1) \subset R^m \), \( B_2(0, R_2) \subset X \) be open balls, and let \( H : [0, 1] \times \overline{B}_1 \times (\overline{B}_2 \cap V) \rightarrow Y \) be an \( A \)-proper homotopy w.r.t. \( \Gamma_m = \{R^m \times X_n, Y_n, Q_n\} \) with \( X_n \subset X \) and such that each \( H_t \) is sectionally proper on bounded and closed sets. Suppose that \( Q_n H_0(0, x) \neq 0 \) on \( \partial B_2 \) and the \( m \)th suspension of \( Q_n H_0 \mid \partial B_2 \cap X_n \) is nontrivial for each large \( n \) and for \( D = B_1 \times B_2 \), conditions (i)–(ii) of Theorem 1.4 hold.

Then there is a minimal connected subset \( \Sigma \) of \( S = H^{-1}_1(f) \) that has dimension at least \( m \) at each point.

**Proof.** (a) Define the map \( F(\lambda, x) = (H(\lambda, x), \lambda - \lambda_0) \) and note that \( Q_n F(\lambda, x) \neq 0 \) on \( \partial D \). Hence,

\[
\deg_G(\tilde{Q}_n F, D \cap (\Lambda \times R^m \times X_n), 0) = \Sigma \deg_G(Q_n H_{\lambda_0}, D_{\lambda_0} \cap (\Lambda \times R^m \times X_n), 0) \neq 0.
\]

Thus, the map \( \tilde{Q}_n F \) is \( G \)-epi on \( D \) and then the map \( \lambda - \lambda_0 \) is \( G \)-epi on the set of zeros of the equation \( H(\lambda, x) = 0 \) with \( \lambda \in \text{Fix}_A(G) \) (cf. \([\text{IMV-1}]\)). Now, the conclusions follow from Theorem 1.2 since the notion of a \( G \)-epi map is more general than that of the \( G \)-degree ([IMV-2]). □

Let \( V \) be a dense subspace of \( X \) and \( D \subset R^m \times X \) be open and bounded. Recall that \([\text{FMP-1}]\) a continuous map \( C : D \rightarrow R^m \) is called a complement of \( T : \overline{D} \cap (R^m \times V) \rightarrow Y \) provided that \( \deg(\tilde{Q}_n(T, C), D \cap X_n, 0) \neq 0 \) for all large \( n \), where \( (T, C)(\lambda, v) = (T(\lambda, v), C(\lambda, v)) \in Y \oplus R^m \) is \( A \)-proper w.r.t. \( \tilde{\Gamma} = \{R^m \times V_n, Y_n \times R^m, \tilde{Q}_n\} \) with \( \tilde{Q}_n(y, \lambda) = (Q_n y, \lambda) \) and \( V_n = X_n \cap V \).

When \( D \) is unbounded but \( (T, C)^{-1}(0) \) is bounded, we define

\[
\deg(\tilde{Q}_n(T, C), D \cap X_n, 0) = \deg(\tilde{Q}_n(T, C), U \cap X_n, 0),
\]

where \( U \) is any bounded neighborhood of \( (T, C)^{-1}(0) \) in \( D \).

If \( H_0 \) is a complementing map, then it is \( A \)-essential and Theorem 1.3 holds for such homotopies.

Next, when \( H_1 \) is just pseudo \( A \)-proper, we can still establish the solvability of \( H(1, \lambda, x) = f \).

**Theorem 1.7.** Let \( V \) be a dense subspace of \( X \), \( D \subset R^m \times X \) be open and bounded, \( f \in Y \), and \( H : [0, 1] \times \overline{D} \cap (R^m \times V) \rightarrow Y \). Suppose that \( H \) is an \( A \)-proper homotopy on \([0, \varepsilon] \times \partial D \cap (R^m \times V) \) w.r.t. \( \Gamma_m = \{R^m \times X_n, Y_n, Q_n\} \) for each \( \varepsilon \in (0, 1) \), \( H(1, \cdot, \cdot, \cdot) \) is pseudo \( A \)-proper w.r.t. \( \Gamma_m \); \( H(1, \lambda, x) \) is continuous at 1 uniformly for \((\lambda, x) \in \overline{D} \cap (R^m \times V) \). Let conditions (i)–(ii) of Theorem 1.4 hold as does either one of the following conditions:

(iii) \( H_0 \) is complemented by \( C : \overline{D} \rightarrow R^m \).

(iv) \( H \) is \( G \)-equivariant, \( \Gamma_m \) is \( G \)-invariant, and

\[
\deg_G(Q_n H_0, D \cap (R^m \times X_n), 0) \neq 0.
\]
(v) \( D = B(0, R) \) and \( Q_nH_0 : \partial B(0, R) \cap (R^n \times X_n) \to Y_n \setminus \{0\} \) has the nontrivial stable homotopy for all large \( n \).

Then the equation \( H(1, \lambda, x) = f \) is solvable in \( R^m \times V \).

**Proof.** In either case we have that \( H_0 \) is \( A \)-essential w.r.t. \( \Gamma_m \) and Theorem 1.4 applies. \( \square \)

Our next result is the following continuation theorem for pseudo \( A \)-proper \( G \)-equivariant maps with \( G \) satisfying

(\( G \)) \( G \) is either finite with \( \mu(X) > 1 \) or is an infinite compact Lie group such that \( \text{Fix}(K) = 0 \) for some subtorus \( K \) of \( G \).

**Theorem 1.8.** Let \( R^m \times X \) be a Banach \( G \)-space, \( R^m \) be \( G \)-invariant, \( \Gamma_0 = \{X_n, P_n\} \) be a \( G \)-invariant scheme for \( X \), and condition (\( G \)) hold on \( R^m \times X \).

(a) If \( m = 0 \) and \( H : [0, 1] \times \overline{B}_r \to X \), then \( H(1, x) = 0 \) is solvable if

(i) \( H \) is pseudo \( A \)-proper at 0 w.r.t. \( \Gamma \);

(ii) \( P_nH(t, x) \neq 0 \) for all \( x \in \partial B_r \cap X_n \), \( t \in [0, 1] \), \( n \geq n_0 \geq 1 \);

(iii) \( H_0 : \overline{B}_r \to X \) is \( G \)-equivariant.

(b) If \( m > 0 \), \( T : R^m \times X \to X \) is \( G \)-equivariant and pseudo \( A \)-proper at 0 w.r.t. \( \Gamma_m = \{R^m \times X_n, X_n, P_n\} \), then the equation \( T(\lambda, x) = 0 \) is solvable. If \( T \) is also \( A \)-proper at 0 w.r.t. \( \Gamma_m \), then

\[
Z_r = \{(\lambda, x) \mid \|\lambda, x\| = r, \ T(\lambda, x) = 0\} \neq \emptyset
\]

for each \( r > 0 \).

**Proof.** (a) Let \( G \) be finite and fix \( n \geq n_0 \). Since \( P_nH_0 : \overline{B}_r \cap X_n \to X_n \) is \( G \)-equivariant and \( P_nH_0(x) \neq 0 \) for \( x \in \partial B_r \cap X_n \), we know that [I]

\[
\text{deg}(P_nH_1, B_r \cap X_n, 0) = \text{deg}(P_nH_0, B_r \cap X_n, 0) = 1 + k|G|.
\]

If \( G \) is infinite, since \( X_n \subset X \), we have that

\[
\text{Fix}(K) \cap X_n = 0, \quad \text{and} \quad \text{deg}(P_nH_1, B_r \cap X_n, 0) = 1
\]

for each \( n \geq n_0 \). Hence, there is an \( x \in B_r \cap X_n \) such that \( P_nH_1(x_n) = 0 \). By the pseudo \( A \)-properness of \( H_1 \), the equation \( H(1, x) = 0 \) is solvable in \( \overline{B}_r \).

