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THE HAUSDORFF DIMENSION OF λ-EXPANSIONS WITH DELETED DIGITS

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ABSTRACT. In this article we examine the continuity of the Hausdorff dimension of the one parameter family of Cantor sets $\Lambda(\lambda) = \{\sum_{k=1}^{\infty} i_k \lambda^k : i_k \in S\}$, where $S \subset \{0, 1, \dots, (n-1)\}$. In particular, we show that for almost all (Lebesgue) $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ we have that $\dim_H(\Lambda(\lambda)) = \frac{\log l}{-\log \lambda}$ where $l = \mathrm{Card}(S)$. In contrast, we show that under appropriate conditions on S we have that for a dense set of $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ we have $\dim_H(\Lambda(\lambda)) < \frac{\log l}{-\log \lambda}$.

0. INTRODUCTION

During the conference on "The dynamics of $\mathbb{Z}^n$ actions" (Warwick, 20–24 September 1993) M. Keane posed the following question.

Question (Keane). Is the Hausdorff dimension $\dim_H(\Lambda(\lambda))$ of the one parameter family of Cantor sets

$$\Lambda(\lambda) = \{\sum_{k=1}^{\infty} i_k \lambda^k : i_k = 0, 1, 3\}$$

continuous on the interval $\lambda \in [\frac{1}{4}, \frac{1}{3}]$?

For $\lambda \leq \frac{1}{4}$ the set $\Lambda(\lambda)$ satisfies a "open cover condition" (in particular, for each $m \geq 1$ the elements of the cover of $\Lambda(\lambda)$ by the projection of cylinders of length m are pairwise disjoint) and so it is easy to show that $\dim_H(\Lambda(\lambda)) = \frac{\log 3}{-\log \lambda}$.

In the subtler case that $\lambda \in [\frac{1}{4}, \frac{1}{3}]$, Keane, Smorodinsky and Solomyak [KSS] pointed out that:

(i) $m(\Lambda(\lambda)) = 0$ (where m denotes the Lebesgue measure); and
(ii) the cylinders of the set $\Lambda(\lambda)$ intersect each other in such a way that one cannot apply the "open set condition".

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In this paper we shall address the natural question of describing the Hausdorff dimension in this more complicated situation.

We shall consider the more general set $\Lambda(\lambda) = \{\sum_{i=1}^{\infty} i_k \lambda^k : i_k \in S\}$, where $S \subset \{0, 1, \dots, (n-1)\}$ and $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ where l denotes the cardinality of the set $S = \{s_1, s_2, \dots, s_l\}$. We shall call $[\frac{1}{n}, \frac{1}{l}]$ the *critical interval* for the set S since on this interval (i) and (ii) hold. For the future reference the special case mentioned above ($n = 4$ and $S = \{0, 1, 3\}$), will be called as "the case $(0, 1, 3)$".

We will show that under suitable conditions on S (which hold in the case $(0, 1, 3)$) we have the following results

(a) for almost all $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ we have the identity $\dim_H(\Lambda(\lambda)) = \frac{\log l}{-\log \lambda}$ (cf. Theorem 1),

(b) there is a dense subset of $[\frac{1}{n}, \frac{1}{l}]$ on which $\dim_H \Lambda(\lambda) < \frac{\log l}{-\log \lambda}$ (cf. Theorem 2).

In particular, we have a negative answer to Keane's question since we see that $\dim_H(\Lambda(\lambda))$ is discontinuous on a dense subset of $[\frac{1}{n}, \frac{1}{l}]$.

In proving (i) we shall make use of an idea first introduced by K. Falconer. A fundamental difference between result (i) above and Falconer's work (see [F1, Theorem 5.3]) is that for the self-similar Cantor sets $\Lambda(\lambda)$ considered in this paper the ratio of the similarity λ is used to parameterize the family of sets $\Lambda(\lambda)$. In Falconer's paper a "for almost all" result is proved for the Hausdorff dimension of Cantor sets which are the invariant sets for families of affinities for which the contractions are fixed and their translation are used to give the parameter space. So one cannot apply the Falconer theorem to prove a result like (i) above, but the main idea of the proof of Falconer's Theorem (the potential theoretic characterization of the Hausdorff dimension, combined with Fubini's theorem) is used in the proof of our Theorem 1.

1. Notation

In this section we shall give the basic definitions and notations. Let $n > 1$ be an arbitrary natural number. Let $l \leq n-1$ and fix a subset

$$S = \{s_1, s_2, \dots, s_l\} \subset \{0, 1, \dots, (n-1)\}.$$

For each $0 < \lambda < 1$ we can define the family of similarities $\{T_k^\lambda\}_{k \in S}$, for $k = 1, \dots, n-1$, on $\mathbb{R}$ (with ratios λ) by $T_k^\lambda(x) := \lambda.(k + x)$.

We define

$$\Lambda(\lambda) = \{\sum_{k=1}^{\infty} i_k \lambda^k : i_k \in S\}.$$

It is easy to see that $\Lambda(\lambda)$ is the unique invariant set for the family $\{T_k^\lambda\}_{k \in S}$ i.e. the only compact set such that $\Lambda(\lambda) = \bigcup_{i \in S} T_k^\lambda(\Lambda(\lambda))$. We denote the Hausdorff dimension of $\Lambda(\lambda)$ by $\dim_H(\Lambda(\lambda))$ and the (upper) box dimension by $\overline{\dim}_B(\Lambda(\lambda))$. (We refer the reader unfamiliar with these terms to [F] for the full definition of Hausdorff dimension and box dimension.)

For each $k \in S$ we define a subset $\Lambda_k(\lambda)$ of $\Lambda(\lambda)$ by

$$\Lambda_k(\lambda) := T_k^\lambda(\Lambda(\lambda)) = \{\sum_{m=1}^{\infty} i_m \lambda^m : i_1 = k \text{ and } i_m \in S, \quad \forall m \in \mathbb{N}\}.$$

We write I^λ for the smallest interval which contains $\Lambda(\lambda)$ (i.e. the convex hull of the set $\Lambda(\lambda)$). It is easy to see that $I^\lambda = [\frac{(\min S)\lambda}{1-\lambda}], \frac{(\max S)\lambda}{1-\lambda}])$. Similarly we write I_k^λ for the smallest interval containing $\Lambda_k(\lambda)$. (That is $I_k^\lambda = T_k^\lambda(I^\lambda)$.) We define $I_{i_1,\dots,i_m}^\lambda := \left(T_{i_1}^\lambda \circ \cdots \circ T_{i_m}^\lambda\right)(I^\lambda)$ and write $E := I^\lambda \setminus \bigcup_{k\in S} I_k^\lambda$

We define a sequence space

$$\mathscr{J}_\infty := \{\mathbf{i} = (i_1, i_2, \dots, i_k, \dots) : i_k \in S, \quad \forall k \in \mathbb{N}\}.$$

and denote by μ the natural (evenly weighted) Bernoulli measure on $\mathscr{J}_\infty$. In particular, the sets

$$[j_1, j_2, \dots, j_m] = \{\mathbf{i} \in \mathscr{J}_\infty : i_k = j_k \text{ for } 1 \le k \le m\}$$

have measure $\mu([j_1, j_2, \dots, j_m]) = \frac{1}{l^m}$.

Let $\Pi^\lambda : \mathscr{J}_\infty \to \Lambda(\lambda)$ denote the projection from $\mathscr{J}_\infty$ onto $\Lambda(\lambda)$ defined by

$$\Pi^\lambda(i_1, i_2, \dots, i_k, \dots) := \sum_{k=1}^{\infty} i_k \lambda^k, \qquad \text{for } (i_1, i_2, \dots, i_k, \dots) \in \mathscr{J}_\infty.$$

The distance between the projections of any two $\mathbf{i}, \mathbf{j} \in \mathscr{J}_\infty$ has an important rôle in our investigation. We shall be particularly intrested in studying the function defined by

$$f(\lambda, \mathbf{i}, \mathbf{j}) := \Pi^\lambda(\mathbf{i}) - \Pi^\lambda(\mathbf{j}) \qquad (\text{ for } \mathbf{i}, \mathbf{j} \in \mathscr{J}_\infty).$$

In Theorem 3 we will use the notion of the "Newhouse thickness". The definition and some important theorems about it can be found in the book [PT, pp. 63–82]. For the convenience of the reader we shall recall the definition of the Newhouse thickness of a Cantor set $K \subset \mathbb{R}$.

