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A DISCRETE TRANSFORM AND TRIEBEL-LIZORKIN SPACES ON THE BIDISC

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ABSTRACT. We use a discrete transform to study the Triebel-Lizorkin spaces on bidisc $\dot{F}_p^{\alpha q}$, $\dot{f}_p^{\alpha q}$ and establishes the boundedness of transform $S_\phi : \dot{F}_p^{\alpha q} \to \dot{f}_p^{\alpha q}$ and $T_\psi : \dot{f}_p^{\alpha q} \to \dot{F}_p^{\alpha q}$. We also define the almost diagonal operator and prove its boundedness. With the use of discrete transform and Journé lemma, we get the atomic decomposition of $\dot{f}_p^{\alpha q}$ for $0 < p \leq 1$, $p \leq q < \infty$. The atom supports on an open set, not a rectangle. Duality $(\dot{f}_1^{\alpha q})^* = \dot{f}_\infty^{-\alpha q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$, $q > 1$, $\alpha \in R$, is established, too. The case for $\dot{F}_p^{\alpha q}$ is similar.

0. INTRODUCTION

In this paper, we use a discrete transform to study the Triebel-Lizorkin spaces on the bidisc.

In §1 we define the sequence space $\dot{f}_p^{\alpha q}$ and the distribution space $\dot{F}_p^{\alpha q}$. Our fundamental formula is $f = \sum_Q \langle f, \phi_Q \rangle \psi_Q$, where Q runs over all dyadic rectangles and ϕ_Q, ψ_Q are translates and dilates of ϕ, ψ associated with Q respectively. Fourier transforms of ϕ, ψ have compact support, and

$$\operatorname{supp} \hat{\phi}_Q, \hat{\psi}_Q \subset \{(\xi_1, \xi_2); 2^{\nu_1 - 1} < |\xi_1| \leq 2^{\nu_1+1}, 2^{\nu_2-1} < |\xi_2| \leq 2^{\nu_2+1}\}$$

for $l_1(Q) = 2^{-\nu_1}$, $l_2(Q) = 2^{-\nu_2}$. Thus $\langle \phi_Q, \psi_P \rangle = 0$ unless

$$\frac{1}{2} \leq \frac{l_i(Q)}{l_i(P)} \leq 2, \qquad i = 1, 2.$$

It will be simpler to study harmonic analysis on product spaces using this kind of expansion rather than the expansion in [2], where ϕ has compact support but its Fourier transform is supported on the whole space. We prove $S_\phi : \dot{F}_p^{\alpha q} \to \dot{f}_p^{\alpha q}$ and $T_\psi : \dot{f}_p^{\alpha q} \to \dot{F}_p^{\alpha q}$ are bounded. In its proof, we use the strong maximal function $M_S f(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\, dy$, where Q runs over all dyadic rectangles. Although M_S is not weak L^1 bounded, it is L^p bounded for $p > 1$ [4].

In §2 we define almost diagonal operators and prove the boundedness of such operators on $\dot{f}_p^{\alpha q}$ by duality. We also define (δ, M) rectangle molecules

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$\{m_Q\}_Q$ and prove the $\dot{F}_p^{\alpha q}$ norm of $f = \sum_Q s_Q m_Q$ is less than the $\dot{f}_p^{\alpha q}$ norm of $\{s_Q\}_Q$.

In §3 we define $\dot{f}_\infty^{\alpha q}$ and $\dot{F}_\infty^{\alpha q}$. For $s = \{s_Q\}_Q$,

$$\|s\|_{\dot{f}_\infty^{\alpha q}} = \sup_\Omega \left(\frac{1}{|\Omega|} \int_\Omega \sum_{Q \subset \Omega} (|Q|^{-\frac{\alpha}{2}} |s_Q| \tilde{\chi}_Q)^q \, dx\, dy \right)^{\frac{1}{q}},$$

where the sup is taken over all open sets Ω, not only dyadic rectangles. This is similar to the characterization of $\mathrm{BMO}(R_+^2 \times R_+^2)$ [2]. Applying the Journé Lemma, we establish the case $p = +\infty$. Finally, we get the atomic decomposition of $\dot{f}_p^{\alpha q}$, $\dot{F}_p^{\alpha q}$ for $0 < p \le 1$, $p \le q < +\infty$. These atoms must be supported on open sets. We cannot get the atomic decomposition supported on rectangles. In fact, L. Carleson gave the counterexample for $H^1(R_+^2 \times H_+^2)$ [1]. Thus combining the discrete transform and Journé Lemma, we give another method by which to obtain the atomic decomposition on product spaces.

1. Discrete transform

Let ϕ_0, ψ_0 satisfy

(A) $\phi_0, \psi_0 \in S(R)$, $S(R)$ is Schwartz space;

(B) $\operatorname{supp} \hat{\phi}_0, \hat{\psi}_0 \subset \{\xi \in R, \frac{1}{2} \le |\xi| \le 2\}$;

(C) $|\hat{\phi}_0(\xi)|, |\hat{\psi}_0(\xi)| \ge c > 0$, if $\frac{3}{5} \le |\xi| \le \frac{5}{3}$;

(D) $\sum_{\nu \in \mathbb{Z}} \overline{\hat{\phi}_0(2^\nu \xi)} \hat{\psi}_0(2^\nu \xi) = 1$, if $\xi \ne 0$.

We put $\phi(x, y) = \phi_0(x)\phi_0(y)$, $\psi(x, y) = \psi_0(x)\psi_0(y)$. Then for $\xi_1 \xi_2 \ne 0$

$$\sum_{\nu_1 \in \mathbb{Z}} \sum_{\nu_2 \in \mathbb{Z}} \overline{\hat{\phi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2)} \hat{\psi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2) = 1.$$

For $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$, $k = (k_1, k_2) \in \mathbb{Z}^2$, we denote by $Q_{\nu k}$ the dyadic rectangle

$$\{(x, y) \in R^2 | k_1 \le 2^{\nu_1} x \le k_1 + 1, k_2 \le 2^{\nu_2} y \le k_2 + 1\},$$

with sidelength $l_1(Q_{\nu k}) = 2^{-\nu_1}$, $l_2(Q_{\nu k}) = 2^{-\nu_2}$ and corner $x_{Q_{\nu k}} = 2^{-\nu_1} k_1$, $y_{Q_{\nu k}} = 2^{-\nu_2} k_2$. For $Q = Q_{\nu k}$, denote

$$\phi_Q(x, y) = |Q|^{-\frac{1}{2}} \phi(2^{\nu_1} x - k_1, 2^{\nu_2} y - k_2),$$

where area $|Q| = l_1(Q) \cdot l_2(Q)$.

