THE DE BRANGES-ROVNYAK MODEL
WITH FINITE-DIMENSIONAL COEFFICIENTS

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ABSTRACT. A characterization in terms of the canonical model spaces of L. de Branges and J. Rovnyak is obtained for Hilbert spaces of formal power series with vector coefficients which satisfy a difference-quotient inequality, thereby extending the closed ideal theorems of A. Beurling and P. D. Lax.

1. Introduction

This paper extends the well-known invariant subspace characterization of A. Beurling [3] and P. D. Lax [11] for the shift on the Hardy space of square summable power series with vector coefficients (cf. [10, 13–15]). The focus is instead on certain (not necessarily orthogonal) complements of contractively contained invariant manifolds of the shift. These are the spaces $H(B)$ of L. de Branges and J. Rovnyak [6–8]. In the Beurling-Lax theory, the key point is a dimension inequality. The inequality is trivial when the coefficient space has infinite dimension, so the essential content is in the finite-dimensional case. Previously only special cases of the more abstract problem have been treated [6, 9], but our methods generalize an argument from [7, Theorem 6]. The main difficulty again comes down to a dimension inequality in the finite-dimensional case. The purpose here is to derive new results on the structure of $H(B)$ spaces which reveal what is needed for the inequality to hold. As a consequence, we obtain a complete characterization of the spaces $H(B)$.

2. $H(B)$ Spaces

A basic concept in the de Branges-Rovnyak theory is complementation: A Hilbert space $F$ is contained contractively in a Hilbert space $H$ if $F$ is a submanifold of $H$ and if the inclusion map of $F$ into $H$ is a contraction. If $F$ is contained contractively in $H$, then the space complementary to $F$ in $H$ is the Hilbert space $G$ of elements $g$ of $H$ with the property that

$$\|g\|^2_G = \sup\{\|g + f\|^2_F - \|f\|^2_F : f \in F\}$$

is finite. The space $G$ is contained contractively in $H$. Moreover, $G$ is the unique Hilbert space such that the inequality $\|k\|^2_H \leq \|f\|^2_F + \|g\|^2_G$ holds whenever $k = f + g$ is a decomposition of $k$ in $H$ into $f$ in $F$ and $g$ in $G$.

Received by the editors May 6, 1993 and, in revised form, January 3, 1994; originally communicated to the Proceedings of the AMS by Palle E. T. Jorgensen.

1991 Mathematics Subject Classification. Primary 46E22, 47A45.

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0002-9947/95 $1.00 + $.25 per page
and such that every element \( k \) in \( \mathcal{H} \) admits a decomposition for which equality holds.

Let \( C \) be a finite-dimensional Hilbert space, and let \( \mathcal{H} \) be a Hilbert space of formal power series \( f(z) \) whose coefficients are in \( C \) such that

\[
(1) \quad \| (f(z) - f(0))/z \|^2 \leq \| f(z) \|^2 - \| f(0) \|^2.
\]

Then \( \mathcal{H} \) is contained contractively in \( C(z) \), the Hilbert space of square summable power series \( \sum a_n z^n \) with \( a_n \) in \( C \) and norm given by \( \| \sum a_n z^n \|_{C(z)}^2 = \sum |a_n|^2 \).

Let \( B(z) \) be a power series whose coefficients are operators on \( C \) such that

\[
\| B(z) f(z) \|_{C(z)} \leq \| f(z) \|_{C(z)} \quad \text{whenever} \quad f(z) \in C(z).
\]

Cauchy multiplication by \( B(z) \) thus defines a contraction operator on \( C(z) \) which will be denoted by \( T_B \). The range \( \mathcal{M}(B) \) of \( T_B \) becomes a Hilbert space in the unique norm with the property that \( \| T_B f \|_{\mathcal{M}(B)} = \| f \|_{C(z)} \) whenever \( f \) is orthogonal to the kernel of \( T_B \). Furthermore, \( \mathcal{M}(B) \) is contained contractively in \( C(z) \), and multiplication by \( z \) is a contraction on \( \mathcal{M}(B) \).