We begin by calling an open set $U \subset \mathbb{R}$ a gap if it is a connected component of $\mathbb{R} - K$. If, moreover, U is a bounded open set in $\mathbb{R}$ we call it a bounded gap. Let U be a bounded gap and let u be a boundary point of U (and hence an element of K). The bridge of K at u is the maximal interval $C \subset \mathbb{R}$ such that:

(i) u is a boundary point of C; and
(ii) C contains no point of a gap U' whose length $|U'|$ is at least the length of U.

The thickness of K at u is defined as $\tau(K, u) = |C|/|U|$. Finally, the *Newhouse thickness* of K (denoted $\tau(K)$) is the infimum over these $\tau(K, u)$ for all boundary points u of bounded gaps.

2. Statement of results

In this section we shall state our main results and the proofs will appear in section 3. Without loss of generality we may assume that $s_1 = 0$ and $s_l = n-1$.

Principal Assumption. In what follows, we shall usually suppose that the following inequality holds between l (the cardinality of the set $S = \{s_1, s_2, \dots, s_l\}$) and n

$$(n-1) < (l-1)^2 \tag{1}$$

Our first theorem shows that we can explicitly compute the Hausdorff dimension of $\Lambda(\lambda)$ on a large set of parameter values λ.

Theorem 1. *Suppose that* (1) *holds. Then for almost all* $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ (*with respect to Lebesgue measure*) *we have that*

$$\dim_H(\Lambda(\lambda)) = \frac{\log l}{-\log \lambda}. \tag{2}$$

We see from this theorem that the Hausdorff dimension can become continuous if we remove a set of zero Lebesgue from the interval $[\frac{1}{n}, \frac{1}{l}]$. The next result characterises the continuity at points in this interval.

Proposition 1. *Suppose that* (1) *holds. Then for any* $\lambda_0 \in [\frac{1}{n}, \frac{1}{l}]$:

(i) *If* $\Lambda_a(\lambda_0) \cap \Lambda_b(\lambda_0) \neq \varnothing$ *for distinct* $a, b \in S$, $(a \neq b)$ *then* λ_0 *is a limit point of the discontinuity places of the function* $\dim_H(\Lambda(\lambda))$;

(ii) *If* $\Lambda_a(\lambda_0) \cap \Lambda_b(\lambda_0) = \varnothing$ *for all distinct* $a, b \in S$, $(a \neq b)$ *then the function* $\dim_H(\Lambda(\lambda))$ *is continuous in a neighbourhood of* λ_0.

This immediately leads to the following corollary.

Corollary 1. *Suppose that* (1) *holds.*

(a) *The set of discontinuity points of the function* $\dim_H(\Lambda(\lambda))$ *for* $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ *is either empty or has no isolated points* (*i.e. there is no isolated discontinuity point*).

(b) *the cardinality of the discontinuity points is either* 0 *or* $\aleph$.

The next theorem is the only statement in which we do not assume that (1) holds. It shows that the equality (2) in Theorem 1 fails on a dense set of points. Let $\overline{\dim}_B(\Lambda(\lambda))$ denote the upper box dimension then we always have $\dim_H(\Lambda(\lambda)) \leq \overline{\dim}_B(\Lambda(\lambda))$ (cf. [F] for definitions and details).

Theorem 2. *Assume that*:

(i) $1 \in S$,

(ii) $\{0, 1, \ldots, (n-1)\} \subset S - S$.

Then there exists a dense set $B \subset [\frac{1}{n}, \frac{1}{l}]$ *such that for every* $\lambda \in B$ *we have*:

$$\dim_H(\Lambda(\lambda)) \leq \overline{\dim}_B(\Lambda(\lambda)) < \frac{\log l}{-\log \lambda} \tag{3}$$

Comparing Theorem 1 and Theorem 2 gives the following immediate corollary.

Corollary 2. *Suppose that*:

(i) $1 \in S$,

(ii) $\{0, 1, \ldots, (n-1)\} \subset S - S$,

(iii) (1) *holds.*

Then the function $\dim_H(\Lambda(\lambda))$ *is discontinuous on a dense subset of* $[\frac{1}{n}, \frac{1}{l}]$.

The next corollary (with the particular choice $S = \{0, 1, 3\}$, $l = 3$, $n = 4$) answers Keane's question.

Corollary 3. *With the choice* $S := \{0, 1, 3, \ldots, 2k+1\}$, $k \geq 1$, *the set of* λ *'s for which* $\overline{\dim}_B(\Lambda(\lambda)) < \frac{\log l}{-\log \lambda}$ *is dense in* $[\frac{1}{n}, \frac{1}{l}]$. *In particular, the function*

$\dim_H(\Lambda(\lambda))$ *is discontinuous on a dense subset of* $[\frac{1}{n}, \frac{1}{l}]$. (*This is precisely the* $(0, 1, 3)$ *case when* $k = 1$.)

Corollary 3 is an immediate consequence of Corollary 2. In addition to Theorem 2, we have another result which gives another condition for the discontinuity (and continuity) of $\dim_H(\Lambda(\lambda))$ at points in the interval $[\frac{1}{n}, \frac{1}{l}]$. We give this as Theorem 3 below.

Theorem 3. *Let* I^λ *be the convex hull of* $\Lambda(\lambda)$, *that is* $I^\lambda = [0, \frac{(n-1)\lambda}{1-\lambda}]$. *Suppose that there exists an interval* $J \subseteqq [\frac{1}{n}, \frac{1}{l}]$ *such that*:

(i) $\forall \lambda \in J \exists a, b \in S, a \neq b$ *such that* $I_a^\lambda \cap I_b^\lambda \neq \varnothing$,
(ii) $\forall \lambda \in J$ *the Newhouse thickness* [PT, pp. 61] *of* $\Lambda(\lambda) > 1$,
(iii) (1) *holds.*

Then the set of λ *'s for which* $\overline{\dim}_B(\Lambda(\lambda)) < \frac{\log l}{-\log \lambda}$ *is dense in* J, *so the function* $\dim_H(\Lambda(\lambda))$ *is discontinuous on a dense subset of* J.

To be able to apply Theorem 3 we need to decide whether the Newhouse thickness is greater than one. For this we rely on the following proposition.

Proposition 2. *Assume that for every connected component* G *of the set* $E := I^\lambda \setminus \bigcup_{k\in S} I_k^\lambda$ *the length of* G *is less than any of the two connected components of* $\bigcup_{k\in S} I_k^\lambda$ *adjacent to* G. *Then* $\tau(\Lambda(\lambda)) > 1$.

As a consequence of Proposition 2 and Theorem 3 one can see that the properties of $\dim_H(\Lambda(\lambda))$ for the set $S = \{0, 1, 3\}$ persist if we instead choose $S = \{0, 2, 3\}$. In contrast, Theorem 3 gives us a method of constructing an example of a set $S = \{0, s_2, \dots, s_{l-1}, n-1\}$ such that the function $\dim_H(\Lambda(\lambda))$ is continuous on one part of its critical interval $[\frac{1}{n}, \frac{1}{l}]$, but on another part on the critical interval the points of its discontinuity are dense. We summarise these examples in the following corollary.

Corollary 4.

(i) *The function* $\dim_H(\Lambda(\lambda))$ *discountinuous on a dense subset of the critical interval* $[\frac{1}{n}, \frac{1}{l}]$ *in the following cases*:
  (a) $S = \{0, 2, 3\}$,
  (b) $S = \{0, 2, 4, \dots, 2i, 2i+1, 2i+2, \dots, 2j\}$ *for any* $i < j$.
(ii) *If* $S = \{0, 3, 6, 9, 15\}$ *then the critical interval is* $[\frac{1}{16}, \frac{1}{5}]$. *The function* $\dim_H(\Lambda(\lambda))$ *is continuous on the interval* $[\frac{1}{16}, \frac{1}{6})$ *and has a dense set of points of discontinuity on the interval* $[\frac{1}{6}, \frac{1}{5}]$.

3. Proofs

The following lemma has a crucial rôle in the proof of the Theorem 1. It quantifies the statement that if for some value λ the function f is close to zero then f' cannot be close to zero.