Let $S'(R^2)$ be the space of tempered distribution space and $\mathfrak{P}$ the space of distributions whose Fourier transform is supported on the x-axis and y-axis. Then $\langle f, \phi_Q \rangle$ is well defined for $f \in S'(R^2)/\mathfrak{P}$. The discrete transform S_ϕ is defined as

$$S_\phi f = \{(S_\phi f)_Q\}_Q, \qquad (S_\phi f)_Q = \langle f, \phi_Q \rangle$$

where Q runs over all dyadic rectangles. Its inverse T_ψ maps every sequence $s = \{s_Q\}_Q$ into a distribution

$$f = \sum_Q s_Q \psi_Q \in S'/\mathfrak{P}.$$

The basis of the discrete transform lies in the following expression (see [6, Lemma 2.1] for the one-parameter case).

Lemma 1.1. *If ϕ, ψ are defined as above and $f \in S'/\mathfrak{P}$, then*

$$f = \sum_Q \langle f, \phi_Q \rangle \psi_Q$$

holds in $S'/\mathfrak{P}$, so $T_\psi \circ S_\phi$ is the identity on $S'/\mathfrak{P}$.

Define the sequence space $\dot{f}_p^{\alpha q}$ as follows. For $s = \{s_Q\}_Q$, where Q runs over all dyadic rectangles, we define the norm

$$\|s\|_{\dot{f}_p^{\alpha q}} = \left\| \left(\sum_Q (|Q|^{-\frac{\alpha}{2}} |s_Q| \tilde{\chi}_Q)^q \right)^{\frac{1}{q}} \right\|_{L^p},$$

where $\tilde{\chi}_Q = |Q|^{-\frac{1}{2}} \chi_Q$. The corresponding distribution space is $\dot{F}_p^{\alpha q}$, and it consists of $f \in S'/\mathfrak{P}$ with

$$\|f\|_{\dot{F}_p^{\alpha q}} = \left\| \left(\sum_{\nu_1, \nu_2 \in \mathbb{Z}} (2^{(\nu_1+\nu_2)\frac{\alpha}{2}} |\phi_{\nu_1, \nu_2} * f(x, y)|)^q \right)^{\frac{1}{q}} \right\|_{L^p} < \infty,$$

where $\phi_{\nu_1, \nu_2}(x, y) = 2^{-\nu_1-\nu_2} \phi(2^{-\nu_1} x, 2^{-\nu_2} y)$. A useful tool to study harmonic analysis on product spaces is the g_λ^* function [4]

$$\begin{aligned}(g_\lambda^* f)^2(x, y) = \int_{R_+^2 \times R_+^2} & |f * \psi_{t_1 t_2}(u, v)|^2 \\ & \cdot \left(\frac{1}{1+|x-u|/t_1} \right)^\lambda \left(\frac{1}{1+|y-v|/t_2} \right)^\lambda du\, dv \frac{dt_1 dt_2}{t_1^2 t_2^2}\end{aligned}$$

where $\psi_{t_1 t_2}(u, v) = \psi(u/t_1, v/t_2)/t_1 t_2$. We define the corresponding sequence $s_r^* = \{(s_Q^*)_Q\}_Q$ for a sequence s by

$$(s_r^*)_Q = \left(\sum_{\substack{l_1(P)=l_1(Q) \\ l_2(P)=l_2(Q)}} |s_P|^r (1 + l_1^{-1}(P)|x_P - x_Q|)^{-\lambda} (1 + l_2^{-1}(P)|y_P - y_Q|^{-\lambda}) \right)^{\frac{1}{r}}$$

for some λ. The main property of s_r^* is

Theorem 1.2. *Let $\alpha \in R$, $0 < p < \infty$, $0 < q \leq \infty$, $\lambda > 1$, and $r = \min(p, q)$. Then*

$$\|s\|_{\dot{f}_p^{\alpha q}} \sim \|s_r^*\|_{\dot{f}_p^{\alpha q}}.$$

We need the following lemma, which can be proved as in [6].

Lemma 1.3. *Let $0 < a \leq r < \infty$, $\lambda > \frac{r}{a}$, $l_1(Q) = 2^{-q_1}$, and $l_2(Q) = 2^{-q_2}$.*

Then

$$\left(\sum_{\substack{l_1(P)=2^{-q_1}\\ l_2(P)=2^{-q_2}}} |s_P|^r(1+l_1^{-1}(P)|x_P-x_Q|)^{-\lambda}(1+l_2^{-1}(P)|y_P-y_Q|^{-\lambda})\right)^{\frac{1}{r}}$$

$$\le C\left(M_S\left(\sum_{\substack{l_1(P)=l_1(Q)\\ l_2(P)=l_2(Q)}} |s_P|^a\chi_P\right)\right)^{\frac{1}{a}}(x), \qquad x\in Q,$$

where $M_S(f)$ is the strong maximal function of f.

Proof of Theorem 1.2. $\|s\|_{\dot{f}_p^{\alpha q}} \le \|s_r^*\|_{\dot{f}_p^{\alpha q}}$ is obvious. To prove the converse, we take the sum for Q with same sidelength in both sides of Lemma 1.3

$$\sum_{\substack{l_1(Q)=2^{-q_1}\\ l_2(Q)=2^{-q_2}}} (s_r^*)_Q\tilde{\chi}_Q \le C\left(M_S\left(\sum_{\substack{l_1(P)=2^{-q_1}\\ l_2(P)=2^{-q_2}}} |s_P|\tilde{\chi}_P\right)^a\right)^{\frac{1}{a}},$$

so

$$\|s_r^*\|_{\dot{f}_p^{\alpha q}} \le C\left\|\left(\sum_{q_1,q_2\in\mathbb{Z}}\left(M_S\left(\sum_{\substack{l_1(P)=2^{-q_1}\\ l_2(P)=2^{-q_2}}} |P|^{-\frac{\alpha}{2}}|s_P|\tilde{\chi}_P\right)^a\right)^{\frac{q}{a}}\right)^{\frac{a}{q}}\right\|_{L^{\frac{p}{a}}}^{\frac{1}{a}}.$$

We will use a vector-valued maximal inequality to control the right side. Suppose $(\sum_k |f_k(x,y)|^q)^{\frac{1}{q}} \in L^p$. Then by Fubini's theorem there exists a set E with zero measure such that for $x\notin E$, $(\sum_k |f_k(x,y)|^q)^{\frac{1}{q}} \in L^p$ as a function of y. Apply the Fefferman-Stein vector-valued maximal inequality [3] to this function to get

$$\left\|\left(\sum_k |M^{(2)}f_k(x,\cdot)|^q\right)^{\frac{1}{q}}\right\|_{L^p} \le C\left\|\left(\sum_k |f_k(x,\cdot)|^q\right)^{\frac{1}{q}}\right\|_{L^p},$$

for $p>1$, $1<q<\infty$, where $M^{(2)}$ is the maximal operator for the second variable. Apply the vector-valued maximal inequality to the first variable, and notice $M_S \le M^{(1)}M^{(2)}$ to get

$$\left\|\left(\sum_k |M_S f_k|^q\right)^{\frac{1}{q}}\right\|_{L^p} \le C\left\|\left(\sum_k |f_k|^q\right)^{\frac{1}{q}}\right\|_{L^p}.$$