(b) Let \( r > 0 \) be fixed and suppose that for some \( n \), \( P_nT : \partial B_r \cap (R^n \times X_n) \to X_n \setminus \{0\} \). Let \( \overline{T}_n : \partial B_1 \cap (R^n \times X_n) \to \partial B_1 \cap X_n \) be given by \( \overline{T}_n(x) = P_nT(rx)\|P_nT(rx)\| \) and \( i_n : \partial B_1 \cap R^n \to \partial B_1 \cap (R^n \times X_n) \) be the natural inclusion. Then \( \overline{T}_n \), \( i_n \), and \( \hat{T}_n = i_n\overline{T}_n : \partial B_1 \cap (R^n \times X_n) \to \partial B_1 \cap (R^n \times X_n) \) are \( G \)-equivariant maps, and \( \text{deg}(\hat{T}_n) \neq 0 \). But, since \( \hat{T}_n \) factors through \( \partial B_1 \cap R^n \subset \partial B_1 \cap (R^n \times X_n) \) and \( R^n \neq 0 \), we have that \( \text{deg}(\hat{T}_n) = 0 \). This contradiction shows that \( \{x \in \partial B_1 \cap (T^m \times X_n) \mid P_nTx = 0\} \neq \emptyset \) for each \( n \).

Then the conclusions follow from the (pseudo) \( A \)-properness of \( T \). \( \square \)

Now, regarding the schemes used above, the following result is useful.

**Proposition 1.1.** If \( X \) is a Banach \( G \)-space and separable (\( \pi_k \)-space, respectively), there are finite dimensional \( G \)-invariant subspaces \( X_n \) of \( X \) (with \( X_1 \subset X_2 \subset \cdots \), respectively) such that \( \text{dist}(x, X_n) \to 0 \) as \( n \to \infty \) for each \( x \in X \).

**Proof.** Let \( U_n \) be a sequence of finite dimensional subspaces of \( X \) such that \( \text{dist}(x, U_n) \to 0 \) as \( n \to \infty \) for each \( x \in X \). Define the linear subspaces of
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by \( X_n = G U_n = \{ T_g x \mid x \in U_n, g \in G \} \). If \( S(U_n) \) is the unit sphere in \( U_n \), then \( S(X_n) = GS(U_n) \) is the unit sphere in \( X_n \) since \( G \) acts by isometries on \( X \). Since \( S(X_n) \) is a compact subset of \( X_n \) as a continuous image of the compact set \( G \times S(U_n) \), it follows that \( X_n \) is finite dimensional. Moreover, each \( X_n \) is \( G \)-invariant and \( \text{dist}(x, X_n) \to 0 \) as \( n \to \infty \) for each \( x \in X \) since each \( T_g : X \to X \) is an isometry. Finally, if \( U_1 \subseteq U_2 \subseteq \cdots \), then \( X_1 \subseteq X_2 \subseteq \cdots \). □

B. On the index of the solution set. Our next step is to connect a given representation of \( G \) on \( X \) with an index theory and then estimate the index of the solution set \( Z_r \) from below. Moreover, for some concrete index theories one can then estimate the covering dimension of \( Z_r \) from below.

By an index theory \( I \) on a Banach space \( X \) we mean a triplet \( \{ \Sigma, M, i \} \), where \( \Sigma \) is a family of closed subsets of \( X \) such that \( A \cup B, A \cap B, \) and \( A \setminus B \in \Sigma \) for all \( A, B \in \Sigma \), \( M \) is a set of continuous maps containing the identity and is closed under composition with \( T(A) \in \Sigma \) for each \( A \in \Sigma \) and each \( T \in M \), and \( i : \Sigma \to \mathbb{N} \cup \{ +\infty \} \) is a map having suitable properties ([FR, Be-1]). Let \( \{ T_g \} \) be a representation of \( G \) on \( X \) and define \( \Sigma(T_g) = \{ A \subseteq X \mid A \text{ is closed and } G\text{-invariant} \} \) and \( M(T_g) = \{ h \in C(X, X) \mid h \text{ is } G\text{-equivariant} \} \). An index theory \( \{ \Sigma, M, i \} \) is related to the representation of \( G \) on \( X \) if \( \Sigma = \Sigma(T_g) \) and \( M = M(T_g) \). It is said to have the \( d \)-dimension property if there is a positive integer \( d \) such that \( i(X_{dk} \cap \partial U) = k \) for all \( dk \)-dimensional subspaces \( X_{dk} \in \Sigma \) such that \( X_{dk} \cap \text{Fix}(G) = \emptyset \) and all closed bounded neighborhoods \( U \in \Sigma \) of zero.

In the literature there are many index theories and we mention only the index theory induced by the genus function of Krasnoselskii ([K], [R-1]), the indices in [FR], [Be-1,2], [FHR], [FH], [M-2], [B-1], [LW], etc. (see the references in these works). Let \([r]\) denote the biggest integer \( n < r \).

We need the following finite dimensional result.

**Theorem 1.9.** Let \( X_n \) be a finite dimensional Banach \( G \)-space, \( X_k \) its proper \( G \)-invariant subspace, \( \text{Fix}(G) = 0 \), and \( \{ \Sigma, M, i \} \) an index theory related to the \( G \) representation on \( X_n \) and such that \( i(K) < \infty \) whenever \( K \in \Sigma \) is compact. Suppose that \( \Omega \) is a bounded closed neighborhood of 0 in \( X_n \), \( \partial \Omega \) is \( G \)-invariant, and \( f : \Omega \to X_k \) is continuous and \( G \)-equivariant on \( \partial \Omega \). Then, if \( Z = \{ x \in \partial \Omega \mid f(x) = 0 \} \), \( i(Z) \geq i(\partial \Omega) - i(S') \), where \( S' = \partial B(0, 1) \subseteq X_k \).

**Theorem 1.10.** (a) Let \( X \) be a Banach \( G \)-space, \( T : X \to X \) be \( G \)-equivariant and pseudo \( A \)-proper at 0 w.r.t. a \( G \)-invariant scheme \( \Gamma_m = \{ X_m = X_0 \oplus V_n, V_n, Q_n \} \) for \( X \) with \( \dim X_0 = m > 0 \) and condition (G) hold. Then the equation \( TX = 0 \) is solvable and, if \( T \) is also \( A \)-proper at 0 w.r.t. \( \Gamma_m \), \( Z_r = \{ x \in \partial B_r \mid TX = 0 \} \neq \emptyset \) for each \( r > 0 \).

(b) Let \( X \) and \( Y \) be Banach \( G \)-spaces, \( \text{Fix}(G) = 0 \) in \( X \), \( K : X \to Y \) be a linear \( G \)-equivariant homeomorphism, and let \( \{ \Sigma, M, i \} \) be an index theory related to the \( G \)-representation on \( X \) and having the \( d \)-dimension property. Suppose that \( \Omega \in \Sigma \) is a bounded closed neighborhood of 0 in \( X \) and \( T : \partial \Omega \to Y \) is \( G \)-equivariant and \( A \)-proper at 0 w.r.t. a \( G \)-invariant scheme \( \Gamma = \{ X_n, Y_n = K(X_{n-m}), Q_n \} \) for \( (X, Y) \) with \( m > 0 \). Then, if \( Z = \{ x \in \partial \Omega \mid TX = 0 \} \), \( i(Z) \geq [m/d] \).

**Proof.** Part (a) follows from Theorem 1.8. Let \( n \geq 1 \) be fixed and \( \Omega_n = \Omega \cap X_n \). Then \( \partial \Omega \cap X_n \) is \( G \)-invariant and \( Q_n T : \partial \Omega \subset X \to Y_n \) is \( G \)-equivariant.
Indeed, let $x \in \partial \Omega$ and $g \in G$ be fixed and $y_k \in Y_k$ be such that $y_k \to Tx$ in $Y$. Then
\[ Q_n T(T_g x) = Q_n \overline{T_g} \left( \lim_k y_k \right) = \lim_k Q_n \overline{T_g}(y_k) = \overline{T_g} Q_n T x. \]
Moreover, $K^{-1}Q_n T : \partial \Omega_n \to X_{n-m}$ is $G$-equivariant since such is $K^{-1}$; i.e., $K^{-1}(\overline{T_g y}) = T_g (K^{-1} y)$ for each $y \in Y$, $g \in G$.