Lemma 1. *Suppose that assumption* (1) *holds. If* $|f(\lambda, \mathbf{i}, \mathbf{j})| < \frac{c}{6n}$ *then it implies that* $|f'(\lambda, \mathbf{i}, \mathbf{j})| > \frac{c}{3}$ (*whenever* $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ *and* $\mathbf{i}, \mathbf{j} \in \mathscr{J}_\infty$ *with* $i_1 \neq j_1$) *where we denote* $f'(\lambda, \mathbf{i}, \mathbf{j}) := \frac{d}{d\lambda} f(\lambda, \mathbf{i}, \mathbf{j})$ *and* $c := 1 - \frac{n-1}{(l-1)^2}$ (*and it follows from the assumption* (1) *that* $c > 0$).

Proof. Fix a value $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ and points $\mathbf{i}, \mathbf{j} \in \mathscr{J}_\infty$. We can consider the power series

$$f(\lambda, \mathbf{i}, \mathbf{j}) = \sum_{k=1}^{\infty} a_k \lambda^k$$

(where $a_k = i_k - j_k$, for $k \geq 1$, and, in particular, $a_1 \neq 0$) and then we observe that since

$$\frac{f(\lambda, \mathbf{i}, \mathbf{j})}{\lambda} = \sum_{k=1}^{\infty} a_k \lambda^{k-1}$$

we have that

$$\begin{aligned} \lambda \frac{d}{d\lambda}\left(\frac{f(\lambda, \mathbf{i}, \mathbf{j})}{\lambda}\right) &= \lambda\left(\sum_{k=1}^{\infty}(k-1)a_k\lambda^{k-2}\right) \\ &= \sum_{k=1}^{\infty}(k-1)a_k\lambda^{k-1}. \end{aligned}$$

In particular, this means that we have

$$\begin{aligned} \frac{2f(\lambda, \mathbf{i}, \mathbf{j})}{\lambda} - f'(\lambda, \mathbf{i}, \mathbf{j}) &= \frac{f(\lambda, \mathbf{i}, \mathbf{j})}{\lambda} - \lambda\frac{d}{d\lambda}\left(\frac{f(\lambda, \mathbf{i}, \mathbf{j})}{\lambda}\right) \\ &= a_1 + \sum_{k=2}^{\infty}(2-k)a_k\lambda^{k-1} \\ &= a_1 + g(\lambda) \end{aligned}$$

where we have introduced

$$g(\lambda) = \sum_{k=2}^{\infty}(2-k)a_k\lambda^{k-1}.$$

Observe that we have the following simple upper bound on the modulus of $g(\lambda)$

$$\begin{aligned} |g(\lambda)| &\leq \sum_{k=3}^{\infty}(k-2)|a_k|\lambda^{k-1} \\ &\leq (n-1)\sum_{k=3}^{\infty}(k-2)\lambda^{k-1} \\ &= (n-1)\lambda^2\sum_{l=0}^{\infty}(l+1)\lambda^l \\ &= (n-1)\frac{\lambda^2}{(1-\lambda)^2} \\ &\leq (n-1)\frac{1/l^2}{(l-1)^2/l^2} \\ &= \frac{n-1}{(l-1)^2} \\ &= 1-c \end{aligned} \tag{4}$$

(where we have used that $\frac{\lambda^2}{(1-\lambda)^2}$ is a monotone increasing function for $\lambda \in [\frac{1}{n}, \frac{1}{l}]$). From (4) we immediately get the following lower bound

$$\begin{aligned} \left|\frac{2f(\lambda, \mathbf{i}, \mathbf{j})}{\lambda} - f'(\lambda, \mathbf{i}, \mathbf{j})\right| &= |a_1 - (-g(\lambda))| \\ &\geq |a_1| - |g(\lambda)| \\ &\geq 1 - |g(\lambda)| \\ &\geq c. \end{aligned} \tag{5}$$

If we assume that $|f(\lambda)| < \frac{c}{6n}$ then we have that $|\frac{2f(\lambda)}{\lambda}| < \frac{c}{3}$ since $\lambda > \frac{1}{n}$. However, we can deduce from this a lower bound on the modulus $|f'(\lambda, \mathbf{i}, \mathbf{j})|$ of the derivative of the form $|f'(\lambda, \mathbf{i}, \mathbf{j})| > \frac{c}{3}$. If this were not the case, then we could write $|f'(\lambda, \mathbf{i}, \mathbf{j})| \leq \frac{c}{3}$, and then

$$\left|\frac{2f(\lambda, \mathbf{i}, \mathbf{j})}{\lambda} - f'(\lambda, \mathbf{i}, \mathbf{j})\right| \leq \left|\frac{2f(\lambda, \mathbf{i}, \mathbf{j})}{\lambda}\right| + |f'(\lambda, \mathbf{i}, \mathbf{j})| \leq \frac{2c}{3}$$

which contrdicts (5).

We can now proceed towards the proof of Theorem 1. This will require two additional preliminary lemmas.

Lemma 2. *Let* $0 < s < 1$. *Then for any* $\mathbf{i}, \mathbf{j} \in \mathcal{J}_\infty$ *with* $i_1 \neq j_1$ *we have*

$$\int_{[\frac{1}{n}, \frac{1}{l}]} \frac{d\lambda}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} \leq K(s)$$

where $K(s) < +\infty$ *and independent of* $\mathbf{i}$ *and* $\mathbf{j}$.

Proof. Fix $\mathbf{i}, \mathbf{j} \in \mathcal{J}_\infty$ with $i_1 \neq j_1$. It is convenient to define a partition of $[\frac{1}{n}, \frac{1}{l}]$ into two subsets $I', I'' \subset [\frac{1}{n}, \frac{1}{l}]$ as follows:

$$I' := \{\lambda \in [\frac{1}{n}, \frac{1}{l}] : |f(\lambda, \mathbf{i}, \mathbf{j})| \geq \frac{c}{6n}\} \qquad \text{and}$$

$$I'' := \{\lambda \in [\frac{1}{n}, \frac{1}{l}] : |f(\lambda, \mathbf{i}, \mathbf{j})| < \frac{c}{6n}\};$$

then we can rewrite the following integral

$$\int_{[\frac{1}{n}, \frac{1}{l}]} \frac{d\lambda}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} = \int_{I'} \frac{d\lambda}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} + \int_{I''} \frac{d\lambda}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s}.$$

For the integral over I' we can estimate

$$\int_{I'} \frac{d\lambda}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} \leq m(I').\left(\frac{6n}{c}\right)^s \leq \left(\frac{6n}{c}\right)^s \tag{6}$$

where $m(I')$ denotes the Lebesgue measure of the set I' and $m(I') \leq m([\frac{1}{n}, \frac{1}{l}]) \leq 1$.

On the set I'' the modulus of the derivative $f'(\lambda, \mathbf{i}, \mathbf{j})$ is bounded from below by $\frac{c}{3} > 0$ (by Lemma 1) and from above by $\frac{(n-1).l^2}{(l-1)^2}$ (by a simple bound on the series). Since this lower bound is nonzero we see that I'' consists of a finite number of (disjoint) intervals on which $f(\lambda, \mathbf{i}, \mathbf{j})$ is monotone as a function of λ. Moreover, by the upper bound on $|f'(\lambda, \mathbf{i}, \mathbf{j})|$ we can see that

the length of each such interval must be at least $\frac{c(l-1)^2}{3n(n-1)l^2}$. This means that the total number of such intervals must be at most $\frac{3n(n-1)l^2}{c(l-1)^2}+2$ (since $I'' \subset [\frac{1}{n}, \frac{1}{l}]$).

Let J be a typical such interval of monotonicity for $f(\lambda, \mathbf{i}, \mathbf{j})$, then since $f'(\lambda, \mathbf{i}, \mathbf{j}) \neq 0$ on $J \subset I''$ we see that it contains at most one zero λ_0, say, for $f(\lambda, \mathbf{i}, \mathbf{j})$ (i.e. $f(\lambda_0, \mathbf{i}, \mathbf{j}) = 0$). If J contains no zero then it must be an exceptional interval containing either $\frac{1}{n}$ or $\frac{1}{l}$.