Let $r=\min(p,q)$, $\varepsilon=\lambda-1>0$, and $a=\frac{r}{1+\varepsilon/2}$. Then $0<a<r$, $\lambda>\frac{r}{a}$, $\frac{p}{a}>1$, $\frac{q}{a}>1$, so we can use the above inequality

$$\|s_r^*\|_{\dot{f}_p^{\alpha q}} \le C\left\|\left(\sum_P (|P|^{-\frac{\alpha}{2}}|s_P|\tilde{\chi}_P)^q\right)^{\frac{1}{q}}\right\|_{L^p} = c\|s\|_{\dot{f}_p^{\alpha q}}.$$

The following result about the operators S_ϕ, T_ψ is similar to [6], so we will not give the details.

Theorem 1.4. *The operator* $S_\phi : \dot{F}_p^{\alpha q} \to \dot{f}_p^{\alpha q}$ *and* $T_\psi : \dot{f}_p^{\alpha q} \to \dot{F}_p^{\alpha q}$ *are bounded, and* $T_\psi \circ S_\phi$ *is the identity.*

Notice

$$\langle f, \phi_Q \rangle = |Q|^{\frac{1}{2}} \tilde{\phi}_{\nu_1 \nu_2} * f(x_Q, y_Q),$$

where $l_1(Q) = 2^{-\nu_1}$, $l_2(Q) = 2^{-\nu_2}$, and $\tilde{\phi}(\cdot) = \overline{\phi(-\cdot)}$. We define

$$\sup(f) = \{\sup_Q f\}_Q, \qquad \sup_Q f = |Q|^{\frac{1}{2}} \sup_{(x,y)\in Q} |\tilde{\phi}_{\nu_1\nu_2} * f(x, y)|,$$
$$\inf_r(f) = \{\inf_{Q,r}(f)\}_Q, \qquad r \text{ is a positive integer,}$$

where

$$\inf_{Q,r}(f) = \max_{\tilde{Q}} \{ |Q|^{\frac{1}{2}} \inf_{(x,y)\in\tilde{Q}} |\tilde{\phi}_{\nu_1\nu_2} * f(x, y)|, \, l_1(\tilde{Q}) = 2^{-r} l_1(Q),$$
$$l_2(\tilde{Q}) = 2^{-r} l_2(Q), \, \tilde{Q} \subset Q\}.$$

These three norms are equivalent.

Lemma 1.5. *If* $f \in S'/\mathfrak{P}$, *then*

$$\|f\|_{\dot{F}_p^{\alpha q}} \sim \|\sup(f)\|_{\dot{f}_p^{\alpha q}} \sim \|\inf_r(f)\|_{\dot{f}_p^{\alpha q}}.$$

Using this lemma we can prove Theorem 1.4 very easily as in [6].

Corollary 1.7. $\dot{F}_p^{\alpha q}$ *is independent of the choice of* ϕ.

2. Almost diagonal operator and smooth rectangle molecules

Similarly to [7], we define $\omega_{QP}(\varepsilon)$ for two rectangles P, Q. It decays rapidly as the distance between these two rectangles or the ratio of their sidelengths becoming large. Suppose $Q = Q_1 \times Q_2$, $P = P_1 \times P_2$,

$$\omega_{QP}(\varepsilon) = \omega_{Q_1P_1}(\varepsilon)\omega_{Q_2P_2}(\varepsilon),$$
$$\omega_{Q_1P_1}(\varepsilon) = \left(\frac{l(Q_1)}{l(P_1)}\right)^{\alpha} \left(1 + \frac{|x_{Q_1} - x_{P_1}|}{\max(l(P_1), l(Q_1))}\right)^{-J-\varepsilon}$$
$$\cdot \min\left(\left(\frac{l(Q_1)}{l(P_1)}\right)^{\frac{1+\varepsilon}{2}}, \left(\frac{l(P_1)}{l(Q_1)}\right)^{\frac{1+\varepsilon}{2}+J-1}\right),$$

where $J = 1/\min(1, p, q)$. An operator A on $\dot{f}_p^{\alpha q}$ is called almost diagonal if its associate matrix $\{a_{QP}\}_{Q,P}$ satisfies

$$\sup_{Q,P} \frac{|a_{QP}|}{\omega_{QP}(\varepsilon)} < \infty,$$

for some $\varepsilon > 0$.

Theorem 2.1. *An almost diagonal operator on* $\dot{f}_p^{\alpha q}$ $(0 < p < \infty,\ 0 < q \le \infty)$ *is bounded.*

Proof. We only need to consider the case $\alpha = 0$, because the general case can be reduced to it as in [7]. Suppose $q \ge 1$, $p \ge 1$, $s = \{s_Q\}$; and denote

$$A = A_1 + A_2 + A_3 + A_4,$$
$$(As)_Q = (A_1 s)_Q + (A_2 s)_Q + (A_3 s)_Q + (A_4 s)_Q$$
$$= \left(\sum_{\substack{l_1(Q)<l_1(P)\\ l_2(Q)<l_2(P)}} + \sum_{\substack{l_1(Q)\ge l_1(P)\\ l_2(Q)<l_2(P)}} + \sum_{\substack{l_1(Q)>l_1(P)\\ l_2(Q)\ge l_2(P)}} + \sum_{\substack{l_1(Q)\le l_1(P)\\ l_2(Q)\ge l_2(P)}} \right) a_{QP} s_P.$$

Notice the dual of $\dot{f}_p^{0q}$ is $\dot{f}_{p'}^{0q'}$ by $(L^p(l^q))^* = L^{p'}(l^{q'})$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Let $t = \{t_Q\}_Q \in \dot{f}_{p'}^{0q'}$, $\|t\|_{\dot{f}_{p'}^{0q'}} \le 1$; and notice $J = 1$ when $p \ge 1$, $q \ge 1$. So

$$|\langle A_4 s, t\rangle| = \left| \sum_Q (A_4 s)_Q t_Q \right| \le C \sum_Q \sum_{\substack{l_1(P)\ge l_1(Q)\\ l_2(P)\le l_2(Q)}} \omega_{QP}(\varepsilon) |s_P|\, |t_Q|$$
$$\le \sum_Q \sum_{\substack{p_1\le q_1\\ p_2\ge q_2}} \sum_{\substack{l_1(P)=2^{-p_1}\\ l_2(P)=2^{-p_2}}} 2^{(-q_1+p_1-p_2+q_2)\frac{1+\varepsilon}{2}} \frac{|s_P|}{(1+l_1^{-1}(P)|x_P - x_Q|)^{1+\varepsilon}}$$
$$\cdot \frac{|t_Q|}{(1+l_2^{-1}(Q)|y_Q - y_P|)^{1+\varepsilon}}.$$