Let $Z_n = \{ x \in \partial \Omega_n | K^{-1} Q_n T x = 0 \}$. By Theorem 1.9 we have that $i(Z_n) \geq i(\partial \Omega_n) - i(S_{n-m}) \geq [r(n)/d] - [r(n) - m/d]^+ = [m/d]$, where $r(n) = \dim X_n$, since $[r(n)/d] \leq k$ for some $k$ with $r(n) = dk + l$ and $l \in [0, d)$ and $[(r(n) - m)/d]^+ \geq k_1$ for some $k_1$ with $r(n) - m = dk_1 + l_1$ and $l_1 \in [0, d)$.

Next, since $T$ is continuous and $A$-proper at 0 w.r.t. a projectionally complete scheme, it follows that $T$ is proper on bounded and closed subsets of $X$, and consequently $Z$ is compact in $X$. Hence, $i(Z) < \infty$ and, for some $\delta$-neighborhood of $Z$, $i(N_\delta(Z)) = i(Z)$. Moreover, there is an $n_0 > 1$ such that $Z_n \subset N_\delta(Z)$ for each $n \geq n_0$. If not, then there would exist $x_n \in Z_n \setminus N_\delta(Z)$ such that for each $k \geq 1$, $Q_n X_{x_n} = 0$ and some subsequence $x_{n_k} \to x \in \partial \Omega$ by the $A$-properness of $T$ at 0 with $Tx = 0$. But, $x \in \partial \Omega \setminus N_\delta(Z)$ since $\partial \Omega \cap X_n \setminus N_\delta(Z) = \partial \Omega \setminus N_\delta(Z)$ for each $n$, in contradiction to $x \in Z$. Hence, such an $n_0 > 1$ exists, and for $n \geq n_0$, $i(Z) \geq i(Z_n) \geq [r(n)/d] - [r(n) - m/d]^+ = [m/d]$. 

Now, let us look at some special cases. Let $G = Z_2 = \{1, -1\}$ and let its representation on a real Banach space $X$ be given by: $T_1 x = x$ and $T_{-1} x = -x$ for all $x \in X$. If $A \in \Sigma(T_g) = \{ \text{closed subsets of } X \text{ symmetric with respect to } 0 \}$, we define the genus of $A$, $\gamma(A) = k$, if $k$ is the smallest integer such that there exists a continuous odd map $\phi : A \to R^k \setminus 0$. If such a map does not exist we set $\gamma(A) = \infty$ and set also $\gamma(\emptyset) = 0$. Then it is well known that $\{\Sigma(T_g), M(T_g), \gamma\}$ is an index theory which possesses the dimension property with $d = 1$ (cf. [K, R-1]) and $\dim(A) \geq \gamma(A) - 1$. In this case Theorem 1.10 was obtained by the author [Mi-5] and extends an earlier result of Holm and Spanier [HS] and Rabinowitz [R-2] for compact perturbations of Fredholm maps of positive index.

Next, consider the multiplicative group of complex numbers $G = S^1 = \{ z \in \mathbb{C} | |z| = 1 \}$ and a unitary representation $\{T_z\}$ of this group on a real Hilbert space. For simplicity, we shall write $T_z$ instead of $T_z$ if $z = e^{is}$, $s \in [0, 2\pi)$. If $A \in \Sigma(T_z) = \{ \text{closed } T_z\text{-invariant subsets of } H \}$, we set (Fadell-Rabinowitz [FR], Benci [Be-1]) $\tau(A) = k$ if $k$ is the smallest integer for which there exist an positive integer $n$ and a continuous map $\phi : A \to C^k \setminus \{0\}$ such that $\phi(T_z x) = e^{ins} \phi(x)$ for all $x \in A$ and $s \in [0, 2\pi)$. If such a map does not exist, we set $\tau(A) = +\infty$ and set also $\tau(\emptyset) = 0$. If $X_k$ is an invariant subspace of $H$ with $X_k \cap \text{Fix}(S^1) = \emptyset$, one can show that its dimension is even and ([FR], [Be-1]) that $\{\Sigma(T_z), M(T_z), \tau\}$ is an index theory having the dimension property with $d = 2$.

Finally, consider a normed linear space $X$ over $F = R$ or $C$ or the quaternions $H$, and let $G$ be the unit sphere in $F$. Let $G$ act freely on $X_\ast = X \setminus \{0\}$; i.e., $\text{Fix}(G) = \{0\}$. For $A \in \Sigma(T_g) = \{ \text{closed } G\text{-invariant subsets of } X_\ast \}$, we let $\text{Ind}_F(A)$ be the Fadell-Rabinowitz index of $A$ [FR, p. 148]. Then $\{\Sigma(T_g), M(T_g), \text{Ind}_F\}$ is an index theory such that $\text{Ind}_F(A) \dim F \leq \dim A$. 
for each \( A \in \Sigma(T_g) \) and, if \( X = F^n \), the Euclidean \( n \)-space over \( F \) and \( \bar{U} \in \Sigma(T_g) \) is a bounded neighborhood of 0 in \( X \), then \( \text{Ind}_F \partial U = n \) (see [FR]). Thus, their index satisfies the dimension property in \( X \) over \( F \) with \( d = 1 \). For other examples, see [FHR], [FH], [M-2], [B-1], [LW].

Remark 1.2. The above results can be used to study the index of the solution set of the \( k \)-parameter eigenvalue problem with symmetries \( A x + N(\lambda, x) = 0 \) where \( A : X \to Y \) is a Fredholm map with \( i(A) \geq 0 \) and \( N : R^k \times X \to Y \) is a nonilinear \( G \)-equivariant map (cf. [Mi-6] for details).

2. Applications

A. Nonlinear perturbations of Fredholm maps of nonnegative index. Throughout this section we assume that \( A : D(A) \subset X \to Y \) is a Fredholm map of nonnegative index, \( X_0 = \ker A \), \( Y = R(A) \), and \( \bar{X} \) and \( Y_0 \) are closed subspaces of \( X \) and \( Y \), respectively, such that \( X = X_0 \oplus \bar{X} \) and \( Y = Y_0 \oplus \bar{Y} \) and \( i(A) = \text{dim} X_0 - \text{dim} Y_0 = m \). Let \( \Gamma_m = \{ X_n, Y_n, Q_n \} \) be a projection scheme for \( (X, Y) \) with \( X_0 \subset X_n \), \( Q_n(\bar{Y}) \subset \bar{Y} \), and \( \text{dim} X_n - \text{dim} Y_n = m \) for each \( n \). Decompose \( X_0 \) as \( X_0 = W \oplus Z \) with \( \text{dim} W = m \) and \( \text{dim} Z = \text{dim} Y_0 \). Let \( Y_0 \subset Y_n \), \( Q_n Ax = Ax \) for \( x \in X_n \), and \( Q_n y \to y \) for each \( y \in Y \). Let \( P : X \to X_0 \) and \( Q : Y \to Y_0 \) be linear projections onto \( X_0 \) and \( Y_0 \) respectively. For each \( x \in X \), we have the unique decomposition \( x = x_0 + x_1 \), \( x_0 \in X_0 \), \( x_1 \in \bar{X} \). Set \( V = Z \oplus \bar{X} \), \( \bar{X} = W \oplus V \), and \( V_n = X_n \cap V \).

If we try to establish continuation results of the type discussed in Section 1, we find that \( \deg(QN, D \cap X_0, 0) = 0 \) since \( QN : D \cap X_0 \to Y_0 \) and \( \text{dim} Y_0 < \text{dim} X_0 \). To overcome this difficulty, we shall present now, in the context of \( A \)-proper maps, the \( A \)-essential mapping approach of the study of semilinear equations as developed by the author [Mi-4-6]. In particular, we shall apply it to complementing and \( G \)-equivariant maps. For compact nonlinearities, we refer to Nirenberg [Ni-1], Berger [Be], [MR].

Now, we shall apply Theorem 1.3 to homotopies of the form \( H(t, x) = Ax + F(t, x) \), where \( A : D(A) \subset X \to Y \) is as above. Let \( D \) be an open and bounded subset of \( X \).