Assume that J contains a zero λ_0 for $f(\lambda, \mathbf{i}, \mathbf{j})$. It follows from Lemma 1 that $|f'(\lambda, \mathbf{i}, \mathbf{j})| > \frac{c}{3}$ on $J \subset I''$ and so $|f(\lambda, \mathbf{i}, \mathbf{j})| \geq \frac{c}{3}|\lambda - \lambda_0|$. This implies that for $\lambda \in J$ we have the inequality

$$\int_J \frac{1}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} d\lambda \leq 2\left(\frac{3}{c}\right)^s \int_0^1 \frac{1}{x^s} dx \leq 2\left(\frac{3}{c}\right)^s \left(\frac{1}{1-s}\right).$$

If J is one of the (at most two) exceptional intervals which do not contain a zero λ_0, then it is easy to see that we can get the same upper bound.

This means that we can estimate

$$\int_{I''} \frac{1}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} d\lambda \leq \frac{6n(n-1)l^2}{c(l-1)^2}\left(\frac{3}{c}\right)^s \left(\frac{1}{1-s}\right). \tag{7}$$

Combining (6) and (7) be have the bound

$$\begin{aligned}\int_{\frac{1}{n}}^{\frac{1}{l}} \frac{1}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} d\lambda &= \int_{I'} \frac{1}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} d\lambda + \int_{I''} \frac{1}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} d\lambda \\ &\leq \left(\frac{6n}{c}\right)^s + \frac{6n(n-1)l^2}{c(l-1)^2}\left(\frac{3}{c}\right)^s \left(\frac{1}{1-s}\right).\end{aligned}$$

This completes the proof of Lemma 2.

Our third lemma involves adapting Falconer's method [F1] to the present context. We now fix $\epsilon > 0$ (for the duration of proof of Lemma 3 and Theorem 1) and define

$$s_\epsilon(\lambda) := \frac{\log(l-\epsilon)}{-\log\lambda}.$$

Observe that this is uniformly less than 1 in the critical interval $[\frac{1}{n}, \frac{1}{l}]$.

Lemma 3.

$$\int_{[\frac{1}{n}, \frac{1}{l}]} \int_{\mathcal{J}_\infty} \int_{\mathcal{J}_\infty} \frac{1}{|f(\lambda, \mathbf{i}, \mathbf{j})|^{s_\epsilon(\lambda)}} d\mu(\mathbf{i}) d\mu(\mathbf{j}) d\lambda < +\infty$$

Proof. We can set $s = \max\{s_\epsilon(\lambda) : \lambda \in [\frac{1}{n}, \frac{1}{l}]\} < 1$. We first observe that if $i_1 \neq j_1$ then we can apply Lemma 2 to deduce that

$$\int_{[\frac{1}{n}, \frac{1}{l}]} \frac{d\lambda}{|f(\lambda, \mathbf{i}, \mathbf{j})|^{s_\epsilon(\lambda)}} \leq \int_{[\frac{1}{n}, \frac{1}{l}]} \frac{d\lambda}{|f(\lambda, \mathbf{i}, \mathbf{j})|^s} \leq K(s) \tag{8}$$

Given $\mathbf{i}, \mathbf{j} \in \mathcal{J}_\infty$ we denote by $\mathbf{i} \wedge \mathbf{j}$ the unique cylinder set $[i_1, \ldots, i_m] \subset \Lambda(\lambda)$ for which:

(i) $i_k = j_k$ for $1 \leq k \leq m$; and
(ii) $i_{m+1} \neq j_{m+1}$,

and we denote $|\mathbf{i} \wedge \mathbf{j}| = m$. We can observe that

$$f(\lambda, \mathbf{i}, \mathbf{j}) = \lambda^{|\mathbf{i}\wedge\mathbf{j}|}.f(\lambda, \mathbf{i}', \mathbf{j}') \tag{9}$$

where $\mathbf{i}', \mathbf{j}'$ are the "tails" of the sequences $\mathbf{i}, \mathbf{j}$ (i.e. $\mathbf{i}$ and $\mathbf{j}$ can be written as concatenations $\mathbf{i} = (\mathbf{i} \wedge \mathbf{j}, \mathbf{i}')$ and $\mathbf{j} = (\mathbf{i} \wedge \mathbf{j}, \mathbf{j}')$, respectively).

Using Fubini's theorem and the identity $\lambda^{s_\epsilon} = \frac{1}{l-\epsilon}$ (which follows from the definition of $s_\epsilon(\lambda)$) we have that

$$\begin{aligned}
&\int_{[\frac{1}{n}, \frac{1}{l}]} \int_{\mathcal{J}_\infty} \int_{\mathcal{J}_\infty} \frac{1}{|f(\lambda, \mathbf{i}, \mathbf{j})|^{s_\epsilon(\lambda)}} d\mu(\mathbf{i}) d\mu(\mathbf{j}) d\lambda \\
&= \int_{\mathcal{J}_\infty} \int_{\mathcal{J}_\infty} \int_{[\frac{1}{n}, \frac{1}{l}]} \frac{d\lambda}{\lambda^{s_\epsilon(\lambda).|\mathbf{i}\wedge\mathbf{j}|} |f(\lambda, \mathbf{i}', \mathbf{j}')|^{s_\epsilon(\lambda)}} d\mu(\mathbf{i}) d\mu(\mathbf{j}) \quad \text{by (9)} \\
&= \int_{\mathcal{J}_\infty} \int_{\mathcal{J}_\infty} (l-\epsilon)^{|\mathbf{i}\wedge\mathbf{j}|} \left(\int_{[\frac{1}{n}, \frac{1}{l}]} \frac{d\lambda}{|f(\lambda, \mathbf{i}', \mathbf{j}')|^{s_\epsilon(\lambda)}} \right) d\mu(\mathbf{i}) d\mu(\mathbf{j}) \\
&\leq K(s) \int_{\mathcal{J}_\infty} \int_{\mathcal{J}_\infty} (l-\epsilon)^{|\mathbf{i}\wedge\mathbf{j}|} d\mu(\mathbf{i}) d\mu(\mathbf{j})
\end{aligned}$$

by using (8) (with $\mathbf{i}'$ replacing $\mathbf{i}$ and $\mathbf{j}'$ replacing $\mathbf{j}$). However, since for each $m \geq 1$ and $\mathbf{j} \in \mathcal{J}_\infty$ the set $A_{\mathbf{j}}^m = \{\mathbf{i} \in \mathcal{J}_\infty : |\mathbf{i} \wedge \mathbf{j}| = m\}$ has measure $\mu(A_{\mathbf{j}}^m) = \frac{1}{l^m}$ and it is easy to bound the integral in the last inequality by

$$\begin{aligned}
\int_{\mathcal{J}_\infty} \int_{\mathcal{J}_\infty} (l-\epsilon)^{|\mathbf{i}\wedge\mathbf{j}|} d\mu(\mathbf{i}) d\mu(\mathbf{j}) &= \int_{\mathcal{J}_\infty} \left(\sum_{m=1}^{\infty} (l-\epsilon)^m \frac{1}{l^m} \right) d\mu(\mathbf{j}) \\
&= \frac{1}{1 - \left(\frac{l-\epsilon}{l}\right)} < +\infty.
\end{aligned}$$

This completes the proof of Lemma 3.

We can now complete the proof of Theorem 1.

Proof of Theorem 1. By Lemma 3 and Fubini's theorem we know that for almost all $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ (with respect to Lebesgue measure) we have that

$$\int_{\mathcal{J}_\infty} \int_{\mathcal{J}_\infty} \frac{1}{|f(\lambda, \mathbf{i}, \mathbf{j})|^{s_\epsilon(\lambda)}} d\mu(\mathbf{i}) d\mu(\mathbf{j}) < +\infty.$$

Fix a value $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ such that the above integral is finite. We can consider the mass distribution $\nu_\lambda := (\Pi^\lambda)^* \mu$ on $\Lambda(\lambda)$ (i.e. for any Borel set $E \subset \Lambda(\lambda)$ we have $\nu_\lambda(E) = \mu(\{\mathbf{i} \in \mathcal{J}_\infty : \Pi^\lambda(\mathbf{i}) \in E\})$. This implies that

$$\begin{aligned}
\int_{\Lambda(\lambda)} \int_{\Lambda(\lambda)} \frac{d\nu_\lambda(x) d\nu_\lambda(y)}{|x-y|^{s_\epsilon(\lambda)}} &= \int_{\mathcal{J}_\infty} \int_{\mathcal{J}_\infty} \frac{d\mu(\mathbf{i}) d\mu(\mathbf{j})}{|\Pi^\lambda(\mathbf{i}) - \Pi^\lambda(\mathbf{j})|^{s_\epsilon(\lambda)}} \\
&= \int_{\mathcal{J}_\infty} \int_{\mathcal{J}_\infty} \frac{d\mu(\mathbf{i}) d\mu(\mathbf{j})}{|f(\lambda, \mathbf{i}, \mathbf{j})|^{s_\epsilon(\lambda)}} < +\infty.
\end{aligned}$$

So using the potential theoretic characterisation of Hausdorff dimension [F, pp. 64–65] we see that this gives

$$\dim_H(\Lambda(\lambda)) \geq s_\epsilon(\lambda) = \frac{\log(l-\epsilon)}{-\log \lambda} \tag{10}$$

for almost $\lambda \in [\frac{1}{n}, \frac{1}{l}]$. Since (10) is true for all $\epsilon > 0$ we get that $\dim_H(\Lambda(\lambda)) \geq \frac{\log l}{-\log \lambda}$ (for almost $\lambda \in [\frac{1}{n}, \frac{1}{l}]$, with respect to Lebesgue measure).