At first we take the sum over P with P_2 fixed and $l_1(P) = 2^{-p_1}$, so

$$|\langle A_4 s, t\rangle| \le C \sum_Q \sum_{\substack{p_1\le q_1\\ p_2\ge q_2}} \sum_{l_2(P)=2^{-p_2}} 2^{(-q_1+p_1-p_2+q_2)\frac{1+\varepsilon}{2}}$$
$$\cdot M^{(1)} \left(\sum_{\substack{l_1(P)=2^{-p_1}\\ P_2 \text{ fixed}}} |s_P| \chi_P \right) \chi_{Q_1}(x) \cdot \frac{|t_Q|}{(1+l_2^{-1}(Q)|y_P - y_Q|)^{1+\varepsilon}}.$$

Then take the sum over Q with $l_2(Q) = 2^{-q_2}$ and Q_1 fixed, so

$$|\langle A_4 s, t\rangle| \le C \sum_{Q_1, P_2} \sum_{\substack{p_1\le q_1\\ p_2\ge q_2}} 2^{(-q_1+p_1-p_2+q_2)\frac{1+\varepsilon}{2}} M^{(1)} \left(\sum_{\substack{l_1(P)=2^{-p_1}\\ P_2 \text{ fixed}}} |s_P| \chi_P \right) \chi_{Q_1}(x)$$
$$\cdot M^{(2)} \left(\sum_{\substack{l_2(Q)=2^{-q_2}\\ Q_1 \text{ fixed}}} |t_Q| \chi_Q \right) \chi_{P_2}(y),$$

for each Q_1, P_2; doing the same thing but averaging over $Q_1 \times P_2$ shows that

$$|\langle A_4 s, t\rangle| \le \sum_{q_1, q_2} \sum_{\substack{p_1 \le q_1 \\ p_2 \ge q_2}} \sum_{Q_1, P_2} \int_{Q_1 \times P_2} 2^{(-q_1+p_1-p_2+q_2)\frac{1+\varepsilon}{2}} M^{(1)} \left(\sum_{l_1(P)=2^{-p_1}, P_2} |s_P| \chi_P \right)$$

$$\cdot M^{(2)} \left(\sum_{l_2(Q)=2^{-q_2}, Q_1} |t_Q| \chi_Q \right) \chi_{Q_1 \times P_2} \cdot 2^{q_1+p_2}\, dx\, dy$$

$$\le C \sum_{\substack{p_1 \le q_1 \\ p_2 \le q_2}} \int_{R^2} 2^{(-q_1+p_1-p_2+q_2)\frac{\varepsilon}{2}} M^{(1)} \left(\sum_{\substack{l_1(P)=2^{-p_1} \\ l_2(P)=2^{-p_2}}} |s_P| \tilde{\chi}_P \right)$$

$$\cdot M^{(2)} \left(\sum_{\substack{l_1(Q)=2^{-q_1} \\ l_2(Q)=2^{-q_2}}} |t_Q| \tilde{\chi}_Q \right) dx\, dy.$$

By Hölder's inequality and summation, we get

$$|\langle A_4 s, t\rangle| \le C \left\| \left(\sum_{p_1, p_2} \left(M^{(1)} \left(\sum_{\substack{l_1(P)=2^{-p_1} \\ l_2(P)=2^{-p_2}}} |s_P| \tilde{\chi}_P \right) \right)^q \right)^{\frac{1}{q}} \right\|_{L^p}$$

$$\cdot \left\| \left(\sum_{q_1, q_2} \left(M^{(2)} \left(\sum_{\substack{l_1(Q)=2^{-q_1} \\ l_2(Q)=2^{-q_2}}} |t_Q| \tilde{\chi}_Q \right) \right)^{q'} \right)^{\frac{1}{q'}} \right\|_{L^{p'}} .$$

By the vector-valued maximal inequality,

$$|\langle A_4 s, t\rangle| \le C \left\| \left(\sum_P (|s_P| \tilde{\chi}_P)^q \right)^{\frac{1}{q}} \right\|_{L^p} \cdot \left\| \left(\sum_Q (|t_Q| \tilde{\chi}_Q) \right)^{q'} \right)^{\frac{1}{q'}} \right\|_{L^{p'}}$$

$$\le C \|s\|_{\dot{f}_p^{0q}}.$$

Hence, A_2 is continuous on $\dot{f}_p^{0q}$. Similarly, A_1, A_2, A_3 are also continuous. For $\min(p, q) = r < 1$, take $\tilde{r} < r$ sufficiently close for and defined matrix $\widetilde{A}$ and sequence $\tilde{t}$ by

$$\widetilde{A} = (\tilde{a}_{QP})_{QP}, \quad \tilde{a}_{QP} = |a_{QP}|^{\tilde{r}} \left(\frac{|Q|}{|P|} \right)^{\frac{1}{2} - \frac{\tilde{r}}{2}},$$

$$\tilde{t} = \{\tilde{t}_Q\}_Q, \quad \tilde{t}_Q = |Q|^{\frac{1}{2} - \frac{\tilde{r}}{2}} |s_Q|^{\tilde{r}}.$$

We have

$$\|s\|_{\dot{f}_p^{0q}} = \|\tilde{t}\|^{1/\tilde{r}}_{\dot{f}^{0, q/\tilde{r}}_{p/\tilde{r}}},$$

and $\widetilde{A}$ is almost diagonal on $\dot{f}^{0,q/\tilde{r}}_{p/\tilde{r}}$ for some other ε. We can deduce the boundness of A from the boundness of $\widetilde{A}$.

Now we generalize the inequality $\|\sum_Q s_Q\psi_Q\|_{\dot{F}^{\alpha q}_p} \le C\|s\|_{\dot{f}^{\alpha q}_p}$ to a more general family of functions $\{m_Q\}_Q$. We prove it holds when $\{m_Q\}_Q$ are smooth rectangle molecules. Let $J = 1/\min(p, q, 1)$, $N = \max([J-1-\alpha], -1)$, $\alpha^* = \alpha-[\alpha]$. For $\alpha^* < \delta \le 1$, $M > J$, we say $\{m_Q\}_Q$ is a family of (δ, M) smooth rectangle molecules for $\dot{F}^{\alpha q}_p$ if for every rectangle $Q = Q_1 \times Q_2$, there exist a_{Q_1}, b_{Q_2} such that $\{a_Q\}_Q$, $\{b_Q\}_Q$ are two families of smooth molecules for $\dot{F}^{\alpha q}_p(R^1)$ (see [7]) and $m_Q(x, y) = a_{Q_1}(x)\cdot b_{Q_2}(y)$.