The de Branges-Rovnyak space \( \mathcal{H}(B) \) is defined to be the complementary space to \( \mathcal{M}(B) \) in \( C(z) \). The space \( \mathcal{H}(B) \) satisfies (1) and is an underlying space for canonical models of contractions on Hilbert space [1, 2, 12, 16, 17].

Multiplication by \( z \) is a contraction on the space \( \mathcal{M} \) complementary to \( \mathcal{H} \) in \( C(z) \). In [6] (cf. [5, Theorem 6]), de Branges extended the Beurling-Lax theorem by showing that if multiplication by \( z \) is isometric on \( \mathcal{M} \), then \( \mathcal{H} \) is isometrically equal to a space \( \mathcal{H}(B) \). It should be further noted that when \( C \) is infinite dimensional, any space \( \mathcal{H} \) which satisfies (1) is isometrically equal to a space \( \mathcal{H}(B) \) [4, Theorem 11].

Let \( \mathcal{H}(B) \) be a given space. Then \( \mathcal{H}(B) \) is also contained contractively in \( \mathcal{H}(zB) \). The space \( \mathcal{H}(zB) \) may be obtained as those elements \( h(z) \) of \( C(z) \) such that \( h(z) - h(0)/z \) is in \( \mathcal{H}(B) \) and \( \| h(z) \|_{\mathcal{H}(zB)} = \| (h(z) - h(0))/z \|_{\mathcal{H}(B)} + \| h(0) \|_{C(z)}^2 \). The complementary space to \( \mathcal{H}(B) \) in \( \mathcal{H}(zB) \) is the space \( B(z) C \) with \( \| B(z) c \|_{B(z) C} = |c|_{C(z)} \) for every \( c \) orthogonal to \( C \cap \ker T_B \). Let us define linear transformations \( J_\pm \) from \( \mathcal{H}(B) \) into \( C \), with ranges denoted \( C_\pm \), as follows:

\[
J_+ f = f(0) \quad \text{and} \quad J_- \text{ is the operator whose adjoint is given by } J_- c = (B(z) - B(0))c/z.
\]

Let \( B(z) = \sum B_n z^n \), and let \( \overline{B}_n \) be the adjoint of \( B_n \) on \( C \). Then \( J_- c = [1 - B(z) \overline{B}(0)]c \); and since \( C \) is finite dimensional, \( C_+ = (1 - B_0 \overline{B}_0) C \) and \( C_- = (\bigvee_{n \geq 1} \overline{B}_n) C \subseteq (1 - \overline{B}_0 B_0) C \).

Let \( R(0) \) denote the difference-quotient transformation on \( \mathcal{H}(B) \), which maps \( f(z) \) into \( [f(z) - f(0)]/z \). Then \( R(0) f(z) = z f(z) - B(z) J_- f \) so that \( [1 - R(0) R(0)^*] f(z) = (B(z) - B(0)) J_- f / z \). Let \( J_+ f = [1 - R(0) R(0)^*] f(z) = (J_- f) + B(z) J_- R(0) f \). Note that if \( [1 - R(0) R(0)^*] f(z) = c + B(z) c_- \) with \( c \) in \( C \) and \( c_- \) in \( C_- \), then necessarily \( c = J_+ f \) and \( c_- = J_- R(0) f \). Therefore, since \( \dim C \) is finite,

\[
\rank [1 - R(0) R(0)^*] = \dim \langle J_+ f, J_- R(0) f \rangle : f \in \mathcal{H}(B) \rangle
\]

\[
(2) \quad \geq \dim C_+ = \rank (1 - \overline{B}_0 B_0)
\]

\[
(3) \quad \geq \dim C_- = \rank [1 - R(0) R(0)^*].
\]

More precisely, the following will turn out to be a defining property of the spaces \( \mathcal{H}(B) \).
Theorem 1. Let $R(0)$ be the difference-quotient transformation on a given space $\mathcal{H}(B)$. Then

$$\text{rank}[1 - R(0) R(0)^*] = \dim\{c \in \mathcal{C} : B(z)c \in \mathcal{H}(B)\} + \text{rank}[1 - R(0) R(0)^*].$$