The reverse inequality $\dim_H(\Lambda(\lambda)) \leq \frac{\log l}{-\log \lambda}$ is a straightforward exercise. We need only observe that for each $\delta > 0$ the cover of $\Lambda(\lambda)$ by the l^m intervals $\Pi^\lambda([i_1, i_2, \dots, i_m])$ of size $\lambda^m = \delta$ satisfy that

$$\sum_{i_1, i_2, \dots, i_m} \operatorname{diam}\left(\Pi^\lambda[i_1, i_2, \dots, i_m]\right)^s = l^m (\lambda^s)^m = 1.$$

The inequality then follows from the definition. (cf. [F, p. 29])

This completes the proof of Theorem 1.

The next lemma states the fact that if the projection of two cylinders coincide then the upper box dimension $\overline{\dim}_B \Lambda(\lambda)$ (cf. [F, p. 38]), and thus the Hausdorff dimension $\dim_H(\Lambda(\lambda))$, is less than $\frac{\log l}{-\log \lambda}$.

Lemma 4. *Assume that there exists* $m \in \mathbb{N}$ *and cylinder sets* $[a_1, \dots, a_m]$, $[b_1, \dots, b_m]$, *with* $a_k, b_k \in S$, $1 \leq k \leq m$, *such that*

$$\Pi^\lambda([a_1, \dots, a_m]) = \Pi^\lambda([b_1, \dots, b_m]). \tag{11}$$

Then

$$\dim_H(\Lambda(\lambda)) \leq \overline{\dim}_B(\Lambda(\lambda)) < \frac{\log l}{-\log \lambda}$$

Proof. For each $k \in \mathbb{N}$ we can cover $\Lambda(\lambda)$ with $(l^m - 1)^k$ sets of diameter $\frac{(n-1)\lambda^{km+1}}{1-\lambda}$. Namely the sets

$$\mathscr{A}_k := \left\{\Pi^\lambda[i_1, \dots, i_m, i_{m+1}, \dots, i_{2m}, \dots, i_{(k-1)m+1}, \dots, i_{km}]\right\},$$

where $i_l \in S$, $1 \leq l \leq km$ and $(i_{jm+1}, \dots, i_{(j+1)m}) \neq (a_1, \dots, a_m)$, $\forall 0 \leq j \leq (k-1)$.

Therefore

$$\overline{\dim}_B(\Lambda(\lambda)) \leq \lim_{k\to\infty} \frac{\log(l^m - 1)^k}{-\log \frac{(n-1)\lambda^{km+1}}{1-\lambda}} = \frac{\log(l^m - 1)}{-m \log \lambda} < \frac{\log l}{-\log \lambda}.$$

Which completes our proof.

Now we are ready to prove Proposition 1.

Proof of Proposition 1(i). Suppose that

$$\Lambda_a(\lambda_0) \cap \Lambda_b(\lambda_0) \neq \varnothing \tag{12}$$

for some $a, b \in S$, (with $a \neq b$) and $\lambda_0 \in [\frac{1}{n}, \frac{1}{l}]$. Fix an arbitrary $0 < \epsilon < 1$. We shall show below that there is a $\lambda \in [\frac{1}{n}, \frac{1}{l}]$ such that $0 < |\lambda - \lambda_0| < \epsilon$ and $(i_1, \dots, i_m, 0, \dots, 0, \dots) \in [a]$ and $(j_1, \dots, j_m, 0, \dots, 0, \dots) \in [b]$ such that

$$\Pi^\lambda(i_1, \dots, i_m, 0, \dots, 0, \dots) = \Pi^\lambda(j_1, \dots, j_m, 0, \dots, 0, \dots). \tag{13}$$

Then from (13) we have that

$$\Pi^{\lambda_0}[i_1, \dots, i_m] = \Pi^{\lambda_0}[j_1, \dots, j_m].$$

Hence Lemma 2 immediately implies that $\dim_H(\Lambda(\lambda)) \le \overline{\dim}_B(\Lambda(\lambda)) < \frac{\log l}{-\log \lambda}$.

To complete the proof of Proposition 1(i) we have to prove that (13) holds. It follows from the assumption (12) that we can find $\mathbf{i}' \in [a]$, $\mathbf{j}' \in [b]$ such that $\Pi^\lambda(\mathbf{i}') = \Pi^\lambda(\mathbf{j}')$. Using that neither $\Lambda_a(\lambda_0)$ nor $\Lambda_b(\lambda_0)$ can have isolated points (since they are Cantor sets) we can choose m sufficiently large that for

$$\mathbf{i} = (i_1, \dots, i_m, 0, \dots, 0, \dots) \in [a],$$
$$\mathbf{j} = (j_1, \dots, j_m, 0, \dots, 0, \dots) \in [b]$$

we have $0 < \Pi^{\lambda_0}(\mathbf{i}) - \Pi^{\lambda_0}(\mathbf{j}) < \frac{c\epsilon}{6n}$, say, where c was defined in Lemma 1. (Here $\mathbf{i}, \mathbf{j}$ correspond to truncations of the sequences $\mathbf{i}', \mathbf{j}'$, respectively.) Thus $0 < f(\lambda_0, \mathbf{i}, \mathbf{j}) < \frac{c\epsilon}{6n}$ and we may apply Lemma 1 to conclude that $|f'(\lambda_0, \mathbf{i}, \mathbf{j})| > \frac{c}{3}$.

Suppose, for definiteness, that $f'(\lambda_0, \mathbf{i}, \mathbf{j}) > \frac{c}{3}$. (The case where $f'(\lambda_0, \mathbf{i}, \mathbf{j}) < -\frac{c}{3}$ is similar.) Using again Lemma 1 we have that if $(\lambda_0 - \epsilon, \lambda_0)$ is an interval which contains no zero for $f(\lambda, \mathbf{i}, \mathbf{j})$ then $f'(\lambda, \mathbf{i}, \mathbf{j}) > \frac{c}{3}$ holds for every $\lambda \in (\lambda_0 - \epsilon, \lambda_0)$. Thus $\exists \lambda \in (\lambda_0 - \epsilon, \lambda_0)$ such that $f(\lambda, \mathbf{i}, \mathbf{j}) = 0$, which shows that (13) holds. This completes the proof of part (i) of Proposition 1.

In the proof of part (ii) of Proposition 1 we will apply the "open set condition" (cf. [F, p. 118]) to calculate $\dim_H(\Lambda(\lambda))$.

Proof of Proposition 1(ii). We begin by setting

$$\epsilon := \min_{\substack{a, b \in S, a \ne b \\ x \in \Lambda_a(\lambda_0), y \in \Lambda_b(\lambda_0)}} |x - y| \tag{14}$$

then by hypothesis we have that $\epsilon > 0$.