Lemma 2.2 [7]. *If $\{a_Q\}_Q$ is a family of smooth molecules for $\dot{F}^{\alpha q}_p(R^1)$, then there exist $\varepsilon = \varepsilon(\alpha, p, q, \delta)$ and a constant C independent on the form of molecules, such that*

$$|a_{QP}| = |\langle a_P, \phi_{0Q}\rangle| \le C\omega_{QP}(\varepsilon),$$

where $\phi_{0Q} = 2^{\frac{\nu}{2}}\phi_0(2^\nu x - k)$, $Q = [2^{-\nu}k, 2^{-\nu}(k+1)]$.

Theorem 2.3. *If $f = \sum_Q s_Q m_Q$ and $\{m_Q\}_Q$ is a family of (δ, M) smooth rectangle molecules, then $\|f\|_{\dot{F}^{\alpha q}_p} \le C\|s\|_{\dot{f}^{\alpha q}_p}$.*

Proof. Expanding m_P as

$$m_P = \sum_Q \langle m_P, \phi_Q\rangle\psi_Q,$$

we get

$$f = \sum_P\sum_Q \langle m_P, \phi_Q\rangle\psi_Q s_P = \sum_Q\left(\sum_P \langle m_P, \phi_Q\rangle s_P\right)\psi_Q.$$

Let the matrix $(a_{QP})_{QP}$ be defined by

$$a_{QP} = \langle m_P, \phi_Q\rangle = \langle a_{P_1}, \phi_{0Q_1}\rangle\langle b_{P_2}, \phi_{0Q_2}\rangle.$$

Applying Lemma 2.2, we get $|a_{QP}|/\omega_{QP}(\varepsilon) \le C^2$ for every two dyadic rectangles. Thus A is almost diagonal and Theorem 2.2 is proved by Theorem 2.1.

3. The case $p = +\infty$ and the atomic decomposition

Let $\dot{F}^{\alpha q}_\infty$ $(0 < q < \infty)$ consist of distributions in $S'/\mathfrak{P}$ satisfying

$$\|f\|_{\dot{F}^{\alpha q}_\infty} = \sup_\Omega\left(\frac{1}{|\Omega|}\int_\Omega \sum_{Q_{\nu k}\subset\Omega}(2^{(\nu_1+\nu_2)\frac{\alpha}{2}}|\tilde{\phi}_{\nu_1\nu_2} * f(x, y)|)^q\, dx\, dy\right)^{\frac{1}{q}} < +\infty,$$

where Ω runs over all open sets. We say $s = \{s_Q\}_Q \in \dot{f}^{\alpha q}_\infty$ if and only if

$$\|s\|_{\dot{f}^{\alpha q}_\infty} = \sup_Q\left(\frac{1}{|\Omega|}\int_\Omega \sum_{Q\subset\Omega}(|Q|^{-\frac{\alpha}{2}}|s_Q|\tilde{\chi}_Q)^q\, dx\, dy\right)^{\frac{1}{q}} < +\infty,$$

where Ω also runs over all open sets. When $\alpha = 0$, $q = 2$,

$$\|s\|_{\dot{f}^{02}_\infty} = \sup_\Omega\left(\frac{1}{|\Omega|}\sum_{Q\subset\Omega}|s_Q|^2\right)^{\frac{1}{2}}.$$

This is similar to the characterization of $\mathrm{BMO}(R_+^2 \times R_+^2)$ in [2]. At first we establish the boundness of S_ϕ, T_ψ.

Lemma 3.1. $\|s_q^*\|_{\dot{f}_\infty^{\alpha q}} \sim \|s\|_{\dot{f}_\infty^{\alpha q}}$, *if* $\lambda > 1$.

Proof. $\|s\|_{\dot{f}_\infty^{\alpha q}} \le \|s_q^*\|_{\dot{f}_\infty^{\alpha q}}$ is obvious. For the converse, we consider Ω a rectangle at first. Let $r = \{r_Q\}$, where $r_Q = s_Q$ if $Q_1 \cap 2\gamma_1 P_1 = \varnothing$ for a fixed rectangle P and otherwise, $r_Q = 0$. Then

$$\frac{1}{|P|}\int_P \sum_{Q\subset P}(|Q|^{-\frac{\alpha}{2}}(r_q^*)_Q\tilde{\chi}_Q)^q\,dx\,dy$$

$$= \frac{1}{|P|}\sum_{Q\subset P}\sum_{l(\tilde{Q})=l(Q)}|Q|(|Q|^{-\frac{\alpha}{2}-\frac{1}{2}}|r_{\tilde{Q}}|)^q\left(1+\frac{|x_Q-x_{\tilde{Q}}|}{l_1(Q)}\right)^{-\lambda}\left(1+\frac{|y_Q-y_{\tilde{Q}}|}{l_2(Q)}\right)^{-\lambda},$$

where $l(Q) = (l_1(Q), l_2(Q)) \in R^2$. Let $P_{(j)} = P + (j_1 l_1(P), j_2 l_2(P))$ be the translate of P. For $l_1(Q) = 2^{-k_1}l_1(P)$, $l_2(Q) = 2^{-k_2}l_2(P)$,

$$1 + l_1(Q)^{-1}|x_Q - x_{\tilde{Q}}| \sim 2^{k_1}|j_1|, \quad 1 + l_2(Q)^{-1}|y_Q - y_{\tilde{Q}}| \sim 2^{k_2}|j_2|, \qquad |j_2| \ge 2,$$

when $\tilde{Q} \subset P_{(j)}$. So the quantity we want to estimate is

$$\le C\sum_{|j_1|\ge\gamma_1,\,|j_2|\ge 2}|j_1|^{-\lambda}|j_2|^{-\lambda}\sum_{k_1,k_2=0}^{\infty}2^{-(k_1+k_2)\lambda}|P|^{-1}\sum_{\tilde{Q}\subset P_{(j)}}|\tilde{Q}|(|\tilde{Q}|^{-\frac{\alpha}{2}-\frac{1}{2}}|r_{\tilde{Q}}|)^q 2^{k_1+k_2}$$

$$+\sum_{Q\subset P}\sum_{|j_2|\le 1}\sum_{\substack{l(Q)=l(\tilde{Q})\\ \tilde{Q}\subset P_{(j)}}}\left(1+\frac{|x_Q-x_{\tilde{Q}}|}{l_1(Q)^{-1}}\right)^{-\lambda}\left(1+\frac{|y_Q-y_{\tilde{Q}}|}{l_2(Q)^{-1}}\right)^{-\lambda}$$