Proof. Suppose that $B(z)c$ is in $\mathcal{H}(B)$. Then $c = (J - f) + d$ where $f$ is in $\mathcal{H}(B)$ and $[B(z) - B(0)]d/z = 0$. Moreover,

$$(1) \quad [1 - R(0) R(0)^*] [R(0)^*f] + B(z)c = (B_0^0 d) + B(z)J_f.$$

Let $J_{-f_1}, \ldots, J_{-f_{s_0}}$ be a basis for the subspace $\mathcal{C}_- = \{c \in \mathcal{C} : B(z)c \in \mathcal{H}(B)\}$, and let $J_{+g_1}, \ldots, J_{+g_t}$ be a basis for $\mathcal{C}_+$ where $f_i$ and $g_j$ are in $\mathcal{H}(B)$ for all $i$ and $j$. Suppose that there are constants $\lambda_1, \ldots, \lambda_{t_0+t}$ such that

$$0 = \sum_{i=1}^{s_0} \lambda_i [1 - R(0) R(0)^*] [R(0)^*f_i] + B(z)J_{-f_i}$$

$$+ \sum_{j=1}^{t} \lambda_{t_0+j} [1 - R(0) R(0)^*] g_j.$$

Equivalently by (1) we have

$$0 = \left( \sum_{i=1}^{s_0} \lambda_{t_0+i} J_{+g_j} \right) + B(z)J_+ \left[ \sum_{i=1}^{s_0} \lambda_{t_0+i} R(0)g_j + \sum_{i=1}^{s_0} \lambda_i f_i \right]$$

so that $\sum_{i=1}^{s_0} \lambda_{t_0+i} J_{+g_j} = 0$ and hence $\lambda_{t_0+j} = 0 \quad (j = 1, \ldots, t)$. It follows that $\sum \lambda_i J_{-f_i} = 0$ and thus $\lambda_i = 0$ for all $i$. Therefore,

$$\text{rank}[1 - R(0) R(0)^*] \geq s_0 + t.$$

Let $c_i = J_{-f_i} (i = 1, \ldots, s_0)$ and expand $\{c_i\}$ to a basis $c_1, \ldots, c_s$ of $\{c \in \mathcal{C} : B(z)c \in \mathcal{H}(B)\}$. For every $j > s_0$ let us write $c_j = (J_{-f_j}) + d_j$ as above where $f_j$ is in $\mathcal{H}(B)$ and $d_j$ is orthogonal to $\mathcal{C}_-$. By (1), $B_0^0 d_j$ is in $\mathcal{C}_+$, so it is in $(B_0^0 \mathcal{C}) \cap (1 - B_0 B_0^0) \mathcal{C}$. But since $\mathcal{C}$ is finite dimensional, it follows that this intersection coincides with $B_0^0 (1 - B_0 B_0^0) \mathcal{C}$, and hence $B_0^0 d_j = B_0^0 e_j$ where $e_j$ is in $(1 - B_0 B_0^0) \mathcal{C}$. Thus $d_j - e_j$ is in ker $B_0$, which is also contained in $(1 - B_0 B_0^0) \mathcal{C}$, and consequently $d_j$ is in $((1 - B_0 B_0^0) \mathcal{C}) \oplus \mathcal{C}_-$. Now $\{d_j : j > s_0\}$ is linearly independent: For suppose $\sum \alpha_j d_j = 0$. Then $\sum_{j > s_0} \alpha_j c_j = \sum_{j > s_0} \alpha_j J_{-f_j}$ is in $\mathcal{C}_+$, so there exist $\beta_i$ such that $\sum_{j > s_0} \alpha_j c_j = \sum_{i \leq s_0} \beta_i c_i$. Since $\{c_i\}$ is linearly independent, $\alpha_j = 0$ for all $j$, and hence

$$t = \dim \mathcal{C}_+ = \text{rank}(1 - B_0 B_0^0) = \text{rank}(1 - B_0 B_0^0)$$