Let K be the maximum of the function $\lambda \to \frac{d(\sum_{k=1}^\infty \lambda^k)}{d\lambda}$ on the critical interval $[\frac{1}{n}, \frac{1}{l}]$ then

$$d(\Lambda(\lambda), \Lambda(\lambda_0)) \le \max_{\mathbf{i} \in \mathscr{I}_\infty} \sum_{k=1}^\infty i_k(\lambda^k - \lambda_0^k) \le K(n-1)(\lambda - \lambda_0),$$

where d is the Hausdorff metric (cf. [F, p. 114]). Thus if $|\lambda - \lambda_0| < \delta := \frac{\epsilon}{5K(n-1)}$ then

$$\min_{\substack{a, b \in S, a \ne b \\ x \in \Lambda_a(\lambda), y \in \Lambda_b(\lambda)}} |x - y| > \frac{\epsilon}{2} > 0.$$

and we get that if $|\lambda - \lambda_0| < \delta$ then $\Lambda_a(\lambda) \cap \Lambda_b(\lambda) = \varnothing$, $\forall a, b \in S, (a \ne b)$ Therefore we may apply the open set condition to conclude that $\dim_H(\Lambda(\lambda)) = \frac{\log l}{-\log \lambda}$ (cf. [F, p. 118]). This completes the proof of Proposition 1.

Corollary 1 immediately follows from Proposition 1.

Proof of Theorem 2. Fix $\lambda > \frac{1}{n}$. It follows from the assumption (ii) of Theorem 2 that the range of the function $(\mathbf{i}, \mathbf{j}) \to f(\lambda, \mathbf{i}, \mathbf{j}) = \sum_{k=1}^\infty a_k \lambda^k$ (where $a_k = i_k - j_k$ can take arbitrary values from $\{-(n-1), \dots, 0, \dots, n-1\}$) is the interval $[-\frac{(n-1)\lambda}{1-\lambda}, \frac{(n-1)\lambda}{1-\lambda}] \ni 1$. Since this interval contains 1 we can find $\mathbf{i}, \mathbf{j} \in \mathscr{I}_\infty$ such that $f(\lambda, \mathbf{i}, \mathbf{j}) = 1$, that is

$$\sum_{k=1}^\infty i_k \lambda^k = 1 + \sum_{k=1}^\infty j_k \lambda^k. \tag{15}$$

We define

$$(a_1, \dots, a_k, \dots) := (0, i_1, \dots, i_k, \dots) \in \mathscr{J}_\infty$$

and similarly

$$(b_1, \dots, b_k, \dots) := (1, j_1, \dots, j_k, \dots) \in \mathscr{J}_\infty.$$

Then it follows from (15) that

$$\Lambda_0(\lambda) \ni \Pi^\lambda(a_1, \dots, a_k, \dots) = \Pi^\lambda(b_1, \dots, b_k, \dots) \in \Lambda_1(\lambda).$$

Thus we get that $\Lambda_0(\lambda) \cap \Lambda_1(\lambda) \neq \varnothing$. Therefore applying Proposition 1 we have that λ is a limit point of the set for which $\overline{\dim}_B(\Lambda(\lambda) < \frac{\log l}{-\log \lambda}$.

Since our argument works for arbitrary $\frac{1}{n} < \lambda < \frac{1}{l}$ this completes the proof of Theorem 2.

To prove Theorem 3 we will make use of the following so-called "Gap Lemma".

Gap Lemma (cf. [PT, p. 63]). *If $K_1, K_2 \subset \mathbb{R}$ are Cantor sets such that*:

(a) *The product of Newhouse thicknesses of K_1 and K_2 is greater than one; and*

(b) *neither K_1 nor K_2 lies in the others gap,*

then $K_1 \cap K_2 \neq \varnothing$

Proof of Theorem 3. Fix an arbitrary value $\lambda \in J$. We first observe that we may apply the Gap Lemma with $K_1 = \Lambda_a(\lambda)$ and $K_2 = \Lambda_b(\lambda)$. The asumption (a) in the Gap Lemma immediately follows from the (ii). Observe that I_a^λ, I_b^λ are the convex hulls of $\Lambda_a(\lambda), \Lambda_b(\lambda)$ respectively. This, together with hypothesis (i), implies assumption (b) in the Gap Lemma. Thus we can apply the Gap Lemma which gives that $\Lambda_a(\lambda) \cap \Lambda_b(\lambda) \neq \varnothing$.

Using Proposition 1(i) we get that λ is a limit point of those λ's for which $\overline{\dim}_B \Lambda(\lambda)) < \frac{\log l}{-\log \lambda}$. Since $\lambda \in J$ was arbitrary this completes the proof of Theorem 3.

To prove Proposition 2 we need some additional notation and a technical lemma. We first set

$$\alpha := \inf\{\frac{|I^\lambda|}{b-a} : b\lambda \in \operatorname{int}(I_a^\lambda), a, b \in S\}$$

if such an a and b exist, and $\alpha := 2$, if not. Obviously $\alpha > 1$ since $[a\lambda, b\lambda] \subsetneq I_a^\lambda$ implies

$$1 < \frac{|I_a^\lambda|}{|b\lambda - a\lambda|} = \frac{|I^\lambda|}{b-a}.$$

We next set $\gamma := \inf\{\frac{|B|}{|G|}\}$, where the infimum ranges over all connected components G of E and B ranges over all connected components of $\bigcup_{k \in S} I_k^\lambda$ adjacent to G. Finally, we set $c := \min\{\alpha, \gamma\}$. It follows from the assumptions of Proposition 2 that $\gamma \geq c > 1$.

Now we can state a preliminary lemma to the proof of Proposition 2.

Lemma 5. *Under the assumption of Proposition* 2, *there is no interval G with the following properties*:

(a) $G \subset I \setminus \Lambda(\lambda)$,

(b) $\frac{\operatorname{dist}(0, G)}{|G|} < c$.

Proof. To get a contradiction suppose that there exists an interval G which satisfies properties (a) and (b) in the statement of the lemma. If $G \subset E$ then we may assume (without loss of generality) that E is a connected component in E since in this case the connected component containing G also satisfies (a), (b).

However, it follows from the definition of γ that the connected components of $\bigcup_{k\in S} I_k^\lambda$ which are adjacent to G are at least $\gamma \geq c$ times longer than G and this contradicts condition (b) (since $\operatorname{dist}(0, G)$ is greater than or equal to any connected components $\bigcup_{k\in S} I_k^\lambda$ adjacent to G).

Let $E_k := I^\lambda \setminus \bigcup I^\lambda_{m_1,\dots,m_k}$ and we recall that $|I^\lambda_{m_1,\dots,m_k}| = \lambda^k |I^\lambda|$. Let j be the smallest number for which $|I^\lambda|\lambda^j < |G|$ and then obviously $G \subset E_j$. The endpoints of the intervals $I^\lambda_{m_1,\dots,m_k}$ lie in $\Lambda(\lambda)$ and so we can find a $1 < k \leq j-1$ such that $G \not\subseteq E_k$ but $G \subset E_{k+1}$. These together imply that $\exists(i_1, \dots, i_k)$ such that $G \subset I^\lambda_{i_1,\dots,i_k}$ furthermore

$$G \subset I^\lambda_{i_1,\dots,i_k} \setminus \bigcup I^\lambda_{m_1,\dots,m_{k+1}} \subset I^\lambda_{i_1,\dots,i_k} \setminus \bigcup_{a\in S} I^\lambda_{i_1,\dots,i_k,a}. \tag{16}$$

Let $T := \left(T^\lambda_{i_1} \circ \cdots \circ T^\lambda_{i_k}\right)^{-1}$ and let u denote the left endpoint of the interval $I^\lambda_{i_1,\dots,i_k}$. Using that $T(u) = 0$, it follows from (16) and assumption (b) above that

(1) $T(G) \subset E$,
(2) $\frac{\operatorname{dist}(0, T(G))}{|T(G)|} = \frac{\operatorname{dist}(u, G)}{|G|} \leq \frac{\operatorname{dist}(0, G)}{|G|} < c$.

This contradicts our observations at the begining of the proof, and so completes the proof of Lemma 5.

Proof of Proposition 2. We shall actually prove that $\tau(\Lambda(\lambda)) \geq c > 1$. Assume for a contradiction that $\tau(\Lambda(\lambda)) < c$. Then there exist

$$G, G_1 \subset I^\lambda \setminus \Lambda(\lambda) \quad \textit{such that} \quad |G| \leq |G_1| \quad \textit{and} \quad \frac{\operatorname{dist}(G, G_1)}{|G|} < c. \tag{17}$$

Without loss of generality we may assume that G_1 lies to the right of G. Let u be the right end point of G. In particular, there exists a $(i_1, \dots, i_m, 0, \dots, 0, \dots) \in \mathscr{J}_\infty$, $i_m \neq 0$ such that $u = \Pi^\lambda(i_1, \dots, i_m, 0, \dots, 0, \dots)$.