$$\cdot\frac{1}{|P|}|\tilde{Q}|(|\tilde{Q}|^{-\frac{\alpha}{2}-\frac{1}{2}}|r_{\tilde{Q}}|)^q$$

$$\le C\gamma_1^{-\lambda+1}\|s\|^q_{\dot{f}_\infty^{\alpha q}}$$

by

$$\sum_{Q\subset P}\sum_{|j_2|\le 1}\sum_{\substack{l(Q)=l(\tilde{Q})\\ \tilde{Q}\subset P_{(j)}}}(1+l_1(Q)^{-1}|x_Q-x_{\tilde{Q}}|)^{-\lambda}(1+l_2(Q)^{-1}|y_Q-y_{\tilde{Q}}|)^{-\lambda}$$

$$\le c2^{k_1\lambda}|j_1|^{-\lambda}2^{k_1}\sum_{m\in\mathbb{Z}}(1+|m|)^{-\lambda} \le c2^{k_1(1-\lambda)}|j_1|^{-\lambda}$$

$$\le c|j_1|^{-\lambda}, \qquad \text{since } k_1 \ge 0.$$

Similarly, for the sequence t defined by $t_Q = s_Q$ when $Q_2 \cap 2\gamma_2 P = \varnothing$ and $t_Q = 0$ otherwise, we have

$$\frac{1}{|P|}\int_P\sum_{Q\subset P}(|Q|^{-\frac{\alpha}{2}}(t_q^*)_Q\tilde{\chi}_Q)^q\,dx\,dy \le C\gamma_2^{-\lambda+1}\|s\|^q_{\dot{f}_\infty^{\alpha q}}.$$

Now we fix an open set Ω. Let $\mu(\Omega)$ be the set of maximal rectangles contained in Ω, and let $\mu^{(1)}(\Omega)$, $\mu^{(2)}(\Omega)$ be the set of maximal rectangles in the x or y direction respectively [5]. Let

$$\Omega^{i+1} = \{(x, y), M_S(\chi_{\Omega^i})(x, y) > \tfrac{1}{2}\},$$

where χ_{Ω^i} is the characteristic function of Ω^i and $\Omega^0 = \Omega$. Take sequences $r = \{r_Q\}_Q$, by

$$r_Q = s_Q \text{ if } Q \subset \Omega^4, \quad \text{and} \quad r_Q = 0 \text{ otherwise,}$$

and $t = \{t_Q\}_Q$ with $t_Q = s_Q - r_Q$. Obviously,

$$\frac{1}{|\Omega|}\int_\Omega \sum_{Q\subset\Omega} (|Q|)^{-\frac{\alpha}{2}}(s_q^*)_Q\tilde{\chi}_Q)^q\,dx\,dy = \frac{1}{|\Omega|}\int_\Omega \sum_{Q\subset\Omega} (|Q|^{-\frac{\alpha}{2}}(r_q^*)_Q\tilde{\chi}_Q)^q\,dx\,dy$$
$$+\frac{1}{|\Omega|}\int_\Omega \sum_{Q\subset\Omega} (|Q|^{-\frac{\alpha}{2}}(t_q^*)_Q\tilde{\chi}_Q)^q\,dx\,dy.$$

The estimate of first term is easy. By M_S being L^2 bounded,

$$|\Omega| \le |\Omega^i| \le C|\Omega|, \qquad i = 1, 2, 3, 4,$$

for some constant C independent of Ω. So

$$\frac{1}{|\Omega|}\int_\Omega \sum_{Q\subset\Omega} (|Q|^{-\frac{\alpha}{2}}(r_q^*)_Q\tilde{\chi}_Q)^q\,dx\,dy \le \frac{1}{|\Omega|}\|r_q^*\|^q_{\dot{f}_q^{\alpha q}}$$
$$\le \frac{C}{|\Omega|}\|r\|^q_{\dot{f}_q^{\alpha q}} \le C'\|s\|_{\dot{f}_\infty^{\alpha q}}.$$

Every rectangle $Q \subset \Omega$ must be contained in a maximal rectangle in Ω, but this maximal rectangle is not unique; therefore

$$\frac{1}{|\Omega|}\int_\Omega \sum_{Q\subset\Omega} (|Q|^{-\frac{\alpha}{2}}(t_q^*)_Q\tilde{\chi}_Q)^q\,dx\,dy$$
$$\le \sum_{R\in\mu(\Omega)} \frac{|R|}{|\Omega|}\frac{1}{|R|}\int_R \sum_{Q\subset R} (|Q|^{-\frac{\alpha}{2}}(t_q^*)_Q\tilde{\chi}_Q)^q\,dx\,dy.$$

Let $R = R_1 \times R_2 \in \mu(\Omega)$, $R_1^i \supset R_1$, be the maximal dyadic interval satisfying $R_1^i \times R_2 \subset \Omega^i$, $i = 1, 2$. Suppose $\gamma_1(R) = |R_1^1|/|R_1|$, γR_1 is the γ dilation of R_1 with the same center. We can get $2\gamma_1(R)R_1 \subset 3R_1^1$ by a simple computation, so

$$2\gamma_1(R)R_1 \times R_2 \subset 3R_1^1 \times R_2 \subset \Omega^2.$$

Doing similar work for R_2, let $\gamma_2(R) = |R_2^1|/|R_2|$; then $R_1 \times 2\gamma_2(R)R_2 \subset R_1 \times 3R_2^1 \subset \Omega^2$.

For $Q = Q_1 \times Q_2 \not\subset \Omega^4$ with $l_i(Q) \le l_i(R)$, $i = 1, 2$, we have either $Q_1 \cap 3R_1^1 = \varnothing$ or $Q_1 \subset 3R_1^1$ by $3R_1^1$ being the union of three dyadic intervals. In the first case, we already have $Q \cap 3R_1^1 \times R_2 = \varnothing$; in the second case, we have either $Q_2 \times 3R_2^1 = \varnothing$ or $Q_2 \subset 3R_2^1$. This is equivalent to either $Q \cap 3R_1^1 \times 3R_2^1 = \varnothing$ or $Q \subset 3R_1^1 \times 3R_2^1$, but the latter case contradicts $Q \not\subset \Omega^4$ by $3R_1^1 \times 3R_2^1 \subset \Omega^4$. So we always have $Q \cap 3R_1^1 \times 3R_2^1 = \varnothing$. That is, either $Q_1 \cap 2\gamma_1(R)R_1 = \varnothing$ or $Q_2 \cap 2\gamma_2(R)R_2 = \varnothing$.