Theorem 2.1. Let \( A : D(A) \subset X \to Y \) be Fredholm of index \( i(A) = m \geq 0 \), and \( F : [0, 1] \times X \to Y \) be nonlinear such that \( H(t, x) = Ax + F(t, x) \) is a sectionally proper \( A \)-proper homotopy on \( [0, 1] \times (\bar{D} \cap D(A)) \) w.r.t. \( \Gamma_m = \{ X_n = W \oplus V_n, Y_n, Q_n \} \). Suppose that

(i) \( A(x) + F(t, x) \neq f \) for \( x \in \partial D \cap D(A), \ t \in [0, 1] \),
(ii) \( F(0, \cdot) : \partial D \cap X_0 \to Y_0 \)
(iii) \( F(0, x) \neq t f_0 \) for \( x \in \partial D \cap X_0, \ t \in [0, 1] \),
and either one of the following conditions holds:

(iv) \( m = 0 \) and \( \deg(F(0, \cdot), D \cap X_0, 0) \neq 0 \).
(v) \( m = 0 \), \( X_0 \) and \( Y_0 \) are \( G \)-spaces, \( F(0, \cdot) : \partial D \cap X_0 \to Y_0 \) is continuous and \( G \)-equivariant with \( G \) satisfying condition (G) in \( X_0 \).

(vi) \( m > 0 \), (i)-(iii) hold with \( D_0 = U \cap (Z \times \bar{X}) \) for some open \( U \subset X \) such that the set \( \{ (t, x) \in [0, 1] \times (D \cap D(A)) \mid Ax + F(t, x) = f \} \) is relatively compact and \( \deg(F(0, \cdot), D_0 \cap Z, 0) \neq 0 \).

(vii) \( m > 0 \), \( \bar{D} = \{ x = x_0 + x_1 \mid \| x_0 \| \leq r, \| x_1 \| \leq R \} \) for some \( r, R > 0 \), and \( F(0, \cdot) : \partial B(0, r) \cap X_0 \to Y_0 \setminus \{ 0 \} \) has nontrivial stable homotopy.
Then the equation \( Ax + F(1, x) = f \) is approximation-solvable in \( \overline{D} \cap D(A) \) when \( m = 0 \). If \( m > 0 \), then there exists a connected subset \( C \) of \( S = \{ x \in D(A) \mid Ax + F(1, x) = f \} \) whose dimension is at least \( m \) at each point and either \( C \) is unbounded or \( C \cap \partial D \neq \emptyset \). Moreover, if \( D = X \) and if \( \{ x_n \} \subset S \) is bounded whenever \( \{ p_n x_n \} \) is bounded, where \( p_n : X = W \oplus V \rightarrow W \) is the projection onto \( W \), then \( C \) covers \( W \) in the sense that \( p_n(C) = W \).

**Proof.** It is clear that conditions (ii) and (iii) imply condition (ii) of Theorem 1.3 for \( H_1 = A + F_t \). It remains only to check condition (iii) of this theorem. Since \( X_0 \subset X_n \) and \( Y_0 \subset Y_n \), we have that \( Q_n H(0, \cdot) = A + F(0, \cdot) \) on \( \overline{D} \cap X_n \).

Define \( F_n(t, x) = A x + F(0, x_0 + t x_1) \) for \( t \in [0, 1] \) and \( x \in \overline{D} \cap X_n \), where \( x = x_0 + x_1 \) with \( x_0 \in X_0 \), \( x_1 \in X \). Since \( \{ x \in \overline{D} \cap D(A) \mid Ax + F(0, x) = 0 \} = \{ x = x_0 \mid x_0 \not\in \partial D \cap X_0 \} \) by (ii)-(iii), it follows that \( F_n(t, x) \neq 0 \) for \( t \in [0, 1] \), \( x \in \partial D \cap X_n \) and therefore

\[
\deg(A + F(0, \cdot), D \cap X_n, 0) = \deg(A + F(0, P_{\cdot}), D \cap X_n, 0).
\]

Next, we have the decomposition \( X_n = X_0 \times X_1_n \), with \( X_1_n = X_n \cap \tilde{X} \), and let \( L_0 : Y_0 \rightarrow X_0 \) be a linear homeomorphism. Then \( L = (A^{-1}, L_0) : Y_1_n \times Y_0 \rightarrow X_1_n \times X_0 \) is a linear homeomorphism and \( LF_{n, 0}(x_0, x_1) = (x_1, L_0 F(0, x_0)) \).

Hence, by the properties of Brouwer's degree we get

\[
\deg(LF_{n, 0}, D \cap (X_1_n \times X_0), 0)
= \deg(L, L^{-1}(D \cap (X_1_n \times X_0)), 0) \deg(F_{n, 0}, D \cap (X_0 \times X_1_n), 0).
\]

Hence, if (iv) holds, then

\[
\deg(A + F(0, P_{\cdot}), D \cap X_n, 0) = \pm \deg(LF_{n, 0}, D \cap (X_0 \times X_1_n), 0)
= \pm \deg(F(0, \cdot), D \cap X_n, 0) \neq 0.
\]

If (v) holds, then it implies (iv) by the generalized Borsuk theorem [N-2]. Hence, condition (iii) of Theorem 1.3 is satisfied.

Let (vi) hold. By [FMP-1], we only need to show that \( H(1, \cdot) = A + F(1, \cdot) - f : D_0 \cap D(A) \subset W \times V \rightarrow Y \) is complemented by \( p_0 : X = W \times V \rightarrow W \) given by \( p(w, v) = w \). By Proposition 3.1 in [FMP-1], this will be so if \( \deg(Q_n H(1, \cdot), D_0 \cap V_n, 0) \neq 0 \) for all large \( n \). We have that

\[
\deg(Q_n H(1, \cdot), D_0 \cap V_n, 0) = \deg(A + Q_n F(0, \cdot), D_0 \cap V_n, 0).
\]

As above, we get, if \( P_0 : X \rightarrow Z \) is a linear projection onto \( Z \), then

\[
\deg(A + F(0, \cdot), D_0 \cap V_n, 0) = \deg(A + F(0, P_0 \cdot), D_0 \cap V_n, 0) \neq 0.
\]

Let (vii) hold. Write \( X_0 = R^k \times Z \) with \( \dim Z = \dim Y_0 \). By (ii)-(iii) we have that \( H(0, x) \neq tf \) for all \( x \in \partial D \cap D(A), t \in [0, 1] \). As above, we have that \( Q_n H_0 = A + F_0 \) is homotopic to \( A + F_0 P \) on \( \overline{D} \cap X_n \) for each large \( n \) with \( A + F_0 P : \partial D \cap X_n \rightarrow Y_n \setminus \{0\} \). As in the proof of Theorem 3.1 in [Mi-5], using (vii) we get the last map is essential on \( \overline{D} \cap X_n \). Hence, \( H_0 \) is \( A \)-essential w.r.t. \( \Gamma_m \) on \( \overline{D} \) and Theorem 1.3 applies. \( \square \)

**Corollary 2.1 [Mi-3,4,5,8].** Let \( A + N : \overline{D} \cap D(A) \subset X \rightarrow Y \) be sectionally proper and \( A \)-proper w.r.t. \( \Gamma \) with \( N \) bounded and nonlinear. Let \( f = f_0 + f_1 \in Y_0 \oplus \tilde{Y} \) and
(i) $A(x) + tNx \neq f$ and either $Ax \neq f_1$ or $QN x \neq f_0$ for $x \in \partial D \cap D(A), t \in [0, 1]$.
(ii) $QN x \neq f_0$ for $x \in \partial D \cap X_0, t \in [0, 1]$.

(a) Let $m = 0$ and either $\deg(QN, D \cap X_0, 0) \neq 0$, or $X_0$ and $Y_0$ are $G$-spaces and $QN : \partial D \cap X_0 \to Y_0$ is continuous and $G$-equivariant. Then (3) is approximation solvable.

(b) Let $i(A) = m > 0$ and $QN : \partial B(0, r) \cap X_0 \to Y_0 \setminus \{0\}$ have a nontrivial stable homotopy. Then there is a minimal connected subset $\Sigma$ of $(A + N)^{-1}(f)$ whose dimension is at least $m$ at each point.

When $A + F_1$ is pseudo $A$-proper, we have

**Theorem 2.2.** Let $A : D(A) \subset X \to Y$ be Fredholm with $i(A) = 0$, $F : [0, 1] \times X \to Y$ be nonlinear, $QnH_t = A + QnFt$ be continuous on $\overline{D} \cap D(A) \cap X_n$ for $t \in [0, 1]$, $n \geq n_0$, and $H_t : \overline{D} \cap D(A) \subset X \to Y$ be pseudo $A$-proper w.r.t. $\Gamma_0$ with $X_0 \subset X_n$, $Y_0 \subset Y_n$, and $Qn(\overline{Y}) \subset \overline{Y}$. Assume

(i) $A(x) + QnF(t, x) \neq tQnF$ for $x \in \partial D \cap X_n, t \in [0, 1], n \geq n_0$.