First we assume that

$$\nexists j \in S \quad \text{such that} \quad j \neq m \text{ and } i_m\lambda \in I_j^\lambda. \tag{18}$$

Let $j := \max\{k \in S : k < i_m\}$ and we write w for the right endpoint of I_j (obviously $w \in \Lambda(\lambda)$). It follows from (18) that $w < i_m\lambda$. Using that $u = T^\lambda_{i_1} \circ \cdots \circ T^\lambda_{i_{m-1}}(i_m\lambda)$, we have

$$G \subset \left(T_{i_1} \circ \cdots \circ T_{i_{m-1}}(w), u\right) \subset I_{i_1,\dots,i_{m-1}}. \tag{19}$$

Now we prove that $G_1 \subset I_{i_1\dots i_{m-1}}$ holds as well. Assume for a contradiction that $G_1 \not\subseteq I_{i_1,\dots,i_{m-1}}$. Then $G_1 \cap I_{i_1\dots i_{m-1}} = \varnothing$ since the endpoints of the interval $I_{i_1\dots i_{m-1}}$ belong to $\Lambda(\lambda)$. Let t be the right endpoint of $I_{i_1,\dots,i_{m-1}}$. Then $\operatorname{dist}(u, G_1) > \operatorname{dist}(u, t)$. Therefore we get that:

$$\frac{\operatorname{dist}(u, G_1)}{|G|} > \frac{\operatorname{dist}(u, t)}{\operatorname{dist}(T_{i_1} \circ \cdots \circ T_{i_{m-1}}(w), u)} = \frac{\operatorname{dist}(i_m\lambda, \frac{(n-1)\lambda}{1-\lambda})}{\operatorname{dist}(w, i_m\lambda)} \geq \gamma \geq c.$$

Because (18) follows that $(w, i_m\lambda)$ is a connected component of E. But this contradicts to the definition of G, G_1. So we have proved that $G_1 \subset I_{i_1,\dots,i_{m-1}}$ holds. We define $T := (T_{i_1} \circ \cdots \circ T_{i_{m-1}})^{-1}$ and $G_2 := T(G_1)$. Then we know the following about G_2:

(1) G_2 lies on the right-hand side of $i_m\lambda$,
(2) $\frac{\operatorname{dist}(i_m\lambda, G_2)}{|G_2|} \le \frac{\operatorname{dist}(u, G_1)}{|G|} < c$,
(3) $G_2 \subset I^\lambda \setminus \Lambda(\lambda)$,
(4) $(w, i_m\lambda)$ is a connected component of E.

Then there are two possibilities:

(i) G_2 lies in a connected component of E,
(ii) $G_2 \subset I_k^\lambda$ where $k \ge i_m$.

First we prove that (i) is impossible. Suppose that $G_3 = (a, b)$ is the connected component of E which contains G_2. Then it follows from the definition of γ that $\frac{\operatorname{dist}(i_m\lambda, a)}{|G_3|} \ge \gamma$, since it follows from (4) above that $(i_m\lambda, a)$ contains a connected component of $I^\lambda \setminus E$. Therefore

$$\frac{\operatorname{dist}(i_m\lambda, G_2)}{|G_2|} \ge \frac{\operatorname{dist}(i_m\lambda, G_3)}{|G_3|} \ge \gamma \ge c$$

which contradicts to (2) above. We next show that (ii) cannot hold either. Suppose for a contradiction that (ii) holds. Then $\frac{\operatorname{dist}(k\lambda, G_2)}{|G_2|} \le \frac{\operatorname{dist}(i_m\lambda, G_2)}{|G_2|} < c$. Thus writing $G_4 := (T_k^\lambda)^{-1} G_2$ we have that:

(a) $G_4 \subset I^\lambda \setminus \Lambda(\lambda)$
(b) $\frac{\operatorname{dist}(0, G_4)}{|G_4|} < c$.

It follows from Lemma 5 that this is impossible, and so this shows that (ii) cannot occur. This completes the proof of the proposition with the additional assumption (18).

It only remains to prove the Proposition in the case that (18) fails, i.e. where we can assume that

$$\exists j \in S \quad \text{such that} \quad j \ne m \text{ and } i_m\lambda \in I_j^\lambda. \tag{20}$$

Then $G_1 \subset I^\lambda_{i_1,\dots,i_m}$. This is so because in this case $G \subset (v, u)$ where v is the left endpoint of the interval $I^\lambda_{i_1,\dots,im-1,j}$. (Obviously $v \in \Lambda(\lambda)$ and $v < u$ follows from (20).) Thus we have that $|G| < u - v = (i_m - j)\lambda^m$. If $G_1 \not\subset I^\lambda_{i_1,\dots,i_m}$ then $G_1 \cap I^\lambda_{i_1,\dots,i_m} = \varnothing$, therefore $\operatorname{dist}(u, G_1) > |I^\lambda_{i_1,\dots,i_m}| = \lambda^m |I^\lambda|$. Therefore

$$\frac{\operatorname{dist}(G, G_1)}{|G|} = \frac{\operatorname{dist}(u, G_1)}{|G|} \ge \frac{\lambda^m |I^\lambda|}{(i_m - j)\lambda^m} \ge \alpha \ge c,$$

which contradicts (17). Thus we have proved that $G_1 \subset I^\lambda_{i_1,\dots,i_m}$ holds. Let us denote $T' := (T_{i_1} \circ, \dots, \circ T_{i_m})^{-1}$ and $G_2 := T'(G_1)$. Then $T'(u) = 0$ and

$$\frac{\operatorname{dist}(0, G_2)}{|G_2|} = \frac{\operatorname{dist}(u, G_1)}{|G_1|} \le \frac{\operatorname{dist}(u, G_1)}{|G|} < c.$$

In particular,

(a) $G_2 \subset I^\lambda \setminus \Lambda(\lambda)$,
(b) $\frac{\operatorname{dist}(0, G_2)}{|G_2|} < c$.

It follows from Lemma 5 that this is impossible. This contradiction completes the proof of Proposition 2.

Proof of Corollary 4(i). We just have to check that the asumptions (i)–(iii) of Theorem 3 hold in both cases (a) and (b) in Corollary 4(i).

Assumption (i) *holds*: Using that

$$\lambda > \frac{1}{n} \quad \Longrightarrow \quad (k+1)\lambda \in I_k^\lambda \tag{21}$$

we have that $I_k^\lambda \cap I_{k+1}^\lambda \neq \varnothing$. Therefore assumption (i) immediately follows from the fact that in both cases (a) and (b) S contains two consecutive numbers.

Assumption (ii) *holds*: It follows from (21) that any "bridge" of E is longer than λ and any "gap" of E is shorter than λ in both cases (a) and (b). This means that we may apply Proposition 2, which immediately follows that (ii) holds.

Assumption (iii) *holds*: We only have to check that $n-1 < (l-1)^2$, but this is immediate in both cases.

Therefore we may apply Theorem 3 which completes the proof.

Proof of Corollary 4(ii). We only need to check that the assumption of Proposition 1(ii) holds on the interval $\lambda \in [\frac{1}{16}, \frac{1}{6})$. This is so because in this case $|\lambda I^\lambda| < 3\lambda$. Thus $I_a^\lambda \cap I_b^\lambda = \varnothing$, $a, b \in S$, $a \neq b$. This implies that we may apply Proposition 1(ii) which completes the proof on the interval $[\frac{1}{16}, \frac{1}{6})$.

On the interval $[\frac{1}{6}, \frac{1}{5}]$ using a similar argument as in the proof (i) we can prove that in this case the assumptions of Theorem 3 hold and thus we may apply Theorem 3 which completes our proof.

4. Comments and open questions

In this section we briefly mention some open questions related to our results. (1). For "most" $\lambda > \frac{1}{l}$ we would expect that the Hausdorff dimension $\dim_H(\Lambda(\lambda))$ would be equal to 1. We can modify the proof of Theorem 1 to prove the following partial result: $\dim_H \Lambda(\lambda) = 1$ for almost all values $\lambda \in [\frac{1}{l}, c_n]$, where $c_n = \frac{(n-1)^{\frac{1}{2}}-1}{(n-2)}$. Towards this end, observe that for $\lambda < c_n$ we have $|g(\lambda)| \leq (n-1)\frac{\lambda^2}{(1-\lambda)^2} < 1$ (from the first four lines of equation (4)) and then $|f(\lambda, \mathbf{i}, \mathbf{j})| < \frac{c}{6n} \implies |f'(\lambda, \mathbf{i}, \mathbf{j})| > \frac{c}{3}$, whenever $\lambda \in [\frac{1}{n}, c_n]$ and $\mathbf{i}, \mathbf{j} \in \mathscr{J}_\infty$ with $i_1 \neq j_1$.