$$= \dim\{(1 - B_0 B_0^0) \mathcal{C} \oplus \mathcal{C}_-\} + \dim \mathcal{C}_-$$

$$\geq (s - s_0) + \text{rank}[1 - R(0) R(0)^*].$$

In conjunction with (5) we have

$$\text{rank}[1 - R(0) R(0)^*] \geq s + \text{rank}[1 - R(0) R(0)^*].$$

To verify the reverse inequality, it suffices to show that there exist $r = \text{rank}[1 - R(0) R(0)^*] - \text{rank}[1 - R(0) R(0)^*]$ linearly independent vectors $a_i$ in $\mathcal{C}$ such that $B(z)a_i$ is in $\mathcal{H}(B)$. By inequalities (2) and (3), it follows that $r = r_0 + r_1$ where $r_0 = \text{rank}[1 - R(0) R(0)] - \dim \mathcal{C}_+$ and $r_1 = \dim\{\text{ran}(1 - B_0 B_0^0) \mathcal{C} \oplus \mathcal{C}_-\}$. 


Suppose that \( r_0 > 0 \) and recall the basis \( \{ J^+ g_j \} \) of \( \mathbb{C}_+ \). As above, \( \{ [1 - R(0)^* R(0)] g_j \} \) is linearly independent, so if \( \mathcal{F} \) is its span, then there are \( r_0 \) vectors \( \{ [1 - R(0)^* R(0)] \hat{g}_i \} \) with \( \hat{g}_i \) in \( \mathcal{H}(B) \), which form a basis of \( \text{ran}[1 - R(0)^* R(0)] \otimes \mathcal{F} \). Now there exist constants \( \lambda_{ij} \) such that \( J^+ \hat{g}_i = \sum_{j=1}^{r_0} \lambda_{ij} J^+ g_j \) for each \( i \). Let us define \( a_i = J_- R(0)(\hat{g}_i - \sum_j \lambda_{ij} g_j) \) for \( i = 1, \ldots, r_0 \). Then \( B(z) a_i = [1 - R(0)^* R(0)](\hat{g}_i - \sum_j \lambda_{ij} g_j) \) is in \( \mathcal{H}(B) \), and \( \{ a_1, \ldots, a_{r_0} \} \) is linearly independent: Suppose that \( \sum \mu_i a_i = 0 \). Then

\[
\sum \mu_i [1 - R(0)^* R(0)] g_i = \sum \mu_i [1 - R(0)^* R(0)] \left( \sum_j \lambda_{ij} g_j \right)
\]

which must be zero since it is in both \( \mathcal{F} \) and \( \mathcal{F}^\perp \). Therefore \( \mu_i = 0 \) for every \( i \).

Next, suppose that \( r_1 > 0 \) and let \( \hat{d}_1, \ldots, \hat{d}_{r_1} \) be a basis of \( \text{ran}(1 - B_0 B_0) \otimes \mathbb{C}_- \). Then \( B(z) \hat{d}_j = B_0 \hat{d}_j \) and \( \hat{d}_j = (1 - B_0 B_0) b_j \) for some \( b_j \) in \( \mathbb{C}_- \). Let \( \hat{f}_j(z) = [1 - B(z) B(0)] B_0 b_j \) and define \( a_{r_0+j} = \hat{d}_j + J_- R(0) \hat{f}_j \) for \( j = 1, \ldots, r_1 \). Then \( B(z) a_{r_0+j} = [1 - R(0)^* R(0)] \hat{f}_j \) is in \( \mathcal{H}(B) \).

Finally, \( \{ a_i : i = 1, \ldots, r = r_0 + r_1 \} \) is linearly independent: Suppose that there are constants \( \nu_1, \ldots, \nu_r \) such that

\[
0 = \sum_{i=1}^{r_0} \nu_i a_i = \sum_{i=1}^{r_0} \nu_i a_i + \sum_{j=1}^{r_1} \nu_{r_0+j} [\hat{d}_j + J_- R(0) \hat{f}_j].
\]

It follows that \( \sum_{i=1}^{r_0} \nu_{r_0+i} \hat{d}_j = 0 \) since \( a_i \) \( (1 \leq i \leq r_0) \) and \( J_- R(0) \hat{f}_j \) \( (1 \leq j \leq r_1) \) are in \( \mathbb{C}_- \), and \( \hat{d}_j \) is orthogonal to \( \mathbb{C}_- \) for every \( j \). Therefore \( \nu_{r_0+j} = 0 \) \( (j = 1, \ldots, r_1) \), and consequently \( \sum_{i=1}^{r_0} \nu_i a_i = 0 \) so that \( \nu_i = 0 \) for all \( i \).