(ii) $F_0(D \cap X_0)$ is $0$.

(a) Let $m = 0$ and either $\deg(QN, D \cap X_0, 0) \neq 0$, or $X_0$ and $Y_0$ are $G$-spaces and $F_0 : \partial D \cap X_0 \to Y_0$ is continuous and $G$-equivariant for some group $G$ satisfying condition (G) in $X_0$.

Then the equation $Ax + F(t, x) = f$ is solvable in $D \cap D(A)$.

**Proof.** As in the proof of Theorem 2.1, we have in either case that

$$\deg(A + F(0, \cdot), D \cap X_n, 0) = \deg(A + F(0, P, \cdot), D \cap X_n, 0) \neq 0$$

for $n \geq n_0$. Hence, for each $n \geq n_0$, $\deg(A + QnF(1, \cdot), D \cap X_n, QnF) \neq 0$ and the conclusion follows from the pseudo $A$-properness of $A = F(1, \cdot)$. □

**B. $k$-parameter semilinear equations.** Let $A : X \to Y$ be a linear Fredholm map, $N : D \subset R^k \times X \to Y$ a nonlinear map, and $m = i(A) + k \geq 0$ with $k \geq 0$. In this section we shall study the solvability and the covering dimension of the solution set of the $k$-parameter equations (2).

As before, we have the direct sums $X = X_0 \oplus \tilde{X}$ and $Y = Y_0 \oplus \tilde{Y}$ with $X_0 = \ker A$ and $\tilde{Y} = R(A)$. Define $A_k : R_k \times X \to Y$ by $A_k(\lambda, x) = Ax$. Then (2) is equivalent to the equation

$$A_k(\lambda, x) + N(\lambda, x) = f.$$ 

Decompose $X_0 = W \oplus Z$ with $\dim Z = \dim Y_0$, $Z \subset X_0$ and set $V = Z \oplus \tilde{X}$. Let $\Gamma_0 = \{V_0, Y_0, Q_0\}$ be an admissible scheme for $(V, Y)$ and $\Gamma_m = \{R^k \oplus W \oplus V_n, Y_n, Q_n\}$, $m = \dim W + k$. Let $Q : Y \to Y_0$ be a linear projection onto $Y_0$, and for $(\lambda, x) = (\lambda, x_0 + x_1) \in R^k \oplus (X_0 \oplus \tilde{X})$, we define $||A_k(\lambda, x)||_1 = \max\{||A + x_0||, ||x_1||\}$. Let $\delta = \sup ||Qn||$. Suppose that

(7) For some $f_0 \in Y_0$ there are constants $M > 0$, $K > 0$, and $\rho \geq 0$ such that for $||x_2|| \leq r, r > K$, $||z|| \geq rM + \rho, z \in Z$, $QN(z + x_1) \neq f_0$.

(8) $S = (I - Q)N : R^k \times X \to Y$ is quasibounded, i.e.,

$$|S| = \lim \sup_{||A_k(\lambda, x)||_1 \to \infty} \frac{||S(\lambda, x)||}{||A_k(\lambda, x)||_1} < \infty$$

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and \( \delta |S| \max\{1, M\} < c \), where \( \|Q_n A x_1\| \geq c \|x_1\| \) for \( x_1 \in \tilde{X} \).

**Theorem 2.3.** Let \( m = i(A) + k \geq 0 \), \( N : R^k \times X \to Y \) be a nonlinear map, and \( \Gamma_m = \{ R^k \oplus W \oplus V_n, Y_n, Q_n \} \) be a scheme for \( (R^k \times X, Y) \) with \( Z \subset V_n \), \( Y_0 \subset Y_n \), and \( Q_n(\tilde{Y}) \subset \tilde{Y} \) for \( n \geq 1 \). Suppose that (7)–(8) hold and

(9) For \( r > K \) sufficiently large and \( \tilde{D}_0 = \{ z \in Z \| \|z\| \leq rM + \rho \} \),
\[ \deg(Q_n, \tilde{D}_0, 0) \neq 0. \]

(a) If \( A + N \) is pseudo A-proper w.r.t. \( \Gamma_m \) and \( m \geq 0 \), then (2) is solvable for each \( f \in f_0 \oplus \tilde{Y} \).

(b) If \( A + N \) is sectionally proper and A-proper w.r.t. \( \Gamma_m \) and \( m > 0 \), then for each \( f \in f_0 \oplus \tilde{Y} \) there exists an unbounded connected closed subset \( C \subset \{ (\lambda, x) \in R^k \times X | Ax + N(\lambda, x) = f \} \) whose dimensional is at least \( m \) at each point and intersects \( W \).

(c) Let \( m = 0 \), \( A + N \) be pseudo A-proper w.r.t. \( \Gamma_m \) and, instead of (9), let \( X_q \) and \( Y_q \) be G-spaces and for \( r > K \) sufficiently large, \( Q_n : \partial B(0, rM + \rho) \cap X_0 \to Y_0 \) be continuous and G-equivariant for some \( G \) satisfying condition (G) in \( X_0 \). Then (3) is solvable for each \( f \in f_0 \oplus \tilde{Y} \).

**Proof.** Let \( f \in f_0 \oplus \tilde{Y} \) be fixed and \( N_f x = N x - f \). We shall show that
\[ \deg(Q_n(A + N_f) | V_n, V_n, 0) \neq 0 \]
for all large \( n \). Arguing by contradiction and using arguments similar to [Mi-2], we get that there is an \( r_0 \) large and \( n_0 \geq 1 \) such that whenever \( Q_n A x + i(Q_n(I - Q)N x) = i Q_n f_t \) for some \( t \in [0, 1] \), \( x = z + x_1 \in X_n \), \( n > n_0 \), with \( \|z\| \leq rM + \rho \) and \( \|x_1\| = r \) for some \( r \) then \( r \leq r_0 \).
Define \( H : [0, 1] \times V \to Y \) by \( H(t, v) = Av + t(I - Q)N v + Q N(z + t x_1) \) and let \( P : V \to Z \) be a linear projection onto \( Z \). For \( r > r_0 \), set \( D_t = \{ v = z + x_1 \| \|x_1\| \leq r, \|z\| \leq rM + \rho, z \in Z \} \). Then \( Q_n H(t, v) \neq t Q_n f \) for \( v \notin D \cap V_n \), \( t \in [0, 1] \), and \( n > n_0 \), and \( \deg(Q_n(A + N_f) | V_n, V_n, 0) \neq 0 \) for \( n \geq n_0 \). Thus, by Proposition 3.1 in [FMP-1], \( T = A + N_f : R^k \oplus W \oplus V \to Y \) is complemented by the projection \( \tilde{P} : R^k \oplus W \oplus V \to R^k \oplus W \) along \( V \). Hence, (a) follows from the pseudo A-properness of \( A + N \), while (b) follows from Theorem 1.3. Part (c) follows from Theorem 2.2. □

The above arguments combined with the arguments of the proof of Theorem 6 in [Mi-2] show that if \( S = (I - Q)N \) has a sub(linear) growth (i.e. there are constants \( \alpha \geq 0 \), \( \beta \geq 0 \), and \( \gamma \in [0, 1] \) with \( \beta \delta < c \) if \( \gamma = 1 \), such that \( \|S(\lambda, x)\| \leq \alpha + \beta \|x\|^\gamma \) for all \( (\lambda, x) \in R^k \times X \) \( ) \), then condition (7) in Theorem 2.3 can be relaxed to

(10) For some \( f_0 \in Y_0 \) there are constants \( M > 0 \) and \( K > 0 \) such that for each \( \|x_1\| \leq r \), \( r > K \), and each \( \rho \geq r \rho(r) \) for some \( \rho(r) \geq M \)
\[ Q N(\rho z + \rho^\gamma x_1) \neq t f_0 \]
for \( z \in \partial B(0, 1) \cap Z \), \( t \in [0, 1] \).