For any $\eta > 0$ we can set $0 < s = 1-\eta < 1$ in the proof of Lemma 2, which can be modified to see that for any $\mathbf{i}, \mathbf{j} \in \mathscr{J}_\infty$ with $i_1 \neq j_1$ and any sufficiently small $\delta > 0$ we have

$$\int_{\frac{1}{l}+\delta}^{c_n} \frac{d\lambda}{|f(\lambda, \mathbf{i}, \mathbf{j})|^{1-\eta}} \leq K(1-\eta)$$

where $K(1-\eta) < +\infty$ and independent of $\mathbf{i}$ and $\mathbf{j}$.

The proof of Lemma 3 can be modified to give

$$\begin{aligned}
&\int_{\frac{1}{l}+\delta}^{c_n}\int_{\mathcal{J}_\infty}\int_{\mathcal{J}_\infty}\frac{1}{|f(\lambda,\mathbf{i},\mathbf{j})|^{1-\eta}}d\mu(\mathbf{i})d\mu(\mathbf{j})d\lambda\\
&=\int_{\mathcal{J}_\infty}\int_{\mathcal{J}_\infty}\int_{\frac{1}{l}+\delta}^{c_n}\frac{d\lambda}{\lambda^{(1-\eta).|\mathbf{i}\wedge\mathbf{j}|}|f(\lambda,\mathbf{i}',\mathbf{j}')|^{1-\eta}}d\mu(\mathbf{i})d\mu(\mathbf{j})\\
&\le\int_{\mathcal{J}_\infty}\int_{\mathcal{J}_\infty}\left(\frac{1}{\frac{1}{l}+\delta}\right)^{(1-\eta)|\mathbf{i}\wedge\mathbf{j}|}\left(\int_{\frac{1}{l}+\delta}^{c_n}\frac{d\lambda}{|f(\lambda,\mathbf{i}',\mathbf{j}')|^{1-\eta}}\right)d\mu(\mathbf{i})d\mu(\mathbf{j})\\
&\le K(1-\eta)\int_{\mathcal{J}_\infty}\int_{\mathcal{J}_\infty}\left(\frac{1}{\frac{1}{l}+\delta}\right)^{(1-\eta)|\mathbf{i}\wedge\mathbf{j}|}d\mu(\mathbf{i})d\mu(\mathbf{j})\\
&=K(1-\eta)\int_{\mathcal{J}_\infty}\left(\sum_{m=1}^{\infty}\left(\frac{1}{\frac{1}{l}+\delta}\right)^{(1-\eta)m}\frac{1}{l^m}\right)d\mu(\mathbf{j})\\
&\le K(1-\eta)\frac{\frac{1}{l}\left(\frac{1}{\frac{1}{l}+\delta}\right)^{1-\eta}}{1-\frac{1}{l}\left(\frac{1}{\frac{1}{l}+\delta}\right)^{1-\eta}}<+\infty
\end{aligned}$$

provided that $\eta>0$ is sufficiently small (for fixed $\delta>0$).

The proof of Theorem 1 then shows that $\dim_H(\Lambda(\lambda))\ge 1-\eta$ for almost all $\lambda\in[\frac{1}{l}+\delta,c_n]$. Since $\eta,\delta>0$ are arbitrary, the claimed result follows.

In the context of Keane's $(0,1,3)$ example we see that $\dim_H(\Lambda(\lambda))=1$ for almost all $\frac{1}{3}<\lambda<\frac{3^{\frac{1}{2}}-1}{2}=0.3660254\ldots$. By a consideration of the way in which the intervals I_k^λ overlap Keane, Smorodinsky and Solomyak observed that for $\lambda>\frac{2}{5}$ the set $\Lambda(\lambda)$ must contain a nontrivial interval, and thus its Hausdorff dimension must be 1.

(2). We know by Theorem 2 and Theorem 3 that we may have $\dim_H\Lambda(\lambda)<\frac{\log l}{-\log\lambda}$ for a dense set of λ's of zero measure in the critical interval $[\frac{1}{n},\frac{1}{l}]$. It would be interesting to know more about the size of this exceptional set. For example, it is not known to the authors if this set is uncountable, or what its Hausdorff dimension might be in the event that it were uncountable. However, we can adapt the proof of Theorem 1 to prove a slightly sharper result on the size of the exceptional set.

We define for each $t\in[\frac{1}{n},\frac{1}{l})$ an exceptional set

$$E_t=\{\lambda\in[\tfrac{1}{n},t):\ \dim_H(\Lambda(\lambda))<\frac{\log l}{-\log\lambda}\}.$$

We set $p:=\frac{\log l}{\log t}$ and choose $s>\frac{1+p}{2}$. Suppose (for a contradiction) that $\mathscr{H}^s(E_t)\neq 0$ (where $\mathscr{H}^s$ denotes the Hausdorff measure for the value s). There exists a Borel set $F\subset E_t$ and $b>0$ such that:

(a) $0<\mathscr{H}^s(F)<+\infty$ [F, Theorem 4.10]; and
(b) $\mathscr{H}^s(F\cap B_r(x))\le br^s$, $\forall x$ [F, Proposition 4.11].

We let $\nu=\mathscr{H}^s|_F$ be the restriction of the measure $\mathscr{H}^s$ to the set F.

We want to repeat the proof of Theorem 1 with the measure ν replacing Lebesgue measure. The only changes required are in the proof of Lemma 2.

More specifically, we can write as before

$$\int_{E_t} \frac{d\nu(\lambda)}{|f(\lambda, \mathbf{i}, \mathbf{j})|^p} = \int_{I' \cap E_t} \frac{d\nu(\lambda)}{|f(\lambda, \mathbf{i}, \mathbf{j})|^p} + \int_{I'' \cap E_t} \frac{d\nu(\lambda)}{|f(\lambda, \mathbf{i}, \mathbf{j})|^p}$$
$$\leq \left(\frac{6n}{c}\right)^p + \int_{I'' \cap E_t} \frac{d\nu(\lambda)}{|f(\lambda, \mathbf{i}, \mathbf{j})|^p}.$$

For each of the (finitely many) λ_0 satisfying $f(\lambda_0) = 0$ we denote $A_n(\lambda_0) = \{\lambda \in F : \frac{1}{n+1} \leq |\lambda_0 - \lambda| \leq \frac{1}{n}\}$, for $n \geq 1$, then we can use (b) above to estimate $\nu(A_n(\lambda_0)) = O(\frac{1}{n^{2s}})$. We can then bound

$$\int_{I'' \cap E_t} \frac{d\nu(\lambda)}{|f(\lambda, \mathbf{i}, \mathbf{j})|^p} = O\left(\sum_{\lambda_0} \sum_{n=1}^{\infty} \int_{A_n(\lambda_0)} \frac{d\nu(\lambda)}{|\lambda - \lambda_0|^p}\right)$$
$$= O\left(\sum_{\lambda_0} \sum_{n=1}^{\infty} \nu(A_n(\lambda_0)) n^p\right)$$
$$= O\left(\sum_{n=1}^{\infty} \frac{1}{n^{2s-p}}\right)$$

which is finite by the choice of s. The proof of Theorem 1 can now be applied to allow us to conclude that $\nu(F) = 0$, giving the required contradiction (to property (a)). In particular, we see that $\mathscr{H}^s(E_t) = 0$ whenever $s > \frac{1+p}{2}$ i.e. $\dim_H E_t \leq \frac{1+p}{2} < 1$.

References

[F1] K. Falconer, *The Hausdorff dimension of self-affine fractals*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 339–350.

[F] ——, *Fractal geometry*, Cambridge Univ. Press, Cambridge, 1990.

[KSS] M. Keane, M. Smorodinsky and B. Solomyak, *Cantor criticality*, Talk delivered by M. Keane (Warwick conference on $\mathbb{Z}^n$-actions), September 1993.

[PT] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge Univ. Press, Cambridge, 1993.

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