Fixing R, let a sequence $t_1^R = \{t_{1Q}^R\}_Q$ be defined by $t_{1Q}^R = s_Q$ if $Q_1 \cap 2\gamma_1(R)R_1 = \varnothing$ and $t_{1Q}^R = 0$ otherwise; also let a sequence $t_2^R = \{t_{2Q}^R\}_Q$ be defined by $t_{2Q}^R = s_Q$ if $Q_2 \cap 2\gamma_2(R)R_2 = \varnothing$ and $t_{2Q}^R = 0$ otherwise. By either $Q_1 \cap 2\gamma_1(R)R_1 = \varnothing$ or $Q_2 \cap 2\gamma_2(R)R_2 = \varnothing$, for $Q \not\subset \Omega^4$, $l_i(Q) \le l_i(R)$, $i = 1, 2$,

$$(t_q^*)_Q^q \le (t_{1q}^{R,*})_Q^q + (t_{2q}^{R,*})_Q^q.$$

Thus the quantity we are considering is

$$\leq \frac{1}{|\Omega|} \sum_{R\in\mu(\Omega)} |R| \left(\frac{1}{|R|} \int_R \sum_{Q\subset R} (|Q|^{-\frac{\alpha}{2}} (t_{1q}^R)_Q^* \tilde{\chi}_Q)^q \, dx \, dy \right.$$

$$\left. + \frac{1}{|R|} \int_R \sum_{Q\subset R} (|Q|^{-\frac{\alpha}{2}} (t_{2q}^R)_Q^* \tilde{\chi}_Q)^q \, dx \, dy) \right)$$

$$\leq \frac{C}{|\Omega|} \sum_{R\in\mu(\Omega)} |R| (\gamma_1^{-\lambda+1}(R) + \gamma_2^{-\lambda+1}(R)) \|t\|_{\dot{f}_\infty^{\alpha q}}$$

by the estimate for the rectangle case, where $\|t\|_{\dot{f}_\infty^{\alpha q}}$ can be controlled by $\|s\|_{\dot{f}_\infty^{\alpha q}}$. Now we need the famous Journé Lemma [5] to control it.

Lemma 3.2. *If $\delta > 0$, γ_i, $\mu^i(\Omega)$, $i = 1, 2$, are assumed as above, then*

$$\sum_{R\in\mu^{(2)}(\Omega)} |R|\gamma_1^{-\delta} \leq C_\delta |\Omega|,$$

$$\sum_{R\in\mu^{(1)}(\Omega)} |R|\gamma_2^{-\delta} \leq C_\delta |\Omega|,$$

where C_δ only depends on δ.

Now apply the Journé Lemma to our case. Having observed that $R \neq \widetilde{R}$ in $\mu^{(1)}(\Omega)$ if $R \neq \widetilde{R}$ in $\mu(\Omega)$,

$$\sum_{R\in\mu(\Omega)} |R|\gamma_i^{-\lambda+1}(R) \leq C_{\lambda-1}|\Omega|, \qquad i = 1, 2,$$

so we get the desired estimate

$$\frac{1}{|\Omega|} \int_\Omega \sum_{Q\subset\Omega} (|Q|^{-\frac{\alpha}{2}} (t_q^*)_Q \tilde{\chi}_Q)^q \, dx \, dy \leq C\|s\|^q_{\dot{f}_\infty^{\alpha q}}.$$

Lemma 3.1 is proved.

Theorem 3.3. *$S_\phi : \dot{F}_\infty^{\alpha q} \to \dot{f}_\infty^{\alpha q}$, $T_\psi : \dot{f}_\infty^{\alpha q} \to \dot{F}_\infty^{\alpha q}$ are bounded operators, and $T_\psi \circ S_\phi$ is identity on $\dot{F}_\infty^{\alpha q}$. The definition of $\dot{F}_\infty^{\alpha q}$ is independent on ϕ.*

The proof is similar to the case $p \neq \infty$.

We say sequence $r = \{r_Q\}_Q$ is a p_1-atom for $\dot{f}_p^{\alpha q}$ $(0 < p \leq 1, p \leq q \leq +\infty, p \leq p_1 < +\infty, \alpha \in R^1)$ if there exists a bounded open set Ω such that $r_Q \neq 0$ only if $Q \subset \Omega$ and $\|r\|_{\dot{f}_{p_1}^{\alpha q}} \leq |\Omega|^{1/p_1 - 1/p}$. We have the following

Theorem 3.4. *Let α, p, q as above; then*

$$\|s\|_{\dot{f}_p^{\alpha, q}} \sim \inf\left\{ \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^p \right)^{\frac{1}{p}} \middle| s = \sum \lambda_k r_k,\ r_k \textit{ is a } p_1\textit{-atom for } \dot{f}_p^{\alpha q} \right\}.$$

Proof. Let

$$G^{\alpha q}(s)(x, y) = \left(\sum_Q (|Q|^{-\frac{\alpha}{2}} |s_Q| \tilde{\chi}_Q)^q \right)^{\frac{1}{q}},$$

$$\Omega_k = \{(x, y) \in R^2 ; G^{\alpha q}(s)(x, y) \geq 2^k\},$$

$$R_k = \{\text{rectangle } R ; |R \cap \Omega_{k+1}| \leq \tfrac{1}{2}|R|, |R \cap \Omega_k| > \tfrac{1}{2}|R|\}.$$

Having observed that $\cdots \supset \Omega_k \supset \Omega_{k+1} \supset \cdots$, there exists one and only one k such that $R \in R_k$. Let the sequence $r_k = \{r_{kQ}\}_Q$ be defined by

$$r_{kQ} = \frac{s_Q}{c2^{k+1}|\widetilde{\Omega}_k|^{\frac{1}{p}}} \quad \text{if } Q \in R_k,$$
$$r_{kQ} = 0 \quad \text{if } Q \notin R_k,$$

where $\widetilde{\Omega}_k = \{(x, y) ; M_S(\chi_{\Omega_k})(x, y) > \frac{1}{2}\}$ and c will be determined later and is independent of k, Q, Ω. By definition, r_k is supported on $\bigcup_{R\in R_k} R \subset \widetilde{\Omega}_k$. Let us estimate the norm of r_k. Putting $E_Q = Q\backslash\Omega_{k+1}$, we get

$$\chi_Q \leq 2^{-\frac{1}{A}} M_S(\chi_{E_Q}^A)^{\frac{1}{A}}.$$

So

$$\|r_k\|_{\dot{f}_{p_1}^{\alpha q}} \leq 2^{-\frac{1}{A}} \left\| \sum_{Q\in R_k} (|Q|^{-\frac{\alpha}{2}-\frac{1}{2}}|r_Q| M_S(\chi_{E_Q}^A)^{\frac{1}{A}})^q)^{\frac{1}{q}} \right\|_{L^{p_1}}$$
$$= 2^{-\frac{1}{A}} \left\| \left(\sum_{Q\in R_k} (M_S(|Q|^{-\frac{\alpha}{2}-\frac{1}{2}}|r_Q|\chi_{E_Q})^A)^{\frac{q}{A}} \right)^{\frac{A}{q}} \right\|_{L^{p_1/A}}^{1/A}.$$