**Remark 2.1.** When \( k = 0 \), \( i(A) > 0 \), and \( N \) is continuous and uniformly bounded (i.e., \( \|N\| \leq C \) for all \( x \)), the solvability of \( Ax + N x = 0 \) in Theorem 2.3 was given in Nirenberg [Ni-1] (cf. also [Cr]). When \( k = i(A) = 0 \), Theorem 2.3 was obtained by the author [Mi-2, 4]. For other special cases, see [Mi-6].

The next resonance conditions proved to be useful in studying the solvability of (3) with \( i(A) = 0 \) [N, F, Fi, Ma-2, Pe-2, Mi-2, 3, 4] and we shall now show
their usefulness in the present situation. Define a continuous bilinear form \([\cdot, \cdot]: Y \times Z \to R\) such that \(y \in \tilde{Y}\) if and only if \([y, z] = 0\) for each \(z \in Z\). As in Mawhin [Ma-2], if \(\phi_1, \ldots, \phi_n\) is a basis for \(Z\), then the linear map \(J: Y_0 \to Z\) given by \(Jy_0 = \sum^n_j [y_0, \phi_j] \phi_j\) is an isomorphism, \([J^{-1}\phi_i, \phi_j] = \delta_{i,j}\), the Kronecker delta, and \([J^{-1}z, \phi_i] = c_i\) for \(1 \leq i \leq n\), where \(z = \sum^n_i c_i \phi_i\). If \((Y, (,))\) is a Hilbert space and \(M: Z \to Y_0\) a linear isomorphism, then we can define \([y, z] = (y, Mz)\).

Let \(f_0 \in Y_0\) be fixed and consider the following conditions.

11) \(N: R^k \times X \to Y\) is asymptotically zero (i.e.,
\[
\|N(\lambda, x)\|/\|\lambda, x\| \to 0 \text{ as } \|\lambda, x\| \to \infty
\]
and either (i) \(\liminf[Nv_n, z] < [f_0, z]\), or (ii) \(\limsup[Nv_n, z] > [f_0, z]\)
whenever \(\{v_n\} \subset V\) is such that \(\|v_n\| \to \infty\) and \(v_n\|v_n\|^{-1} \to z \in Z\).

12) \(N\) has a sublinear growth (i.e., for some \(\alpha \geq 0, \beta \geq 0, \) and \(\gamma \in (0, 1)\),
\[
\|N(\lambda, x)\| \leq \alpha + \beta \|\lambda, x\|^{\gamma}
\]
and either (i)
\[
\liminf[N(\rho_n z_n + \rho_n^2 v_n), z] < [f_0, z],
\]
or (ii)
\[
\limsup[N(\rho_n z_n + \rho_n^2 v_n), z] > [f_0, z]
\]
whenever \(\rho_n \to \infty, \{v_n\} \subset \tilde{X}\) is bounded, and \(\{z_n\} \subset Z\) is such that \(z_n \to z \in \partial B(0, 1)\).

13) \(N\) has a sublinear growth and either (i) \(\liminf[Nv_n, u_n] < [f_0, z]\), or (ii) \(\limsup[Nv_n, u_n] > [f_0, z]\)
whenever \(\{v_n\} \subset V\) is such that \(\|Pv_n\| \to \infty\), \(\|(I - P)v_n\|/\|Pv_n\|^{\gamma}\) is bounded, and \(u_n = Pv_n/\|Pv_n\| \to z \in Z\), where \(P\) is a linear projection of \(V\) onto \(Z\).

14) Suppose that \(X\) is a vector subspace of \(Y\), \([\cdot, \cdot]: V \times Y \to R\) is a positive bilinear form such that if \(\{v_n\} \subset V\) and \(v_n \to v_0\), then \(\liminf[v_n, x_0] \geq [v_0, x_0]\) and \(Y = Z \oplus \tilde{Y}\), where \(\oplus\) denotes orthogonal direct sum with respect to \([\cdot, \cdot]\). Let \(N\) have a linear growth, i.e., \(\gamma = 1\) in (12), and either (i) \(\liminf[Nv_n, u_n] < [f_0, z]\), or (ii) \(\limsup[Nv_n, u_n] > [f_0, z]\)
whenever \(\{v_n\} \subset V\), \(\|v_n\| \to \infty\), and \(\limsup[\|(I - P)v_n\|/\|Pv_n\| \leq \beta (c^2 - \beta^2)^{-1/2}, where\)
\[
u_n = Pv_n/\|Pv_n\| \to z \in Z\) and \(c\) is the largest positive constant such that \(c\|x\| \leq \|Ax\| \text{ for } x \in R(A)\) and \(\beta \in (0, c)\).

15) (Antipodes condition) For a given \(f_0 \in Y_0\) there exist constants \(M \geq 0, K \geq 0, \) and \(\rho_0 \geq 0\) such that for each \(\|x_1\| \leq r, r > K, z \in \partial B(0, 1) \cap Z, \) and \(\rho \geq r + M\rho_0\)
\[
QN(\rho z + x_1) - f_0 \neq \mu QN(-\rho z - x_1) - \mu f_0\text{ for } \mu \in [0, 1].
\]

We shall also need that the scheme \(\Gamma_m\) for \((R^k \times X, Y)\) has the following
Property (P) [Mi-4] (i) \(Y_0 \subset Y_n\) and \(Q_n(A + C)x = (A + C)x\) for \(x \in V_n, n \geq 1, \) where \(C: V \to Y\) is linear and \(B = A + C: V \to Y\) is bijective;
(ii) \([Q_n y, z] = [y, z]\) for each \(y \in Y, z \in Z\).

Various schemes having Property (P) have been discussed in [Mi-3, 4]. For maps satisfying the above conditions we have

**Theorem 2.4.** Let \(A: X \to Y\) and \(N: R^k \times X \to Y\) with \(m = i(A) + k \geq 0, \Gamma_m\) be as in Theorem 2.3, and either one of conditions (11)–(14) holds or (8) and (15) hold for some \(f_0 \in Y_0\). Let \(f \in f_0 \oplus \tilde{Y}\).
(a) If $A + tN$ is sectionally proper and $A$-proper w.r.t. $\Gamma_m$ for $t \in [0, 1]$, $m > 0$ and either one of conditions (11)–(14) holds, or if $A + N$ is $A$-proper and (8) and (15) hold, then there is an unbounded connected closed subset $K$ of \{$(\lambda, x)|Ax + N(\lambda, x) = f$\} whose dimension is at least $m$ at each point and intersects $W$.

(b) If $A + N$ is pseudo $A$-proper w.r.t. $\Gamma_m$ and $m \geq 0$, then (2) is solvable if either $A|_V$ is $A$-proper w.r.t. $\Gamma_m$ having Property (P) and either one of conditions (11)–(14) holds, or (8) and (15) hold.

**Proof.** Let $f \in f_0 + \hat{Y}$ be fixed, $N_f(\lambda, x) = N(\lambda, x) - f$, $C = \pm J^{-1}P: V \to Y_0$, and $H(t, v) = Av + tCv + (1 - t)Nv$ for $t \in [0, 1]$ and $v \in V = Z \oplus \hat{X}$.

Assume first that one of (11)–(14) holds. Arguing by contradiction as in [Mi-2] and using Property (P) we find an $R > 0$ such that for each $r \geq R$ there are $r_0 > 1$ such that for $n \leq n_0$

$$\|Q_nf(t, v) - (1 - t)Q_n f\| \geq \gamma \quad \text{for} \quad v \in \partial B(0, r) \cap V_n, \quad t \in [0, 1].$$

Hence, $\deg(Q_n(A + N_f)\mid V_n, 0) \neq 0$ for each $n \geq n_0$. If $A + N$ is $A$-proper, it follows as before that $A + N$ is complemented by the projection $\hat{P}$ of $R^m \oplus W \oplus V$ onto $R^k \oplus W$ and so $A$-essential. Thus, (a) follows from Theorem 1.3. If $A + N$ is pseudo $A$-proper, then by Proposition 3.1 in [FMP-1] we get that $\deg(Q_n(\hat{P}, A + N_f), B(0, r) \cap V_n, 0) \neq 0$ for each $n \geq n_0$ and therefore there exists a $v_n \in B(0, r) \cap V_n$ such that $Q_n A v_n + Q_n N v_n = Q_n f$.