\section{3. The characterization}

Let \( \mathcal{H} \) be a space which satisfies (1), and let \( \mathcal{H}' \) be the Hilbert space of all power series \( h(z) \) such that \( [h(z) - h(0)]/z \) is in \( \mathcal{H} \) with \( \| h(z) \|_{\mathcal{H}'}^2 = \| [h(z) - h(0)]/z \|_{\mathcal{H}}^2 + \| h(0) \|_{\mathcal{H}}^2 \). Then \( \mathcal{H}' \) satisfies (1), and \( \mathcal{H} \) is contained contractively in \( \mathcal{H}' \). Let \( \mathcal{R} \) be the complementary space to \( \mathcal{H} \) in \( \mathcal{H}' \), and let \( i_{\mathcal{H}} \) and \( i_{\mathcal{R}} \) denote the respective inclusion maps of \( \mathcal{H} \) and \( \mathcal{R} \) into \( \mathcal{H}' \). Then every \( h \) in \( \mathcal{H}' \) admits the unique decomposition \( h = (i_{\mathcal{H}}^* h) + (i_{\mathcal{R}}^* h) \) where \( \| h \|_{\mathcal{H}'}^2 = \| i_{\mathcal{H}}^* h \|_{\mathcal{H}}^2 + \| i_{\mathcal{R}}^* h \|_{\mathcal{R}}^2 \).

A fundamental result from the theory of \( \mathcal{H}(B) \) spaces is: \( \mathcal{H} \) is isometrically equal to a space \( \mathcal{H}(B) \) if and only if the dimension of \( \mathcal{R} \) does not exceed the dimension of \( \mathcal{C} \) [6]. More generally, if \( \mathcal{C} \subset \mathbb{C} \) and \( \dim \mathcal{R} \leq \dim \mathcal{C} \), then \( \mathcal{H} \) is a space \( \mathcal{H}(B) \) where the coefficients of \( B(z) \) act on \( \mathcal{C} \).

\textbf{Lemma.} Let \( \mathcal{I} \) be the subspace of elements of \( \mathcal{H} \) for which equality holds in (1). Then \( \mathcal{R} \) and \( \mathcal{R} \cap \mathcal{I} \) are contained in \( \mathcal{H}' \oplus \mathcal{I} \) and \( \mathcal{H} \oplus \mathcal{I} \) respectively. Moreover, \( \dim \mathcal{R} = \dim \mathcal{H}' \oplus \mathcal{I} \) and \( \dim \mathcal{H} \oplus \mathcal{R} = \dim \mathcal{H} \oplus \mathcal{I} \).

\textbf{Proof.} As in [9], \( \mathcal{I} \) is a (closed) subspace of \( \mathcal{H} \) and is contained isometrically in \( \mathcal{H}' \). Therefore for any \( f \) in \( \mathcal{I} \) and \( g \) in \( \mathcal{R} \), we have

\[
(f, g)_{\mathcal{H}'} = (f, i_{\mathcal{R}} g)_{\mathcal{H}'} = (i_{\mathcal{R}}^* f, g)_{\mathcal{R}} = (0, g)_{\mathcal{R}} = 0.
\]

Hence \( \mathcal{R} \) is a subset of \( \mathcal{H}' \oplus \mathcal{I} \).
The restriction of $i_{\mathcal{H}'}$ to $\mathcal{H}' \oplus \mathcal{I}$ is linear and continuous and has trivial kernel: if $i_{\mathcal{H}'}^* h = 0$ for some $h$ in $\mathcal{H}' \oplus \mathcal{I}$, then $i_{\mathcal{H}'}^* h = h$, so $h$ is also in $\mathcal{I}$, and thus $h = 0$. It follows that $\dim \mathcal{H}' \oplus \mathcal{I} = \dim i_{\mathcal{H}'}^*(\mathcal{H}' \oplus \mathcal{I}) \leq \dim \mathcal{H}'$, and hence $\dim \mathcal{H}' = \dim \mathcal{H}' \oplus \mathcal{I}$.