Choose A such that $p_1/A > 1$, $q/A > 1$; then by the vector-valued maximal inequality,

$$\|r_k\|_{\dot{f}_{p_1}^{\alpha q}} \leq C'2^{-\frac{1}{A}} \left\| \left(\sum_{Q\in R_k} (|Q|^{-\frac{\alpha}{2}-\frac{1}{2}}|r_Q|\chi_{E_Q})^q \right)^{\frac{1}{q}} \right\|_{L^{p_1}}$$
$$\leq \frac{c'2^{-\frac{1}{A}}}{c2^{k+1}|\widetilde{\Omega}_k|^{\frac{1}{p}}} \left\| \left(\sum_{Q\in R_k} (|Q|^{-\frac{\alpha}{2}-\frac{1}{2}}|s_Q|\chi_{E_Q})^q \right)^{\frac{1}{q}} \right\|_{L^{p_1}}.$$

Now choose $c = c'2^{-\frac{1}{A}}$ and by the definition of $G^{\alpha q}$

$$\|r_k\|_{\dot{f}_{p_1}^{\alpha q}} \leq |\widetilde{\Omega}_k|^{1/p_1 - 1/p}.$$

In the above, we have written r_{kQ} as r_Q. From this, we already have an atomic decomposition of s,

$$s = \sum \lambda_k r_k, \quad \lambda_k = c2^{k+1}|\widetilde{\Omega}_k|^{\frac{1}{p}},$$

with

$$\begin{aligned}\sum |\lambda_k|^p &= c^p \sum_k 2^{(k+1)p} |\widetilde{\Omega}_k| \\ &\le C'' \sum_k 2^{(k+1)p} (|\Omega_k \backslash \Omega_{k+1}| + |\Omega_{k+1} \backslash \Omega_{k+2}| + \cdots) \\ &= c'' \sum_k \left(\sum_{j \le k} 2^{(j+1)p} \right) |\Omega_k \backslash \Omega_{k+1}| \\ &\le c'' \sum_k 2^{(k+1)} |\Omega_k \backslash \Omega_{k+1}|.\end{aligned}$$

By the definition of $G^{\alpha q}$, this is less than $c'' \|G^{\alpha q}(s)\|_{L^p}^p$. So the right side in the theorem is less than $c'' \|s\|_{\dot{f}_p^{\alpha q}}$. For the converse, we use the following inequality whose proof is exactly similar to [7],

$$\|s + t\|_{\dot{f}_p^{\alpha q}}^p \le \|s\|_{\dot{f}_p^{\alpha q}}^p + \|t\|_{\dot{f}_p^{\alpha q}}^p.$$

Theorem 3.5. *Let* $\alpha \in R$, $1 < q < +\infty$; *then we have*

$$(\dot{f}_1^{\alpha q})^* \sim \dot{f}_{+\infty}^{-\alpha q'}, \qquad \frac{1}{q} + \frac{1}{q'} = 1.$$

Generally, $t = \{t_Q\}_Q \in \dot{f}_{+\infty}^{-\alpha q'}$; *then* $l_t : s \to \langle s, t\rangle = \sum_Q s_Q \overline{t_Q}$ *defines a continuous functional on* $\dot{f}_1^{\alpha q}$, $\|l_t\|_{(\dot{f}_1^{\alpha q})^*} \sim \|t\|_{\dot{f}_{+\infty}^{-\alpha q'}}$ *and every* $l \in (\dot{f}_1^{\alpha q})^*$ *has the form of* $l(s) = \langle s, t\rangle$ *for some* $t \in \dot{f}_{+\infty}^{-\alpha q'}$.

Proof. Let $s \in \dot{f}_1^{\alpha q}$, $t \in \dot{f}_{+\infty}^{-\alpha q'}$. Then

$$\begin{aligned}|\langle s, t\rangle| &= \left| \sum_Q s_Q \overline{t_Q} \right| = \left| \sum_k \sum_{Q \in R_k} s_Q \overline{t_Q} \right| \\ &\le 2 \sum_k \int \sum_{Q \in R_k} |Q|^{-\frac{\alpha}{2} - \frac{1}{2}} |s_Q| \chi_{E_Q} |Q|^{\frac{\alpha}{2} - \frac{1}{2}} |t_Q| \chi_Q \, dx\, dy,\end{aligned}$$

where the definitions of R_k, $G^{\alpha q}(s)$, Ω_k, E_Q are as above. Using Hölder's inequality

$$|\langle s, t\rangle| \le 2 \sum_k \int \left(\sum_{Q \in R_k} (|Q|^{-\frac{\alpha}{2} - \frac{1}{2}} |s_Q| \chi_{E_Q})^q \right)^{\frac{1}{q}} \left(\sum_{Q \in R_k} (|Q|^{\frac{\alpha}{2} - \frac{1}{2}} |t_Q| \chi_Q)^{q'} \right)^{\frac{1}{q'}} dx\, dy.$$

By $(\sum_{Q \in R_k} (|Q|^{\frac{\alpha}{2} - \frac{1}{2}} |s_Q| \chi_{E_Q})^q)^{\frac{1}{q}} \le G^{\alpha q}(s) \chi_{\widetilde{\Omega}_k \backslash \Omega_{k+1}} \le 2^{k+1} \chi_{\widetilde{\Omega}_k}$ and

$$\begin{aligned}|\langle s, t\rangle| &\le 2 \sum_k 2^{k+1} \left(\int_{\widetilde{\Omega}_k} \sum_{Q \in R_k} (|Q|^{-\frac{\alpha}{2} - \frac{1}{2}} |t_Q| \chi_Q)^{q'} \, dx\, dy \right)^{\frac{1}{q'}} \left(\int_{\widetilde{\Omega}_k} dx\, dy \right)^{\frac{1}{q}} \\ &\le 2 \|t\|_{\dot{f}_{+\infty}^{-\alpha q'}} \sum_k 2^{k+1} |\widetilde{\Omega}_k|^{\frac{1}{q'}} |\widetilde{\Omega}_k|^{\frac{1}{q}}\end{aligned}$$

we have estimate

$$\sum_k 2^k |\widetilde{\Omega}_k| \le C \|s\|_{\dot{f}_1^{\alpha q}},$$

as in the proof of Theorem 3.4. Thus $\dot{f}^{-\alpha q}_{+\infty} \subset (\dot{f}^{\alpha q}_{1})^*$. For the other inclusion, the proof is the same as [7].

Using the techniques for sequence spaces in [7], we can also get the atomic decomposition of $\dot{F}^{\alpha q}_{p}$ and the duality properties, but we will not write the details.

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