Hence, $A v + N v = f$ for some $v$ by the pseudo $A$-properness of $A + N$.

Now, assume that (8) and (15) hold. Let $f = f_0 + f_1$ be fixed and $R_0 > K$ such that $\|Sv\| < S \|v\|_1$ for each $\|v\|_1 \geq R_0$. Again, arguing by contradiction [Mi-2], we find an $R > R_0$ such that if, for some $I \in [0, 1]$, $n \geq n_0$, and $v \in V_n$ with $v = v_0 + v_1$, $\|v_0\| \leq rM + \rho_0$ and $\|v_1\| \leq k$ we have that

$$Q_n(I - Q)(Av - Nv - f) = \mu Q_n(I - Q)(-Av - N(-v) - f)$$

then $r \leq R$. For $k \geq R$, set $D = \{v = v_0 + v_1 | v_0 \in Z, \|v_0\| \leq kM + \rho_0, \|v_1\| \leq k\}$. Let $H: [0, 1] \times D \to Y$ be given by

$$H(t, v) = Av - 1/(1 + t)Nfv - t/(1 + t)Nf(-v)$$

with $Nfv = Nv - f$. Then $Q_nH(t, v) \neq tQ_nf$ for $v \in V_n \setminus D$, $t \in [0, 1]$, and $n \geq n_0$. We get that $\deg(Q_n(A + N_f), D \cap V_n, 0) = \deg(Q_nH_1, D \cap V_n, 0) \neq 0$ for each $n \geq n_0$ since $H_1$ is an odd map. Hence, $A + N_f$ is complemented by the projection $\hat{P}: R^k \oplus W \oplus V \to R^k \oplus W$ if $A + N$ is $A$-proper. Then the conclusions of the theorem follow as for Theorem 2.3. \qed

When $k = i(A) = 0$, variants of (12) were first used by Necas [N], de Figueiredo [F], and Mawhin [Ma-2], while (11) and (13)–(14) were used by Fitzpatrick [Fi] in his study of (3) involving condensing maps and are improvements of the earlier conditions of Necas [N], Fucik [Fu-1], and Fucik-Kucera-Necas [FKN]. Still in this case, (11)–(12) and/or a variant of (13) were used in the study of (3) in the $A$-proper setting by the author [Mi-2-4] and Petryshyn [Pe-2]. Theorem 2.4 extends some of the results of these authors when $k = i(A) = 0$.

If $S = (I - Q)N$ has a (sub)linear growth, i.e. satisfies the inequality in (12) with $\delta \beta < c$, if $\gamma = 1$, then (15) in Theorem 2.4 can be relaxed to
For some $t_0 \leq t_0$ there are constants $M > 0$ and $K \geq 0$ such that for each $\|x_1\| < r$, $r > K$, each $\rho \geq r\rho(r)$ for some $\rho(r) \geq M$, and $z \in \partial B(0, 1) \cap N$

\begin{equation}
QN(\rho z + \rho^2 x_1) - f_0 \neq \mu QN(-\rho z - \rho^2 x_1) - \mu f_0 \quad \text{for } \mu \in [0, 1].
\end{equation}

**Remark 2.2.** Inequalities (15) and (17) are valid in particular if, for all such $\rho$, $z$, $x_1$, $\gamma$, and $\mu$ and some linear isomorphism $J : Z \to Y_0^*$ (with $\gamma = 0$ in case of (15))

\begin{equation}
(18) \quad (QN(\rho z + \rho^2 x_1) - f_0, Jz) \neq 0.
\end{equation}

It is clear that (18) is implied by either one of the inequalities

\begin{align}
(19) &\quad (QN(\rho z + \rho^2 x_1), Jz) < (f_0, Jz), \\
(20) &\quad (QN(\rho z + \rho^2 x_1), Jz) > (f_0, Jz).
\end{align}

When $i(A) = k = 0$ and (19) or (20) hold, the theorem was proved by the author [Mi-2] using the Leray-Schauder fixed point theorem in finite dimensions (see also Jarusek-Necas [JN]). Still in this case, the approximation solvability of (3) was proved in [Pe-2] under much stronger conditions on $H(t, x)$, which extends the earlier results of Mawhin [Ma-1] and Hetzer [H-1] when $Y = X$ and $N$ is $A$-compact (condensing, respectively). The solvability of (3) with $i(A) = 0$ and $S$ sublinear and $N$ $k$-set-contractive, $k < 1$, was proved by Tarafdar [T] using different types of arguments.

**Remark 2.3.** Using the full force of Theorem 1.2 in [FMP-1], if we also know that the projection $\tilde{P}$ of $W \oplus V$ onto $W$ is proper, then in the results of this section with $m > 0$ and $A + N$ $A$-proper we can also assert that $\tilde{P}(C) = W$ for the connected component of solutions of (3). This observation will be useful for our applications in subsection C.

### C. Applications to BVP’s for differential equations.

Let $Q \subset \mathbb{R}^n$ be an open and bounded region with smooth boundary. For $p \in (1, \infty)$ and some integers $k$, $m \geq 1$, we denote by $W_p^m(Q, \mathbb{R}^k)$ the Sobolev space of $\mathbb{R}^k$-valued functions. In this section we shall illustrate how some of the abstract results can be applied to the solvability of the BVP

\begin{equation}
\sum_{|\alpha| \leq m} A_\alpha(x)D^\alpha u(x) + F(x, u, Du, \ldots, D^m u) = 0 \quad \text{in } Q,
\end{equation}

\begin{equation}
B_j(u) = 0 \quad \text{on } \partial Q, \quad j = 1, 2, \ldots, r.
\end{equation}

Assume that the linear part is elliptic. We shall consider BVP (21) with symmetries as well as when the associated operator can be complemented.

(A) Let us look at BVP (21) with symmetries. Suppose that $V = \mathbb{R}^k$ is an orthogonal real representation of a compact Lie group $G$, whose action is $\{A_g | A_g = \text{orthogonal real } k \times k\text{-matrix}, \ g \in G\}$. Then $W_p^m(Q, V)$ has a structure of a Banach $G$-space with the linear action of $G$ on it being given by $(T_g u)(x) = A_g(u(x))$, $x \in Q$, $g \in G$. We can identify $W_p^m(Q, V)$ with a tensor product of Banach $G$-spaces $W_p^m(Q, R) \otimes V$, and the left factor is a trivial $G$-space.

Suppose that for each $|\alpha| \leq m$, $A_\alpha(x)$ are sufficiently smooth maps from $Q$ into Hom$_R(V, V)$, the space of $k \times k$ real matrices, and the $B_j$’s are boundary operators on $\partial Q$ of order less than $m$. Denote by $s_m$ the number of different derivatives $D^\alpha$ with $|\alpha| \leq m$ and $s_m' = s_m - s_{m-1}$. Define $X = \ldots$
$W^m_p(Q, V; \{B_j\}) = \{u \in W^m_p(Q, V) \mid B_j u = 0 \text{ on } \partial Q \text{ for } j = 1, \ldots, r\}$ and $Y = L_p(Q, V)$. Suppose that

(22) For each $|\alpha| \leq m$, the matrix-valued function $A_\alpha(x)$ is $G$-equivariant; i.e., $A_\alpha(x)A_g = A_gA_\alpha(x)$ for each $x \in Q$, $g \in G$.

(23) The boundary conditions $\{B_j(u) = 0\}$ are $G$-equivariant; i.e., $B_j(A_gu) = A_gB_j(u)$ for all $g \in G$, $j = 1, \ldots, r$.

(24) For each $v \in W^m_p$, the map $u \rightarrow F(x, u, \ldots, D^{m-1}u, D^mv)$ is continuous from $W^m_p$ to $L_p$.

(25) For all $x \in Q$, $\eta \in R^{i-1}$, $\xi, \zeta' \in R^s$, and some $k > 0$ sufficiently small

$$|F(x, \eta, \zeta) - F(x, \eta, \zeta')| \leq k \sum_{|\alpha|=m} |\xi_\alpha - \zeta'_\alpha|.$$

(26) The map $N: X \rightarrow Y$, given by $N(u)(x) = F(x, u(x), \ldots, D^m u(x))$ is $G$-equivariant; i.e., $N(T_gu)(x) = T_gN(u)(x)$ for all $g \in G$.