Next, let $g$ be in $\mathcal{H} \cap \mathcal{A}$. Then $g$ is in $\mathcal{H}' \oplus \mathcal{I}$ but also in $\mathcal{H} \oplus \mathcal{I}$ since for any $f$ in $\mathcal{I}$

$$\langle f, g \rangle_{\mathcal{H}} = \langle i_{\mathcal{H}'}^* f, g \rangle_{\mathcal{H}'} = \langle f, i_{\mathcal{H}'} g \rangle_{\mathcal{H}'} = \langle f, g \rangle_{\mathcal{I}} = 0.$$ Therefore $(\mathcal{H} \cap \mathcal{A}) \subseteq (\mathcal{H} \oplus \mathcal{I})$. Finally $\dim \mathcal{H} \cap \mathcal{A} = \dim \mathcal{H} \oplus \mathcal{I}$ as above since $i_{\mathcal{H}'}(\mathcal{H} \oplus \mathcal{I})$ is contained in $\mathcal{H} \cap \mathcal{A}$. □

The following will distinguish the spaces $\mathcal{H}(B)$.

**Corollary 1.** Let $\mathcal{I}(B)$ be the subspace of elements of a given space $\mathcal{H}(B)$ for which equality holds in (1). Then

$$\dim \mathcal{H}(B) = \dim \mathcal{H}(B)_{\mathcal{I}(B)} + \text{rank}[1 - R(0)^* R(0)].$$

**Proof.** Since $B(z)c$ is finite dimensional, the lemma implies that $\mathcal{H}(B) \oplus \mathcal{I}(B)$ coincides with $\mathcal{H}(B) \cap B(z)c$. By (1), the kernel of $1 - R(0)^* R(0)$ is contained in $\mathcal{I}(B)$ and is exactly the kernel of the restriction of $J_+$ to $\mathcal{I}(B)$. Thus since $1 - R(0)^* R(0)$ has finite rank and

$$J_+ \mathcal{I}(B) = J_+ \{\text{ran}[1 - R(0)^* R(0)] \cap \mathcal{I}(B)\},$$

it follows that

$$\text{rank}[1 - R(0)^* R(0)] = \dim \{\text{ran}[1 - R(0)^* R(0)] \cap \mathcal{I}(B)\} + \dim \mathcal{I}(B) \oplus \mathcal{I}(B)$$

$$= \dim \mathcal{I}(B)_{\mathcal{I}(B)} + \dim[\mathcal{H}(B) \cap B(z)c].$$

The corollary now follows from Theorem 1 since we also have

$$\text{rank}[1 - R(0)^* R(0)] = \dim \mathcal{H}(B)_{\mathcal{I}(B)} + \text{dim}[\mathcal{H}(B) \cap B(z)c]$$

$$+ \text{rank}[1 - R(0) R(0)^*].$$ □

By [7, Lemma 4], equality holds in (1) for a given space $\mathcal{H}(B)$ if and only if $\mathcal{H}(B)$ contains no nonzero element of the form $B(z)c$ with $c$ in $\mathcal{C}$. An immediate consequence of the above results is

**Corollary 2.** Let $\mathcal{H}(B)$ be a given space. Then $\text{rank}[1 - R(0)^* R(0)] = \text{rank}[1 - R(0) R(0)^*]$ if and only if equality holds in (1) for every $f(z)$ and there is no nonzero vector $c$ such that $B(z)c = 0$.

We now have the proposed characterization.