Using the classical results (Nirenberg [Ni-1]), one shows that $A: X \rightarrow Y$ defined by the linear part of (21), is a $G$-equivariant Fredholm map and its $G$-index $[\ker A] - [\text{coker } A] \in RO^+(G)$, the semiring of isomorphism classes of real representations of $G$ with direct sum and tensor product of representations as addition and multiplication, respectively of $G$. Then, Theorem 1.10 implies (see [Mi-6])

**Theorem 2.5.** Suppose that conditions (22)-(26) hold and $\text{ind}_G(A) \in RO^+(G)$. Then

(a) If $V$ is such a representation that $G$ satisfies condition (G), then BVP (21) has a solution $u \in X$ with $\|u\| = r$ for each $r > 0$.

(b) If $G = S^1$ and $\text{Fix}(S^1) = \{0\}$ in $V$, then for each $r > 0$ and $Z_r = \{u \in X \mid u \text{ is a solution of (21)}\}$, the Fadell-Rabinowitz-index $\text{Ind}_G(Z_r) \geq \text{ind } A$.

A special case of Theorem 2.5(a) is when $F$ is odd, i.e., $G = Z_2$. It extends the results of Rabinowitz [R-2] with $G = Z_2$ and Marzantowicz [M-1] when $F$ depends only on the derivatives up to order of $m - 1$, and the author [Mi-5].

In particular, let $Q = I = [0, 1]$, $V$ be as above, and $C^k(V)$ be the space of $V$-valued $C^k$-functions on $I$ with the norm

$$\|u\|_k = \sup_I (\|u(t)\|^2 + \cdots + \|u^{(k)}(t)\|^2)^{1/2}.$$

If $G$ acts on $C^k(V)$ as above, then $Au = u^{(m)}(t): X = C^m(V) \rightarrow \tilde{Y} = C(V)$ is a linear continuous $G$-equivariant epimorphism. Let $W$ be a proper $G$-invariant subspace of $U$, the direct sum of $m$ copies of $V$. If $i: \tilde{Y} \rightarrow Y = W \oplus \tilde{Y}$ is the $G$-equivariant inclusion, then $\tilde{A} = iA: X \rightarrow Y$ is a $G$-equivariant Fredholm map with index $[V \oplus \cdots \oplus V] - [W] \in RO^+(G)$. Let the $G$-boundary conditions be given by a $G$-equivariant map $B: X \rightarrow W$. Suppose that $F: I \times U \rightarrow V$ is continuous and $G$-equivariant and for all $t \in I$, $\eta \in U \oplus V$, $\xi, \zeta' \in V$, and a sufficiently small $k > 0$

$$|F(t, \eta, \zeta) - F(t, \eta, \zeta')| \leq k|\zeta - \zeta'|.$$

Then, under these assumptions, one shows that the conclusions of Theorem 2.5 hold in $C^m(V)$ for the BVP (which extends a result in [M-1])

$$u^{(m)}(t) = F(t, u(t), \ldots, u^{(m)}(t)), \quad Bu = 0.$$
Note that if $V$ and $V'$ are two representations of $G$, of dimensions $n$ and $n'$, respectively, then a $G$-equivariant polynomial map $F: V \to V'$ is called a $(V, V')$-invariant. Thus, the invariant theory, which describes and classifies $(V, V')$-invariants, gives us examples of $G$-equivariant maps (see [We]).

(B) Let us now look at BVP (21) without symmetries when $k = 1$ and $p = 2$. Suppose that $Au = \sum_{|\alpha| \leq m} A_\alpha(x)D^\alpha u$ is elliptic and of index $m > 0$ from $X = W_2^m(Q, R; B_j)$ into $Y = L_2(Q, R)$, with $B_j$ as before. As usual, if $X_0 = \ker A$ and $\tilde{Y} = R(A)$, then $X = X_0 \oplus \tilde{X}$ and $Y = Y_0 \oplus \tilde{Y}$ with $\dim X_0 - \dim Y_0 = m$.

We now impose the following assumptions on $F$:

(27) $F: Q \times R^{2m} \to R$ satisfies the Carathéodory conditions and there are $a, b \in L_2(Q)$ and $\gamma \in [0, 1]$ such that

$$|F(x, \xi)| \leq a(x) + b(x)|\xi|^\gamma$$

for $x \in Q$, $\xi \in R^{2m}$.

(28) There exists $c > 0$ such that $\|Au\| \geq c\|u\|$ for $u \in \tilde{X}$ and for $x \in Q$, $\eta \in R^{2m-1}$, and $\xi, \xi' \in R^{2m}$

$$|F(x, \eta, \xi) - F(x, \eta, \xi')| \leq c \sum_{|\alpha|=m} |\xi_\alpha - \xi'_\alpha|.$$

(29) Suppose that $F(x, \xi) = G(x, \xi) + H(x, \xi)$ for $x \in Q$, $\xi \in R^{2m}$ and

(i) $\lim_{|s| \to \pm \infty} G(x, s, \eta)/|s|^\gamma = g_\pm(x)$ uniformly in $\eta \in R^{2m-1}$,

(ii) there exist $c, d \in L_2(Q)$ and $e > 0$ such that for $\gamma_1 < \min\{1, \gamma\}$

$$|H(x, \xi)| \leq \gamma_1[c(x) + d(x)|\xi|^\gamma_1-e]$$

for $x \in Q$, $\xi \in R^{2m}$,

(iii) there exists a subspace $Z \subset X_0$, $\dim Z = \dim Y_0$, and a linear bijection

$J: Z \to Y_0$ such that $Ju = 0$ a.e. on $\{x| x \in Q, u(x) = 0\}$ and

$$\int_{u>0} g_+|u|^\gamma J(u) \, dx - \int_{u<0} g_-|u|^\gamma J(u) \, dx$$

$$\neq \mu \left[ \int_{u>0} g_+|u|^\gamma J(u) \, dx - \int_{u<0} g_-|u|^\gamma J(u) \, dx \right]$$

for $u \in \partial B(0, 1) \cap Z$ and $\mu \in [0, 1]$.

**Theorem 2.6.** Under the above conditions on $A$ and $F$ there exists a connected closed subset $C$ in $X$ of solutions of BVP (21) whose dimension at each point is at least $m$ and such that the projection of $X$ onto $X_0 \oplus Z = W$ maps $C$ onto $W$.

**Proof.** Define $N: X \to Y$ by $Nu = F(x, u, Du, \ldots, D^{2m}u)$. Then there are constants $a$ and $b$ such that

$$\|Nu\| \leq a + b\|u\|^\gamma$$

for $u \in X$.

and $A + N: X \to Y$ is $A$-proper with respect to a suitable scheme $\Gamma$ as shown in [Mi-5]. Since it is continuous, it is proper on each bounded and closed set. Hence, the conclusions of the theorem follow from Theorem 2.4 and Remarks 2.2–2.3 provided we can show that (18) holds; i.e., there are constants $M \geq 0$, $K > 0$, and $\rho_0 \geq 0$ such that for each $v \in R(A)$ with $\|v\| \leq r$, $r > K$, $z \in \partial B(0, 1) \cap Z$, $\rho \geq rM + \rho_0$, and $\mu \in [0, 1]$

$$(N(\rho z + \rho^2 v) - \mu N(-\rho z - \rho^2 v), Jz) \neq 0$$
where $(\, , \,)$ is the $L_2$-inner product. This can be shown by arguing by contradiction and extending suitably the corresponding arguments in Tarafdar [T] for the case $i(A) = 0, \gamma < 1, H = 0$, and $F = F(x, u)$. □

When $i(A) = 0, \gamma < 1, H = 0, F = F(x, u)$ the existence assertion of Theorem 2.6 was obtained by Tarafdar [T]. For other resonance conditions with $i(A) > 0$, see [FMP-1, Mi-5, Ni-1].

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DEPARTMENT OF MATHEMATICS AND CAMS, NEW JERSEY INSTITUTE OF TECHNOLOGY, NEWARK, NEW JERSEY 07102

E-mail address: pemilo@m.njit.edu