**Theorem 2.** Let $\mathcal{H}$ be a Hilbert space of formal power series which satisfies (1), and let $\mathcal{I}$ be the subspace of those series for which equality holds in (1). Then $\mathcal{H}$ is isometrically equal to a space $\mathcal{H}(B)$ if and only if the dimension of the space of constant coefficients of elements of $\mathcal{I}$ is at least the rank of $1 - TT^*$ where $T$ is the difference-quotient transformation on $\mathcal{H}$.

**Proof.** Any space $\mathcal{H}(B)$ has the stated property by Corollary 1.

Conversely, suppose that $\mathcal{H}$ is a space which satisfies (1) and the dimension hypothesis. Let $\mathcal{H}'$, $\mathcal{A}$, $i_{\mathcal{H}}$ and $i_{\mathcal{A}}$ be defined as above, and let $f(z)$ and $g(z)$ be in $\mathcal{H}$. Since
it follows that \( T^*f(z) = i_{\mathcal{H}}^*zf(z) \).

Let \( S \) denote the difference-quotient transformation on \( \mathcal{H}' \). Then
\[
(1 - TT^*)f(z) = f(z) - Ti_{\mathcal{H}}^*zf(z) = f(z) - S[zf(z) - i_{\mathcal{H}}^*zf(z)] = Si_{\mathcal{H}}^*zf(z).
\]

More generally, \( \mathcal{R} \) is contained in the range of \( 1 - TT^* \): Let \( g(z) \) be in \( \mathcal{R} \) such that \( g(z) \) is orthogonal to \( i_{\mathcal{H}}^*zf(z) \) for every \( f(z) \) in \( \mathcal{H} \). Then
\[
0 = (g(z), i_{\mathcal{H}}^*zf(z))_{\mathcal{H}} = (g(z), zf(z))_{\mathcal{H}'} = (Sg(z), f(z))_{\mathcal{H}}
\]
for every \( f(z) \) in \( \mathcal{H} \). Letting \( f(z) = Sg(z) \), we conclude that \( g(z) \) is constant. Hence \( S\mathcal{R} = S \cup \{i_{\mathcal{H}}^*zf(z) : f(z) \in \mathcal{H}\} \), which is contained in \( (1 - TT^*)\mathcal{H} \) since the rank of \( 1 - TT^* \) is finite by the hypothesis.

It follows that \( \mathcal{R} \) is finite dimensional since
\[
\dim \mathcal{R} \leq \dim S\mathcal{R} + \dim \ker S \leq \text{rank}(1 - TT^*) + \dim \mathcal{C}.
\]

Thus by the lemma \( \mathcal{R} = \mathcal{H}' \oplus \mathcal{F} \).

Furthermore, since \( \mathcal{H}' \) contains \( \mathcal{C} \), the kernel of the restriction of \( S \) to \( \mathcal{H}' \oplus \mathcal{F} \) is \( \mathcal{C} \oplus \{f(0) : f(z) \in \mathcal{F}\} \). Hence, we have that
\[
\dim \mathcal{R} = \dim(\mathcal{C} \oplus \{f(0) : f(z) \in \mathcal{F}\}) + \dim S\mathcal{R} \\
\leq \dim \mathcal{C} - \dim \{f(0) : f(z) \in \mathcal{F}\} + \text{rank}(1 - TT^*) \\
\leq \dim \mathcal{C}
\]
by the hypothesis. Therefore, \( \mathcal{H} \) is isometrically equal to a space \( \mathcal{H}(B) \).

Finally, any space which satisfies (1) is at least a reducing subspace of \( R(0) \) on some space \( \mathcal{H}(B) \).

**Corollary 3.** Let \( \mathcal{H} \), \( \mathcal{F} \) and \( T \) be defined as in Theorem 2, but assume on the other hand that
\[
\delta = \text{rank}(1 - TT^*) - \dim \{f(0) : f(z) \in \mathcal{F}\}
\]
is finite and positive. If \( \mathcal{C} \) is any Hilbert space with dimension at least \( \delta \), then \( \mathcal{H} \oplus \mathcal{C}(z) \) is isometrically equal to a space \( \mathcal{H}(B) \).

**Acknowledgment**

I am grateful to the referee for suggesting many improvements in this article.

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